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ORBIFOLD GROUPS, QUASI-PROJECTIVITY AND COVERS

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Abstract. We discuss properties of complex algebraic orbifold groups, their characteristic varieties, and their abelian covers. In particular, we deal with the question of (quasi)-projectivity of orbifold groups. We also prove a structure theorem for the variety of characters of normal-crossing quasi-projective orbifold groups. Finally, we extend Sakuma's formula for the first Betti number of abelian covers of orbifold fundamental groups. Several examples are presented, including a compact orbifold group which is not projective and a Zariski pair of plane curves in $\mathbb{P}^2$ that can be told by considering an unbranched cover of $\mathbb{P}^2$ with an orbifold structure.

Introduction

Any finitely presented group G is the fundamental group of a closed oriented 4-manifold. If we ask these manifolds to have extra-properties, some restrictions may apply. For example, such a group is said to be *Kähler* (resp. *projective*) if it is the fundamental group of a compact Kähler manifold (resp. a projective manifold). Note that projective groups are Kähler groups, but the converse is still an open question posed by Serre. In this direction, it is worth mentioning that there exist compact Kähler manifolds whose homotopy type cannot be realized by a smooth projective manifold (cf. [25]).

The family of projective groups is a subfamily of *quasi-projective* groups. Recall that a quasi-projective manifold is the difference of two projective varieties. The study of Kähler, projective and quasi-projective groups is closely related to orbifold groups, or more precisely to orbicurve groups, i.e. orbifold fundamental groups of complex 1-dimensional orbifolds. Recently, orbifold groups (in any complex dimension) have been considered (cf. [24, 9] also [17] for real orbifolds).

The first purpose of this paper is to define and study the properties of the different classes of complex orbifold fundamental groups such as compact, locally finite, and normal crossing. In particular, we prove that orbifold fundamental groups are quasi-projective, but compact orbifold groups in general are not projective (see §1). In this context, we develop in §2 the concept of *saturated* orbifolds, which will allow one to transform orbifolds without altering their fundamental group.

Our second purpose (see §3) is to extend two classical results regarding the variety of characters on smooth quasi-projective fundamental groups (due to Arapura [1] and the authors [4]) and normal-crossing compact Kähler orbiface groups (due to Campana [9]) to the general case of normal-crossing quasi-projective orbifold groups.

Finally in §5, we extend Sakuma's formula (cf. [21, 15]) to orbifold fundamental groups and their abelian covers in terms of their orbifold characteristic varieties. In order to do so, in §4 we present the concepts of unbranched and branched coverings as well as the possible uniformizations (Galois, regular, and virtually regular). Such formulas are illustrated with examples in dimensions one and two.

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1. Orbifold groups

Definition 1.1. Let $\bar{X}$ be a projective Riemann surface and let $\varphi : \bar{X} \to \mathbb{Z}_{\geq 0}$ be a function such that $S_\varphi := \{p \in X \mid \varphi(p) \neq 1\}$ is finite. The pair $(\bar{X}, \varphi)$ is said to be a 1-dimensional *orbifold* or an *orbicurve*. The *positive part* of the orbicurve is $X_\varphi^+ := \bar{X} \setminus \varphi^{-1}(0)$ and we say that the orbifold is *compact* if $X_\varphi^+ = \bar{X}$. The set $S_\varphi^{>1} := X_\varphi^+ \cap S_\varphi$ is called the *singular part* and $\varphi(p)$ is the *orbifold index* of $p \in X_\varphi^+$.

The geometrical interpretation is the following. The source of the charts centered at $p \in X_\varphi^+$ are of the type $\Delta/\mu_{\varphi(p)}$ where $\mu_n := \{z \in \mathbb{C} \mid z^n = 1\}$, Δ is an open disk centered at 0 and $\mu_{\varphi(p)}$ acts on Δ by multiplication. This interpretation suggests the following definition.

Definition 1.2. Let $(\bar{X}, \varphi)$ be an orbicurve. Let $X_\varphi := \bar{X} \setminus S_\varphi$ and $G := \pi_1(X_\varphi; p_0)$ for some $p_0 \in X_\varphi$. For each $p \in S_\varphi$ choose a meridian $x_p \in G$ (its conjugacy class is well defined). Then, the *orbifold fundamental group* of (X, φ) is defined as

$$\pi_1^{\rm orb}(\bar{X}, \varphi; p_0) := G/\langle x_p^{\varphi(p)} \rangle.$$

A group is said to be an *orbicurve group* if it is the orbifold fundamental group of an orbicurve.

Remark 1.3. If the group can be described as the orbifold group of a compact orbicurve then we will refer to it as a *compact orbicurve group*.

Proposition 1.4. *Any orbicurve group is quasi-projective.*

In order to prove this result we introduce the following concept.

Definition 1.5. Let X be a smooth quasi-projective surface, let $\mathcal{D}$ be a divisor in X and let $D \subset X$ an irreducible component of $\mathcal{D}$. An *n-fold blow-up* ρ of $(X, \mathcal{D})$ on D is a composition of blowing-ups $\rho_j : X_j \to X_{j-1}$, $1 \leq j \leq n$, such that $X_0 := X$, the center of ρ_1 is a point of D which is smooth in $\mathcal{D}$ and if $E_j \subset X_j$ is the exceptional component of ρ_j then the center of ρ_{j+1} is the intersection of E_j with the strict transform of D. The component E_j is called the *j-th exceptional component* of ρ.

Remark 1.6. As a general comment, consider a double point on a smooth surface, i.e. a point of local equations $\mathcal{D} := \{x^2 - y^2 = 0\}$ in a small ball $\mathbb{B}$ around $P = (0, 0)$. Perform a blow-up $\rho : \tilde{\mathbb{B}} \to \mathbb{B}$ centered at P and consider $\gamma(t) = (e^{2\pi\sqrt{-1}t}, 0)$ which is a product of the meridians around the two components of $\mathcal{D}$. Note that $G_P := \pi_1(\mathbb{B} \setminus \mathcal{D}) = \pi_1(\tilde{\mathbb{B}} \setminus (E \cup \tilde{\mathcal{D}}))$ where E is the exceptional divisor and $\tilde{\mathcal{D}}$ is the strict transform of $\mathcal{D}$. Using both affine charts and using $\gamma(t) = (e^{2\pi\sqrt{-1}t}, 1)$ it is easy to see that $\gamma = \gamma_E$ as elements of G_P, where γ_E is a meridian around E.

Analogously, if we consider $G := \pi_1(X \setminus \mathcal{D}) = \pi_1(X \setminus (E_1 \cup \mathcal{D}))$, ρ an n-fold blow-up ρ of $(X, \mathcal{D})$ on an irreducible divisor D and μ is a meridian around D, then μ is also a meridian around E_1. Using the property discussed in the previous paragraph, μ^2 is a meridian around E_2 and by induction μ^j is a meridian around the j-th exceptional component of ρ in G.

Proof of Proposition 1.4. Let $(\bar{X}, \varphi)$ be an orbicurve. A quasi-projective surface Z will be constructed satisfying $\pi_1(Z) \cong \pi_1^{\rm orb}(\bar{X}, \varphi)$.

Let $Y := X_\varphi^+ \times \mathbb{P}^1$ be a surface, and let $\mathcal{D} := S_\varphi^{>1} \times \mathbb{P}^1 \subset Y$. For each $p \in S_\varphi^{>1}$ consider a $\varphi(p)$-fold blow-up $\rho_p : \tilde{Y} \to Y$ on the divisor $F_p := \{p\} \times \mathbb{P}^1$. Let E_j^p be the j-th exceptional component of ρ_p. Let x_p be a meridian of $\{p\} \times \mathbb{P}^1$ in $\pi_1(\tilde{Y})$. Following the previous remark,

x_p^j is a meridian of E_j^p in $\pi_1(\tilde{Y})$. The surface

$$Z := \tilde{Y} \setminus \bigcup_{p \in S_\varphi^{>1}} \left(F_p \cup \bigcup_{j=1}^{\varphi(p)-1} E_j^p \right)$$

is quasi-projective. The groups $\pi_1(X_\varphi^+)$, $\pi_1(Y)$ and $\pi_1(\tilde{Y})$ are naturally isomorphic. The kernel of the epimorphism $\pi_1(\tilde{Y}) \twoheadrightarrow \pi_1(Z)$ is normally generated by the meridians $x_p^{\varphi(p)}$ of $E_{\varphi(p)}^p$. Then, $\pi_1(Z)$ is isomorphic to $\pi_1(\tilde{Y})/\langle x_p^{\varphi(p)} \rangle$ which, by the definition of orbicurve group, is nothing but $\pi_1^{\text{orb}}(\bar{X}, \varphi)$. □

Remark 1.7. As shown in [14, Theorem II.2.3], compact orbicurve groups are projective groups.

We will define orbifolds and orbifold groups following Campana (cf. [9] and bibliography therein). Since we are mostly interested in quasi-projective groups, after using Zariski-Lefschetz theory we can restrict our attention to the curve and surface case. However, since we will deal with orbifold covers (see §4), orbifolds with abelian quotient singularities will also be allowed.

Definition 1.8. Let $\bar{X}$ be a projective variety with only abelian quotient singularities and let $\mathcal{D} = \bigcup_{j=1}^r D_i$ be the decomposition of a hypersurface in irreducible components. Let us consider a function $\varphi : \{D_1, \ldots, D_r\} \to \mathbb{Z}_{\geq 0}$, $n_i := \varphi(D_i)$. An *orbifold* is simply a pair $(\bar{X}, \varphi)$. The *positive part* of the orbifold is defined as $X_\varphi^+ := \bar{X} \setminus \varphi^{-1}(0)$. The orbifold is said to be *compact* if $\bar{X} = X_\varphi^+$. The orbifold will be a *normal-crossing orbifold* (NC for short) if $\mathcal{D}$ is a normal crossing divisor with smooth components.

Remark 1.9. Note that, for technical reasons, the components of $\mathcal{D}$ are allowed to have index one (that is, $n_j = 1$). However, this plays no important role in the definition of an orbifold. Hence, if no ambiguity seems likely to arise, we denote by the same symbols an orbifold and its analogous where $\varphi^{-1}(1)$ is disregarded. What is really important in the definition is the quasi-projective variety X_φ^+ and the components D_j with $n_j > 1$. Following the definitions for the orbicurve case we also define

$$S_\varphi := \{D_j \mid n_j \neq 1\},\ S_\varphi^{>1} := \{D_j \mid n_j > 1\},\ X_\varphi := \bar{X} \setminus \left(\bigcup S_\varphi \right),\ \mathring{X}_\varphi := \bar{X} \setminus \mathcal{D}.$$

Note that $\mathring{X}_\varphi \subset X_\varphi \subset X_\varphi^+$. In $\pi_1(X_\varphi)$ and $\pi_1(\mathring{X}_\varphi)$ one has special conjugacy classes: for each D_i we consider the meridians of D_i in either $\pi_1(X_\varphi)$ or $\pi_1(\mathring{X}_\varphi)$. Note that the kernel of the epimorphism $\pi_1(X_\varphi) \twoheadrightarrow \pi_1(\bar{X})$ is the subgroup generated by the meridians of D_j, $n_j \neq 1$ whereas the kernel of the epimorphism $\pi_1(\mathring{X}_\varphi) \twoheadrightarrow \pi_1(\bar{X})$ is the subgroup generated by the meridians of $D_1, \ldots, D_r$.

Definition 1.10. Under the notation above, given an orbifold $(\bar{X}, \varphi)$ we define its *orbifold fundamental group* as the group $\pi_1^{\text{orb}}(\bar{X}, \varphi; p_0)$, $p_0 \in \operatorname{Reg}(\mathring{X}_\varphi) := \mathring{X}_\varphi \setminus \operatorname{Sing}(\mathring{X}_\varphi)$ obtained as the quotient of $\pi_1(\mathring{X}_\varphi; p_0)$ by the subgroup normally generated by $\{\mu_j^{n_j}\}_{1 \leq j \leq r}$, where μ_j is a meridian of D_j. Note that $\pi_1(\mathring{X}_\varphi)$ can also be replaced by $\pi_1(X_\varphi)$ in this definition.

For $p \in X_\varphi^+$ one can define the *local orbifold fundamental group* $\pi_1^{\text{orb}}(\bar{X}, \varphi)_p$ as the quotient of $\pi_1(\operatorname{Reg}(\mathring{X}_\varphi))_p$ by the subgroup normally generated by the appropriate powers $\mu_j^{n_j}$ of the meridians μ_j of $\mathcal{D}$ in a small ball around p. The orbifold $(\bar{X}, \varphi)$ shall be called *locally finite at* p if $\pi_1^{\text{orb}}(\bar{X}, \varphi)_p$ is a finite group, and *locally finite* (or simply LF) if it is locally finite at p, $\forall p \in X_\varphi^+$.

We need to extend the notion of the orbifold index of a point in an orbifold as we did for orbicurves in Definition 1.1.

Definition 1.11. Let $(\bar{X},\varphi)$ be an NC-orbifold and let $p\in\bar{X}$. We define the *orbifold index* $\nu(p)=\nu_{(\bar{X},\varphi)}(p)$ of p as follows:

$$\nu(p):=\begin{cases}\iota_p & \text{if } p\in\bar{X}\setminus\mathcal{D},\\ n_j\cdot\iota_p & \text{if } p\in D_j\setminus\bigcup_{i\neq j}D_i,\\ n_i\cdot n_j\cdot\iota_p & \text{if } p\in D_i\cap D_j, i\neq j,\end{cases}$$

where $\iota_p=|A|$ if $(\bar{X},p)\cong(\mathbb{C}^2/A,0)$, the quotient by the linear action of a small abelian subgroup $A\subset\mathrm{GL}(2;\mathbb{C})$ (note that $\iota_p=1$ iff $p\in\mathrm{Reg}(\bar{X})$).

Remark 1.12. If $p\in\mathrm{Reg}(X_\varphi)$ (or $\mathrm{Reg}(\mathring{X}_\varphi)$) then $\pi_1^{\mathrm{orb}}(X,\varphi)_p$ is a trivial group.

Proposition 1.13. *If $p\in X_\varphi^+$ then $\nu(p)=\#\pi_1^{\mathrm{orb}}(X,\varphi)_p$.*

Proof. We distinguish several cases for p such that $(\bar{X},p)\cong(\mathbb{C}^2/A,0)$ where A is a small abelian group (hence cyclic). Let $\mathbb{B}_p$ be a small neighborhood of p (a quotient of a ball $\mathbb{B}_0$ in $\mathbb{C}^2$).

Let us suppose that $p\in\mathring{X}_\varphi$. In this case $\pi_1^{\mathrm{orb}}(\bar{X},\varphi)_p$ is isomorphic to $\pi_1(K_p)$, where K_p is the link of the singularity $(\bar{X},p)$ which is a lens space with fundamental group A and the result follows.

Let us assume now that p belongs only to one irreducible component $D_i\subset\mathcal{D}$ where D_i is the image of $Y:=\{y=0\}\subset\mathbb{C}^2$. We have a short exact sequence

$$0\to\pi_1(\mathbb{B}_0\setminus Y)\to\pi_1(\mathbb{B}_p\setminus D_i)\to A\to 0.$$

Both $\pi_1(\mathbb{B}_0\setminus Y)$ and $\pi_1(\mathbb{B}_p\setminus D_i)$ are isomorphic to $\mathbb{Z}$ (written with multiplicative notation), which is generated by an element t which projects to a generator of A. By the definition of the action, the image of a generator of $\pi_1(\mathbb{B}_0\setminus Y)$ is a meridian μ_i of D_i which equals t^{ι_p}. Hence, we obtain $\pi_1^{\mathrm{orb}}(X,\varphi)_p$ from $\pi_1(\mathbb{B}_p\setminus D_i)$ by killing $x_i^{n_i}=t^{i_p n_i}=t^{\nu(p)}$ and the result follows.

Finally, let us assume that p belongs to two irreducible components $D_i,D_j\subset\mathcal{D}$ where D_i is the image of $Y:=\{y=0\}\subset\mathbb{C}^2$ and D_j is the image of $X:=\{x=0\}$. The covering induces the following short exact sequence:

$$0\to\pi_1(\mathbb{B}_0\setminus(X\cup Y))\to\pi_1(\mathbb{B}_p\setminus(D_i\cup D_j))\to A\to 0. \tag{1.1}$$

Both $\pi_1(\mathbb{B}_0\setminus Y)$ and $\pi_1(\mathbb{B}_p\setminus D_i)$ are isomorphic to $\mathbb{Z}^2$ (written with multiplicative notation as above). The group $\pi_1(\mathbb{B}_0\setminus(X\cup Y))$ is generated by commuting meridians of X and Y whose images are x_i and x_j. We can choose an element $t\in\pi_1(\mathbb{B}_p\setminus(D_i\cup D_j))$ which projects to a generator of A. With a suitable choice of t, we have $t^{\iota_p}=x_i x_j^k$ (k depends on the specific action and is coprime with ι_p). Hence (1.1) induces the following short exact sequence

$$0\to\langle x,y\mid[x,y]=1,x^{n_j}=x^{n_i}=1\rangle\to\pi_1^{\mathrm{orb}}(X,\varphi)_p\to A\to 0$$

and the result follows. □

Remark 1.14. Note that if p is an orbifold point of index m, then $\pi_1^{\mathrm{orb}}(\bar{X},\varphi)_p$ is cyclic of order m. If p is an ordinary double point of $\mathcal{D}$ belonging to two components D_i,D_j with $n_i,n_j>1$, then $\pi_1^{\mathrm{orb}}(X,\varphi)_p$ is the product of two finite cyclic groups. As a consequence, if $(\bar{X},\varphi)$ is a normal crossing orbifold then it is in particular a locally finite orbifold.

Definition 1.15. A group G is said to be an *orbifold group* if it is isomorphic to $\pi_1^{\mathrm{orb}}(\bar{X},\varphi;p_0)$ for some orbifold $(\bar{X},\varphi)$. If one can choose $(\bar{X},\varphi)$ to be such that $n_i>0$, $\forall i$, then we say that G is a *compact orbifold group.* If, moreover $(\bar{X},\varphi)$ is a locally finite (resp. normal crossing orbifold), we say that G is an LF (resp. NC) *compact orbifold group.*

Remark 1.16. Note that an orbifold group as defined below is also the fundamental group of an orbifold $(\bar{X},\varphi)$ where $\bar{X}$ is smooth.

Remark 1.17. We do not define the more general concept of LF *or* NC *orbifold groups* since they coincide immediately with the concept of *orbifold group* by the following fact. If we blow up a point in $p \in \mathcal{D}$, we obtain a new surface $\bar{Y}$ and a new divisor $\hat{\mathcal{D}}$ with $r+1$ irreducible components (the strict transforms of the components D_i, with the same notation, and the exceptional component D_{r+1}). We can define a map $\hat{\varphi}$ such that $\hat{\varphi}(D_i) = n_i$, $1 \leq i \leq r$, and $\hat{\varphi}(D_{r+1}) = 0$ and the orbifold fundamental group does not change. An iterated application of this procedure will give us a normal crossing divisor.

Proposition 1.18. *Let us consider in $\mathbb{P}^2$ the arrangement of lines $\mathcal{L}$ given by the equation $xyz(x^2-z^2)(y^2-z^2)$ and consider the orbifold structure $\varphi_{\mathcal{L}}$ given by assigning 2 to each line in $\mathcal{L}$. Let $G := \pi_1^{\text{orb}}(\mathbb{P}^2, \varphi_{\mathcal{L}})$. The meridians in G of the exceptional components of the blowing-ups of the quadruple points of $\mathcal{L}$ are of infinite order.*

Proof. It is easy to see that

$$G := \langle x_i, y_j, \gamma_z : x_i^2 = y_j^2 = \gamma_z^2 = [x_i, y_j] = 1, \gamma_z = (XY)^{-1} \rangle_{i,j=1,2,3},$$

where $X = x_1x_2x_3$, $Y = y_1y_2y_3$ and x_i (resp. y_j) $i,j = 1,2,3$ are meridians around the vertical (resp. horizontal) lines and γ_z is a meridian around the line at infinity $\{z=0\}$. Denote by γ_{E_x} (resp. γ_{E_y}) the meridian in G around the exceptional divisor E_x (resp. E_y) after blowing up the point $[0:1:0]$ (resp. $[1:0:0]$). Note that $\gamma_{E_x} = \gamma_z X = Y^{-1}$, $\gamma_{E_y} = \gamma_z Y = X^{-1}$. By symmetry, it is enough to show that X has infinite order in G or equivalently $c := X^2 \in G'$ has infinite order. Using Reidemeister-Schreier method it is easily seen that

$$G' = \langle a_i, b_j, c \mid \ [a_i, b_j] = 1, [a_1, a_2] = [b_1, b_2] = c^4, \ c \text{ central} \rangle_{i,j=1,2}. \tag{1.2}$$

It is straightforward that c has infinite order. □

Using the same ideas as in Proposition 1.4 we obtain the following result.

Proposition 1.19. *Any orbifold group is a quasi-projective group.*

Proof. As above, consider $(\bar{X}, \varphi)$ an orbifold for which $\mathcal{D} = D_1 \cup \cdots \cup D_r$. For each divisor $D_j \in S_\varphi^{>1}$ let ρ_j be the n_j-fold blow-up on D_j and denote by $\rho : \bar{Y} \to \bar{X}$ the composition of all of them. Let us denote by $E_{k,j}$, $1 \leq k \leq n_j$, $1 \leq j \leq r$ the k-th exceptional component of ρ_j. Define $Y := \bar{Y} \setminus \bigcup_{D_j \in S_\varphi^{>1}} \left(D_j \cup \bigcup_{k=1}^{n_j-1} E_{k,j} \right)$, where D_j here denotes the strict transform of D_j by ρ and similarly with $E_{k,j}$. Note that $\bar{Y}$ is the result of a finite process of blow-ups of a projective variety $\bar{X}$, hence Y is quasi-projective variety. Moreover, using Remark 1.6 it is straightforward to check that Y satisfies the required property $\pi_1(Y) \cong \pi_1^{\text{orb}}(\bar{X}, \varphi)$. □

In light of Remark 1.7 and Proposition 1.19, the following question arises:

Question 1.20. Is any compact orbifold group (or NC-compact orbifold group) a projective group?

A negative answer to the first part is provided by the ideas given in Example 2.5 and Proposition 1.18. This seems to suggest that NC-compact orbifolds are a reasonable class of orbifolds to work with for our purposes.

Proposition 1.21. *Compact orbifold groups are not necessarily* NC*-compact orbifold groups, and thus not projective groups.*

Proof. We are going to prove that the compact orbifold group G presented in Proposition 1.18 is not an NC-compact orbifold group. We will proceed by contradiction. Assume that G is an NC-compact orbifold group. Since the subgroup G' is of finite index, it is also an NC-compact orbifold group. The group G' is described as a central extension of $\mathbb{Z}^4$ by $\mathbb{Z}$ as it is deduced from the presentation (1.2). Since G' is torsion free, the group G' is in fact projective and thus Kähler. The group

$$H = \langle a_i, b_j, d \mid [a_i, b_j] = 1, [a_1, a_2] = [b_1, b_2] = d, \ d \text{ central}\rangle_{i,j=1,2}. \tag{1.3}$$

is an index-four subgroup of G' ($d = c^4$) and hence, it is also projective. Moreover, H is the Heisenberg group $H(2)$ (following the notation in [10]). This group is nilpotent, but not *almost abelian* (i.e. no finite-index subgroup is abelian). Since the rank of its abelianization is 4, it cannot be Kähler using [10, Corollary 4.5] (one can also use [7, Corollary 3.8] to obtain this statement). This contradicts the original asumption and thus G cannot be an NC-compact orbifold group and thus not a projective group. □

Remark 1.22. From another point of view, Proposition 1.18 implies that the local fundamental group $\pi_1^{\text{orb}}(\mathbb{P}^2, \varphi_{\mathcal{L}})_{[0:1:0]}$ is infinite and thus the orbifold $(\mathbb{P}^2, \varphi_{\mathcal{L}})$ has no uniformization in the sense of [24, Theorem 2.4].

Remark 1.23. Note that Propositions 1.19 and 1.21 partially answer questions posed by Simpson [22, §8].

2. Saturated orbifolds

Since we are mainly interested in orbifold groups it is sometimes useful to replace in $(\bar{X}, \varphi)$ the function φ by another function $\tilde{\varphi}$ where $\tilde{\varphi}(D_i)$ is defined by the actual order of μ_i in $\pi_1^{\text{orb}}(\bar{X}, \varphi; p_0)$; we may perform this operation only when $n_i > 0$ in order to have $X^+_\varphi = X^+_{\tilde{\varphi}}$. This notion is somehow related with [20, Condition (1.3.3)].

Definition 2.1. Given an orbifold $(\bar{X}, \varphi)$ (for a fixed $\mathcal{D}$), we say that φ is a *saturated orbifold structure* if for any meridian μ_i of D_i (with $n_i > 0$), the order of μ_i in $\pi_1^{\text{orb}}(\bar{X}, \varphi; p_0)$ is exactly n_i.

There is a natural way to saturate an orbifold. Unless otherwise stated we will consider only saturated orbifolds in the sequel. Sometimes an extra saturation can be performed; even if $n_i = 0$, it may happen that μ_i is of finite order in $\pi_1^{\text{orb}}(\bar{X}, \varphi; p_0)$. Note that in that case if we define $\tilde{\varphi}(n_i)$ to be this order, then $X^+_\varphi \subsetneq X^+_{\tilde{\varphi}}$.

We are going to study different kinds of saturation and their relationship with the concept of NC-orbifolds. Let $(\bar{X}, \varphi)$ be an orbifold; if $\mathcal{D}$ is not a normal crossing divisor there is a sequence $\pi : \bar{Y} \to \bar{X}$ of blowing-ups (which is an isomorphism outside $\mathring{X}$) such that $\pi^{-1}(\mathcal{D})$ becomes a normal crossing divisor. An orbifold structure ψ can be endowed to $\bar{Y}$ as in Remark 1.17, i.e. ψ vanishes on any exceptional component of π. This procedure does not change the orbifold fundamental group but in general X^+_φ and Y^+_ψ are not isomorphic; in particular, when $(\bar{X}, \varphi)$ is not NC, $(\bar{Y}, \psi)$ is not a compact orbifold even if $(\bar{X}, \varphi)$ is.

We are going to consider now a more general class of saturations where X^+_φ may change without modifying $\pi_1^{\text{orb}}(\bar{X}, \varphi)$.

Definition 2.2. Let $(\bar{X}, \varphi)$ be an orbifold and let $p \in \bigcup S^+_\varphi$. Let $\pi : \bar{Y} \to \bar{X}$ be the blowing-up of p and keep the notation of Remark 1.17. We say that p is an LF-*point at first level* if the order of the meridian μ_{r+1} is finite in $\pi_1(\bar{X}, \varphi)_p$.

Let $\pi : \bar{Y} \to \bar{X}$ be the blowing-up of an LF-point at first level; let $\hat{\mathcal{D}} := \pi^{-1}(\mathcal{D})$; with the notation of Remark 1.17, we consider a saturation ψ such that $\psi(D_i) := n_i$, $1 \leq i \leq r$ and $\psi(D_{r+1})$ is the order of the meridian μ_{r+1} in $\pi_1^{\text{orb}}(\bar{X}, \varphi)_p$.

Definition 2.3. A point p is an LF-*point* if all of its infinitely near points are LF-points at first level (in particular, if an orbifold is locally finite at a point p then this point is an LF-point).

Example 2.4. By the very construction $\pi_1^{\mathrm{orb}}(\bar{X},\varphi) \cong \pi_1^{\mathrm{orb}}(\bar{Y},\psi)$. Hence if p is an LF-point we can obtain a sequence of blow-ups such that the divisor $\mathcal{D}$ becomes a normal crossing divisor *over* p and such that all the exceptional divisors have non-zero orbifold indices.

Example 2.5. If $p \in X_\varphi^+$ is an ordinary double point $p \in D_i \cap D_j$ of $\mathcal{D}$ then $\psi(D_{r+1}) = \mathrm{lcm}(n_i, n_j)$; if $p \in D_i$ is a smooth point of $\mathcal{D}$ then $\psi(D_{r+1}) = n_i$.

Examples 2.5 and 2.4 show that LF and NC compact orbifold fundamental groups are the same class of groups.

Example 2.6. Locally finiteness may not happen for more complicated singular points. As a simple example if $p \in \mathcal{D}$ is an ordinary triple singular point with orbifold indices for each branch $u, v, w \in \mathbb{N}$ such that $\frac{1}{u} + \frac{1}{v} + \frac{1}{w} \leq 1$, then p is not an LF-point. In the same way if p is an ordinary cusp and the orbifold index is ≥ 6 then p is not an LF-point.

We set the global version of Definitions 2.2 and 2.3.

Definition 2.7. Let $(\bar{X},\varphi)$ be an orbifold and let $p \in \bigcup S_\varphi^+$. Let $\pi : \bar{Y} \to \bar{X}$ be the blowing-up of p (keeping again the notation of Remark 1.17). We say that p is a *finite-type point at first level* if the order of the meridian μ_{r+1} is finite in $\pi_1(\bar{X},\varphi)$. A point p is a finite-type point if all of its infinitely near points are finite-type points at first level.

Remark 2.8. Let us consider $\bar{X} = \mathbb{P}^2$, $\mathcal{D}$ the union of three lines through a point p with orbifold indices $u, v, w \in \mathbb{N}$ such that $\frac{1}{u} + \frac{1}{v} + \frac{1}{w} \leq 1$. It is clear that p is not an LF-point at first level and it is easy to see that it is a finite-type point, since the meridian around the exceptional component is in fact trivial. The quadruple points of the arrangement in Proposition 1.18 are not of finite type. Hence the classes of *compact orbifold* and NC-*compact orbifold* groups do not coincide.

Let us start from a saturated orbifold structure. Hence, if all the points of $\mathcal{D}$ (it is enough to check it for singular points of $\mathcal{D}$ worse than nodal points) are LF-points (or finite-type point) we can replace $(\bar{X},\varphi)$ by an NC-orbifold structure in a surface after successive blowing-ups without changing the fundamental group. In the first case we call this structure *locally saturated*; in the second case it is called *globally saturated*. Moreover, this can be done respecting the compactness.

We finish this section with a new saturation procedure which modifies $\pi_1^{\mathrm{orb}}(\bar{X},\varphi)$. An interesting object of study associated with $\pi_1^{\mathrm{orb}}(\bar{X},\varphi)$ is the set of its characteristic varieties, see §3, which is a stratification of the space of characters. Since $H_1^{\mathrm{orb}}(\bar{X},\varphi;\mathbb{Z})_p$ is generated by the meridians of the components of $\mathcal{D}$ passing through p, we can associate to D_{r+1} the order of μ_{r+1} in $H_1^{\mathrm{orb}}(\bar{X},\varphi;\mathbb{Z})_p$ (or in $H_1^{\mathrm{orb}}(\bar{X},\varphi;\mathbb{Z})$). The orbifold structure is called *locally homologically saturated* or *globally homologically saturated*.

Example 2.9. If we consider an ordinary triple point where all the components have index 2, the local homological saturation is given by assigning 2 to the exceptional component. It is easily seen that the local saturation assigns index 4.

3. Orbifolds and characteristic varieties

The relationship between orbifolds and characteristic varieties (or similar invariants) appear implicitly in the works of Beauville [6] and Arapura [1] and explicitly in the works of Campana, e.g. [9], Simpson-Corlette [12], Delzant [13] and ourselves [4], among others. Except in Campana's work, the relationship comes from the following fact: given a smooth variety (projective, quasi-projective or Kähler) the positive-dimensional components of the characteristic varieties can be obtained as pull-back by mappings whose targets are orbifolds. Campana's work focuses on the study of characteristic varieties of compact Kähler orbifolds (more precisely, NC-projective orbifolds in the language of §1). In this section we will study the characteristic varieties of quasi-projective orbifolds. For a detailed exposition of the concept of characteristic varieties (or Green-Lazarsfeld invariant), the reader can check any of the above references. Some definitions will also be given in §5.

Before we state the aforementioned results in the context and language of orbifolds we need to recall the concept of orbifold morphism, which as it occurs in the classical case, allows one to define a morphism of fundamental groups.

Definition 3.1. Let $(\bar{X},\varphi),(\bar{Y},\psi)$ be orbifolds with divisors $\mathcal{D}:=\bigcup_{y=1}^{r} D_j\subset\bar{X}$, $n_j:=\varphi(D_j)$, $\mathcal{E}:=\bigcup_{k=1}^{s} E_j\subset\bar{Y}$, $m_k:=\psi(E_k)$. A dominant holomorphic map $\Phi: X_\varphi^+\to Y_\psi^+$ defines an *orbifold map* $\Phi^{\text{orb}}:(\bar{X},\varphi)\to(\bar{Y},\psi)$ if for each $k\in\{1,\dots,s\}$, the divisor $\Phi^*(E_k)$ can be written as $\sum_{j=1}^{r} h_{j,k}D_j+m_kH_k$ where m_k divides n_jh_{jk} and H_k is a divisor in X_φ^+.

Proposition 3.2 ([11, 3])**.** *Let $\Phi^{\text{orb}}:(\bar{X},\varphi)\to(\bar{Y},\psi)$ be an orbifold map. This map induces (in a functorial way) a morphism $\Phi_*^{\text{orb}}:\pi_1^{\text{orb}}(\bar{X},\varphi)\to\pi_1^{\text{orb}}(\bar{Y},\psi)$. Moreover, if $(\bar{Y},\psi)$ is an orbicurve and the generic fiber of Φ^{orb} is irreducible then Φ_*^{orb} is surjective.*

There are two main examples of orbifold morphisms: either the target is an orbicurve or the orbifolds have the same dimension. The last case (étale or branched covers) is specially interesting when all the fibers are finite.

Let us compare the following results. We use the language of §1 if needed. The natural definition of $\mathcal{V}_k^{\text{orb}}$ (which is the orbifold analogue of $\mathcal{V}_k$, the k-th characteristic variety) can be found in §5.

Theorem 3.3 ([4, Theorem 1])**.** *Let X be a smooth quasi-projective variety and let $\mathcal{V}_k(X)$ be the k-th characteristic variety of X. Let V be an irreducible component of $\mathcal{V}_k(X)$. Then one of the two following statements holds:*

(1) *There exists an orbicurve $(\bar{C},\psi)$, an orbifold morphism $\Phi^{\text{orb}}:X\to(\bar{C},\psi)$ and an irreducible component W of $\mathcal{V}_k^{\text{orb}}(\bar{C},\psi)$ such that $V=(\Phi^{\text{orb}})^*(W)$.*
(2) *V is an isolated torsion point not of type* (1)*.*

Theorem 3.4 ([8, Théorème 3.1])**.** *Let $(\bar{X},\varphi)$ be an* NC*-compact Kähler orbifold surface. Let V be an irreducible component of $\mathcal{V}_k^{\text{orb}}(\bar{X},\varphi)$. Then, one of the following statements holds:*

(1) V is an isolated torsion point.
(2) There exists a compact hyperbolic orbicurve $(\bar{C},\psi)$, where the genus of $\bar{C}$ is at least 1*, an orbifold map $\Phi^{\text{orb}}:(\bar{X},\varphi)\to(\bar{C},\psi)$ and an irreducible component W of $\mathcal{V}_k^{\text{orb}}(\bar{C},\psi)$ such that $V=(\Phi^{\text{orb}})^*(W)$.*

The goal of this section is to state and prove a combination of the above theorems.

Theorem 3.5. *Let $(\bar{X},\varphi)$ be an* NC*-quasi-projective orbifold surface. Let V be an irreducible component of $\mathcal{V}_k^{\text{orb}}(\bar{X},\varphi)$. Then, one of the following statements holds:*

(1) *There exists an orbicurve $(\bar{C},\psi)$, an orbifold map $\Phi^{\text{orb}}:(\bar{X},\varphi)\to(\bar{C},\psi)$ and an irreducible component W of $\mathcal{V}_k^{\text{orb}}(\bar{C},\psi)$ such that $V=(\Phi^{\text{orb}})^*(W)$.*
(2) *V is an isolated torsion point.*

Proof. Let $(\bar{X},\varphi)$ be an NC-quasi-projective orbifold surface. Let $\mathcal{D}$ be the hypersurface defining the orbifold structure where we assume that $\mathcal{D}=\bigcup_{j=1}^{r+s}D_j$, where $n_j\geq 2$ if $1\leq j\leq r$ and $n_{r+k}=0$ if $1\leq k\leq s$. We may assume the orbifold structure is saturated.

We proceed as in the proof of Proposition 1.19. Let $\pi:\bar{Y}\to\bar{X}$ the composition of the $\sum_{j=1}^{r}n_j$ blow-ups indicated in that proof. We denote by D_i the strict transforms of D_i and by $E_{k,j}$,

$1 \leq k \leq n_j$, $1 \leq j \leq r$, the exceptional components of π. Let $Y := \bar{Y} \setminus \bigcup_{j=1}^{r} \left(D_j \cup \bigcup_{k=1}^{n_j-1} E_{k,j} \right)$. Recall that $\pi_1^{\text{orb}}(\bar{X}, \varphi) \cong \pi_1(Y)$.

We can apply Theorem 3.3 to Y. Let us consider a component V of $\mathcal{V}_k(Y)$ of type (1) and consider the orbifold map given in the statement. Let us write this orbifold map in the language of §1. We consider in $\bar{Y}$ the hypersurface

$$\hat{\mathcal{D}} = \bigcup_{j=1}^{r} \left(D_j \cup \bigcup_{k=1}^{n_j} E_{k,j} \right) \cup \bigcup_{\ell=1}^{s} D_{r+\ell},$$

and the map $\hat{\varphi}$ given by:

$$\underset{1\leq j\leq r+s}{\hat{\varphi}(D_j) = 0}, \quad \underset{1\leq k<n_j, 1\leq j\leq r}{\hat{\varphi}(E_{k,j}) \quad = 0}, \quad \underset{1\leq j\leq r}{\hat{\varphi}(E_{n_j,j}) = 1}.$$

Since $Y_{\hat{\varphi}}^{+} = Y$, the map given by Theorem 3.3 can be written as $\hat{\Phi}^{\text{orb}} : (\bar{Y}, \hat{\varphi}) \to (\bar{C}, \psi)$. Let us consider $\hat{\Phi} : Y \to \bar{C}$ the underlying dominant holomorphic mapping.

Note that $\check{E}_j := E_{n_j,j} \cap Y$ is isomorphic to $\mathbb{C}^*$. Let us assume that $\hat{\Phi}_{\check{E}_j}$ is not constant and hence dominant on $\bar{C}$; in particular, it determines an orbifold morphism $\hat{\Phi}^{\text{orb}} : (E_{n_j,j}, \varphi_j) \to (\bar{C}, \psi)$ where φ_j is the induced orbifold structure, which is the trivial one. The only possible choices for $(\bar{C}, \psi)$ are either $\mathbb{C}^*$ (with smooth structure) or $\mathbb{C}_{2,2}$; the characteristic varieties of these orbifolds are finite and we are led to a contradiction.

Then, we have proven that $\hat{\Phi}_{\check{E}_j}$ is constant and denote by $p_j \in \bar{C}$ its image. Let us consider a small neighborhood $\mathcal{U}_j$ of $\bigcup_{k=1}^{n_j-1} E_{k,j}$; this curve is a linear chain of rational smooth curves with self-intersection -2 and the space $\tilde{\mathcal{U}}_j$ obtained from $\mathcal{U}_j$ by contracting the curves is isomorphic to the quotient of a neighborhood $\tilde{\mathcal{U}}_j$ of the origin in $\mathbb{C}^2$ by the action of a cyclic group of order k_j. We may lift $\hat{\Phi}$ to a dominant morphism $\hat{\Phi}_j : \tilde{\mathcal{U}}_j \setminus \{0\} \to \bar{C}$; it is easily seen that if $\hat{\Phi}_j$ cannot be extended to the origin, then $\bar{C} \cong \mathbb{P}^1$ and the characteristic varieties of $(\bar{C}, \psi)$ are finite. Since this is not possible, $\hat{\Phi}_j$ can be extended and $\hat{\Phi}$ can be extended to $\bigcup_{k=1}^{n_j-1} E_{k,j}$ by sending the curve to p_j.

A similar argument allows us to extend $\hat{\Phi}$ to the regular part of $\hat{\mathcal{D}}$ in D_j; moreover it is also possible to extend it to $D_j \cap E_{n_j,j}$ (with image p_j). Finally we can extend it to the double points $D_i \cap D_j$, $1 \leq i < j \leq r$. Moreover, since this map is constant on $\bigcup_{k=1}^{n_j} E_{k,j}$, we can contract these divisors (the exceptional divisors of π) and we obtain a holomorphic map $\Phi : X_{\varphi}^{+} \to \bar{C}$.

All we are left to do is to check that Φ defines the required orbifold morphism. Before we prove this, note that Φ_* induces a morphism of orbifold fundamental groups. To see this, let μ_j be a meridian around D_j; note that $\mu_j^{n_j}$ is a meridian of $E_{n_j,j}$ whose image by Φ_* is trivial and hence the map induces a morphism of the orbifold fundamental groups.

Let us assume that D_j is contained in the preimage of p_j and let us compute its multiplicity in $\Phi^*(p_j)$, say a_j. If we compose Φ and π the multiplicity of $E_{k,j}$ in the divisor defined by p_j equals ka_j. Let b_j the multiplicity of p_j by ψ; the condition of Definition 3.1 for orbifold morphism implies that $n_j a_j$ divides b_j which is exactly the needed condition for Φ. Hence, the required $\Phi^{\text{orb}} : (\bar{X}, \varphi) \to (\bar{C}, \psi)$ is constructed. □

4. Unbranched and branched orbifold covers

One of the advantages of using orbifold fundamental groups is that we can study standard ramified covers as unbranched orbifold covers. For technical reasons, we restrict our attention to NC-orbifolds.

Definition 4.1. We call an orbifold morphism $\pi : (\bar{Y}, \varphi_Y) \to (\bar{X}, \varphi_X)$ an *orbifold unbranched covering* if the fibers of π are finite and the following equality holds

$$\nu_{(\bar{Y},\varphi_Y)}(y) \cdot \deg \pi_y = \nu_{(\bar{X},\varphi_X)}(x)$$

$\forall x \in X^+_\varphi$, $\forall y \in \pi^{-1}(x)$ (see Definition 1.11).

Remark 4.2. For the shake of simplicity we will often refer to *orbifold unbranched covering* as *unbranched covering.* Note that usual *unbranched covering* are actually *orbifold unbranched covering.*

The main point in Definition 4.1 is that *orbifold unbranched coverings* behave for orbifold fundamental groups as unbranched coverings behave for fundamental groups. In particular, the monodromy action completely determines the orbifold unbranched coverings.

Proposition 4.3. *An orbifold unbranched covering induces an injective morphism on orbifold fundamental groups. Moreover, let $(\bar{X}, \varphi_X)$ be an orbifold and let us denote $G := \pi_1^{\text{orb}}(\bar{X}, \varphi_X)$. Let $H \subset G$ be a finite-index subgroup; then there is an orbifold unbranched covering $\pi : (\bar{Y}, \varphi_Y) \to (\bar{X}, \varphi_X)$ such that $\pi_*(\pi_1^{\text{orb}}(\bar{Y}, \varphi_Y)) = H$. Moreover, $(\bar{Y}, \varphi_Y)$ is essentially unique (i.e., both $Y^+_{\varphi_Y}$ and $S^+_{\varphi_Y}$ are unique up to isomorphism).*

As in the standard case, the cover is said to be *regular* or *Galois* if $H \trianglelefteq G$; in that case the group G/H acts on $Y^+_{\varphi_Y}$ with quotient $X^+_{\varphi_X}$.

Proposition 4.4. *An orbifold unbranched covering satisfies both the path and homotopy lifting properties and are determined by the monodromy representation $\rho : \pi_1^{\text{orb}}(X, \varphi_X) \to \Sigma_n$, $n := \#G/H$.*

A proof of these results can be found basically rewriting [20, Theorem 1.3.9] in the language of orbifolds instead of in the language of branched coverings.

Example 4.5. Consider $\mathbb{P}^1_{2,3,5}$ and the subgroup of $G := \pi_1^{\text{orb}}(\mathbb{P}^1_{2,3,5}) = \langle \mu_2, \mu_3, \mu_5 : \mu_2^2 = \mu_3^3 = \mu_5^5 = (\mu_2\mu_3\mu_5) = 1, \rangle$ given by the kernel of

$$\begin{array}{rccl} \rho: & G & \to & \Sigma_5 \\ & \mu_2 & \mapsto & (1,5)(2,3) \\ & \mu_3 & \mapsto & (1,4,3) \\ & \mu_5 & \mapsto & (1,2,3,4,5). \end{array} \tag{4.1}$$

Note that the preimage of the orbifold point of order 2 has three points. For two of them, the local degree of the map is 2 (and hence their index is 1) whereas on the remaining point the local degree of ρ is 1 (and hence it should become a point of index 2). Analogously, around the orbifold point of order 3, the preimage has three points: two of which will have orbifold index 3 and one with orbifold index 1. Finally, around the orbifold point of index 5, the preimage is a local uniformization. Hence the local conditions on the orbifold points of the covering are given to satisfy Definition 4.1. A simple Euler characteristic computation shows that ρ induces in fact a (non-regular) unbranched covering from $\mathbb{P}^1_{2,3,3}$ to $\mathbb{P}^1_{2,3,5}$ of order 5.

Example 4.6. Consider the following morphism:

$$\begin{array}{rccl} \pi: & \mathbb{P}^1 & \to & \mathbb{P}^1 \\ & [x:y] & \mapsto & [(x^3-y^3)^2 : (x^2+y^2)^3] \end{array} \tag{4.2}$$

Generically, fibers have 6 different preimages. The special fibers are at $[1:0]$ (the roots of $(x^2+y^2)^3$), $[0:1]$ (the roots of $(x^3-y^3)^2$), $[1:1]$ (the roots of $y^2x^2(2xy+3x^2+3y^2)$), and $[2:1]$ (the roots of $(x^4-2yx^3-2xy^3+y^4)(y+x)^2$). Therefore this induces a non-regular unbranched covering from $\mathbb{P}^1_{6(2),2(3)}$ to $\mathbb{P}^1_{3(2),3}$ of order 6 (where the subindex $k(m)$ stands for k points of index m).

Definition 4.7. An orbifold unbranched cover $\pi : (\bar{Y},\varphi_Y) \to (\bar{X},\varphi_X)$ is a *uniformization* of $(\bar{X},\varphi_X)$ if $\bar{Y}$ does not contain points of orbifold index greater than 1. The uniformization will be *Galois* or *regular* if π realizes a quotient of Y^+_φ by the action of a finite group (which may not act freely).

Remark 4.8. As in the standard case, a uniformization (or more generally an unbranched cover) is Galois if and only if the image G_Y of $\pi_1^{\text{orb}}(\bar{Y},\varphi_Y)$ in $G_X := \pi_1^{\text{orb}}(\bar{X},\varphi_X)$ is a normal subgroup (the group action is carried by G_X/G_Y). Recall that if π is a finite Galois uniformization, then the image of a meridian of a component with orbifold index $n_i > 1$ by the monodromy action is a product of cycles of the same length n_i (with no fixed points). This is not a characterization of Galois uniformization as Example 4.9 shows. This condition is called *virtual regularity* in [19].

Also note that saturation (see Definition 2.1) is trivially a necessary condition for the existence of a uniformization.

Example 4.9. Let us consider an orbicurve $(\bar{C},\varphi)$ where $\bar{C}$ is an elliptic curve and the divisor contains two points of index 2. Recall that

$$\pi_1^{\text{orb}}(\bar{C},\varphi) = \langle a,b,x,y \mid a^2 = b^2 = 1, ab = [x,y]\rangle$$

Consider the morphism:

$$\begin{array}{rrcl} \rho : & \pi_1^{\text{orb}}(\bar{C},\varphi) & \to & \Sigma_4 \\ & a,x & \mapsto & (1,2)(3,4) \\ & b & \mapsto & (1,3)(2,4) \\ & y & \mapsto & (1,2,3). \end{array} \tag{4.3}$$

This morphism defines an unbranched orbifold cover; using Riemann-Hurwitz formula the source of this cover is a Riemann surface of genus 3 (with no point of orbifold index greater than 1). This is an example of a uniformization which is virtually regular, but not regular.

For our purposes, a more global and regular definition of unbranched covering will be enough.

Definition 4.10 ([17, 24])**.** Let $(\bar{X},\varphi)$ be an orbifold. We say $(\bar{X},\varphi')$ is a *suborbifold* of $(\bar{X},\varphi)$ (or equivalently $(\bar{X},\varphi)$ is a *superorbifold* of $(\bar{X},\varphi')$) if $\varphi'(D_i)|\varphi(D_i)$ (meaning there exists $k \in \mathbb{Z}\setminus\{0\}$ such that $\varphi(D_i) = k\varphi'(D_i)$ in particular, if $\varphi(D_i) = 0$, then $\varphi'(D_i) = 0$).

On the other side branched orbifold coverings can also be defined. The definitions will be straightforward for the orbicurve case.

Definition 4.11. A Galois covering $\pi : \bar{Y} \to \bar{X}$ between two orbifolds $(\bar{X},\varphi_X)$ and $(\bar{Y},\varphi_Y)$ is a *branched orbifold covering* if there exists a superorbifold structure $(\bar{X},\varphi_s)$ for which π defines an unbranched orbifold covering.

5. Sakuma's formulæ

Given an orbifold $(\bar{X},\varphi_X)$, we will define $b_1^{\text{orb}}(\bar{X},\varphi_X)$ as the rank of the abelianization of $G_{\varphi_X} := \pi_1^{\text{orb}}(\bar{X},\varphi_X)$, that is, $\operatorname{rank}(G_{\varphi_X}/G'_{\varphi_X})$. After taking a superorbifold, all branched orbifold coverings can be assumed to be unbranched. Consider $\pi : (\bar{Y},\varphi_Y) \to (\bar{X},\varphi_X)$ an unbranched covering.

Note that, any unbranched covering $\pi : (\bar{Y},\varphi_Y) \to (\bar{X},\varphi_X)$ produces the action of the group of deck transformations G_φ over $H_1^{\text{orb}}(\bar{Y},\varphi_Y)$ by conjugation, that is, consider $\bar{g} \in G_\varphi$ the class of $g \in G_{\varphi_X}$ and $\bar{x} \in H_1^{\text{orb}}(\bar{Y},\varphi_Y)$ the class of $x \in G_{\varphi_Y}$, then $\bar{g}\cdot\bar{x} = \overline{gxg^{-1}}$. Since

$\overline{ghaxh^{-1}g^{-1}} = \overline{g[h,ax]ag^{-1}gxg^{-1}} = \overline{gxg^{-1}}$, the action is well defined. This action endows $H_1^{\text{orb}}(\bar{Y},\varphi_Y)$ with a module structure over the group ring $\mathbb{Z}[G_\varphi]$. After tensoring by $\mathbb{C}$, the group $H_1^{\text{orb}}(\bar{Y},\varphi_Y)$ acquires a $\mathbb{C}[G_\varphi]$-module structure.

Recall the definition of the characteristic variety of a finitely presented group G. Consider a free resolution of a $\mathbb{C}[H_1(G)]$-module M

$$\mathbb{C}[H_1(G)]^m \xrightarrow{\phi} \mathbb{C}[H_1(G)]^n \to M,$$

then $\mathcal{V}_k(M) := V(F_k(M))$, where $F_k(M)$ is the k-th Fitting ideal (or elementary ideal) of M and $V(I)$ denotes the zero set of the ideal I. Recall that F_k is defined as 0 if $k \le \max\{0, n-m\}$, 1 if $k > n$. Otherwise F_k is the set of minors of order $(n-k+1)\times(n-k+1)$ of a presentation matrix A_ϕ, which is an $n \times m$ matrix with coefficients in $\mathbb{C}[H_1(G)]$. Note that $\mathcal{V}_{k+1}(M) \subseteq \mathcal{V}_k(M)$ and $\mathcal{V}_{n+1}(M) = \emptyset$. For any $\xi \in \mathbb{C}[H_1(G)]$, it is common to define as $\text{null}(M,\xi)$ (nullity of ξ) or $d_\xi(M)$ (depth of ξ) as the maximum $k \in \mathbb{Z}$ such that $\xi \in \mathcal{V}_k(M)$.

We will denote by $\mathcal{V}_k^{\text{orb}}(\bar{X},\varphi_X)$ and $\text{null}^{\text{orb}}(\bar{X},\varphi_X)$, the invariants of the $\mathbb{C}[H_1(G)]$-module M described above where G is the orbifold fundamental group $G := \pi_1^{\text{orb}}(\bar{X},\varphi_X)$.

Unless otherwise stated, all groups orbifold homology groups H_1^{orb} will be considered as $\mathbb{C}[G_\varphi]$-modules. Sakuma's formulæ [21, Theorem 7.3] (see also [15, Proposition 2.5.6]) can be combined and extended in the following result.

Theorem 5.1. *Under the above conditions, if* $\pi : (\bar{Y},\varphi_Y) \to (\bar{X},\varphi_X)$ *is a*

$$b_1^{\text{orb}}(\bar{Y},\varphi_Y) = b_1^{\text{orb}}(\bar{X},\varphi_X) + \sum_{\xi\in\text{Hom}(G_\varphi,\mathbb{C}^*)\setminus\{1\}} \text{null}^{\text{orb}}(\bar{X},\xi) \tag{5.1}$$

where $G_\varphi := G_{\varphi_X}/G_{\varphi_Y}$, $\text{null}^{\text{orb}}(\bar{X},\xi)$ *is the depth of* ξ *considered as a character in* $\pi_1^{\text{orb}}(\bar{X},\varphi_X)$.

Remark 5.2. Note that there is a connection between b_1^{orb} and b_1, namely

$$b_1^{\text{orb}}(\bar{X},\varphi_X) = b_1(X^+_{\varphi_X}) = b_1(\bar{X} \setminus \varphi_X^{-1}(0)).$$

Proof of Theorem 5.1. The proof offered in [21] also works in this context. We will briefly outline the original proof.

Step 1. From representation theory one has

$$H_1^{\text{orb}}(\bar{Y},\varphi_Y) \cong \bigoplus_{\xi\in\text{Hom}(G_\varphi,\mathbb{C}^*)} \left[H_1^{\text{orb}}(\bar{Y},\varphi_Y)\right]_\xi,$$

where

$$\left[H_1^{\text{orb}}(\bar{Y},\varphi_Y)\right]_\xi = \{x \in H_1^{\text{orb}}(\bar{Y},\varphi_Y) \mid g(x) = \xi(g)\cdot x \ \forall g \in G_\varphi\}.$$

Step 2. Using Proposition 4.3 there exists an orbifold $(\bar{X}_\xi,\varphi_\xi)$ such that

$$\left[H_1^{\text{orb}}(\bar{Y},\varphi_Y)\right]_\xi \cong \left[H_1^{\text{orb}}(\bar{X}_\xi,\varphi_\xi)\right]_\xi.$$

The orbifold $(\bar{X}_\xi,\varphi_\xi)$ corresponds to the kernel of $G_{\varphi_X} \to G_\varphi \xrightarrow{\xi} \mathbb{C}^*$, which is of finite index in G_{φ_X}.

Step 3. For $\xi \ne 1$ one has

$$\dim_{\mathbb{C}} \left[H_1^{\text{orb}}(\bar{X}_\xi,\varphi_\xi)\right]_\xi = \text{null}^{\text{orb}}(\bar{X},\xi). \qquad \square$$

Remark 5.3. Note that even if $(\bar{X},\varphi_X)$ is not an NC-orbifold, the notion of unbranched covers may easily be defined as long as orbifolds with (normal) arbitrary singularities are allowed. In that case we may also consider the orbifold $(\hat{X},\hat{\varphi}_X)$ obtained after a sequence of blow-ups such that the transform of $\mathcal{D}$ by this sequence of blow-ups is a normal crossing divisor and $\hat{\varphi}_X$ is defined by homological saturation. The pull-back of π defines another orbifold $(\hat{Y},\hat{\varphi}_Y)$. Note that $\hat{Y} = \hat{Y}^+_\varphi$ and it has only abelian quotient singularities; it is a resolution of $\bar{Y}$ which may have more complicated singularities. There is a natural surjection $\pi_1^{\text{orb}}(\hat{Y}, \hat{\varphi}_Y) \twoheadrightarrow \pi_1^{\text{orb}}(\bar{Y}, \varphi_Y)$ which is not in general an isomorphism. Nevertheless, generalizing Libgober's arguments in [18], it can be proved that the first Betti numbers coincide.

To illustrate Theorem 5.1, we can compute the genus of the uniformization of $\bar{X} := \mathbb{P}^1_{d_1,\dots,d_{n+1}}$ in some cases where for instance the abelianization map $\pi : (\bar{X}_{\text{ab}}, \varphi_{\text{ab}}) \to (\bar{X}, \varphi_X)$ is a uniformization. According to [20, Theorem 1.3.43] this is the case whenever d_i divides

$$\operatorname{lcm}(d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_{n+1}).$$

On the one hand one can directly use the Riemann-Hurwitz formula to obtain

$$\chi(\bar{X}_{\text{ab}}) = 2 - 2g(\bar{X}_{\text{ab}}) = (1-n)\frac{d_1 \cdot \ldots \cdot d_{n+1}}{d} + \sum_{k=1}^{n+1} \frac{d_1 \cdot \ldots \cdot d_{n+1}}{dd_k},$$

where $d := \operatorname{lcm}(d_1, \dots, d_{n+1})$. This implies

$$g(\bar{X}_{\text{ab}}) = \frac{d_1 \cdot \ldots \cdot d_{n+1}}{2d} \left[-1 + \sum_{k=1}^{n+1} \left(1 - \frac{1}{d_k} \right) \right] + 1. \tag{5.2}$$

Example 5.4. Consider the case $1 \leq d_1 \leq \cdots \leq d_n \leq d_{n+1} = d = \operatorname{lcm}(d_1, \dots, d_n)$. This is a particular case of the result mentioned above and hence the universal abelian covering $\pi : \bar{X}_{\text{ab}} \to \bar{X} := \mathbb{P}^1_{d_1,\dots,d_{n+1}}$ is in fact a uniformization. Using Theorem 5.1 one can obtain $b_1(\bar{X}_{\text{ab}}) = b_1^{\text{orb}}(\bar{X}_{\text{ab}})$ by counting the characters in the orbifold characteristic variety of $\bar{X}$. Note that the space of characters on $\pi_1^{\text{orb}}(\bar{X})$ is a union of $(n+1)$-tuples

$$\mathbb{T} := \{(\xi_1, \dots, \xi_n, \xi_{n+1}) \mid \xi_j \in \mu_{d_j}, \prod_{k=1}^{n+1} \xi_k = 1\} \subset (\mathbb{C}^*)^{n+1},$$

where $\mu_n \subset \mathbb{C}^*$ is the subgroup of n-th roots of unity. Since the equation in the definition of $\mathbb{T}$ can always be solved for ξ_{n+1} one has that $\mathbb{T} \cong \mu_{d_1} \times \cdots \times \mu_{d_n}$. Denote by $\ell(\xi)$ the length of $\xi \in \mathbb{T}$, that is, the number of non-trivial coordinates of ξ. From [4, Proposition 3.11] one deduces that $\operatorname{depth}(\xi) = \ell(\xi) - 2$. Denote by $\ell'(\xi)$ the length of ξ in the first n coordinates, that is, its length as an element of $\mu_{d_1} \times \cdots \times \mu_{d_n}$. Note that $\ell(\xi) = \ell'(\xi) + 1$ unless its last coordinate is 1, in which case $\ell(\xi) = \ell'(\xi)$. Therefore if we define

$$b_1'(\bar{X}_{\text{ab}}) := \sum_{\xi \in \mathbb{T}} \ell'(\xi).$$

Then $b_1'(\bar{X}_{\text{ab}}) - b_1(\bar{X}_{\text{ab}}) = \frac{D}{d}$, where $D := d_1 \cdot \ldots \cdot d_n$ which is the order of the kernel of the map $\mu_{d_1} \times \cdots \times \mu_{d_n} \to \mu_d$ given by multiplication. Hence,

$$b_1'(\bar{X}_{\text{ab}}) = \sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (\#I - 1) \prod_{i \in I} (d_i - 1)$$

and thus,

$$b_1(\bar{X}_{\text{ab}}) = \left[\sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (\#I - 1) \prod_{i \in I} (d_i - 1) \right] - \frac{D}{d}.$$

Using (5.2) this implies

$$\frac{D}{d^2} \left[-1 + \sum_k \left(1 - \frac{1}{d_k} \right) \right] + 2 = \left[\sum_{\emptyset \neq I \subseteq \{1,\dots,n\}} (\#I - 1) \prod_{i \in I} (d_i - 1) \right] - \frac{D}{d}.$$

For computational purposes, the depth $\operatorname{null}^{\text{orb}}(\bar{X},\xi)$ can also be obtained from $\mathring{X}_\varphi$.

Proposition 5.5. *Under the above conditions,*

$$\mathcal{V}_k^{\text{orb}}(X,\varphi)\setminus\{1\}=\mathcal{V}_k(\mathring{X}_\varphi)\cap\mathbb{T}_\varphi\setminus\{1\},$$

where $\mathbb{T}_\varphi$ is the inclusion of $\mathbb{T}(X,\varphi)$ into $\mathbb{T}(\mathring{X}_\varphi)$ given by the surjection $\pi_1(\mathring{X}_\varphi)\twoheadrightarrow\pi_1^{\text{orb}}(X,\varphi)$.

Proof. The proof is analogous to the one shown in [5, Proposition 2.26] for $k=1$. ☐

Example 5.6. Consider the space $\mathcal{M}$ of sextics with the following combinatorics:

(1) $\mathcal{C}$ is a union of a smooth conic $\mathcal{C}_2$ and a quartic $\mathcal{C}_4$.
(2) $\operatorname{Sing}(\mathcal{C}_4)=\{P,Q\}$ where Q is a cusp of type $\mathbb{A}_4$ and P is a node of type $\mathbb{A}_1$.
(3) $\mathcal{C}_2\cap\mathcal{C}_4=\{Q,R\}$ where Q is a $\mathbb{D}_7$ on $\mathcal{C}$ and R is a $\mathbb{A}_{11}$ on $\mathcal{C}$.

The space $\mathcal{M}=\mathcal{M}^{(1)}\cup\mathcal{M}^{(2)}$ is a union of two connected components. Any such sextics $\mathcal{C}_6^{(i)}=\mathcal{C}_2^{(i)}\cup\mathcal{C}_4^{(i)}$ in $\mathcal{M}^{(i)}$ can be characterized by the fact that the conic q passing through R and Q such that $\operatorname{mult}_R(q,\mathcal{C}_2^{(i)})=\operatorname{mult}_R(q,\mathcal{C}_4^{(i)})=3$, and $\operatorname{mult}_Q(q,\mathcal{C}_2^{(i)})=1$ satisfies $\operatorname{mult}_Q(q,\mathcal{C}_4^{(i)})=3+i$. The following example is presented in [2], we refer to it for details. Consider the orbifolds $(\mathbb{P}^2,\varphi_i)$, where $\varphi_i(\mathcal{C}_4^{(i)})=0$ and $\varphi_i(\mathcal{C}_2^{(i)})=2$. Using Proposition 5.5 and [5, Proposition 3.1] it can be checked that

$$\mathcal{V}_1^{\text{orb}}(\mathbb{P}^2,\varphi_i)\setminus\{1\}=\begin{cases}\emptyset & \text{if } i=1\\ \{(1,-1)\} & \text{if } i=2,\end{cases}\quad \mathcal{V}_2^{\text{orb}}(\mathbb{P}^2,\varphi_i)\setminus\{1\}=\emptyset$$

and hence, using Sakuma's formula 5.1 one has

$$b_1^{\text{orb}}(Y_i,\varphi_{Y_i})=\begin{cases}0 & \text{if } i=1\\ 1 & \text{if } i=2,\end{cases}$$

where (Y_i,φ_{Y_i}) denotes the unramified covering of $(\mathbb{P}^2,\varphi_i)$, since $b_1^{\text{orb}}(\mathbb{P}^2,\varphi_i)=0$. This provides an alternative way to show that $\mathcal{C}_6^{(1)}$ and $\mathcal{C}_6^{(2)}$ form a Zariski pair, that is, two curves with the same combinatorics but different embedding in $\mathbb{P}^2$. In other words, we prove that $(\mathbb{P}^2,\mathcal{C}_6^{(1)})$ and $(\mathbb{P}^2,\mathcal{C}_6^{(2)})$ are not homeomorphic by showing that $\pi_1^{\text{orb}}(\mathbb{P}^2,\varphi_1)$ and $\pi_1^{\text{orb}}(\mathbb{P}^2,\varphi_2)$ are not isomorphic. Note that any homeomorphism of $\mathbb{P}^2$ sending $\mathcal{C}_6^{(1)}$ to $\mathcal{C}_6^{(2)}$ should send a meridian around $\mathcal{C}_2^{(1)}$ to a meridian around $\mathcal{C}_2^{(2)}$ and hence $\pi_1^{\text{orb}}(\mathbb{P}^2,\varphi_1)$ and $\pi_1^{\text{orb}}(\mathbb{P}^2,\varphi_2)$ should be isomorphic.

We can readily recover, using Theorem 5.1 and Proposition 5.5, known computations of the first Betti number of Hirzebruch congruence covers associated to line arrangements in $\mathbb{P}^2$, see [16, 23].

For example, consider the orbifold $X=(\mathbb{P}^2,\varphi)$ associated to the 6 lines Ceva arrangement, where φ takes value n for all lines. Then let Y be the orbifold cover associated to the abelianization $\pi_1^{\text{orb}}(X)\to(\mathbb{Z}/n\mathbb{Z})^5$. A straightforward counting argument shows that $b_1^{\text{orb}}(Y)=5(n-1)(n-2)$.

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