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Quasi-projectivity, Artin-Tits Groups, and Pencil Maps

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Abstract. We consider the problem of deciding if a group is the fundamental group of a smooth connected complex quasi-projective (or projective) variety using Alexander-based invariants. In particular, we solve the problem for large families of Artin-Tits groups. We also study finiteness properties of such groups and exhibit examples of hyperplane complements whose fundamental groups satisfy F_{k-1} but not F_k for any k.

To Anatoly Libgober

1. Introduction

The interest in characterizing or finding properties of (quasi)projective groups, that is, groups that can be obtained as fundamental groups of (quasi)projective varieties, has been known since J.-P. Serre [**51**] (who raised questions about general characterization of such groups) and O. Zariski [**58**] (who asked whether or not they are residually finite). The latter question was negatively answered by D. Toledo [**54**]. Only recently, after the work published by A. Libgober [**42**] and D. Arapura [**3**], the question about characterization has experienced an increasing interest.

Serre's original question is far from being solved, but at least new effective obstructions have been found and used for showing that certain groups cannot be (quasi)projective in a series of papers by Dimca-Suciu-Papadima [**33, 35, 36, 34**]. Such obstructions are mostly based on Alexander invariants and finiteness conditions.

In this paper, we are mainly concerned about determining which Artin groups are quasi-projective and which ones are not by exhibiting quasi-projective varieties that realize them in the first case and by using obstructions in the second case. Finally, we are also interested in finiteness conditions by means of studying certain subgroups of quasi-projective groups that appear as kernel of pencil based maps. The reason for considering Artin groups has to do with two facts: first of all, it is a large class of groups that can be easily described with a decorated 1-graph, but

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most importantly, the way these groups are built is by using *algebraic* relations, i.e. relations appearing in local fundamental groups of algebraic singularities of type $\mathbb{A}_k$.

The Alexander invariant of a (quasi)projective variety M is the first homology group of the universal abelian cover considered as a module over the group of deck transformations. As it turns out, an important invariant of such a module is related with the space of rank-one representations of the (quasi)projective fundamental group G of M. In fact one has a stratification of the space of local systems of rank 1 whose strata are called *characteristic varieties* of M (or G). Characteristic varieties of quasi-projective varieties were first considered by D. Arapura (cf.[**4**]) who gave a structure theorem on the biggest stratum, namely, he proved that it is a finite union of translated subtori (by a torsion element) and a finite union of unitary elements. This was completed by A. Libgober [**45**] (in the local case) and N. Budur [**19**] (in the global case following the work of C. Simpson [**52**] in the projective case) showing that the unitary elements should in fact be torsion elements. This structure theorem and new properties presented in several papers (see [**30, 37, 37, 6**] among others) impose restrictions on G. Our purpose here is to show how these restrictions are enough to prove the non quasi-projectivity of a large family of Artin-Tits groups. Also, an infinite number of realizable triangular Artin-Tits groups are shown.

As for the finiteness conditions, in a series of papers [**56, 55**] from 1960's C.T.C. Wall studied general finiteness properties of groups and CW-complexes. A group G is said to be of type F_n if it has an Eilenberg-MacLane complex $K(G,1)$ with finite n-skeleton. Clearly G is finitely generated if and only if it is F_1 and finitely presented if and only if it is F_2. An interesting example of a finitely presented group which is not finitely presented was given by J. Stallings in [**53**]. A group G is said to be of type FP_n if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ admits a projective resolution which is finitely generated in dimensions $\leq n$. Note that G is of type FP_1 if and only if it is finitely generated. In general, the property F_n implies the FP_n property, and they are equivalent in the presence of F_2. But, as shown by Bestvina-Brady [**11**] FP_2 does not imply finite presentation. The first example of a group which is F_2 but not FP_3 was given by J. Stallings in [**53**]. Afterwards R. Bieri [**13, 12**] generalized Stallings' examples to the following family: Let $G_n = \mathbb{F}_2 \times \cdots \times \mathbb{F}_2$ be the direct product of n free groups, each of rank 2. Then the kernel of the map taking each generator to $1 \in \mathbb{Z}$ is F_{n-1} but not F_n. Stallings' examples mentioned above are the cases $n = 2$ and $n = 3$. Analogous, more general kernels, now known as Bestvina-Brady groups, coming from arbitrary right-angled Artin groups, were considered in [**11**].

In our work, we propose an approach towards an answer to Serre's question based on the cohomology associated to rank 1 complex representations of $G = \pi_1(M)$ and its relationship with the geometric properties of pencils of hypersurfaces on M. This point of view will lead us naturally into considering the finiteness properties of G. In this respect, we will pay particular attention to the groups G which can be realized as $\pi_1(\mathbb{P}^r \setminus \mathcal{A})$ for $\mathcal{A}$ a hypersurface arrangement in $\mathbb{P}^r$.

We find that the Bieri-Stallings groups appear as $\pi_1(\mathbb{P}^r \setminus \mathcal{A})$ for some arrangements of hyperplanes $\mathcal{A}$. In this way, we exhibit a family of hyperplane arrangement groups that are F_{n-1} but not F_n, for any $n \geq 3$. The entire class of quasi-projective

Bestvina-Brady groups was characterized in [**35**]. We will refine elsewhere that result by showing that all quasi-projective Bestvina-Brady groups are in fact hyperplane arrangement groups. Here we only give the argument for the Bieri-Stallings groups.

The paper is organized as follows: in the first three sections we give an expository presentation of the objects, techniques, and tools necessary. In particular section §2 will deal with definitions and properties of (quasi)projective groups, sections §3 and §4 with definitions and properties of two of the most important invariants of finitely presented groups (in this context) such as the Characteristic Varieties and Alexander Invariants respectively. In section §5 we will study the question of what Artin groups are (quasi)projective. Finally, in section §6 we will study some finiteness properties of quasi-projective groups via the pencil map construction.

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2. (Quasi)-projective groups

In this section we will describe the class of finitely presented groups we are interested in. It is known that any finitely presented group G can be realized as $\pi_1(M)$ of a smooth connected complex manifold M. If M is allowed to be compact, then its complex dimension cannot be in general less than 4. In terms of $\pi_1(M)$, if one allows M to be open, then M might be chosen just to be 2-dimensional. If in addition, M is required to be Stein, then again its dimension cannot be less than 4.

However, if M is required to be a smooth complex algebraic variety, then not any finitely presented G can be realized as $\pi_1(M)$. In the compact case there is a well-known restriction, namely G must be 1-formal (cf. [**28**]). Rational homotopy theory imposes restrictions on $\pi_1(M)$ even in the non-compact case, where M is quasi-projective, as shown in [**46**], but 1-formality is no longer guaranteed.

Definition 2.1. We call a group G *projective* (resp. *quasi-projective*) if $G = \pi_1(M)$ for M a smooth connected complex projective (resp. quasi-projective) variety.

We denote by $\mathcal{QP}$ the set of all quasi-projective groups, and by $\mathcal{P}$ the set of all projective groups. Obviously we have an inclusion, $\mathcal{P} \subset \mathcal{QP}$, which is in fact strict, since groups G with odd first Betti number $b_1(G)$ cannot be projective, whereas for instance, free groups of odd rank are clearly quasi-projective.

Remark 2.2. Standard Lefschetz-Zariski-Van Kampen principle guarantees that all groups in $\mathcal{QP}$ are finitely presented. Moreover, any (quasi)projective group is the fundamental group of a (quasi)projective surface.

Let $\mathcal{F}_1$ stand for the set of all finitely presented 1-formal groups. We pointed out above that there is an inclusion $\mathcal{P} \subset \mathcal{F}_1$. This inclusion is also strict by an easy odd first Betti number argument. However, in general, quasi-projectivity and 1-formality are independent properties (cf. [**37, 33**]). An important class of groups which are both 1-formal and quasi-projective is that of fundamental groups

of complements to hypersurfaces in a projective space. We denote by $\mathcal{H}$ the set of such groups, that is, $G = \pi_1(M)$ where $M = \mathbb{P}^N \setminus V$ for a hypersurface V in $\mathbb{P}^N$; we call these groups *hypersurface* or *curve* groups (again because of Lefschetz-type arguments). Then $\mathcal{H} \subset \mathcal{F}_1 \cap \mathcal{QP}$, as shown in [**40**] using techniques from [**46**].

PROPERTIES 2.3. The following properties have been proved in the literature:

- ($\mathcal{P}1$) If G_1, G_2 are in $\mathcal{QP}$, then the direct product $G_1 \times G_2$ is also in $\mathcal{QP}$.
- ($\mathcal{P}2$) If G_1, G_2 are in $\mathcal{F}_1$, then the direct product $G_1 \times G_2$ is also in $\mathcal{F}_1$.
- ($\mathcal{P}3$) The free product $G_1 * G_2$ is not necessarily in $\mathcal{QP}$ (cf. [**37, 33**]). For example, $\pi_1 * \pi_2$ is not quasi-projective if π_1, π_2 are fundamental groups of Riemann surfaces of non-zero genus.
- ($\mathcal{P}4$) By M. Gromov [**39**], a non-trivial free product $G_1 * G_2$ is never projective.
- ($\mathcal{P}5$) If a group G has more than one end, then $G \notin \mathcal{P}$. Also if G surjects with finitely generated kernel onto a group with infinitely many ends, then again $G \notin \mathcal{P}$ (cf. [**5**]).
- ($\mathcal{P}6$) If G_1, G_2 are in $\mathcal{F}_1$, then so is $G_1 * G_2$, see [**37, 33**].
- ($\mathcal{P}7$) The classes $\mathcal{P}$ and $\mathcal{QP}$ are closed under taking finite index subgroups. The same is not true in general for the class $\mathcal{F}_1$, see [**32**, Example 2.9].

EXAMPLE 2.4. The fundamental group π_g of a smooth compact projective curve C_g of genus g, which admits a presentation of the form $\pi_g = \langle a_i, b_i, 1 \leq i \leq g \mid \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \rangle$, is the most basic projective group. Maybe the simplest infinite quasi-projective groups are the free groups, both abelian $\mathbb{Z}^n$, and non-abelian $\mathbb{F}_n$. Of these, only $\mathbb{Z}^{2n}$ are projective. Note that $\mathbb{F}_{2g+p-1} = \pi_1(C_{g,p})$, where $C_{g,p}, p \geq 1$ is C_g punctured p times.

We are going to extend these examples to *orbifold groups*.

DEFINITION 2.5. An *orbifold* X_φ is a quasi-projective Riemann surface X with a function $\varphi : X \to \mathbb{N}$ with value 1 outside a finite number of points.

DEFINITION 2.6. For an orbifold X_φ, let $p_1, \ldots, p_n$ the points such that $\varphi(p_j) := m_j > 1$. Then, the *orbifold fundamental group* of X_φ is

$$\pi_1^{\mathrm{orb}} := \pi_1(X \setminus \{p_1, \ldots, p_n\}) / \langle \mu_j^{m_j} = 1 \rangle$$

where μ_j is a meridian of p_j in X. We oftentimes denote X_φ simply by $X_{m_1,\ldots,m_n}$.

DEFINITION 2.7. A dominant algebraic morphism $\rho : Y \to X$ defines an *orbifold morphism* $Y \to X_\varphi$ if for all $p \in X$, the divisor $\rho^*(p)$ is a $\varphi(p)$-multiple.

REMARK 2.8. An orbifold morphism $\rho : S \to C_\varphi$ with connected generic fiber F defines an exact sequence

$$\pi_1(F) \to \pi_1(S) \to \pi_1^{\mathrm{orb}}(C_\varphi) \to 1.$$

EXAMPLE 2.9. Any orbifold fundamental group $G := \pi_1^{\mathrm{orb}}(C_\varphi)$ can be realized as $\pi_1(S)$ for S a ruled affine surface, as shown by J. Bertin [**10**]. If the underlying curve is not projective, it is easily seen that G is not projective, since it admits a finite-index free subgroup.

On the other hand, if C is projective, one can show that G is projective, using arguments used by Morgan-Friedman in [**38**]. They show that, for an elliptic fibration $\pi : S \to C$ with at least one fiber having singular reduction, the exact sequence in Remark 2.8 induces an isomorphism of $\pi_1(S)$ onto G. Such an elliptic fibration

can be found using logarithmic transformations. Moreover, it is proved in [**38**] that such a surface can be deformed to become projective if and only if $b_1(S)$ is even, and thus the result follows.

In light of the discussion above, it seems legitimate to ask the following:

QUESTION 2.10. Which finitely presented groups G are quasi-projective?

This question, originally posed by J.-P. Serre in [**51**], was taken up by A. Libgober in [**42**] using Alexander invariants. This is the point of view that will be considered here as well.

3. Characteristic varieties

Throughout this paper G will represent a finitely presented group. Characteristic varieties are invariants of G which can be computed using any connected topological space X (having the homotopy type of a finite CW-complex) such that $G = \pi_1(X, x_0)$, $x_0 \in X$. Let us consider a character $\xi : G \to \mathbb{C}^*$; recall that the space of characters is

$$\mathbb{T}_G = \operatorname{Hom}(G, \mathbb{C}^*) = \operatorname{Hom}(H_1(X;\mathbb{Z}), \mathbb{C}^*) = H^1(X;\mathbb{C}^*). \tag{1}$$

Given such a character ξ, one can construct a local system of coefficients over X as follows. Let $\rho : \tilde{X} \to X$ be the universal abelian covering of X. The group $H_1(X;\mathbb{Z})$ acts freely (on the right) on $\tilde{X}$. The local system of coefficients $\mathbb{C}_\xi$ is defined as

$$\pi_\xi : \tilde{X} \times_{H_1(X;\mathbb{Z})} \mathbb{C} \to X \text{ where } \tilde{X} \times_{H_1(X;\mathbb{Z})} \mathbb{C} := \Big(\tilde{X} \times \mathbb{C}\Big) \Big/ (x,t) \sim (x^h, \xi(h^{-1})t).$$

DEFINITION 3.1. The k-th *characteristic variety* of G is the subvariety of $\mathbb{T}_G$, defined by:

$$\operatorname{Char}_k(G) = \{\xi \in \mathbb{T}_G \mid \dim H^1(G, \mathbb{C}_\xi) \geq k\},$$

where $H^1(G, \mathbb{C}_\xi)$ is the twisted cohomology with coefficients in the local system ξ.

REMARK 3.2. There is a simple way to describe this cohomology. Let us suppose that X is a CW-complex. Then, $\tilde{X}$ inherits also a CW-complex structure. For each cell σ in X we fix a cell $\tilde{\sigma}$ in $\tilde{X}$ such that $\rho(\tilde{\sigma}) = \sigma$. Then, the set of cells of $\tilde{X}$ is

$$\{\tilde{\sigma}^h \mid \sigma \text{ cell of } X, h \in H_1(X;\mathbb{Z})\}.$$

In particular, the chain complex $C_*(\tilde{X};\mathbb{C})$ is a free Λ-module with basis $\{\tilde{\sigma}\}$, where $\Lambda := \mathbb{C}[H_1(X;\mathbb{Z})]$ is the group algebra of $H_1(X;\mathbb{Z})$. Evaluating the elements in $H_1(X;\mathbb{Z})$ by ξ we obtain a chain complex $C_*(X;\mathbb{C})^\xi$, which as a vector space, is isomorphic to $C_*(X;\mathbb{C})$ but whose differential is twisted.

REMARK 3.3. Let us assume that G is finitely generated; then so is $H_1(X;\mathbb{Z})$. Let $n := \operatorname{rank} H_1(X;\mathbb{Z})$ and let Tors_G be the torsion subgroup of $H_1(X;\mathbb{Z})$. Then $\mathbb{T}_G$ is an abelian complex Lie group with $|\operatorname{Tors}_G|$ connected components (each one isomorphic to $(\mathbb{C}^*)^n$) satisfying the following exact sequence:

$$1 \to \mathbb{T}_G^{\mathbf{1}} \to \mathbb{T}_G \to \operatorname{Tors}_G \to 1,$$

where $\mathbb{T}_G^{\mathbf{1}}$ is the connected component containing the trivial character $\mathbf{1}$.

EXAMPLE 3.4. Let $G := \langle x, t \mid xt^2x = t^2x^2t \rangle$. In this case $\mathbb{T}_G = \mathbb{C}^*$ and $\operatorname{Char}_1(G)$ is defined by $z^2 - 2z + 2 = 0$.

These invariants are very close to other ones, like the Green-Lazarsfeld's invariant. They were studied by A. Beauville [**9**] for *projective surfaces* and his approach was extended by D. Arapura [**4**] to quasi-projective varieties. There are also important contributions from C. Simpson [**52**], N. Budur [**19**], T. Delzant [**29**], and A. Dimca [**30**] for the structure of characteristic varieties (for compact Kähler or quasi-projective manifolds). In the hypersurface case, A. Libgober [**43, 44**] proposed a computation method where the knowledge of the group is not required and showed that they reflect deep algebraic properties of the manifolds.

The properties of characteristic varieties of quasi-projective groups provide strong obstructions for a group to be in $\mathcal{QP}$. Following Remark 3.2, it is not difficult to prove that the characteristic varieties of a quasi-projective group G are algebraic subvarieties of $\mathbb{T}_G$. The following result can be found in one form or another in the literature:

THEOREM 3.5 (Arapura [**4**], Budur [**19**]). *If $G \in \mathcal{QP}$, then all irreducible components of $\mathrm{Char}_k(G)$ are subtori of the character torus $\mathbb{T}_G$ (possibly translated by a torsion character).*

Note that, in particular, the group of Example 3.4 cannot be quasi-projective.

The main part of Theorem 3.5 was proved by D. Arapura, the only part missing in his result is the fact that isolated points are not only unitary, but in fact torsion characters. The main result of D. Arapura involves more obstructions.

THEOREM 3.6 (Arapura [**4**]). *Let Σ be an irreducible component of $\mathrm{Char}_1(G)$. Then,*

1. *If $\dim \Sigma > 0$ then there exists a surjective morphism $\rho : X \to C$, C algebraic curve, and a torsion element σ such that $\Sigma = \sigma\rho^*(H^1(C;\mathbb{C}^*))$.*
2. *If $\dim \Sigma = 0$ then Σ is unitary.*

We give here a refined statement which will be proved in a forthcoming paper [**7**].

THEOREM 3.7. *Let $G \in \mathcal{QP}$ and Σ be an irreducible component of $\mathrm{Char}_k(G)$, $k \geq 1$. Then one of the two following statements holds:*

- *There exists a surjective orbifold morphism $\rho : X \to C_\varphi$ and an irreducible component Σ_1 of $\mathrm{Char}_k(\pi_1^{orb}(C_\varphi))$ such that $\Sigma = \rho^*(\Sigma_1)$.*
- *Σ is an isolated torsion point.*

REMARK 3.8. In many cases, isolated torsion points come from orbifold morphisms. It will be proved in a forthcoming paper [**6**] that this is not always the case.

DEFINITION 3.9. Let Σ be an irreducible component of $\mathrm{Char}_k(G)$ and $\dim_{\mathbb{C}} \Sigma \geq 1$, consider $\mathrm{Shd}\,\Sigma$ (not necessarily in $\mathrm{Char}_k(G)$) parallel to Σ (that is, $\Sigma = \rho\,\mathrm{Shd}\,\Sigma$ for some $\rho \in \mathbb{T}_G$) and such that $\mathbf{1} \in \mathrm{Shd}\,\Sigma$. Such a subtorus $\mathrm{Shd}\,\Sigma$ will be referred to as the *shadow* of Σ.

The result and the proof of Theorem 3.7 provide many obstructions which will be used in §5.

PROPOSITION 3.10. *Let $G \in \mathcal{QP}$ and Σ_1, Σ_2 different irreducible components of $\mathrm{Char}_k(G)$, $k \geq 1$ of positive dimension. Then*

(1) *If the intersection* $\Sigma_1 \cap \Sigma_2$ *is non-empty, it consists of isolated torsion points, which are of torsion type.*
(2) *Their shadows are either equal or have* $\mathbf{1}$ *as an isolated intersection point.*
(3) *If* Σ_1 *is not a component of* $\mathrm{Char}_{k+1}(G)$ *and* $p \in \Sigma_1 \cap \mathrm{Char}_{k+1}(G)$ *then* p *is a torsion point.*
(4) Σ_1 *is an irreducible component of* $\mathrm{Char}_\ell(G)$, $1 \leq \ell \leq k$.

REMARK 3.11. Parts (1) and (2) in Proposition 3.10 can be found in [**34**], Part (3) is proved in [**31**], and Part (4) is immediate.

PROPOSITION 3.12. *Let* $G \in \mathcal{QP}$ *and consider* Σ *an irreducible component of* $\mathrm{Char}_k(G)$, $k \geq 1$ *of positive dimension* d. *Then:*

(1) *If* $\mathbf{1} \in \Sigma$, *then* $k \leq d-1$. *Moreover, one can ensure that* Σ *is a component of* $\mathrm{Char}_{d-2}(G)$ *(resp.* $\mathrm{Char}_{d-1}(G)$*) if* d *even (resp. odd).*
(2) *If* $\mathbf{1} \notin \Sigma$, *then* Σ *is a component of* $\mathrm{Char}_d(G)$.
(3) *If* $\mathbf{1} \notin \Sigma$ *and* $d > 2$, *then its shadow is an irreducible component of* $\mathrm{Char}_1(G)$.
(4) *If* $\mathbf{1} \notin \Sigma$ *and* $d = 2$, *then its shadow is an irreducible component of* $\mathrm{Char}_1(G)$ *if and only if it is for* $\mathrm{Char}_2(G)$.
(5) *If* $\mathbf{1} \notin \Sigma$ *and* $d = 1$, *then its shadow is not an irreducible component of* $\mathrm{Char}_1(G)$.

REMARK 3.13. The results in Proposition 3.12(4), and (5) can be found in [**31**]. All of them will appear in [**6**]. The cases where the shadow is not in the characteristic variety correspond, according to Theorem 3.7, to either orbifold pencils over $\mathbb{C}^*$ or elliptic pencils.

DEFINITION 3.14. A subspace $0 \neq V \subset H^1(G, \mathbb{C})$ is called 0-*isotropic* (resp. 1-*isotropic*) if the restriction $\cup_V$ of the cup-product map $\cup_G : \bigwedge^2 H^1(G, \mathbb{C}) \to H^2(G, \mathbb{C})$ is equivalent to $\cup_C : \bigwedge^2 H^1(C, \mathbb{C}) \to H^2(C, \mathbb{C})$ for C a non-compact (resp. compact) smooth connected complex curve (see also [**37**, Definition 6.5]).

PROPOSITION 3.15 ([**37, 34**]). *Let* Σ *be an irreducible component of* $\mathrm{Char}_1(G)$, $G \in \mathcal{QP}$. *Let* $V \subset H^1(G, \mathbb{C})$ *be the tangent space of* $\operatorname{Shd}\Sigma$ *at* $\mathbf{1}$. *Then* V *is* p*-isotropic,* $p \in \{0, 1\}$.

We finish with a new obstruction which will appear in a forthcoming paper [**7**].

PROPOSITION 3.16. *Let* Σ_1 *be an irreducible component of* $\mathrm{Char}_k(G)$, $G \in \mathcal{QP}$, *and let* Σ_2 *be an irreducible component of* $\mathrm{Char}_\ell(G)$. *If* $\xi \in \Sigma_1 \cap \Sigma_2$ *a torsion point, then* $\xi \in \mathrm{Char}_{k+\ell}(G)$.

4. Alexander Invariants

Let G be a finitely generated group, consider $H := H_1(G; \mathbb{Z}) = G/G'$ and ψ a surjective homomorphism from H onto a group A (note A has to be abelian). We are mostly interested in the case $H = A$ (as in §3) but it is not more difficult to work in this more general situation. Note that there is a short exact sequence

$$0 \ \to \ K \ \stackrel{\varphi}{\to} \ G \ \stackrel{\tilde{\psi}}{\to} \ A \ \to \ 0.$$

The group A acts on $M_G^A := H_1(K; \mathbb{Z})$ by conjugation. This makes M_G^A a module over the group algebra $\Lambda_{\mathbb{Z}}^A := \mathbb{Z}[A]$. Note that if $A = H$ then $K = G'$

DEFINITION 4.1. . The $\Lambda_{\mathbb{Z}}^A$-module M_G^A is the *first Alexander Invariant* of G with respect to A. Analogously, one can define the *first Alexander Invariant* $M_{\mathbb{K},G}^A$ of G with respect to A over $\mathbb{K}$ as a $\Lambda_{\mathbb{K}}^A$-module, where $\Lambda_{\mathbb{K}}^A := \mathbb{K} \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z}}^A$. We will drop the superscript A when $A = H$.

This is a powerful invariant, but it is not easy to deal with it directly. A classical object of study is the set of Fitting invariants of M_G^A. Consider

$$\left(\Lambda_{\mathbb{Z}}^A\right)^m \xrightarrow{\phi} \left(\Lambda_{\mathbb{Z}}^A\right)^r \to M_G^A \to 0$$

a finite free presentation of the $\Lambda_{\mathbb{Z}}^A$-module M_G^A. Let $\mathrm{Mat}(\phi)$ be the $(r \times m)$ matrix, with coefficients in $\Lambda_{\mathbb{Z}}^A$, of ϕ.

DEFINITION 4.2. The k-th *Fitting ideal* of M_G^A is defined as the ideal generated by

$$f_k^A := \begin{cases} 0 & \text{if } k \leq \max\{0, r-m\} \\ 1 & \text{if } k > r \\ \text{minors of } \mathrm{Mat}(\phi) \text{ of order } (r-k+1) & \text{otherwise.} \end{cases}$$

Such an ideal does not depend on the free presentation of M_G^A and it is denoted by F_k^A if no ambiguity seems likely to arise. Analogously, one can define $F_k(M_{\mathbb{K},G}^A)$.

REMARK 4.3. For computational purposes (when A is a free abelian group, for simplicity), if one writes a presentation of G of the following type

$$G = \langle x_1, ..., x_r, y_1, ..., y_s : \bar{R}(\bar{x}, \bar{y}) = 0 \rangle,$$

where $x_1, ..., x_r$ freely generate A and $y_1, ..., y_s$ are trivial after the abelianization morphism, then $M_{\mathbb{Z}}^A$ admits a presentation

$$R\Lambda_{\mathbb{Z}}^A \oplus J\Lambda_{\mathbb{Z}}^A \xrightarrow{\phi} \left(\bigoplus_{1 \leq i < j \leq r} x_{ij} \Lambda_{\mathbb{Z}}^A \right) \bigoplus \left(\bigoplus_{k=1}^{s} y_k \Lambda_{\mathbb{Z}}^A \right) \to M_{\mathbb{Z}}^A \to 0 \tag{2}$$

as a $\Lambda_{\mathbb{Z}}^A$-module generated by $x_{ij} := [x_i, x_j]$, $1 \leq i < j \leq r$ and $y_1, ..., y_s$. A complete set of relations can be given by the *Jacobi relations*:

$$(t_i - 1)x_{jk} + (t_j - 1)x_{ki} + (t_k - 1)x_{ij} = 0$$

and the rewriting of $\bar{R}(\bar{x}, \bar{y}) = 0$ in terms of x_{ij} and y_k. For a detailed description of this module see [**8**, § 2.5].

The ring $\Lambda_{\mathbb{C}}^A$ as a $\mathbb{C}$-algebra of finite type, is the ring of functions of an affine variety $\mathbb{T}_G^A$ which is a (maybe non-connected) complex torus (like in (1), replacing H by A)

DEFINITION 4.4. The reduced zero locus of $F_k(M_{\mathbb{C},G}^A)$ is the *k-th characteristic variety of G with respect to A* and is denoted by $\mathrm{Char}_k^A(G)$.

REMARK 4.5. The Definitions 4.4 of $\mathrm{Char}_k^H(G)$ and 3.1 of $\mathrm{Char}_k(G)$ agree with the convention of forgetting the superscript (with the exception of the behavior of **1**), see, for instance, [**43**] or a sketch in Remark 4.12(M3).

In case A is a free abelian group of rank r, then $\mathbb{Z}[A] = \mathbb{Z}[t_1^{\pm 1}, ..., t_r^{\pm 1}]$ is the ring of Laurent polynomials in n variables and $\mathbb{Q}[A]$ is a UFD.

DEFINITION 4.6. For A a free abelian group, the *multi-variable Alexander polynomial of G with respect to A* is the Laurent polynomial Δ_G^A in $\mathbb{Q}[A]$ defined by $\Delta_G^A = \gcd F_1(M_G^A)$. Analogously, one has the *multi-variable Alexander polynomial of G with respect to A* over $\mathbb{K}$.

REMARK 4.7. Note that $\Delta_{\mathbb{K},G}$ is only well defined up to multiplication by a unit in $\Lambda_{\mathbb{K}}$, and hence any equality involving $\Delta_{\mathbb{K},G}$ has to be considered up to a unit.

EXAMPLE 4.8 ([**8**]). Consider the group $G := \langle a, b \mid aba = bab, [a, a^2b^2] = [b, a^2b^2] = 1\rangle$, the abelianization morphism $G \to \mathbb{Z}$, and a field $\mathbb{K}$ of characteristic q, then

$$M_{\mathbb{K},G} = \begin{cases} \frac{\mathbb{K}[t^{\pm 1}]}{(t+1)} & \text{if } q = 3 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\mathrm{Char}_k(\mathbb{K}, G) = \begin{cases} \{-1\} \subset \mathbb{K}^* & \text{if } q = 3 \\ \emptyset & \text{otherwise,} \end{cases} \text{ and } \Delta_{\mathbb{K},G}(t) = \begin{cases} t+1 & \text{if } q = 3 \\ 1 & \text{otherwise.} \end{cases}$$

As in §3, another fundamental approach to these invariants takes place when G is considered as the fundamental group of X, a connected CW-complex of finite type (which we can assume has a single 0-cell e_0). In this case M_G^A is nothing but $H_1(X_K; \mathbb{Z})$ where X_K is the covering of X associated with the subgroup $K = \ker(G \to A)$. In this scenario, we can define new invariants; let $\tilde{e}_0$ be the preimage of the 0-cell in X_K.

DEFINITION 4.9. The *Alexander module of X with respect to A* is the relative homology $H_1(X_K, \tilde{e}_0; \mathbb{Z})$ (as a $\Lambda_{\mathbb{Z}}^A$-module) and we will denote it by $\tilde{M}_G^A$. Starting off with $\tilde{M}_G^A$ and analogously to the previous discussion, one can define characteristic varieties $\widetilde{\mathrm{Char}}_k(\mathbb{K}, G)$ and multi-variable Alexander polynomials $\tilde{\Delta}_{\mathbb{K},G}^A$ associated with $\tilde{M}_{\mathbb{K},G}^A$.

REMARK 4.10. The relationship between $\tilde{M}_G^A$ and M_G^A is clearly given by the exact sequence of the pair $(X_K, \tilde{e}_0)$, that is,

$$0 = H_1(\tilde{e}_0) \to M_G^A \to \tilde{M}_G^A \to \ker\left(H_0(\tilde{e}_0) \to H_0(X; \mathbb{Z})\right) \to 0. \tag{3}$$

Since $H_0(\tilde{e}_0) = \mathbb{Z}Ae_0$, $H_0(X; \mathbb{Z})) = \mathbb{Z}e_0$, and the map is given by $ae_0 \mapsto 1$, its kernel is nothing but the augmentation ideal $I_{\mathbb{Z}[A]} = \left\{\sum_{a\in A} n_a \mid \sum_{a\in A} n_a = 0\right\}$.

A free presentation of the Alexander module can be given from the following chain map using Fox derivation with respect to A:

$$C_2^A(X) \otimes \mathbb{Z}[A] \xrightarrow{\tilde{\delta}_2} C_1^A(X) \otimes \mathbb{Z}[A] \to \tilde{M}_G^A \to 0. \tag{4}$$

In order to describe the boundary map $\tilde{\delta}_2$, let us fix for any 2-cell $e_2 \in C_2(X)$ a certain closed path ∂e_2 representing its boundary as induced by the cell map on the boundary. It might happen that such map is constant. In that case $\tilde{\delta}_2(e_2 \otimes a) = 0$. Otherwise ∂e_2 can be written as a composition of closed paths (1-cells), say $\partial e_2 = x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot \ldots \cdot x_n^{\varepsilon_n}$, where $\varepsilon_1 = \pm 1$. Then $\tilde{\delta}_2$ can be described recursively as a function of its boundary $\tilde{\delta}_2(e_2 \otimes a) = D(\partial e_2, a)$ where

$$D(x^\varepsilon \cdot y, a) = \begin{cases} x \otimes a + D\left(y, \psi(x)a\right) & \text{if } \varepsilon = 1 \\ -x \otimes \psi^{-1}(x)a + D\left(y, \psi^{-1}(x)a\right) & \text{if } \varepsilon = -1. \end{cases} \tag{5}$$

REMARK 4.11. The main computational advantage of $\tilde{M}_G^A$ over M_G^A is the size of the representation matrices, see Remark 4.3. The number of rows (resp. columns) of the matrix $\mathrm{Mat}(\tilde{\delta}_2)$ is linear with respect to the number of generators (resp. relations) of G, whereas the number of rows (resp. columns) of the matrix $\mathrm{Mat}(\phi)$ described in (2) is quadratic (resp. cubic) with respect to the number of generators (resp. relations) of G.

From (3) one can easily see that

$$\mathrm{Char}_k(G) \setminus \{\mathbf{1}\} = \widetilde{\mathrm{Char}}_{k+1}(G) \setminus \{\mathbf{1}\}. \tag{6}$$

(see for instance [**21**] for a proof).

Analogously, one can easily check that

$$\tilde{\Delta}_G^A = \begin{cases} \Delta_G^A & \text{if } \operatorname{rank} A > 1 \\ (t-1)^k \Delta_G^A & \text{if } \operatorname{rank} A = 1 \end{cases} \tag{7}$$

REMARK 4.12. Let us compare these arguments with the chain complex introduced in Remark 3.2. For $A = H$, $X_K = \tilde{X}$ and $C_*(\tilde{X}; \mathbb{C})$ is

$$0 \to C_2(\tilde{X}; \mathbb{C}) \xrightarrow{\delta_2} C_1(\tilde{X}; \mathbb{C}) \xrightarrow{\delta_1} C_0(\tilde{X}; \mathbb{C}) \to 0;$$

the map δ_2 is the complexification of $\tilde{\delta}_2$ in (4). For $\xi \in \mathrm{Hom}(G; \mathbb{C}^*)$, $\xi \neq \mathbf{1}$, $\mathbb{C}$ has a natural structure of $\Lambda_{\mathbb{C}}$-module denoted by $\mathbb{C}_\xi$. Given any $\Lambda_{\mathbb{C}}$-module V we can produce a *twisted* $\mathbb{C}$-vector space $V_\xi := V \otimes_{\Lambda_{\mathbb{C}}} \mathbb{C}_\xi$. We have the following properties:

- (M1) The complexified Alexander Invariant $M_{\mathbb{C},G}$ is the homology in degree 1 of $C_*(\tilde{X}; \mathbb{C})$.
- (M2) The complexified Alexander Module $\tilde{M}_{\mathbb{C},G}$ is $\operatorname{coker} \delta_2$.
- (M3) For $\xi \neq \mathbf{1}$, $\xi \in \mathrm{Char}_k^H M$ if and only if $\dim(M_{\mathbb{C},G})_\xi \geq k$, i.e., if and only if $\xi \in \mathrm{Char}_k(G)$. This follows from the fact that the operations *taking homology* and $\otimes_{\Lambda_{\mathbb{C}}} \mathbb{C}_\xi$ commute.

The systematic use of the Alexander polynomial to distinguish quasi-projective groups has been known since Zariski [**57**], even though technically the invariant was defined later by Libgober [**41**].

The following property will be strongly used in this paper as an obstruction to quasi-projective groups:

THEOREM 4.13 ([**34**, Theorem 4.3]). *Let $G = \pi_1(M) \in \mathcal{QP}$ be the fundamental group of a smooth, connected, complex quasi-projective variety.*

- ($\mathcal{A}1$) *If $b_1(G) = r \neq 2$, then the Alexander polynomial Δ_G has a single essential variable, that is, there exist a Laurent polynomial $P \in \mathbb{Z}[t^{\pm 1}]$ such that $\Delta_G(t_1, ..., t_r)) = P(t_1^{n_1} \cdots t_r^{n_r})$.*
- ($\mathcal{A}2$) *If $b_1(G) \geq 2$, and Δ_G has a single essential variable, then either*
  - (a) *$\Delta_G = 0$, or*
  - (b) *$\Delta_{\mathbb{C},G}(t_1, \ldots, t_r) = P(u)$ in $\mathbb{C}[t_1^{\pm 1}, ..., t_r^{\pm 1}]$, where P is a product of cyclotomic polynomials (possibly equal to 1), and $u = t^{n_1} \cdots t^{n_r}$, with $\gcd(n_1, \ldots, n_r) = 1$.*
  - (c) *If $G \in \mathcal{P}$, then $\Delta_G = const.$*

Theorem 4.13($\mathcal{A}2$) is a generalization of the single-variable result due to Libgober [**41**]. A generalization of the Alexander polynomial for the twisted case can be found in [**22**].

5. Artin-Tits groups

Let Γ be a simplicial graph with vertices V, edges E (an element of E is a subset of V with two elements) and a labeling $\ell : E \to \mathbb{N}_{\geq 2}$ of the edges. The *Artin-Tits group* A_Γ associated to $\Gamma = (V, E, \ell)$ is given by the presentation:

$$A_\Gamma = \langle v \in V \mid \underbrace{uvu\dots}_{\ell(e) \text{ times}} = \underbrace{vuv\dots}_{\ell(e) \text{ times}} \quad \text{if } e := \{u, v\} \in E \rangle.$$

Artin groups associated with the constant map $\ell = 2$ are called *right-angled* Artin groups and their graphs will be denoted simply by $\Gamma = (V, E)$.

The *Coxeter group* W_Γ associated with $\Gamma = (V, E, \ell)$ is the quotient of A_Γ obtained by factoring out by the squares of the generators:

$$W_\Gamma = A_\Gamma / \langle v^2,\, v \in V \rangle.$$

Remark 5.1. It is useful to think of the Artin-Tits groups and Coxeter groups as particular instances of a more general class. Let $\Gamma = (V, E)$ be a graph as above. To each vertex $v \in V$ we associate a group G_v. To each edge $e = \{u, v\} \in E$ we associate a group $G_e = G_{u,v}$ of the form $G_e = G_u * G_v/(R_e)$, where (R_e) is the subgroup of the free product $G_u * G_v$ normally generated by a set of words R_e. Then the *Pride group* P_Γ is defined as

$$P_\Gamma = *_{v \in V} G_v / (R_e,\, e \in E).$$

For example, in the case of A_Γ the vertex groups are all infinite cyclic $G_v = \mathbb{Z}$ and the edge group of $e = uv$ labeled by m is the two generators, one-relator group

$$G_e = \langle u, v \mid \underbrace{uvu\dots}_{m \text{ times}} = \underbrace{vuv\dots}_{m \text{ times}} \rangle.$$

In the case of W_Γ the vertex groups are cyclic of order 2, $G_v = \mathbb{Z}_2$. Some of the facts that are proved here easily generalize to Pride groups.

Definition 5.2. An Artin-Tits group A_Γ is called *spherical* or of *finite type* if its Coxeter group W_Γ is finite. Otherwise A_Γ is of *infinite type*.

Among Artin-Tits groups of infinite type we distinguish certain classes.

Definition 5.3. The Artin-Tits group A_Γ is called *euclidean* (or *affine*) if its Coxeter group W_Γ is euclidean. If W_Γ is of infinite non-euclidean type then A_Γ is of *general type*. Furthermore, A_Γ is called *hyperbolic* if W_Γ is hyperbolic.

Example 5.4. Let us show some examples of Artin-Tits groups.

(1) If Γ, Γ' are two arbitrary graphs, then $A_{\Gamma \sqcup \Gamma'} = A_\Gamma * A_{\Gamma'}$, where $\Gamma \sqcup \Gamma'$ is the disjoint union.
(2) If $\Gamma_1, \dots, \Gamma_k$ are the connected components of Γ then A_Γ is the free product $A_{\Gamma_1} * \dots * A_{\Gamma_k}$.
(3) $A_{\Gamma * \Gamma'} = A_\Gamma \times A_{\Gamma'}$, where $\Gamma * \Gamma'$ is the join graph with vertices $V \cup V'$, edges $E \cup E' \cup \{\{v, v'\} \mid v \in V, v \in V'\}$ with new edges having labels $\mu(\{v, v'\}) = 2$.
(4) If Γ is the graph with no edges and n vertices, then $\mathcal{A}_\Gamma = \mathbb{F}_n$, the free group on n generators.
(5) If $\Gamma = K_n$, the complete graph on n vertices, then the right-angled Artin group $\mathcal{A}_\Gamma$ is $\mathbb{Z}^n$, the free abelian group of rank n.

(6) If $\Gamma = K_{n_1,\dots,n_r}$, the complete multipartite graph on n vertices partitioned into subsets of sizes $n_1, \dots, n_r$, then right-angled group $\mathcal{A}_\Gamma$ is $\mathbb{F}_{n_1} \times \cdots \times \mathbb{F}_{n_r}$.
(7) If Γ is the complete graph K_n with labeling $\ell(\{i,j\}) = 2$ if $|i-j| \geq 2$, and 3 otherwise, then $\mathcal{A}_\Gamma = B_n$, the braid group on n strings.
(8) Let Γ be the one edge graph on two vertices with labeling $m \geq 2$. It is well known that $\mathcal{A}_\Gamma = \langle u, v \mid uvu \cdots = vuv \dots \rangle$ is isomorphic to the group $\pi_1(\mathbb{C}^2 \setminus C)$ of the affine curve $C = \{x^2 = y^m\}$ having an $\mathbb{A}_{m-1}$ singular point at the origin.
(9) If Γ has two vertices and no edges, then the free group $\mathcal{A}_\Gamma = \mathbb{F}_2$ is isomorphic to $\pi_1(\mathbb{C}^2 \setminus C)$ for C the union of two parallel lines in $\mathbb{C}^2$.

REMARK 5.5. Thus $A_\Gamma \in \mathcal{QP}$ for Γ a graph on two vertices.

EXAMPLE 5.6. Another interesting case is that of graphs with three vertices. All such graphs will be described by graphs $\Gamma(p,q,r)$ as in Figure 1, with $p \geq q \geq r$, where $p, q, r \in \mathbb{N} \cup \{\infty\}$ means that there is no edge. These groups will be denoted by $A(p,q,r)$.

(1) If Γ has no edges, then $A_\Gamma \in \mathcal{H} \subset \mathcal{QP}$ since it is the group of the complement of three parallel lines in $\mathbb{C}^2$.
(2) The case $\Gamma(\infty,\infty,k)$ can also be disregarded as follows. In fact, one has $\mathrm{Char}_1(A(\infty,\infty,k)) = \mathbb{T}_{A_\Gamma}$, whereas $\mathrm{Char}_2(A_\Gamma)$ contains irreducible components of proper dimension, that is, positive but strictly less than the maximal. This contradicts Proposition 3.10(4).
(3) For the case where Γ has two edges there are partial results in Remark 5.24 and Example 5.30.
(4) For the general case $p, q, r \in \mathbb{N}$ there are partial results in Theorem 5.27 and Examples 5.10-5.13, 5.29, and 5.31.

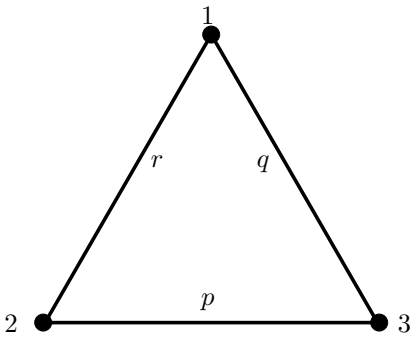

FIGURE 1. $\Gamma(p,q,r)$

We are interested in the following general question (see also [**37**]):

PROBLEM 5.7. Which Artin groups A_Γ are (quasi)projective?

This problem is partially solved for quasi-projective right-angled Artin groups as follows.

THEOREM 5.8 ([**37, 33**]). *The quasi-projective right-angled Artin groups A_Γ are precisely those associated to a complete multipartite graph $\Gamma = K_{n_1,\dots,n_r}$. In particular, $A_\Gamma \in \mathcal{QP}$ if and only if $A_\Gamma = \mathbb{F}_{n_1} \times \cdots \times \mathbb{F}_{n_r}$.*

As a consequence of classical invariant theory for spherical and euclidean Coxeter groups we have the following result.

THEOREM 5.9. *All spherical and euclidean Artin-Tits groups A_Γ are hypersurface groups.*

PROOF. It is enough to prove it for irreducible Artin-Tits groups. We first deal with the spherical case.

We will follow E. Brieskorn [**18**]. Let W be a finite Coxeter group. Realize W as a reflection group in $\mathrm{GL}(V_\mathbb{R})$, for a real vector space $V_\mathbb{R}$ of dimension n. Let $V = V_\mathbb{R} \otimes \mathbb{C}$ be its complexification. We have a proper action of W on V with quotient map $\pi : V \to V/W$ a branched cover with branch locus Δ, the hypersurface defined by the union of all reflecting hyperplanes of W (we abuse the notation by using the same symbol for a hypersurface and a defining polynomial). The action of W is free on $V \setminus \Delta$, and the image of the W-invariant polynomial D^2 is the discriminant locus $\Delta = \pi(D^2)$ of π.

Thus the restriction $\pi : V \setminus \Delta \to V/W \setminus D$ is an unbranched cover with group W. In fact, V/W is isomorphic to the complex affine space $\mathbb{C}^n$ as provided by the Chevalley's classical invariant theory result: $\mathbb{C}[V]^W$ is a polynomial algebra $\mathbb{C}[f_1, \ldots, f_n]$, and the isomorphism $V/W \to \mathbb{C}^n$ is given by $[v] \to (f_1(v), \ldots, f_n(v))$. The fundamental group $\pi_1(\mathbb{C}^n \setminus D)$ of the discriminant complement is nothing but the Artin-Tits group A_W associated to W.

We now consider the euclidean case. We will follow N. Bourbaki [**14**]. Let R be an irreducible root system of rank n in a real vector space $V_\mathbb{R}$, with root lattice Q and Weyl group group W. The affine Weyl group $\widetilde{W} = Q \rtimes W$ acts on $V = V_\mathbb{R} \otimes \mathbb{C}$ by affine reflections. Let T for be complex torus with rational character lattice the weight lattice P of R. For $\lambda \in P$ write e^λ for the corresponding character in T. The exponential map $e : V \to T$ determines a short exact sequence $0 \to Q \to V \to T \to 1$ that provides an identification of the orbit spaces $V/\widetilde{W} = T/W$. Classical exponential invariant theory tells us that the invariants $\mathbb{C}[T]^W$ of the algebra $\mathbb{C}[T]$ of Fourier polynomials on T, under the natural W-action is a polynomial algebra $\mathbb{C}[z_1, \ldots, z_n]$. Hence the varieties $V/\widetilde{W} = T/W$ are in fact isomorphic to the affine space $\mathbb{C}^n$. If $\lambda_1, \ldots, \lambda_n$ are the fundamental weights in P then we may take $z_j = \sum_{\mu \in W\cdot\lambda_j} e^\mu$, where $W\cdot\lambda$ is the W-orbit of λ. In this way we obtain a branched cover $\tilde{\pi} : T \to T/W$ with branch locus the Weyl denominator $\Delta = \prod_{\alpha \in R_{>0}} (e^{\alpha/2} - e^{-\alpha/2})$ and discriminant locus $D = \tilde{\pi}(\Delta^2)$. In this way we obtain an unbranched cover $V \setminus \widetilde{\Delta} \to V/\widetilde{W} \setminus \widetilde{D} = T/W \setminus D$, where $\widetilde{\Delta}$ is the union of all reflecting hyperplanes of $\widetilde{W}$. Recall that the affine Weyl groups are precisely the euclidean Coxeter groups. Now according to [**26**] the euclidean Artin-Tits group $A_{\widetilde{W}}$ is isomorphic to the the fundamental group $\pi_1(V/\widetilde{W} \setminus \widetilde{D}) = \pi_1(T/W \setminus D) = \pi_1(\mathbb{C}^n \setminus D)$. □

The procedures described in the proof of Theorem 5.9 lead to concrete equations for the hypersurfaces whose complement have the Artin-Tits groups as fundamental groups. This has been carried out by E.M. Opdam [**47**] in the euclidean rank 2 case. We will just give the results here.

EXAMPLE 5.10. The type $\widetilde{\mathbb{A}}_2$ corresponds to the Artin-Tits group $A(3,3,3)$. This is the complement in $\mathbb{C}^2$ of the quartic $D = \{z_1^2 z_2^2 - 4z_1^3 - 4z_2^3 + 18 z_1 z_2 - 27 = 0\}$. This curve is the tricuspidal quartic where the line at infinity is the bitangent line.

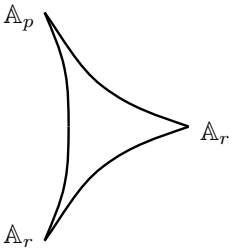

FIGURE 2. Triangle curve

EXAMPLE 5.11. The type $\widetilde{\mathbb{C}}_2$ corresponds to the Artin-Tits group $A(4,4,2)$. This is the complement in $\mathbb{C}^2$ of the reducible quartic $D = \{(z_1^2 - 4z_2)(2z_1 + z_2 + 4)(-2z_1 + z_2 + 4) = 0\}$ having 1 node and 2 tacnodes as singularities: a parabola with two tangent lines intersecting outside the conic.

EXAMPLE 5.12. The type $\widetilde{\mathbb{G}}_2$ corresponds to the Artin-Tits group $A(6,3,2)$. This is the complement in $\mathbb{C}^2$ of the reducible quintic $D = \{(z_1^2 - 4z_2)(-4z_1^3 + z_2^2 + 12z_1z_2 + 24z_2 + 36z_1 + 36) = 0\}$; this is a cuspidal cubic (where the line at infinity is the tangent at the inflection point) and a parabola (with the same point at infinity) intersecting at two smooth points, with intersection multiplicities 1 and 3. The curve D has singularities of type $\mathbb{A}_1$, $\mathbb{A}_2$, and $\mathbb{A}_5$.

EXAMPLE 5.13. The spherical Artin triangle groups $A(p,q,r)$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ can also be obtained in an easy way by curves having singularities of type $\mathbb{A}_{p+1}, \mathbb{A}_{q+1}, \mathbb{A}_{r+1}$ as follows:

- $A(n,2,2)$**:** The curve of equation $(y^2 - x^n)(x-1) = 0$.
- $A(3,3,2)$**:** The curve of equation $8y^3 + 3y^2 - 6x^2y - x^2(2+x^2) = 0$. The curve has two ordinary cusps $\mathbb{A}_2$ and one ordinary node $\mathbb{A}_1$; there is only one branch at infinity, which is a non-ordinary inflection point.
- $A(4,3,2)$**:** The curve of equation $(y^2 - x^3)(2y - 3x + 1) = 0$ the line being an ordinary tangent to the cubic.
- $A(5,3,2)$**:** The curve of equation $3125y^3 + 16(16 - 125x)y^2 - 4x^2(32 - 225x)y + 4x^4(4 - 27x) = 0$. It is a rational affine curve with three singular points of type $\mathbb{A}_1$, $\mathbb{A}_2$, and $\mathbb{A}_4$. There is only one place at infinity; at this place the projectivized curve has a singular point of type $\mathbb{A}_4$ such that the tangent line (the line at infinity) has intersection number 5.

REMARK 5.14. All the curves presented in the examples above have a common structure. In suitable coordinates they can be presented as in Figure 2, where the *vertices* are singular double points. Note that for odd indices the picture should continue till infinity. These curves are defined by a braid monodromy; if we choose a vertical line close to the leftmost singular points, it is defined by four braids in four strings: $\sigma_1^{p+1} \cdot \sigma_3^{q+1}$, σ_2 and $\sigma_1^{-1} \cdot \sigma_3^{-1} \cdot \sigma_2^{r+1} \cdot \sigma_1 \cdot \sigma_3$. Following ideas of L. Rudolph [**49**], S. Orevkok [**48**] proved that there exists an analytic curve in $\Delta \times \mathbb{C}$ (Δ a disk) realizing such a braid monodromy. For the elliptic and euclidean case the above Examples show that these curves can be extended to algebraic curves in $\mathbb{C}^2$ such that the complements in $\Delta \times \mathbb{C}$ and $\mathbb{C}^2$. One can construct for small values of p, q, r (in the hyperbolic case) algebraic curves realizing this braid monodromy over the disk, but the monodromy of the curves is not trivial outside the disk.

In fact, any attempt to construct hyperbolic triangle curves in $\mathbb{C}^2$ violates the Riemann-Hurwitz principle. On the other side, using ideas of L. Rudolph in [**50**], any hyperbolic triangle curve is the fundamental group of a Stein surface properly embedded in $\mathbb{C}^4$.

Now we turn to the question of projectivity of the Artin groups.

THEOREM 5.15. *All spherical and euclidean Artin-Tits groups A_Γ are not projective.*

PROOF. The abelianization of A_Γ is calculated for example in [**27**]. Following [**37**, §11.9], let Γ_{odd} be the graph obtained from Γ by removing all even labeled edges. Then A_Γ/A'_Γ is isomorphic to $\mathbb{Z}^m$, where m is the number of connected components of Γ_{odd}. Thus, if Γ_{odd} is connected the abelianization of A_Γ is $\mathbb{Z}$.

Recall that the simplest necessary condition for group to be projective is to have even first Betti number. If we go over the list of irreducible spherical and euclidean Artin groups A_Γ, we quickly discover that: $b_1(A_\Gamma) = 1$ for the types $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_k, \mathbb{H}_l, \mathbb{I}_2(m)$, with m odd, $\widetilde{\mathbb{A}}_n, \widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_k$, $b_1(A_\Gamma) = 1$ for type $\widetilde{\mathbb{C}}_n$, and $b_1(A_\Gamma) = 2$ for the rest of the cases.

Now, for the types with small number of nodes $\mathbb{F}_4, \mathbb{G}_2, \widetilde{\mathbb{F}}_4, \widetilde{\mathbb{G}}_2$ and $\mathbb{I}_2(m)$, with m even, one can easily exhibit low index subgroups of A_Γ that have and odd first Betti number. Although the types $\mathbb{B}_n$ and $\widetilde{\mathbb{B}}_n$ can also be dealt with that way, we take here a different road.

We are going to use an obstruction to projectivity (in fact to *Kähler-ness*) obtained in [**5**], following ideas of M. Gromov in [**39**], see also [**2**] for more background. More precisely, if a group G fits into an extension $1 \to K \to G \to Q \to 1$ such that K is finitely generated and Q has an infinite number of ends $e(Q)$, then G cannot be Kähler, hence projective. In fact, the group G has also $e(G) = \infty$, which is not possible for a projective group (cf. [**5**] using the results in [**39**]). In [**3**] this fact is used to show that pure braid groups $P_n, n \geq 3$ are not projective. Indeed we have a surjection $P_n \to P_3$ with finitely generated kernel, and $e(P_3) = \infty$ as $P_3 = \mathbb{F}_2 \times \mathbb{Z}$. We will follow the same line of reasoning.

First, it is known that the Artin group $A(B_n)$ surjects onto $A(A_{n-1})$ with kernel a free group of rank n, see [**24, 25**]. Then Arapura's discussion in [**3**] gives that the braid group $A(A_{n-1})$ has infinitely many ends.

Secondly, it is proved by D. Allcock [**1**] that $\widetilde{\mathbb{B}}_n, n \geq 3$ is an index 2 subgroup of the braid group $B_n(\Sigma)$ of a 2-dimensional orbifold Σ, so it is enough to show that this group has infinitely many ends. The orbifold $\mathbb{C}^*_\varphi$ where $\varphi(\frac{1}{2}) = 2$ and it is the only point with value different from 1. Its associated pure braid group $P_n(\Sigma)$ is the kernel of the canonical surjection $B_n(\Sigma) \to S_n$ onto the symmetric group. As with the ordinary pure braid groups we have a surjection $P_n(\Sigma) \to P_3(\Sigma)$ with finitely generated kernel. It is readily seen that $P_3(\Sigma)$ has infinitely many ends.

In fact, the general irreducible case can be treated as above by showing that they have an infinite number of ends. This way one shows the result for the non-irreducible case, since the product of two groups with infinite number of ends also has an infinite number of ends. □

CONJECTURE 5.16. *All general type Artin groups A_Γ are not quasi-projective.*

This was already known for right-angled Artin groups (cf. [**37**]).

DEFINITION 5.17. A graph $\Gamma = (V, E, \ell)$ is called *even* if the labeling ℓ takes only even values. In that case the Artin group A_Γ is called of *even type*. Suppose $\Gamma = (V, E, \ell)$ is an arbitrary graph. Then Γ_{ev} is the graph obtained from Γ by identifying the endpoints of each odd labeled edge and then removing the loops.

REMARK 5.18. Note that Γ_{ev} is not the same as the odd contraction of Γ considered in [**37**, §11.9] since the even-labeled edges are kept in Γ_{ev}.

Since Theorem 5.8 solves the problem for right-angled groups, we say that an Artin group is of *strictly even type* if it is of even type and not right-angled.

REMARK 5.19. Note that if Γ has connected components $\Gamma_1, \ldots, \Gamma_r$ then the Alexander matrix M_Γ of A_Γ can be seen to be, for a suitable ordering of the vertices of Γ, a block matrix with blocks $M_{\Gamma_1}, \ldots, M_{\Gamma_r}$.

In [**37**, Theorem 9.9] it is proved that a free product $G_1 * G_2$ of two 1-formal groups G_i with $b_1(G_i) > 0, i = 1, 2$ and presented by commutator relators is in $\mathcal{QP}$ if and only if both G_i are free. We apply this result to the case of even Artin groups.

PROPOSITION 5.20. *A free product $A_{\Gamma_1} * A_{\Gamma_2}$ of even Artin groups is not quasi-projective unless it is free.*

Thus if we focus on the even graphs Γ, we only need to study the quasi-projectivity of A_Γ for connected graphs Γ.

PROPOSITION 5.21. *For a connected graph $\Gamma = (V, E, \ell)$ let n_v be the valence of the vertex v. Then the Fitting ideal $F_1 := F_1(\tilde{M}_{A_\Gamma})$ is of the following form:*

$$F_1 = I^\varepsilon \cdot \left(\prod_{v \in T_{ev}} (t_v - 1)^{n_v - 1} \cdot \prod_{e \in T} \Delta_{\ell(e)}(t_e), T \text{ spanning tree of } \Gamma \right),$$

where:

(1) *I is the augmentation ideal,*
(2) *$\varepsilon = 1$ if there exists an even spanning tree for Γ, otherwise $\varepsilon = 0$,*
(3) *$t_e = t_u t_v$ if $\ell(e)$ is even, otherwise $t_e = t_u = t_v$,*
(4) *$\Delta_{2k}(t) := \dfrac{t^k - 1}{t - 1}$ and $\Delta_{2k+1}(t) := \dfrac{t^{2k+1} + 1}{t + 1}$.*

PROOF. The proof is a pleasant exercise. The Alexander matrix $M = M_\Gamma$ of A_Γ has $|E|$ rows and $|V|$ columns. Let $n = |V|$ and identify V with $[n]$ in increasing order, and E with a subset of $\binom{[n]}{2}$, ordered lexicographically, where $[n] := \{1, \ldots, n\}$. The entry of M on row jk and column i is denoted as $m_{jk,i}$. As Γ is connected, it has at least $n-1$ edges. Let $m = |E| - n + 1$. The proof is carried out by double induction on $n \geq 1$ and $m \geq 0$.

Suppose $m = 0$. Thus Γ is a tree. Then the Fitting ideal F_1 is generated by the n minors of M_Γ of codimension 1. Let M_i be the minor obtained by removing the i-th column of M. Let n_i be the valence of the vertex $v = i$ and $ij_1, \ldots, ij_{n_i}$ the edges at i. Then the entry $m_{ij_p,i}$ is up to sign of the form $(t_i - 1)^{((-1)^{\ell(ij_p)}+1)/2} \cdot \Delta_{ij_p}$. Moreover $m_{ij_p,i}$ is the only non-zero entry on the row ij_p of the minor M_i. Expanding M_i along the rows ij_p, $1 \leq p \leq n_i$, and using Remark 5.19, we obtain

$$M_i = \prod_{p=1,n_i} (t_i - 1)^{((-1)^{\ell(ij_p)}+1)/2} \cdot \Delta_{ij_p} \cdot \prod_{p=1,n_i} M_{j_p}(T_p),$$

where T_p is the sub-tree of Γ not containing vertex i and edges $ij_1, \dots, ij_{n_i}$, and $M_{j_p}(T_p)$ is the codimension 1 minor of the Alexander matrix of A_{T_p} obtained by removing its column corresponding to vertex j_p.

The last formula, combined with the induction hypothesis on n finishes the proof for $m = 0$. If $m \geq 1$, we proceed as follows.

Let $E' \subset E$ be a set of rows of M of size $n-1$, and denote by Γ' the full subgraph of Γ on vertices V and edges E'. Let $M_{E',i}$ be the codim 1 minor of M obtained by removing its i-th column and the rows in $E \setminus E'$. Then $M_{E',i}$ is the minor M'_i of $M' = M(A_{\Gamma'})$. By Remark 5.19, M' can be seen as a block matrix with blocks $M_{\Gamma'_1}, \dots, M_{\Gamma'_r}$, where $\Gamma'_1, \dots, \Gamma'_r$ are the connected components of Γ'. If $r = 1$ then Γ' is a spanning tree for Γ, and we are done. If $r > 1$ then $M'_i = 0$. □

As an immediate corollary we have the following:

COROLLARY 5.22. *For a connected tree T the Alexander polynomial $\Delta(A_\Gamma)$ of A_Γ is given by:*

$$\Delta(A_\Gamma) = \prod_{v \in T_{ev}} (t_v - 1)^{n_v - 1} \cdot \prod_{e \in T} \Delta_{\ell(e)}(t_e).$$

PROPOSITION 5.23. *If Γ is an even tree with at least 3 vertices, then $A_\Gamma \notin \mathcal{QP}$.*

PROOF. Its Alexander polynomial $\Delta(A_\Gamma)$ has at least 2 essential variables and hence Theorem 4.13($\mathcal{A}1$) applies. □

REMARK 5.24. In particular, the even graphs with three vertices and two edges are not in $\mathcal{QP}$.

PROPOSITION 5.25. *If Γ is a strictly even n-cycle $\mathbb{C}_n$, $n \geq 4$, then A_Γ is not quasi-projective.*

PROOF. The first characteristic variety $\mathrm{Char}_1(A_\Gamma)$ does not pass the quasi-projectivity test. Let $2m_1, \dots, 2m_n$ be the weights.

If $n \geq 5$, then $V_{i,i+1} = \{t_i = t_{i+1} = 1\}$ is an irreducible component. In particular, V_{12} and V_{23} intersect in a codimension 3 variety contradicting Proposition 3.10(1).

For $n = 4$ we may assume that $m_1 > 1$ and we proceed in a similar way. □

THEOREM 5.26. *If Γ is a strictly even non-complete graph $n \geq 3$, then A_Γ is not quasi-projective.*

PROOF. The first characteristic variety $\mathrm{Char}_1(A_\Gamma)$ does not pass the quasi-projectivity test. It contains two irreducible components of $\mathrm{Char}_1(A_\Gamma)$ intersecting in a subvariety of dimension greater than zero. Since Γ not complete, there exists a subset $W \subset V$ that disconnects V. Then from Proposition 5.21 it follows that both

$$C_W = \cap_{w \in W}\{t_w - 1 = 0\},\ C_{W,w_0,v} = \cap_{w \in W \setminus \{w_0\}}\{t_w - 1 = 0\} \cap \{t_{w_0} t_v + 1 = 0\},$$

are components of $\mathrm{Char}_1(A_\Gamma)$, where $v \in W$ is any neighbor of w_0 in Γ. Clearly, their intersection $C_W \cap C_{W,w_0,v} = \cap_{w \in W}\{t_w - 1 = 0\} \cap \{t_v + 1 = 0\}$ is of strictly positive dimension since $n \geq 3$. By Proposition 3.10(1) we are done. □

Theorem 5.26 leaves the case of a strictly even complete graph $\Gamma = K_n$, $n \geq 3$ open. Nevertheless one can still use properties of the characteristic varieties to rule out quasi-projectivity for such graphs in many instances. For simplicity, we consider here just the case $\Gamma = K_3$.

THEOREM 5.27. *The hyperbolic groups $A(2p,2q,2r)$ are not quasi-projective if $(p,q,r)\neq(k,k,k)$ with k odd.*

PROOF. From the Alexander matrix of $G=A(2p,2q,2r)$ we readily obtain the Fitting ideals F_i of $\tilde{M}_G$:

$$\begin{aligned}F_1 =& I\cdot((t_1-1)\Delta_r(t_1t_2)\Delta_q(t_1t_3),(t_2-1)\Delta_r(t_1t_2)\Delta_p(t_2t_3),\\ &(t_3-1)\Delta_q(t_1t_3)\Delta_p(t_2t_3)),\\ F_2 =& ((t_1-1)\Delta_r(t_1t_2),(t_2-1)\Delta_r(t_1t_2),(t_2-1)\Delta_p(t_2t_3),(t_3-1)\Delta_p(t_2t_3),\\ &(t_1-1)\Delta_q(t_1t_3),(t_3-1)\Delta_q(t_1t_3)),\end{aligned}$$

see Figure 1 and Proposition 5.21 for the notation.

We may restrict to $G=A(2p,2q,2r)$, $p\geq 2, q\geq 2, r\geq 1$ since $A(2p,2,2)$, $p\geq 1$ are spherical and $A(4,4,2)$ is euclidean. There are two cases $r=1$ and $r\geq 2$.

NOTATION 5.28. Denote by μ_N^* the set of the N-th roots of unity distinct from 1.

CASE 1. *Suppose $r=1$, $p\geq 3$, and p,q are coprime.*

Then $\mathrm{Char}_1(G)$ has pq irreducible components, one of dimension zero $(1,1,1)$, and $pq-1$ of dimension one:

$$\begin{aligned}C_{1,\zeta} =& \{t_1-1=t_2t_3-\zeta=0\},\zeta\in\mu_q^*\\ C_{2,\zeta,\xi} =& \{t_2t_3-\zeta=t_1t_2-\xi=0\},\zeta\in\mu_q^*,\xi\in\mu_p^*,\\ C_{3,\xi} =& \{t_3-1=t_1t_2-\xi=0\},\xi\in\mu_p^*.\end{aligned}$$

Furthermore $\mathrm{Char}_2(G)$ consists of just $(1,1,1)$. Now the 1-dimensional components of $\mathrm{Char}_1(G)$ intersect as follows:

$$C_{1,\zeta}\cap C_{2,\zeta,\xi}=\{(1,\xi,\zeta\xi^{-1})\},\ C_{3,\xi}\cap C_{2,\zeta,\xi}=\{(\xi\zeta^{-1},\zeta,1)\}.$$

Clearly $(1,\xi,\zeta\xi^{-1})$ and $(\xi\zeta^{-1},\zeta,1)\}$ do not belong to $\mathrm{Char}_2(G)$. This case follows from Proposition 3.16.

CASE 2. *Suppose $r=1$, $p\geq 3$ and $\gcd(p,q)=d>1$.*

The types of components of $\mathrm{Char}_1(G)$ are identical to the ones when p,q are coprime. Furthermore $\mathrm{Char}_2(G)$ consists of $p+q-1$ points: $(1,1,1),(1,\xi,1)$, with $\xi\in\mu_p^*\cap\mu_q^*$. Nevertheless, some of the intersections

$$C_{1,\zeta}\cap C_{2,\zeta,\xi}=\{(1,\xi,\zeta\xi^{-1})\}\text{ and }C_{3,\xi}\cap C_{2,\zeta,\xi}=\{(\xi\zeta^{-1},\zeta,1)\}$$

will not be in $\mathrm{Char}_2(G)$, unless $\xi=\zeta\in\mu_p^*\cap\mu_q^*$. If p and q are distinct then one can always find a $\xi\in\mu_p^*$ and a $\zeta\in\mu_q^*$ that are not equal. If $p=q$ then $\zeta\xi^{-1}$ must be 1 for all pairs $\xi,\zeta\in\mu_p^*$, which is only possible when $p=q=2$ (excluded).

Thus, there will always be a pair of components intersecting outside $\mathrm{Char}_2(G)$ and thus the group is not in $\mathcal{QP}$, by Proposition 3.16.

CASE 3. *Suppose $p,q,r\geq 2$, not all equal.*

Let $\alpha\in\mu_p^*,\beta\in\mu_q^*,\gamma\in\mu_r^*$. Then $\mathrm{Char}_1(G)$ has $(1,1,1)$ as an isolated point, and $pq+qr+rp-p-q-r$ components of dimension one, of two types:

$$C_{1,\beta}=\{t_1-1=t_2t_3-\beta=0\}\text{ and }C_{1,\alpha,\gamma}=\{t_1t_2-\alpha=t_1t_3-\gamma=0\},$$

and similarly $C_{2,\gamma},C_{3,\alpha}$, respectively $C_{2,\alpha,\beta},C_{3,\beta,\gamma}$. Furthermore $\mathrm{Char}_2(G)$ consists of $(1,1,1)$ and possibly points of the following three types:

- having two trivial coordinates, namely either $(\xi, 1, 1)$ with $\xi \in \mu_p^* \cap \mu_r^*$, or $(1, \xi, 1), \xi \in \mu_p^* \cap \mu_q^*$, or $(1, 1, \xi), \xi \in \mu_q^* \cap \mu_r^*$);
- having one trivial coordinate, namely either $(1, \xi, \zeta)$ with $\xi \in \mu_p^*, \zeta \in \mu_r^*$ or $\xi\zeta \in \mu_q^*$, and the corresponding ones with 1 in the other variables;
- having no trivial coordinates, namely $(t^{-1}, t\xi, t\zeta)$ with $\xi \in \mu_p^*, \zeta \in \mu_r^*$ and $t^2\xi\zeta \in \mu_q^*$.

Consider the following intersections of 1-dimensional components of $\mathrm{Char}_1(G)$:

$$C_{1,\beta} \cap C_{2,\alpha,\beta} = \{(1, \alpha, \beta\alpha^{-1})\},\ C_{1,\beta} \cap C_{3,\beta,\gamma} = \{(1, \gamma\beta^{-1}, \beta)\}.$$

In order for $(1, \alpha, \beta\alpha^{-1})$ to be in $\mathrm{Char}_2(G)$ either $\alpha = \beta \in \mu_p^* \cap \mu_q^*$, or $\beta\alpha^{-1} \in \mu_r^*$. Similarly $(1, \gamma\beta^{-1}, \beta) \in \mathrm{Char}_2(G)$ if either $\beta = \gamma \in \mu_q^* \cap \mu_r^*$, or $\gamma\beta^{-1} \in \mu_p^*$.

Since p, q, r are not all equal, we have $p > 2$; we choose α a primitive root in μ_p^*, and $\beta \in \mu_q^*$, also primitive, distinct from α. This implies that $\beta\alpha^{-1} \in \mu_r^*$. If $\alpha^2 \neq \beta$ then we must also have $\beta\alpha^{-2} \in \mu_r^*$. It follows that $\alpha \in \mu_r^*$, which is also the case if $\alpha^2 = \beta$. We deduce that which implies $p \mid r$, which leads to a contradiction. Using Proposition 3.16, we deduce that the groups in this Case are not in $\mathcal{QP}$.

Case 4. *Suppose $p = q = r \geq 2$, $p = 2k$ even.*

We show that $A(2p, 2p, 2p), p \geq 2$ is not in $\mathcal{QP}$, by testing the characteristic varieties of its index 2 subgroup N defined by the homomorphism $x_1, x_2 \to 0, x_3 \to 1 \mod 2$. Then N is generated by $x_i, y_i := x_3 x_i x_3^{-1}, i = 1, 2$, and $z := x_3^2$. The relations are:

$$R = \{(x_1x_2)^{2k} = (x_2x_1)^{2k}, (y_1y_2)^{2k} = (y_2y_1)^{2k}, (x_iy_iz)^k = (y_izx_i)^k = (zx_iy_i)^k\}.$$

By abuse of notation we denote the generators of $H_1(N) = \mathbb{Z}^5$ by x_1, x_2, y_1, y_2, z. We see that certain irreducible components of $\mathrm{Char}_1(N)$ intersect in dimension 1. More precisely, for $\xi \in \mu_k^*$ the following 2-dimensional tori

$$C_{1,2} = \{x_1 - 1 = y_2 - 1 = z - 1 = 0\},\ C_{2,1} = \{x_2 - 1 = y_1 - 1 = z - 1 = 0\},$$
$$C_{\xi,1} = \{x_1x_2 - \xi = y_1 - 1 = z - 1 = 0\},\ C_{\xi,2} = \{x_1x_2 - \xi = y_2 - 1 = z - 1 = 0\},$$
$$C_{1,\xi} = \{y_1y_2 - \xi = x_1 - 1 = z - 1 = 0\},\ C_{2,\xi} = \{y_1y_2 - \xi = x_2 - 1 = z - 1 = 0\},$$

are components of $\mathrm{Char}_1(N)$, and we have that

$$C_{1,2} \cap C_{1,\xi} = \{(1, t, \xi, 1, 1) \mid t \in \mathbb{C}^*\},\ C_{1,2} \cap C_{\xi,2} = \{(1, \xi, t, 1, 1) \mid t \in \mathbb{C}^*\},$$
$$C_{2,1} \cap C_{2,\xi} = \{(1, t, \xi, 1, 1) \mid t \in \mathbb{C}^*\},\ C_{2,1} \cap C_{\xi,2} = \{(\xi, 1, 1, t, 1) \mid t \in \mathbb{C}^*\}.$$

Using Proposition 3.10(1), we conclude that these groups are not quasi-projective. □

Example 5.29. The case excluded in Theorem 5.27 cannot be treated as Case 4. We sketch a proof for the simplest case $A(6, 6, 6)$. Let $N = \ker \psi$, where ψ is the morphism sending each generator to the same non-trivial element of $\mathbb{Z}_3$. One can check that $b_1(N) = 7$ and that $\mathrm{Char}_4(N)$ has three irreducible components of dimension 2 through $\mathbf{1}$. The group N, and thus $A(6, 6, 6)$, is not quasi-projective by Proposition 3.12(1).

Example 5.30. We will consider the case where Γ has three vertices and two edges labeled 2, 3, say $\Gamma(\infty, 3, 2)$ (see Figure 1). The representation $A(\infty, 3, 2) \to \Sigma_3$ given by $1 \mapsto (1, 2)$, $2 \mapsto 1$, and $3 \mapsto (2, 3)$ has as kernel the Artin group of a bamboo with 4 vertices and labels $(2, 4, 2)$, which is not quasi-projective by Proposition 5.23.

The case $\Gamma(\infty,3,3)$ can be treated analogously, since the natural representation $A(\infty,3,3)\to\Sigma_3$ given by $1\mapsto(1,2)$, $2\mapsto(2,3)$, and $3\mapsto(2,3)$ has as kernel the Artin group of a triangle $\Gamma(\infty,4,4)$, which is not quasi-projective by Proposition 5.23.

EXAMPLE 5.31. We will prove that $A(5,4,2)\notin\mathcal{QP}$ (note that $b_1(A(5,4,2))=2$). Consider the following representation $A(5,4,2)\to\Sigma_5$ given by $1\mapsto(2,3)$, $2\mapsto(2,3)(4,5)$, and $3\mapsto(1,2)(3,4)$. Its kernel K has Betti number 4, and $\mathrm{Char}_1(K)$ has three irreducible components V_1^2,V_2^1,V_3^1 ($\dim V_i^j=j$). However $V_1^2\cap V_2^1\notin\mathrm{Char}_2(K)$, which contradicts Proposition 3.16.

The case $A(5,5,4)$ can be treated similarly (note that $b_1(A(5,5,4))=1$). Consider the following representation $A(5,5,4)\to\Sigma_5$ given by $1\mapsto(2,3)(4,5)$, $2\mapsto(2,4)(3,5)$, and $3\mapsto(1,2)(3,4)$. Its kernel K has Betti number 3, and $\mathrm{Char}_1(K)$ has three 1-dimensional irreducible components V_1,V_2,V_3. However $V_1\cap V_2\notin\mathrm{Char}_2(K)$, which contradicts Proposition 3.16.

6. Pencil map construction

We fix a quasi-projective group $G=\pi_1(M)$. We assume that $b_1(G)>1$ and that the first characteristic variety $\mathrm{Char}_1(G)\subset\mathbb{T}_G$ is not 0-dimensional. Then, according to Theorem 3.7, each irreducible component W of dimension $d>0$ of $\mathrm{Char}_1(G)$ is of the form $W=f^*(V)$, where $f:M\to C$ is an orbifold morphism (see Definition 2.6), $b_1(C)=d$, and $V\subset\mathbb{T}_C$ is an irreducible component of $\mathrm{Char}_1(C)$. In light of these facts, we make the following construction.

Suppose that $f_i:M\to C_i$, $1\le i\le k$ are the pencils determined by the irreducible components of $\mathrm{Char}_1(G)$ passing through the trivial character $\mathbf{1}\in\mathbb{T}_G$. Define $F:M\to C_1\times\cdots\times C_k$ to be the product map and let $\phi=F_\#$ be the induced homomorphism

$$\phi:=\phi_1\times\cdots\times\phi_k:G\to D=\pi_1\times\cdots\times\pi_k,$$

where $\pi_i:=\pi_1(C_i)$, and $\phi_i=(f_i)_\#:G\to\pi_i$. Note that none of the factors of D is abelian.

If $N:=\ker\psi$ and $S:=\operatorname{im}\phi$, then we have exact sequences: $1\to N\to G\to S\to 1$ and $1\to T\to D\to A\to 1$, where $T=\bar{S}$ is the normal closure of S in D and $A:=D/T$. Note that, in the terminology of Bridson-Miller [**17**], S is a subdirect product of D, due to the surjectivity of each ϕ_i. Note that if a subgroup H of a direct product of $\pi\times\sigma$ is such that $\pi\cap H$ is trivial, then H is isomorphic to a subgroup of σ. In view of this we will assume that S intersects each factor π_i non-trivially, that is, S is a full subdirect product of G (cf. [**17**]).

We have the following lemma. The proof is immediate and will be omitted, see also [**17**, Proposition 1.2].

LEMMA 6.1. *The group A is free abelian of finite rank.*

Note that, since S is finitely presented, it is also of type $FP_2(\mathbb{Q})$, by Bridson-Howie-Miller-Short [**15**, Theorem B]. It follows, for all $n\ge1$, that S is type F_n if and only if is type $FP_n(\mathbb{Q})$. Moreover, again by [**15**, Theorem D], the normal subgroup T shares the finiteness properties of S.

REMARK 6.2. Recall that if $\mathbf{1}\notin W$, then $\operatorname{Shd}W$ is also an irreducible component of $\mathrm{Char}_1(G)$, unless C is supported either over $\mathbb{C}^*$ or over an elliptic curve. The orbifold version of ϕ is obtained by considering all orbifold morphisms of M

(and not only those whose associated components of $\mathrm{Char}_1(G)$ pass through the trivial character). Note that in this case A may have torsion.

PROPOSITION 6.3. *If ϕ is injective, then either G is finite index in D, or G is of infinite index in D and not of type $FP_k(\mathbb{Q})$.*

PROOF. Suppose G has infinite index in D. The injectivity of ϕ implies $G = S$, making G a direct subproduct of D. Now Bridson-Howie-Miller-Short [**16**, Theorem C] ensures the existence of a finite index subgroup S_0 of $S = G$ which is not type $FP_k(\mathbb{Q})$, and we are done. □

In the case where ϕ is injective and G is finite index in D, then G is clearly of type $FP_k(\mathbb{Q})$.

COROLLARY 6.4. *If G is an infinite index subgroup of D, then M is not a $K(G, 1)$.*

REMARK 6.5. By F. Catanese [**20**], ϕ_i has finitely generated kernel N_i if and only if either $g(C_i) = 0$, or ≥ 1 and f_i has no multiple fibers. Note that $N = N_1 \cap \cdots \cap N_k$.

CONJECTURE 6.6. *The group G is of type $FP_k(\mathbb{Q})$ if and only if either ϕ is not injective or G has a finite index subgroup which is a direct product of at most k smooth curve groups.*

REMARK 6.7. Note that D is linear and residually finite (resp. nilpotent). Thus, if ϕ is injective, then G is linear and residually finite (resp. nilpotent) as well.

In the following examples we will exhibit some hyperplane arrangement groups and study their pencil maps. In the context of line arrangements, a construction similar to our pencil map was considered by Cohen-Falk-Randell in [**23**], where analogous finiteness results and examples are given.

EXAMPLE 6.8. Let $M = \mathbb{P}^2 \setminus \mathcal{A}$, with $\mathcal{A}$ the arrangement of 6 lines in $\mathbb{P}^2$ with coordinates x, y, z, given by $Q = xyz(x - y)(y - z)(z - x)$. There is a total of five pencil morphisms: four of them coming from the triple points of Q and one coming from the pencil of conics through the triple points. The pencil map $\phi : G \to (\mathbb{F}_2)^5$ is not injective. In fact, for each pencil ϕ_i, G fits into a split exact sequence of the form $1 \to N_i \to G \to \mathbb{F}_2 \to 1$, where N_i is a free group of rank 3 generated by meridians. Consider three meridians a_1, a_2, a_3 such that two of them are meridians of two lines intersecting at a double point. Then the commutator $[[a_1, a_2], a_3]$ is in fact a non-trivial element of $N = N_1 \cap N_2 \cap N_3$.

EXAMPLE 6.9. Let $M = \mathbb{P}^2 \setminus \mathcal{A}$, where $\mathcal{A}$ is the arrangement of 6 lines in $\mathbb{P}^2$ given by $Q = xyz(x - y)(y - z)(z - \lambda x)$, $\lambda \neq 0, 1$. Then $\phi : G \to (\mathbb{F}_2)^3$ is injective and $G = \pi_1(M)$ is the Stallings group, fitting into the exact sequence: $1 \to G \to (\mathbb{F}_2)^3 \to \mathbb{Z} \to 1$.

The fact that Stallings' group is realizable as an arrangement group has been first discovered by the third author and A. Suciu in 2004, but no satisfactory extension of this observation to other arrangements was achieved then. We can now give a generalization of the previous example, by realizing all the Stallings-Bieri groups as arrangement groups. Further connections between Bestvina-Brady type groups and fundamental groups of hypersurface complements will be pursued elsewhere.

PROPOSITION 6.10. *Let $M_k := \mathbb{P}^{k-1} \setminus \mathcal{A}$ and $k > 3$, where $\mathcal{A}$ is the hyperplane arrangement in $\mathbb{P}^{k-1}$ with coordinates $x_1, \ldots, x_k$, given by*

$$Q = x_1 \cdots x_k (x_1 - x_2)(x_2 - x_3) \cdots (x_{k-2} - x_{k-1})(x_{k-1} - x_k)(x_k - \lambda x_1),$$

$\lambda \neq 0, 1$. Then the group $G_k = \pi_1(M_k)$ has type F_{k-1}, but not F_k.

PROOF. Consider the following map

$$(\mathbb{C} \setminus \{0, 1\})^k \stackrel{\psi}{\to} \mathbb{C}^*, \quad (x_1, \ldots, x_k) \mapsto x_1 \cdots x_k.$$

At the level of fundamental groups it induces Bieri's map $(\mathbb{F}_2)^k \to \mathbb{Z}$ (cf. [**13**]). It is easy to see that ψ is a locally trivial fibration outside the fiber of 1. The generic fiber is homeomorphic to M_k. Since the critical locus is of real codimension greater than 2, the kernel of Bieri's map is G_k and the result follows from Bieri's arguments. □

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