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Pro-p link groups and p-homology groups

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Contents

Introduction

Let L be a tame link in the 3-sphere S^3 consisting of n knots $K_1, \cdots, K_n$ and let G_L be the link group $\pi_1(X_L), X_L = S^3 \setminus L$. For a prime number p, let $\widehat{G_L}$ denote the pro-p completion of the group G_L, $\widehat{G_L} = \varprojlim G_L/N$ where N runs over normal subgroups of G_L having p-power indices. By a theorem of J. Milnor [Mi], it is shown that $\widehat{G_L}$ has the following simple presentation as a pro-p group

$$\widehat{G_L} = \langle x_1, \cdots, x_n \mid [x_1, y_1] = \cdots = [x_n, y_n] = 1 \rangle$$

where x_i and y_i represent the meridian and longitude around K_i respectively (Theorem 1.1.4). The purpose of this paper is to use the pro-p link group $\widehat{G_L}$ and the associated group-theoretic invariants for the study of the p-homology groups of p^m-fold cyclic branched covers of S^3 along L, following the analogies between link theory and number theory [Mo1$\sim$4],[Rez1,2]. The invariants we derive from $\widehat{G_L}$ are the p-adic Milnor invariants and the completed Alexander module over the formal power series ring $\widehat{\Lambda_n} = \mathbf{Z}_p[[X_1, \cdots, X_n]]$ with coefficients in the ring $\mathbf{Z}_p$ of p-adic integers. The tool involved here is the Fox differential calculus on a free pro-p group [Ih]. Although these invariants are simply p-adic analogues of the usual Milnor invariants and Alexander modules, it is natural to work over $\widehat{\Lambda_n}$ since the completed Alexander module can be presented over $\widehat{\Lambda_n}$ by a sort of *universal p-adic higher linking matrix* $\widehat{T_L}$, called the *p-adic Traldi matrix*. This is defined in terms of the p-adic Milnor numbers and we can derive from $\widehat{T_L}$ systematically the "p-primary" information on the homology of p^m-fold branched covers of L. This is an idea analogous to Iwasawa theory [Iw] which may also be regared as a p-adic strengthening of the method employed by W. Massey [Mas] and L. Traldi [T]. We note that the method using the truncated Traldi matrices was considered in [Mat] to study the homology of unbranched covers.

The homology of cyclic branched covers of a link L is one of the basic invariants of L and has been extensively investigated by many authors. The Betti number and the order have been determined in terms of the Alexander (Hosokawa) polynomial ([HK],[MM],[S1] etc) and further the (Galois) module structure has been studied ([Da],[HS],[S2] etc), however most results are concerned mainly with the part which is prime to the covering degree. In [Rez1,2], A. Reznikov studied the p-homology of p-fold branched covers after the model of the classical problem on p-ideal class groups in number theory (see also [Mo1]). In this paper, we push this line of study in *arithmetic topology* further and determine the Galois module structure of the p-homology of a p-fold branched cover along a link completely in terms of the p-adic higher linking matrices. To be precise, let M be the p-fold cyclic branched cover of S^3 along L obtained from the completion of the p-fold total linking cover of X_L and let σ denote a generator of the Galois group of M over S^3. The homology group $H_1(M, \mathbf{Z}_p) = H_1(M, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}_p$ is then a module over the complete discrete valuation ring $\widehat{\mathcal{O}} := \mathbf{Z}_p[\langle\sigma\rangle]/(\sigma^{p-1} + \cdots + \sigma + 1) = \mathbf{Z}_p[\zeta]$, $\zeta := \sigma \bmod (\sigma^{p-1} + \cdots + \sigma + 1)$. Assume that $H_1(M, \mathbf{Z})$ is finite. Then the p-primary part $H_1(M, \mathbf{Z}_p)$ has the $\mathfrak{p}$-rank $n-1$ ([Mo1],[Rez2]) so that it has form

$$H_1(M, \mathbf{Z}_p) = \bigoplus_{i=1}^{n-1} \widehat{\mathcal{O}}/\mathfrak{p}^{a_i} \quad (a_i \geq 1)$$

as $\widehat{\mathcal{O}}$-module where $\mathfrak{p} := (\zeta - 1)$ is the maximal ideal of $\widehat{\mathcal{O}}$. Hence the determination of the Galois module structure of $H_1(N, \mathbf{Z}_p)$ is equivalent to that of the $\mathfrak{p}^k$-rank

$$e_k := \#\{i \mid a_i \geq k\} \quad (k \geq 1).$$

Our main result is to give formulas for e_k's in terms of the higher linking matrices obtained by specializing the truncated p-adic Traldi matrices at $X_1 = \cdots = T_n = \zeta - 1$ (Theorem 4.1.3). For the simplest case of $k = 2$, our formula reads

$$e_2 = n - 1 - \mathrm{rank}_{\mathbf{F}_p}(C \bmod p)$$

where $C = (C_{ij})$ is the linking matrix defined by $C_{ij} = \mathrm{lk}(K_i, K_j)$ for $i \neq j$ and $C_{ii} = -\sum_{j \neq i} \mathrm{lk}(K_i, K_j)$. In view of the analogy between the linking number and the power residue symbols [Mo2,3], this is seen as a link-theoretic analog of L. Rédei's formula for the 4-rank of the class group of a quadratic field ([Réd1]), and our general result was partly suggested by the relation between Rédei's triple symbol and the 8-rank of a class group [Réd2]. In fact, the whole argument here can be translated into arithmetic [Mo5]. In the last section, we study the asymptotic behavior of the order $|H_1(M_m, \mathbf{Z}_p)|$ for the p^m-fold cyclic branched cover M_m as $m \to \infty$, following Iwasawa theory on $\mathbf{Z}_p$-extensions [Iw]. Though our results obtained in this paper are rather elementary, they seem to indicate further possibilities of our arithmetic approach to link theory.

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Notation. Throughout this paper, we fix a prime number p. We denote by $\mathbf{F}_p$ the field with p elements and by $\mathbf{Z}_p$ the ring of p-adic integers. Let ord_p denote the additive p-adic valuation extended on the algebraic closure $\overline{\mathbf{Q}_p}$ of the p-adic field

$\mathbf{Q}_p$ with $\mathrm{ord}_p(p) = 1$ and set $|x|_p = p^{-\mathrm{ord}_p(x)}$, $x \in \overline{\mathbf{Q}_p}$. We use the letter q to denote p or 0. For a topological (possibly discrete) group G, we denote by $G^{(k,q)}$ the k-th term of lower central q-series defined by $G^{(1,q)} = G, G^{(k+1,q)} = (G^{(k,q)})^q[G^{(k,q)}, G]$ where for closed subgroups A, B of G, $[A, B]$ stands for the closed subgroup of G generated by $[a, b] = aba^{-1}b^{-1}, a \in A, b \in B$. We simply write $G^{(k)}$ for $G^{(k,0)}$, the k-th term of the lower central series of G. For a pro-finite group G and a complete local ring R, we denote by $R[[G]]$ the completed group ring of G over R [Ko,§7].

1. Pro-p completion of a link group

1.1. *The pro-p completion of a link group.* Let L be a tame link in the 3-sphere S^3 consisting of n component knots $K_1, \cdots, K_n$ and let G_L be the link group $\pi_1(X_L)$, $X_L = S^3 \setminus L$. After the work of K.T. Chen, J. Milnor [Mi] derived the following information about the presentation of the nilpotent quotient $G_L/G_L^{(k,q)}$. Let F be the free group on the n words $x_1, \cdots, x_n$ where x_i represents the meridian m_i around K_i and let $\pi : F \to G_L$ be the meridianal homomorphism defined by $\pi(x_i) = m_i$ $(1 \leq i \leq n)$.

Theorem 1.1.1 ([Mi]). *For each $k \geq 1$ and i $(1 \leq i \leq n)$, there is a word $y_i^{(k)}$ in $x_1, \cdots, x_n$ representing the image of the i-th longitude in the quotient $G_L/G_L^{(k,q)}$ such that*

$$y_i^{(k)} \equiv y_i^{(k+1)} \mod F^{(k,q)} \tag{1.1.2}$$

and such that $\pi : F \to G_L$ induces the isomorphism

$$F/N_k F^{(k,q)} \simeq G_L/G_L^{(k,q)} \tag{1.1.3}$$

where N_k is the subgroup of F generated normally by $[x_i, y_i^{(k)}]$ $(1 \leq i \leq n)$.

Let $\widehat{G_L}$ be the pro-p completion of G_L, namely the inverse limit $\varprojlim G_L/N$ of the tower of quotients G_L/N which are finite p-groups. Since the quotients by the lower central p-series of G_L are cofinal in this tower, we have

$$\widehat{G_L} = \varprojlim_k G_L/G_L^{(k,p)}.$$

Since $\{y_i^{(k)} F^{(k,p)}\}_{k \geq 1}$ forms an inverse system in $\{F/F^{(k,p)}\}_{k \geq 1}$ by (1.1.2), we define the pro-p word y_i to be $(y_i^{(k)} F^{(k,p)})$ in the free pro-p group $\widehat{F} := \varprojlim F/F^{(k,p)}$ which represents the i-th "longitude" in $\widehat{G_L}$ under the map $\widehat{\pi} : \widehat{F} \to \widehat{G_L}$ induced by π. By taking the inverse limit with respect to k in the isomorphism (1.1.3) of finite p-groups for $q = p$, we have the following

Theorem 1.1.4. *The map $\widehat{\pi}$ induces the isomorphism of pro-p groups*

$$\widehat{F}/\widehat{N} \simeq \widehat{G_L}$$

where $\widehat{N}$ is the closed subgroup of $\widehat{F}$ generated normally by $[x_i, y_i]$ $(1 \leq i \leq n)$. In particular, we have $\widehat{G_K} \simeq \mathbf{Z}_p$ for a knot K.

Remark 1.1.5. (1) By the construction above, we note $y_i \equiv y_i^{(k)} \bmod \widehat{F}^{(k,q)}$ (F is embedded in $\widehat{F}$).

(2) In view of the analogy between knots and primes, the pro-p link group $\widehat{G_L}$ is regarded as an analog of the maximal pro-p Galois group over the rational number field $\mathbf{Q}$ unramified outside prime numbers $p_1, \cdots, p_n$, $p_i \equiv 1 \bmod p$ [Mo2].

Theorem 1.1.4 tells us that from the group-theoretic point of view, any link of n components looks like a pure braid link with n strings after the pro-p completion. In particular, by applying the method of D. Anick [A] to determine the graded quotients of the lower central series of a pure braid link group to our pro-p link group $\widehat{G_L}$, we see that the pro-p analog of Murasugi's conjecture holds (cf. [L]). We define the *mod p linking diagram* of L to be the graph with vertices the components of L and an edge joining K_i and K_j if and only if the linking number $\mathrm{lk}(K_i, K_j) \not\equiv 0 \bmod p$.

Theorem 1.1.6. *If the mod p linking diagram of L is connected, we have the isomorphisms*

$$\widehat{G_L}^{(q)}/\widehat{G_L}^{(q+1)} \simeq \widehat{F}_1^{(q)}/\widehat{F}_1^{(q+1)} \times \widehat{F}_{n-1}^{(q)}/\widehat{F}_{n-1}^{(q+1)} \ \text{for} \ q \geq 1,$$

where $\widehat{F}_r$ denotes the free pro-p group of rank r.

1.2. *The p-goodness of a link group.* Let G be a group and $\widehat{G}$ be the pro-p completion of G. We then call G *p-good* if the natural map $G \to \widehat{G}$ induces the isomorphisms on cohomology $H^q(\widehat{G}, M) \stackrel{\sim}{\to} H^q(G, M)$ for all $q \geq 0$ for any finite p-primary $\widehat{G}$-modules M (cf. [Se]).

Theorem 1.2.1. *A link group G_L is p-good.*

Proof. We shall say that a subgroup G of finite index in G_L is *open* if $[G_L : G]$ is a power of p. Let M be a finite p-primary $\widehat{G_L}$-module. We shall show by induction on the length of M that if G is an open subgroup of G_L then there is a smaller open subgroup G_1 such that restriction from $H^2(G, M)$ to $H^2(G_1, M)$ is trivial.

Suppose first that $M = \mathbf{F}_p$ with trivial G_L-action and let $H^*(G)$ denote $H^*(G, \mathbf{F}_p)$ for ease of reading. Since $[G_L : G]$ is finite and $G_L/G_L^{(2)} \cong \mathbf{Z}^n$ there is an epimorphism $\tau : G \to C = \mathbf{Z}/p\mathbf{Z}$. Then $K = \mathrm{Ker}(\tau)$ is another open subgroup of G_L. The Hochschild-Serre spectral sequence for G as an extension of C by K has E_2 term $E_2^{p,q} = H^p(C, H^q(K))$, r^{th} differential d_r of bidegree $(r, 1-r)$ and converges to $H^*(G)$. Since $H^p(K) = 0$ for $p > 2$ there are only three nonzero rows, and since $H^*(G) = 0$ for $* > 2$ we see that $d_3^{p,2}$ is an isomorphism for all $p \geq 1$. The spectral sequence is an algebra over the ring $H^*(C) = E_2^{*,0}$. Since C has cohomological period 2, the cup product with a generator of $H^2(C) \cong \mathbf{F}_p$ induces isomorphisms $\gamma_2^{p,q} : E_2^{p,q} \cong E_2^{p+2,q}$ such that $d_2^{p+2,q}\gamma_2^{p,q} = \gamma_2^{p+2,q-1}d_2^{p,q}$ for all $p, q \geq 0$. Therefore we have the isomorphisms $\mathrm{Ker}(d_2^{p,q}) \simeq \mathrm{Ker}(d_2^{p+2,q})$, $\mathrm{Im}(d_2^{p,q}) \simeq \mathrm{Im}(d_2^{p+2,q})$ for any $p, q \geq 0$. In particular we have the isomorphisms $\gamma_3^{0,2} : E_3^{0,2} = \mathrm{Ker}(d_2^{0,2}) \cong E_3^{2,2} = \mathrm{Ker}(d_2^{2,2})$ and $\gamma_3^{3,0} : E_3^{3,0} = E_2^{3,0}/\mathrm{Im}(d_2^{1,1}) \cong E_3^{5,0} = E_2^{5,0}/\mathrm{Im}(d_2^{3,1})$ with $d_3^{2,2} \circ \gamma_3^{0,2} = \gamma_3^{2,1} \circ d_3^{0,2}$. It follows that $d_3^{0,2}$ is also an isomorphism, and so $E_\infty^{0,2} = 0$. But the edge homomorphism from $H^2(G)$ to $H^2(K)$ factors through $E_\infty^{0,2} \leq E_2^{0,2} = H^2(K)^C$, and so is 0.

In general, M has a finite composition series whose factors are copies of the simple module $\mathbf{F}_p$. Suppose that M_1 is a maximal proper submodule of M, with quotient $M/M_1 \cong \mathbf{F}_p$. Restriction from G to K induces a homomorphism from the exact sequences of cohomology corresponding to the coefficient sequence $0 \to M_1 \to M \to \mathbf{F}_p \to 0$. The result for $\mathbf{F}_p$ implies that the image of $H^2(G; M)$ lies in the image of $H^2(K, M_1)$. By the hypothesis of induction we may assume the result is true for M_1, and so there is an open subgroup $K_1 < K$ such that restriction from $H^2(K, M_1)$ to $H^2(K_1, M_1)$ is trivial. Hence restriction from $H^2(G, M)$ to $H^2(K_1, M)$ is also trivial. This establishes the inductive step.

In particular, restriction from $H^2(G_L, M)$ to $H^2(J, M)$ is trivial, for some open subgroup J, and so the result follows, as in Exercise 1 of Chapter I.§2.6 of [Se]. (This exercise is stated in terms of profinite completions, but extends easily to the pro-p case). □

Since the cohomological dimension $cd(G_L) \leq 2$, with equality if and only if L is nontrivial, Theorem 1.2.1 gives the corresponding bound for the pro-p completion $\widehat{G_L}$.

Corollary 1.2.2. *The cohomological p-dimension* $cd_p(\widehat{G_L}) \leq 2$.

If the Milnor invariants of L are all 0 mod p (cf. Section 2), then $\widehat{G_L}$ is a free pro-p group and so $cd_p(\widehat{G_L})$ may be strictly less than $cd(G_L)$. In particular, this is so if L is a nontrivial knot.

2. p-adic Milnor invariants

2.1. *The pro-p Fox differential calculus.* Let $\widehat{F}$ be the pro-p completion of the free group F on n generators $x_1, \cdots, x_n$. Y. Ihara [Ih] extended the Fox differential calculus on the abstract free group F ([F]) to that on $\widehat{F}$. The basic result is stated as the following

Theorem 2.1.1 ([Ih]). *There is a unique continuous* $\mathbf{Z}_p$*-homomorphism*

$$\partial_i = \frac{\partial}{\partial x_i} \; : \; \mathbf{Z}_p[[\widehat{F}]] \longrightarrow \mathbf{Z}_p[[\widehat{F}]]$$

for each i $(1 \leq i \leq n)$ *such that any element* $\alpha \in \mathbf{Z}_p[[\widehat{F}]]$ *is expressed uniquely in the form*

$$\alpha = \epsilon(\alpha)1 + \sum_{i=1}^{n} \partial_i(\alpha)(x_i - 1)$$

where ϵ *is the augmentation map* $\mathbf{Z}_p[[\widehat{F}]] \to \mathbf{Z}_p$.

The higher order derivatives are defined inductively by

$$\partial_{i_1} \cdots \partial_{i_r}(\alpha) = \partial_{i_1}(\partial_{i_2} \cdots \partial_{i_r}(\alpha)).$$

Here are some basic rules (cf. [Ih,2]).

(2.1.2) 1. If one restricts ∂_i to $\mathbf{Z}[F]$ under the natural embedding $\mathbf{Z}[F] \to \mathbf{Z}_p[[\widehat{F}]]$, we get the usual Fox derivative on $\mathbf{Z}[F]$ ([F]).
2. $\partial_i(x_j) = \delta_{ij}$ (Kronecker delta).
3. $\partial_i(\alpha\beta) = \partial_i(\alpha)\epsilon(\beta) + \alpha\partial_i(\beta)$ $(\alpha, \beta \in \mathbf{Z}_p[[\widehat{F}]])$.
4. $\partial_i(f^{-1}) = -f^{-1}\partial_i(f)$ $(f \in \widehat{F})$.
5. For $f \in \widehat{F}$ and $a \in \mathbf{Z}_p$, $\partial_i(f^a) = b\partial_i(f)$, where b is any element of $\mathbf{Z}_p[[\widehat{F}]]$ such that $b(f-1) = f^a - 1$.

6. Let $\widehat{F'}$ be another free pro-p group on $x'_1, \cdots, x'_m$ and let $\varphi : \widehat{F} \to \widehat{F'}$ be a continuous surjective homomorphism. Then one has $\partial'_i(\varphi(\alpha)) = \sum_{j=1}^{n} \varphi(\partial_j(\alpha))\partial'_i(\varphi(x_j))$, where $\partial'_i = \frac{\partial}{\partial x'_i}, \alpha \in \mathbf{Z}_p[[\widehat{F}]]$.
7. For $f \in \widehat{F}$, $\epsilon(\partial_i^r(f)) = \binom{a}{r}$ where $a = \epsilon(\partial_i(f))$ and $\binom{a}{r} = \frac{a(a-1)\cdots(a-r+1)}{r!} \in \mathbf{Z}_p$.

Let $\mathbf{Z}_p\langle\langle X_1, \cdots, X_n\rangle\rangle$ be the formal power series ring over $\mathbf{Z}_p$ in non-commuting variables $X_1, \cdots, X_n$ which is compact in the topology taking the ideals $I(r)$ of power series with homogeneous components of degree $\geq r$ as the system of neighborhood of 0. The pro-p Magnus embedding M of $\widehat{F}$ into $\mathbf{Z}_p\langle\langle X_1, \cdots, X_n\rangle\rangle^{\times}$ is defined by

$$M(x_i) = 1 + X_i,\ M(x_i^{-1}) = 1 - X_i + X_i^2 + \cdots$$

and it is extended to the isomorphism $\mathbf{Z}_p[[\widehat{F}]] \simeq \mathbf{Z}_p\langle\langle X_1, \cdots, X_n\rangle\rangle$ of compact $\mathbf{Z}_p$-algebras. The resulting expansion of $\alpha \in \mathbf{Z}_p[[\widehat{F}]]$ is given by the Fox derivatives:

$$M(\alpha) = \epsilon(\alpha) + \sum_{I=(i_1\cdots i_r)} \epsilon_I(\alpha)X_{i_1}\cdots X_{i_r}, \qquad (2.1.3)$$
$$\epsilon_I(\alpha) = \epsilon(\partial_{i_1}\cdots\partial_{i_r}(\alpha)),\ \ I = (i_1\cdots i_r).$$

Finally, we recall the following fact ([Ko],7.14): Let $\widehat{J}_q$ be the two-sided ideal of $\mathbf{Z}_p[[\widehat{F}]]$ generated by q and the augmentation ideal $I_{\mathbf{Z}_p[[\widehat{F}]]}$. Then for $f \in \widehat{F}$ and $k \geq 1$, we have

(2.1.4)

$$f \in \widehat{F}^{(k,q)} \Longleftrightarrow f - 1 \in \widehat{J}_q$$
$$\Longleftrightarrow M(f) = 1 + (\text{term of degree} \geq k)\ \ (\text{for case } q = 0).$$

2.2. *p-adic Milnor invariants.* We keep the same notation as in Section 1. Let $\widehat{F}$ be the free pro-p group on $x_1, \cdots, x_n$ where each x_i represents the i-th meridian. For a multi-index $I = (i_1\cdots i_r), r \geq 1$, we define the *p-adic Milnor number* $\hat{\mu}(I)$ by

$$\hat{\mu}(I) := \epsilon_{I'}(y_{i_r})\ \ I' = (i_1\cdots i_{r-1}) \qquad (2.2.1)$$
$$= \epsilon(\partial_{i_1}\cdots\partial_{i_{r-1}}(y_{i_r}))$$

where $y_j \in \widehat{F}$ represents the j-th "longitude" in $\widehat{G_L}$ (cf. Section 1.1). By convention, we set $\hat{\mu}(I) = 0$ for $|I| = r = 1$. We let $\hat{\Delta}(I)$ denote the ideal of $\mathbf{Z}_p$ generated by $\hat{\mu}(J)$ where J runs over all cyclic permutations of proper subsequences of I. We then define the *p-adic Milnor invariant* $\overline{\hat{\mu}}(I)$ by

$$\overline{\hat{\mu}}(I) := \hat{\mu}(I) \bmod \hat{\Delta}(I). \qquad (2.2.2)$$

Since the usual Milnor number $\mu(I), I = (i_1\cdots i_r)$ is defined by $\epsilon(\partial_{i_1}\cdots\partial_{i_{r-1}}(y_{i_r}^{(r)}))$ by (2.1.2),1, Remark 1.1.5, (1) and (2.2.1) yield

$$\hat{\mu}(I) = \mu(I)\ \text{ and }\ \hat{\Delta}(I) = \Delta(I) \qquad (2.2.3)$$

as elements in $\mathbf{Z}_p$. Hence, we have

(2.2.4) $\overline{\hat{\mu}}(I)$'s are isotopy invariants of L and satisfy the same properties such as the cycle symmetry and Shuffle relations $\overline{\mu}(I)$'s enjoy ([Mi]).

Theorem 1.1.4 and (2.1.4) implies the following:

(2.2.5) All $\overline{\mu}(I) = 0$ for $|I| < r$ if and only if $\widehat{\pi} : \widehat{F} \to \widehat{G_L}$ induces an isomorphism $\widehat{F}/\widehat{F}^{(r,q)} \simeq \widehat{G_L}/\widehat{G_L}^{(r,q)}$. In particular, if all p-adic Milnor invariants of L are zero, one has $\widehat{G_L} \simeq \widehat{F}$. This is the case for boundary links.

Finally, we remark that (2.1.2), 7 implies

$$(2.2.6) \qquad \hat{\mu}(\underbrace{i \cdots i}_{r} j) = \binom{\mathrm{lk}(K_i, K_j)}{r} \text{ for } i \neq j.$$

The Milnor invariant is also given by the Massey products in the cohomology of $\widehat{G_L}$. For the normalized Massey system in profinite group cohomology and sign convention, we refer to [Mo3]. Let $\xi_1, \cdots, \xi_n$ be the $\mathbf{Z}_p$-basis of $H^1(\widehat{G_L}, \mathbf{Z}_p)$ dual to the meridians m_i's, and let $\eta_j \in H_2(\widehat{G_L}, \mathbf{Z}_p)$ be the image of $[x_j, y_j]$ under the transgression $H_1(\widehat{N}, \mathbf{Z}_p)_{\widehat{G_L}} \to H_2(\widehat{G_L}, \mathbf{Z}_p)$. Then for $I = (i_1 \cdots i_r), r \geq 2$, there is a normalized Massey system M for the product $\langle \xi_{i_1}, \cdots, \xi_{i_r} \rangle \in H_2(\widehat{G_L}, \mathbf{Z}_p/\hat{\Delta}(I))$ so that

$$(2.2.7) \qquad \langle \xi_{i_1}, \cdots, \xi_{i_r} \rangle (\eta_j) = \begin{cases} (-1)^r \overline{\mu}(I) & j = i_r \neq i_1, \\ (-1)^{r+1} \overline{\mu}(I) & j = i_1 \neq i_r, \\ 0 & \text{otherwise.} \end{cases}$$

3. Completed Alexander modules

3.1. *The Alexander module of* $\widehat{G_L}$. The Alexander module of a finitely presented pro-p group was introduced in several modes in [Mo2]. As a particular case, the Alexander module of the pro-p link group $\widehat{G_L}$ is defined using the pro-p Fox free differential calculus as follows. We keep the same notation as in Section 1 and 2. Let $\widehat{\psi}$ be the abelianization map $\widehat{G_L} \to \widehat{H} := \widehat{G_L}/\widehat{G_L}^{(2)} = \mathbf{Z}_p^n$ and denote by the same $\widehat{\psi}$ the continuous $\mathbf{Z}_p$-homomorphism $\mathbf{Z}_p[[\widehat{G_L}]] \to \mathbf{Z}_p[[\widehat{H}]]$ on the completed group rings. We identify $\mathbf{Z}_p[[\widehat{H}]]$ with the commutative formal power series ring $\widehat{\Lambda_n} := \mathbf{Z}_p[[X_1, \cdots, X_n]]$ over $\mathbf{Z}_p$ by setting $t_i := \widehat{\psi} \circ \widehat{\pi}(x_i) = 1 + X_i$. By Theorem 1.1.4, we call the matrix over $\widehat{\Lambda_n}$

$$(3.1.1) \qquad \widehat{P_L} = \left(\widehat{\psi} \circ \widehat{\pi}(\partial_j([x_i, y_i]))\right)$$

the *Alexander matrix* of $\widehat{G_L}$. We then define the *Alexander module* $\widehat{A_L}$ of $\widehat{G_L}$ by the compact $\widehat{\Lambda_n}$-module presented by $\widehat{P_L}$:

$$(3.1.2) \qquad \widehat{A_L} := \mathrm{Coker}(\widehat{\Lambda_n}^n \xrightarrow{\widehat{P_L}} \widehat{\Lambda_n}^n)$$

and call $\widehat{A_L}$ the *completed Alexander module* of L over $\widehat{\Lambda_n}$. Let $\widehat{\Lambda} = \mathbf{Z}_p[[X]]$, the *Iwasawa algebra* and $\tau : \widehat{\Lambda_n} \to \widehat{\Lambda}$ the reducing homomorphism defined by $\tau(X_i) = X$ $(1 \leq i \leq n)$. Then the *reduced completed Alexander module* $\widehat{A_L}^{red}$ is defined by the compact $\widehat{\Lambda}$-module

$$(3.1.3) \qquad \widehat{A_L}^{red} := \tau(\widehat{A_L}) = \widehat{A_L} \otimes_{\widehat{\Lambda_n}} \widehat{\Lambda}$$

which is presented by $\tau(\widehat{P_L}) = \widehat{P_L}(X, \cdots, X)$. For a knot K, $\widehat{P_L} = O_n$ (zero matrix) and so we have

$$\widehat{A_L} = \widehat{A_L}^{red} = \widehat{\Lambda}. \tag{3.1.4}$$

We also define the *i-th completed Alexander ideal* $\widehat{E_i}(L)$ of L by the i-th elementary ideal $E_i(\widehat{A_L})$ of $\widehat{A_L}$ and *i-th p-adic Alexander series* $\widehat{\Delta_i}(L)$ of L by the greatest common divisor $\Delta_i(\widehat{E_i}(L))$ of generators of the ideal $\widehat{E_i}(L)$:

$$\widehat{E_i}(L) := E_i(\widehat{A_L}), \ \ \widehat{\Delta_i}(L) := \Delta_i(\widehat{E_i}(L)) \tag{3.1.5}$$

The relation with the usual Alexander module is given as follows. Let $\psi : G_L \to H = H_1(X_L, \mathbf{Z})$ be the abelianization map which induces the ring homomorphism $\psi : \mathbf{Z}[G_L] \to \mathbf{Z}[H]$ on the group rings where $\mathbf{Z}[H]$ is identified with the Laurent polynomial ring $\Lambda_n = \mathbf{Z}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]$, $t_i = \psi \circ \pi(x_i)$. Given a presentation $G_L = \langle x_1, \cdots, x_m \mid r_1 = \cdots r_l = 1\rangle$ $(m \geq n)$, the Alexander module of L is given as the Λ_n-module presented by the Alexander matrix $(\psi \circ \pi(\partial_j(r_i)))$. By Theorem 1.1.1, we can take $m = n$ and the relators to be $[x_i, y_i^{(k+1)}]$ $(1 \leq i \leq n)$ and some finite number of generators $f_i^{(k+1)}$ of $F^{(k+1,p)}$ $(k \geq 1)$. We let $J_p = \widehat{J_p} \cap \mathbf{Z}[F]$ and $J_{\Lambda_n,p} = \psi \circ \pi(J_p)$. Since $\partial_j(f_i^{(k+1)}) \in {J_p}^k$ by (2.1.4), passing to the quotients modulo $(J_{\Lambda_n,p})^k$, $A_L/(J_{\Lambda_n,p})^k A_L = A_L \otimes_{\Lambda_n} \Lambda_n/(J_{\Lambda_n,p})^k$ is presented by the matrix $\left(\psi \circ \pi(\partial_j([x_i, y_i^{(k+1)}])) \bmod (J_{\Lambda_n,p})^k\right)$. Here we see by Remark 1.1.5, (1) that the elements $\{\psi \circ \pi(\partial_j([x_i, y_i^{(k+1)}])) \bmod (J_{\Lambda_n,p})^k\}$ form an inverse system with respect to k and its limit is given as $\widehat{\psi} \circ \widehat{\pi}(\partial_j([x_i, y_i]))$ under the identification $\varprojlim_k \Lambda_n/(J_{\Lambda_n,p})^k = \widehat{\Lambda_n}$ defined by $t_i = 1 + X_i$. Hence by (3.1.2) we have

$$\widehat{A_L} = \varprojlim_k A_L \otimes_{\Lambda_n} (\Lambda_n/(J_{\Lambda_n,p})^k) = A_L \otimes_{\Lambda_n} \widehat{\Lambda_n}. \tag{3.1.6}$$

Similarly, $\widehat{A_L}^{red}$ is related with the usual reduced Alexander module $A_L^{red} = \tau(A_L)$ by

$$\widehat{A_L}^{red} = A_L \otimes_{\Lambda} \widehat{\Lambda} \tag{3.1.7}$$

where $\Lambda = \mathbf{Z}[t^{\pm 1}]$ is embedded into $\widehat{\Lambda}$ by $t = 1 + X$.

3.2. *The p-adic Traldi matrix.* The Alexander matrix $\widehat{P_L}$ of $\widehat{G_L}$ (3.1.1) is computed explicitly as a *universal p-adic higher linking matrix* in terms of p-adic Milnor numbers. This is regarded as a p-adic strengthening of the work by L. Traldi [Tr].

Definition 3.2.1. The *p-adic Traldi matrix* $\widehat{T_L} = (\widehat{T_L}(i,j))$ of L over $\widehat{\Lambda_n}$ is defined by

$$\widehat{T_L}(i,j) = \begin{cases} -\displaystyle\sum_{r\geq 1} \sum_{\substack{1 \leq i_1, \cdots, i_r \leq n \\ i_r \neq i}} \hat{\mu}(i_1 \cdots i_r i) X_{i_1} \cdots X_{i_r} & i = j \\ \hat{\mu}(ji)X_i + \displaystyle\sum_{r \geq 1} \sum_{1 \leq i_1 \cdots i_r \leq n} \hat{\mu}(i_1 \cdots i_r j i) X_i X_{i_1} \cdots X_{i_r} & i \neq j. \end{cases}$$

and we also define the *reduced p-adic Traldi matrix* $\widehat{T_L}^{red}$ of L over $\widehat{\Lambda}$ by

$$\widehat{T_L}^{red} := \tau(\widehat{T_L}) = \widehat{T_L}(X, \cdots, X).$$

Our theorem is then stated as

Theorem 3.2.2. *The p-adic Traldi matrix $\widehat{T_L}$ gives a presentation matrix for the completed Alexander module $\widehat{A_L}$ over $\widehat{\Lambda_n}$, and the reduced p-adic Traldi matrix $\widehat{T_L}^{red}$ gives a presentation matrix for the reduced completed Alexander module $\widehat{A_L}^{red}$ over $\widehat{\Lambda}$.*

Proof. By (3.1.1),(3.1.2) and (3.1.3), it suffices to show $\widehat{\psi} \circ \widehat{\pi}(\partial_j([x_i, y_i])) = \widehat{T_L}(i,j)$. By the rules $2 \sim 4$ of (2.1.2), we have

$$\partial_j([x_i, y_i]) = (1 - x_i y_i x_i^{-1})\delta_{ij} + x_i(1 - y_i x_i^{-1} y_i^{-1})\partial_j(y_i).$$

Here $y_i = 1 + \sum_{r \geq 1} \sum_{1 \leq i_1, \cdots, i_r \leq n} \hat{\mu}(i_1 \cdots i_r i)(x_{i_1} - 1) \cdots (x_{i_r} - 1)$ by (2.1.3) and (2.2.1). Hence we get

(3)
$$\begin{aligned} \widehat{\psi} \circ \widehat{\pi}\,(\partial_j([x_i, y_i])) = \delta_{i,j} &- \sum_{r \geq 1} \sum_{1 \leq i_1, \cdots, i_r \leq n} \hat{\mu}(i_1 \cdots i_r i) X_{i_1} \cdots X_{i_r} \\ &+ \hat{\mu}(ji) X_i + \sum_{r \geq 1} \sum_{1 \leq i_1, \cdots, i_r \leq n} \hat{\mu}(i_1 \cdots i_r j i) X_i X_{i_1} \cdots X_{i_r}. \end{aligned}$$

which yields the assertion. □

The following is an extension of (3.1.4).

Corollary 3.2.3. *For a link whose p-adic Milnor invariants are all zero, we have*

$$\widehat{A_L} \simeq \widehat{\Lambda_n}^n, \quad \widehat{A_L}^{red} \simeq \widehat{\Lambda}^n.$$

This is the case for boundary links.

Finally, we introduce the *truncated p-adic Traldi matrices.*

Definition 3.2.4. For $k \geq 2$, the *k-th truncated p-adic Traldi matrix* $\widehat{T_L}^{(k)} = (\widehat{T_L}^{(k)}(i,j))$ is defined by

$$\widehat{T_L}^{(k)}(i,j) = \begin{cases} -\sum_{r=1}^{k-1} \sum_{\substack{1 \leq i_1, \cdots, i_r \leq n \\ i_r \neq i}} \hat{\mu}(i_1 \cdots i_r i) X_{i_1} \cdots X_{i_r} & i = j \\ \hat{\mu}(ji) X_i + \sum_{r=1}^{k-2} \sum_{1 \leq i_1, \cdots, i_r \leq n} \hat{\mu}(i_1 \cdots i_r j i) X_i X_{i_1} \cdots X_{i_r} & i \neq j \end{cases}$$

and we also define the *k-th truncated reduced p-adic Traldi matrix* $\widehat{T_L}^{red,(k)}$ by

$$\widehat{T_L}^{red,(k)} := \tau(\widehat{T_L}^{(k)}) = \widehat{T_L}^{(k)}(X, \cdots, X).$$

We note that $\widehat{T_L}^{red,(2)}$ is the *linking matrix* multiplied by X, where the linking matrix $C = (C_{ij})$ is defined by $C_{ii} = -\sum_{j \neq i} \mathrm{lk}(K_i, K_j)$ and $C_{ij} = \mathrm{lk}(K_i, K_j)$ for

$i \neq j$. Thus the p-adic Traldi matrix $\widehat{T_L}$ is regarded as a *universal higher linking matrix* over $\widehat{\Lambda_n}$ which contains all information on the completed Alexander module. In the following section, we derive from $\widehat{T_L}$ the information on the p-homology groups of p^m-fold cyclic branched covers along L.

4. Galois module structure for the p-homology group of a p-fold cyclic branched cover

4.1. *Galois module structure of the p-homology of a p-fold cover.* Let X_∞ be the infinite cyclic cover of X_L associated to the kernel of the homomorphism $G_L \to \langle t \rangle$ sending each meridian to t. Let M be the the completion of the p-fold subcover of X_∞ over X_L so that M is a p-fold cyclic cover of S^3 branched along L. We set $\nu_d(t) = t^{d-1} + \cdots + t + 1$ for $d \geq 1$. Let $\phi : M \to S^3$ be the covering map and σ a generator of its Galois group. Since $\nu_p(\sigma) = tr \circ \phi_* = 0$ where $tr : H_1(S^3, \mathbf{Z}) \to H_1(M, \mathbf{Z})$ is the transfer, we can regard $H_1(M, \mathbf{Z})$ as a module over the Dedekind ring $\mathcal{O} = \mathbf{Z}[\langle \sigma \rangle]/(\nu_p(\sigma)) = \mathbf{Z}[\zeta]$, $\zeta = \sigma \bmod (\nu_p(\sigma))$ is a primitive p-th root of 1. Hence $H_1(M, \mathbf{Z}_p) = H_1(M, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}_p$ is regarded as a module over the complete discrete valuation ring $\widehat{\mathcal{O}} = \mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}_p = \mathbf{Z}_p[\zeta]$. Note that $\widehat{\mathcal{O}}$ is the completion of $\mathcal{O}$ with respect to the maximal ideal $\mathfrak{p}$ generated by the prime element $\pi := \zeta - 1$ and the residue field $\widehat{\mathcal{O}}/\mathfrak{p}$ is $\mathbf{F}_p$. By Theorem 2.2.1, we can derive the following information on a presentation matrix for the $\widehat{\mathcal{O}}$-module $H_1(M, \mathbf{Z}_p)$. Note that the evaluation of a power series $F(X) \in \mathbf{Z}_p[[X]]$ at $s = \pi$ makes sense in the $\mathfrak{p} = (\pi)$-adically complete ring $\widehat{\mathcal{O}}$.

Theorem 4.1.1. *A presentation matrix for $H_1(M, \mathbf{Z}_p) \oplus \widehat{\mathcal{O}}$ over $\widehat{\mathcal{O}}$ is given by $\widehat{T_L}^{red}(\pi)$. Further, for any integer $k \geq 2$, a presentation matrix for $(H_1(M, \mathbf{Z}_p) \otimes_{\widehat{\mathcal{O}}} \widehat{\mathcal{O}}/\mathfrak{p}^k) \oplus \widehat{\mathcal{O}}/\mathfrak{p}^k$ over $\widehat{\mathcal{O}}/\mathfrak{p}^k$ is given by $\widehat{T_L}^{red,(k)}(\pi)$. Here $\widehat{T_L}^{red}$ (resp. $\widehat{T_L}^{red,(k)}$) is the reduced (resp. reduced truncated) Traldi matrix defined in Section 3.2.*

Proof. Note that the well-known relation ([S1,Theorem 6])

$$H_1(M, \mathbf{Z}) \simeq H_1(X_\infty, \mathbf{Z})/\nu_p(t) H_1(X_\infty, \mathbf{Z})$$

is an $\widehat{\mathcal{O}}$-isomorphism since σ acts on the r.h.s by t. Hence we have the following isomorphisms over $\widehat{\mathcal{O}}$

$$\begin{aligned} H_1(M, \mathbf{Z}_p) &\simeq (H_1(X_\infty, \mathbf{Z})/\nu_p(t) H_1(X_\infty, \mathbf{Z})) \otimes_{\mathbf{Z}} \mathbf{Z}_p \\ &\simeq H_1(X_\infty, \mathbf{Z}) \otimes_\Lambda (\Lambda \otimes_{\mathbf{Z}} \mathbf{Z}_p)/(\nu_p(t))) \\ &\simeq H_1(X_\infty, \mathbf{Z}) \otimes_\Lambda (\widehat{\Lambda}/(\nu_p(1+X))) \\ &\simeq H_1(X_\infty, \mathbf{Z}) \otimes_\Lambda \widehat{\mathcal{O}}. \end{aligned}$$

Since $A_L^{red} \simeq H_1(X_\infty, \mathbf{Z}) \oplus \Lambda$ as Λ-module ([H, 5.4]), tensoring with $\widehat{\mathcal{O}}$ over Λ, we have an isomorphism of $\widehat{\mathcal{O}}$-modules

$$A_L^{red} \otimes_\Lambda \widehat{\mathcal{O}} \simeq H_1(M, \mathbf{Z}_p) \oplus \widehat{\mathcal{O}}.$$

Since the l.h.s is same as $\widehat{A_L}^{red} \otimes_{\widehat{\Lambda}} \widehat{\mathcal{O}}$, the first assertion follows from Theorem 3.2.4. The second assertion is obtained from the first one by taking modulo $\mathfrak{p}^k$. □

Now, we asuume that $H_1(M,\mathbf{Z})$ is finite so that $H_1(M,\mathbf{Z}_p)$ is the p-primary part of $H_1(M,\mathbf{Z})$. Using Theorem 3.1.1, we will see the $\widehat{\mathcal{O}}$-module structure of $H_1(M,\mathbf{Z}_p)$ more precisely. First, we recall the following result on the $\mathfrak{p}$-rank of $H_1(M,\mathbf{Z}_p)$ (cf. [Mo1],[Rez]).

Lemma 4.1.2. *$H_1(M,\mathbf{Z}_p)\otimes_{\widehat{\mathcal{O}}}\mathbf{F}_p$ has dimension $n-1$ over $\mathbf{F}_p$.*

Proof. By [Mo1], the map $\Phi : H_1(M,\mathbf{Z})\to\mathbf{F}_p^n$ defined by

$$\Phi(c) := (\mathrm{lk}(\phi_*(c),K_1) \bmod p)$$

induces an isomorphism

$$H_1(M,\mathbf{Z})/(\sigma-1)H_1(M,\mathbf{Z})\simeq\{(\xi_i)\in\mathbf{F}_p^n \mid \sum_{i=1}^{n}\xi_i=0\}\simeq\mathbf{F}_p^{n-1}$$

where the l.h.s is $H_1(M,\mathbf{Z})\otimes_{\mathcal{O}}\mathcal{O}/\mathfrak{p}=H_1(M,\mathbf{Z}_p)\otimes_{\widehat{\mathcal{O}}}\mathbf{F}_p$, and hence we are done. □

By Lemma 4.1.2, $H_1(M,\mathbf{Z}_p)$ has the form

$$H_1(M,\mathbf{Z}_p)=\bigoplus_{i=1}^{n-1}\widehat{\mathcal{O}}/\mathfrak{p}^{a_i}\qquad(a_i\geq 1)$$

as $\hat{\mathcal{O}}$-module. Hence, the determination of $\widehat{\mathcal{O}}$-module structure of $H_1(M,\mathbf{Z}_p)$ is equivalent to the determination of the $\mathfrak{p}^k$-rank

$$e_k := \#\{i \mid a_i\geq k\}\ \ (k\geq 1).$$

By Theorem 4.1.1, we can describe the $\mathfrak{p}^k$-rank e_k in terms of $\widehat{T_L}^{red,(k)}(\pi)$.

Theorem 4.1.3. *For $k\geq 2$, let $\varepsilon_1^{(k)},\cdots,\varepsilon_{n-1}^{(k)},\varepsilon_n^{(k)}=0$ be the elementary divisors of $\widehat{T_L}^{red,(k)}(\pi)$, where $\varepsilon_i^{(k)}|\varepsilon_{i+1}^{(k)}$ $(1\leq i\leq n-1)$. Then we have*

$$e_k=\#\{i\,|\,\varepsilon_i^{(k)}\equiv 0 \bmod \mathfrak{p}^k\}-1.$$

Here we may call $\widehat{T_L}^{red,(k)}(\pi)$ the *k-th higher linking matrix* in view of the following

Corollary 4.1.4. *For $k=2$, we have*

$$e_2=n-1-\ \mathrm{rank}_{\mathbf{F}_p}(C \bmod p)$$

where $C=(C_{ij})$ is the linking matrix of L defined by $C_{ii}=-\sum_{j\neq i}\mathrm{lk}(K_i,K_j)$ and $C_{ij}=\mathrm{lk}(K_i,K_j)$ for $i\neq j$.

Proof. In fact, we have, by definition,

$$\widehat{T_L}^{red,(2)}(\pi)=\pi C$$

and hence $e_2=n-1-\mathrm{rank}_{\mathbf{F}_p}(C \bmod p)$ by Theorem 4.1.3. □

4.2. *2-component case.* We suppose $n=2$ and keep to assume $H_1(M,\mathbf{Z})$ is finite. By Lemma 4.1.2, $H_1(M,\mathbf{Z}_p)$ has the $\mathfrak{p}$-rank 1 so that we have

$$H_1(M,\mathbf{Z}_p)=\widehat{\mathcal{O}}/\mathfrak{p}^a,\ \ a\geq 1.$$

Hence $e_k = 0$ or 1 for $k \geq 2$, and by Theorem 4.1.3 we have

$$e_k = 1 \Longleftrightarrow \widehat{T_L}^{red,(k)}(\pi) \equiv O_2 \bmod \pi^k.$$

As $\widehat{T_L}^{red,(k)}(1,2)(\pi) = -\widehat{T_L}^{red,(k)}(1,1)(\pi)$, $\widehat{T_L}^{red,(k)}(2,2)(\pi) = -\widehat{T_L}^{red,(k)}(2,1)(\pi)$, we have the following

Theorem 4.2.1. *Suppose $n = 2$. For each $k \geq 1$, assuming $e_k = 1$, we have*

$$e_{k+1} = 1 \Longleftrightarrow \begin{cases} \displaystyle\sum_{r=1}^{k} \sum_{i_1,\cdots,i_{r-1}=1,2} \hat{\mu}(i_1 \cdots i_{r-1}21)\pi^r \equiv 0 \bmod \pi^{k+1}, \\ \displaystyle\sum_{r=1}^{k} \sum_{i_1,\cdots,i_{r-1}=1,2} \hat{\mu}(i_1 \cdots i_{r-1}12)\pi^r \equiv 0 \bmod \pi^{k+1}. \end{cases}$$

We give the condition in Theorem 4.2.1 in more concise forms for lower k. In the following computation, we use simply the usual Milnor number $\mu(I)$ instead of $\hat{\mu}(I)$ by (2.2.3).

Example 4.2.2. e_2: Since $e_1 = 1$, we have by Theorem 4.2.1

(4.2.2.1) $$e_2 = 1 \Longleftrightarrow \mu(12)\pi \equiv 0 \bmod \pi^2 \Longleftrightarrow \mathrm{lk}(K_1, K_2) \equiv 0 \bmod p.$$

e_3: Assume $\mathrm{lk}(K_1, K_2) \equiv 0 \bmod p$. By Theorem 4.2.1, we have

$$e_3 = 1 \Longleftrightarrow \begin{cases} \mu(21)\pi + (\mu(121) + \mu(221))\pi^2 \equiv 0 \bmod \pi^3, \\ \mu(12)\pi + (\mu(112) + \mu(212))\pi^2 \equiv 0 \bmod \pi^3. \end{cases}$$

By cycle symmetry, $\mu(121) \equiv \mu(112)$, $\mu(221) \equiv \mu(212) \bmod \mu(12)$. Here $\mu(112) \equiv \mu(221) \equiv \binom{\mu(12)}{2} \bmod \mu(12)$ by (2.2.6). Thus we have $\mu(121) + \mu(221) \equiv \mu(112) + \mu(212) \equiv \mu(12) \equiv 0 \bmod p$. Hence, we have

(4.2.2.2) $$e_3 = 1 \Longleftrightarrow \mathrm{lk}(K_1, K_2) \equiv 0 \bmod p^2.$$

As one easily sees, this condition is also equivalent to

(4.2.2.3) $$\mu(112) \equiv \mu(221) \equiv 0 \bmod p.$$

e_4: Assume $\mathrm{lk}(K_1, K_2) \equiv 0 \bmod p^2$. By Theorem 4.2.1, we have

$$e_4 = 1 \Longleftrightarrow \begin{cases} \mu(21)\pi + (\mu(121) + \mu(221))\pi^2 \\ \quad +(\mu(1121) + \mu(1221) + \mu(2121) + \mu(2221))\pi^3 \equiv 0 \bmod \pi^4, \\ \mu(12)\pi + (\mu(112) + \mu(212))\pi^2 \\ \quad +(\mu(1112) + \mu(1212) + \mu(2112) + \mu(2212))\pi^3 \equiv 0 \bmod \pi^4. \end{cases}$$

As in case of e_3, we have $\mu(121)+\mu(221) \equiv \mu(12) \equiv 0 \bmod p^2$. Similarly, $\mu(1121) \equiv \mu(1112) \equiv \binom{\mu(12)}{3} \bmod \Delta(1121)$ and $\mu(2221) \equiv \binom{\mu(12)}{3} \bmod \Delta(2221)$ by (2.2.6). Since $\Delta(1121) \equiv \Delta(2221) \equiv 0 \bmod p$, $\mu(1121) + \mu(2221) \equiv \mu(12)/3 \equiv 0 \bmod p$. Finally, by shuffle relation, $\mu(1221) + \mu(2121) + \mu(2211) \equiv \mu(2121) + 2\mu(2211) \equiv 0 \bmod p$. Thus the first condition is equivalent to $\mu(21)\pi - \mu(2211)\pi^3 \equiv 0 \bmod \pi^4$. Similarly, we see that the second condition is equivalent to $\mu(12)\pi - \mu(1122)\pi^3 \equiv 0 \bmod \pi^4$ which is same as the first one. Hence, we obtain

$$e_4 = 1 \Longleftrightarrow \mathrm{lk}(K_1, K_2) - \mu(1122)\pi^2 \equiv 0 \bmod \pi^3.$$

For case $p = 2$, this is equivalent to the following condition*

$$(4.2.2.4)\quad \begin{cases} \mathrm{lk}(K_1, K_2) \equiv 0 \bmod 8,\ \mu(1122) \equiv 0 \bmod 2 \\ \text{or} \\ \mathrm{lk}(K_1, K_2) \equiv 0 \bmod 4, \mathrm{lk}(K_1, K_2) \not\equiv 0 \bmod 8, \mu(1122) \equiv 1 \bmod 2. \end{cases}$$

For example, the Whitehead link $L = K_1 \cup K_2$ satisfies $\mathrm{lk}(K_1, K_2) = 0, \mu(1122) = 1$ and so $e_3 = 1, e_4 = 0$, hence $H_1(M, \mathbf{Z}_p) = \widehat{\mathcal{O}}/\mathfrak{p}^3$. For the 2-bridge link of type (48,37), the latter condition of (4.2.2.4) is satisfied and so $H_1(M, \mathbf{Z}_2) = \mathbf{Z}/2^k\mathbf{Z}$, $k \geq 4$.

5. Iwasawa type formulas for the p-homology groups of p^m-fold cyclic branched covers

5.1. *Asymptotic formula for the p-homology of p^m-fold covers.* For $m \geq 1$, let M_m be the completion of the p^m-fold subcover of X_∞ over X_L so that M_m is a p^m-fold cyclic cover of S^3 branched along L. In this last Section, we are concerned with the asymptotic behavior of the order of $H_1(M_m, \mathbf{Z}_p)$ as $m \to \infty$ using the standard argument in Iwasawa theory. As in Section 4, we start again with the following isomorphisms

$$(5.1.1)\qquad \widehat{A_L}^{red} \simeq (H_1(X_\infty, \mathbf{Z}) \otimes_\Lambda \widehat{\Lambda}) \oplus \widehat{\Lambda},$$

$$(5.1.2)\qquad H_1(M_m, \mathbf{Z}_p) \simeq (H_1(X_\infty, \mathbf{Z}) \otimes_\Lambda \widehat{\Lambda})/(\nu_{p^m}(1+X)).$$

From these, we get immediately an extension of a theorem of M. Dellomo [D] for a knot.

Proposition 5.1.3 *For a link L whose p-adic Milnor invariants are all zero, for example a boundary link, we have $H_1(M_m, \mathbf{Z}_p) = \mathbf{Z}_p^{(p^m-1)(n-1)}$ for $m \geq 1$. In particular, $H_1(M_m, \mathbf{Z}_p) = 0$ for $m \geq 1$ if L is a knot.*

Proof. In fact, $\widehat{A_L}^{red} = \widehat{\Lambda}^n$ for such a link L by Corollary 3.2.5. Hence $H_1(X_\infty, \mathbf{Z}) \otimes_\Lambda \widehat{\Lambda} = \widehat{\Lambda}^{n-1}$ by (5.1.1) and so $H_1(M_m, \mathbf{Z}_p) = \mathbf{Z}_p^{(p^m-1)(n-1)}$ by (5.1.2). □

In the following, we assume that $n \geq 2$. By (5.1.1), the 0-th elementary ideal $E_0(H_1(X_\infty, \mathbf{Z}) \otimes_\Lambda \widehat{\Lambda})$ over $\widehat{\Lambda}$ is same as the 1st completed Alexander ideal $\widehat{E_1}(L)$ (3.1.5). Note that the 1st p-adic Alexander series $\widehat{\Delta_1}(L)$ is given as the greatest common divisors of all $n-1$ minors of the reduced p-adic Traldi matrix and so it is written by the form

$$\widehat{\Delta_1}(L) = X^{n-1} \cdot \widehat{\nabla_L}$$

where we call $\widehat{\nabla_L}$ the *p-adic Hosokawa series* of L. Then by (5.1.2), we have the following formula on the order $|H_1(M_m, \mathbf{Z}_p)|$ which is seen as the p-primary part of the well known formula by S. Kinoshita and F. Hosokawa [KT] (See also [MM]). Here we interpret $|H_1(M_m, \mathbf{Z}_p)| = 0$ to mean $H_1(M_m, \mathbf{Z}_p)$ is infinite.

Proposition 5.1.4. $|H_1(M_m, \mathbf{Z}_p)| = p^{m(n-1)} \prod_{\substack{\zeta^{p^m}=1 \\ \zeta \neq 1}} |\widehat{\nabla_L}(\zeta - 1)|_p^{-1}.$

*K. Murasugi informed us of this condition and examples which are obtained by the relation between the Alexander polynomial and Milnor invariants [Mu].

Now, we assume $H_1(M_m, \mathbf{Z}_p)$ is finite for any m and see the asymptotic behaviour of the order $|H_1(M_m, \mathbf{Z}_p)|$ as $m \to \infty$. For this, we recall the following standard facts from Iwasawa theory. We call a polynomial $g(X) \in \mathbf{Z}_p[X]$ *distinguished* if $g(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_0$, $a_i \equiv 0 \bmod p$ for $0 \le i \le d-1$.

Lemma 5.1.5 (p-adic Weierstrass preparation theorem [W, Theorem 7.3]). *A non-zero element $f(X) \in \widehat{\Lambda}$ is written uniquely as*

$$f(X) = p^{\mu} g(X) u(X)$$

where μ is a non-negative integer, $g(X)$ is a distinguished polynomial and $u(X) \in \widehat{\Lambda}^{\times}$.

Lemma 5.1.6 ([W,Theorem 7.14]). *Let $f(X) \in \widehat{\Lambda}$ and assume $f(\zeta - 1) \neq 0$ for any primitive p^m-th root ζ of 1 for $k \geq 1$. Write $f(X) = p^{\mu} g(X) u(X)$ according to Lemma 5.5 and define λ by the degree of $g(X)$. Then there is an integer ν independent of k such that we have the equality*

$$\mathrm{ord}_p\Big(\prod_{\substack{\zeta^{p^m}=1 \\ \zeta \neq 1}} f(\zeta - 1)\Big) = \lambda m + \mu p^m + \nu$$

for sufficiently large m.

For the convenience of the reader, we include herewith a proof of Lemma 5.1.6.

Proof of Lemma 5.1.6. Since $u(x) \in \widehat{\Lambda}^{\times}$, we have

$$\mathrm{ord}_p\Big(\prod_{\substack{\zeta^{p^m}=1 \\ \zeta \neq 1}} f(\zeta - 1)\Big) = (p^m - 1)\mu + \mathrm{ord}_p\Big(\prod_{\substack{\zeta^{p^m}=1 \\ \zeta \neq 1}} g(\zeta - 1)\Big).$$

Write $g(X) = X^{\lambda} + a_{\lambda-1}X^{\lambda-1} + \cdots + a_0$, $a_i \equiv 0 \bmod p$ $(0 \le i \le \lambda - 1)$. For a primitive p^l-th root ζ of 1 $(1 \le l \le k)$, one has the equality $(p) = (\zeta - 1)^{\phi(p^l)}$ of ideals of $\mathbf{Z}[\zeta]$ and so $\mathrm{ord}_p((\zeta-1)^{\lambda}) = \frac{\lambda}{\phi(p^l)}$ where $\phi(x)$ is the Euler function. Therefore, if l is large enough, $\mathrm{ord}_p((\zeta-1)^{\lambda}) < \mathrm{ord}_p(a_i(\zeta-1)^i)$ for $0 \le i \le \lambda - 1$ and so $\mathrm{ord}_p(g(\zeta - 1)) = \mathrm{ord}_p((\zeta-1)^{\lambda})$. Hence, there is a constant C independent of k such that for sufficiently large m, we have

$$\mathrm{ord}_p\Big(\prod_{\substack{\zeta^{p^m}=1 \\ \zeta \neq 1}} g(\zeta - 1)\Big) = \mathrm{ord}_p\Big(\prod_{\substack{\zeta^{p^m}=1 \\ \zeta \neq 1}} (\zeta - 1)^{\lambda}\Big) + C = \mathrm{ord}_p(p^{m\lambda}) + C = \lambda m + C. \square$$

To apply Lemma 5.1.6 to $\widehat{\nabla_L}$, write

$$\widehat{\nabla_L} = p^{\mu(L;p)} g(L;p) u(L;p)$$

where $\mu(L;p)$ is a nonnegative integer, $g(L;p)$ is a distinguished polynomial and $u(L,p) \in \widehat{\Lambda}^{\times}$ according to Lemma 5.1.5 and set $\lambda(L;p) = \deg(g(L;p))$. Then Proposition 5.1.4 and Lemma 5.1.6 yield the following

Theorem 5.1.7. *Notation and assumption being as above, there is a constant $\nu(L;p)$ depending only on L and p such that we have*

$$\mathrm{ord}_p(|H_1(M_m, \mathbf{Z}_p)|) = (n - 1 + \lambda(L;p))m + \mu(L;p)p^m + \nu(L;p)$$

for sufficiently large m.

We call the invariants $\lambda(L;p), \mu(L;p)$ the *Iwasawa* λ, μ*-invariants* of L with respect to p respectively after the model of the Iwasawa invariants in the theory of $\mathbf{Z}_p$-extensions [Iw].

5.2. *Examples.*

1. Let L be the Whitehead link. We then have

$$\widehat{T_L} = \begin{pmatrix} X_1X_2^2 & -X_1^2X_2 \\ X_1^2X_2 & -X_1X_2^2 \end{pmatrix}$$

and $\widehat{\nabla_L} = X^2$. Hence, we have

$$\lambda(L;p) = 2, \mu(L;p) = 0 \text{ and } \mathrm{ord}_p(H_1(M_m, \mathbf{Z}_p)) = 3m \text{ for } m \geq 1.$$

2. Let $L = K_1 \cup K_2 \cup K_3$ be the Borromean rings so that we can take $y_1 = [x_3, x_2], y_2 = [x_3, x_1], y_3 = [x_1, x_2]$. Then we can compute all Milnor number needed to get the reduced Traldi matrix

$$\widehat{T_L}^{red} = \begin{pmatrix} X + X^2 & -X^2 & -X \\ -X & X + X^2 & -X^2 \\ -X^2 & -X & X + X^2 \end{pmatrix}$$

and so $\widehat{\nabla_L} = 1 + X + X^2$. Hence, we have

$$\lambda(L;p) = \mu(L;p) = 0 \text{ and } \mathrm{ord}_p(H_1(M_m, \mathbf{Z}_p)) = 2m \text{ for } m \geq 1.$$

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