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Birkhoffian formulation of the dynamics of LC circuits

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Abstract. We present a formulation of general nonlinear LC circuits within the framework of Birkhoffian dynamical systems on manifolds. We develop a systematic procedure which allows, under rather mild non-degeneracy conditions, to write the governing equations for the mathematical description of the dynamics of an LC circuit as a Birkhoffian differential system. In order to illustrate the advantages of this approach compared to known Lagrangian or Hamiltonian approaches we discuss a number of specific examples. In particular, the Birkhoffian approach includes networks which contain closed loops formed by capacitors, as well as inductor cutsets. We also extend our approach to the case of networks which contain independent voltage sources as well as independent current sources. Also, we derive a general balance law for an associated "energy function".

Keywords. Conservative dynamical systems, Birkhoffian differential systems, Birkhoffian vector fields, electrical networks, geometric theory.

1. Introduction

In this paper we give a formulation of the dynamics of LC circuits within the framework of Birkhoffian systems [3]. Based on the constitutive relations of the involved inductors and capacitors and on Kirchhoff's laws, we define a configuration space and a corresponding Birkhoffian that describes the "elementary work" done by a set of "generalized forces". As a matter of fact, in order to cover circuits for which the topological assumptions usually imposed in the literature, are not satisfied, we are forced to describe a single circuit by a whole family of Birkhoffian systems parameterized by a finite number of real parameters. Relevant values of these parameters correspond to initial values for the time evolution of certain state variables of the circuit. The dimension of each configuration space is given by the cardinality of a selection of loops that cover the whole circuit.

In order to study the dynamics of LC circuits, various Lagrangian and Hamiltonian formulations have been considered in the literature (see for example [2], [4], [5], [6], [9], [10]).

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In the Lagrangian approach, a central issue is the selection of suitable coordinates and corresponding velocities in terms of which the Lagrangian function is expressed. A specific technique for the sometimes difficult task of choosing the proper Lagrangian variables is presented in paper [6].

More often Hamiltonian formulations have been used to describe circuit equations. In [2] it is shown how to construct, based on the circuit topology, canonical variables and a Hamiltonian, so that the circuit equations attain canonical form.

For a more general approach including also resistors, the RLC circuits, see Brayton–Moser's approach [5]. In [5], under the hypothesis that the currents through the inductors and the voltages across the capacitors determine all currents and voltages in the circuit via Kirchhoff's law, is proved the existence of the mixed potential function with the aid of which the system of differential equations describing the dynamics of such a network is written into a special form (see §4 in [5]). The mixed potential function is constructed explicitly only for the networks whose graph possesses a tree containing all the capacitor branches and none of the inductive branches, that is, the network does no contain any loops of capacitors or cutsets of inductors, each resistor tree branch corresponds to a current-controlled resistor, each resistor co-tree branch corresponds to a voltage-controlled resistor (see §13 in [5]).

In [9], the dynamics of a nonlinear LC circuit is shown to be of Hamiltonian nature with respect to a certain Poisson bracket which may be degenerate, that is, non-symplectic. In this formalism, the constitutive relations of the inductors and capacitors are used to define the Hamiltonian function in terms of capacitor charges and inductor fluxes, while the topological constraints of the network graph and Kirchhoff's laws define the Poisson bracket on the space of capacitor charge and inductor flux variables.

But for all those formulations, a certain topological assumption on the electrical circuit appears to be crucial, that is, the circuit is supposed to contain neither loops of capacitors nor cutsets of inductors.

In [12], [10] and [4] the Poisson bracket is replaced by the more general notion of a Dirac structure on a vector space, leading to implicit Hamiltonian systems. The Hamiltonian function is the total electromagnetic energy of the circuit and the vectorial state space is defined by the inductors' fluxes and capacitors' charges. The Dirac structure on the state space is obtained from Kirchhoff's laws. In this formalism, it is possible to include networks which do not obey the topological assumption mentioned before.

In the paper at hand we will see that the restricted class of networks involving capacitor loops and inductor cut sets are naturally captured by the Birkhoffian approach. We are going to discuss explicit examples in order to demonstrate the advantages of the Birkhoffian approach in the analysis of the resulting systems. Another advantage of the Birkhoffian approach is the possible inclusion of dissipative effects caused by resistors included in a network. It is a straight-forward matter to extend the approach presented here to the case of RLC circuits, that is circuits containing resistors in addition to capacitors and inductors. However, to start it appears to be more instructive to restrict the theory to the case of LC circuits. The investigation of RLC circuits will be presented in another paper.

The following parts of the paper are organized as follows. In Section 2 we recall the basics of Birkhoffian systems (see [3]) presented from the point view of differential geometry using the formalism of jets (see [8]). Birkhoffian formalism is a global formalism of the dynamics of implicit systems of second order ordinary differential equations on a manifold. In particular, we extend the approach in [8] to non-autonomous systems in order to be able to treat the case of networks with independent voltage and current sources later on in Section 4. In Section 3, our Birkhoffian formulation of the dynamic equations of a nonlinear LC circuit is introduced. Properties of the corresponding Birkhoffian such as its regularity and its conservativeness are also discussed in this section. For a nonlinear LC electric network each Birkhoffian of the family is conservative. If there exists in the network some loop which contains only capacitors the Birkhoffian is never regular. For such electrical circuits, we present a systematic procedure to reduce the original configuration space to a lower dimensional one, thereby regularizing the Birkhoffian. On the reduced configuration space the reduced Birkhoffian will still be conservative. In case the LC circuit has loops which contain only linear inductors, the original configuration space can be further reduced to a lower dimensional one. Inductor loops can be regarded as some conservative quantities of the network. In Section 4 we give a Birkhoffian formulation of a nonlinear LC circuit with independent sources and discuss in this context the concepts of regularity and conservativeness. For instance, it turns out that voltage sources do not destroy conservativeness, even in the nonlinear case, while current sources might do so. Finally, in Section 5 we consider some specific examples. These examples are supposed to serve our purpose of demonstrating the power of the Birkhoffian approach. In particular, we can allow capacitor loops as well as inductor cutsets, as already mentioned before. Also, we investigate the question of conservativeness of the underlying Birkhoffian in case of a circuit with independent current and voltage sources.

2. Birkhoffian systems

For a smooth m-dimensional differentiable connected manifold M, we consider the tangent bundles (TM, π_M, M) and (TTM, π_{TM}, TM). Let $q = (q^1, q^2, \ldots, q^m)$ be a local coordinate system on M. This induces natural local coordinate systems on TM and TTM, denoted by $(q, \dot{q})$, respectively $(q, \dot{q}, dq, d\dot{q})$. The *2-jets manifold* $J^2(M)$ is a $3m$-dimensional submanifold of TTM defined by

$$J^2(M) = \{z \in TTM \ / \ T\pi_M(z) = \pi_{TM}(z)\} \tag{2.1}$$

where $T\pi_M : TTM \to TM$ is the tangent map of π_M. We write $\pi_J := \pi_{TM}|_{J^2(M)} = T\pi_M|_{J^2(M)}$. $(J^2(M), \pi_J, TM)$, called the *2-jet bundle* (see [8]), is an affine bundle modelled on the vertical vector bundle $(V(M), \pi_{TM}|_{V(M)}, TM)$, $V(M) = \bigcup_{v\in TM} V_v(M)$, where $V_v(M) = \{z \in T_vTM \,|\, (T\pi_M)_v(z) = 0\}$. In [1], [11] this bundle is denoted by $T^2(M)$ and named *second-order tangent bundle*. In natural local coordinates, the equality in (2.1) yields $(q,\, \dot q,\, \dot q,\, d\dot q|_{J^2(M)})$ as a local coordinate system on $J^2(M)$. We set $\ddot q := d\dot q|_{J^2(M)}$. Thus, a local coordinate system q on M induces the natural local coordinate system $(q,\, \dot q,\, \ddot q)$ on $J^2(M)$. For further details on this affine bundle see [1], [8], [11].

A ***Birkhoffian*** corresponding to the configuration manifold M is a smooth 1-form ω on $J^2(M)$ such that, for any $x \in M$, we have

$$i_x^*\omega = 0 \tag{2.2}$$

where $i_x : \beta^{-1}(x) \to J^2(M)$ is the embedding of the submanifold $\beta^{-1}(x)$ into $J^2(M)$, $\beta = \pi_M \circ \pi_J$. From this definition it follows that, in the natural local coordinate system $(q,\, \dot q,\, \ddot q)$ of $J^2(M)$, a Birkhoffian ω is given by

$$\omega = \sum_{j=1}^{m} Q_j(q,\, \dot q,\, \ddot q) dq^j \tag{2.3}$$

with certain functions $Q_j : J^2(M) \to \mathbf{R}$.

The pair $(M,\, \omega)$ is said to be a ***Birkhoff system*** (see [8]).

The ***differential system associated to a Birkhoffian*** ω (see [8]) is the set (maybe empty) $D(\omega)$, given by

$$D(\omega) := \left\{ z \in J^2(M) \mid \omega(z) = 0 \right\}. \tag{2.4}$$

The manifold M is the *space of configurations* of $D(\omega)$, and $D(\omega)$ is said to have m 'degrees of freedom'. The Q_i are the 'generalized external forces' associated to the local coordinate system (q). In the natural local coordinate system, $D(\omega)$ is characterized by the following implicit system of second order ODE's

$$Q_j(q,\, \dot q,\, \ddot q) = 0 \text{ for all } j = \overline{1, m}. \tag{2.5}$$

We conclude that the Birkhoffian formalism is a global formalism for the dynamics of implicit systems of second order differential equations on a manifold.

Let us now associate a vector field to a Birkhoffian ω.

A *vector field* Y on the manifold TM is a smooth function $Y : TM \to TTM$ such that $\pi_{TM} \circ Y$=id. Any vector field Y on TM is called a *second order vector field on* TM if and only if $T\pi_M(Y_v) = v$ for all $v \in TM$.

A cross section X of the affine bundle $(J^2(M),\, \pi_J,\, TM)$, that is, a smooth function $X : TM \to J^2(M)$ such that $\pi_J \circ X$=id, can be identified with a special vector field on TM, namely, the second order vector field on TM associated to X. Indeed, because $(J^2(M),\, \pi_J,\, TM)$ is a sub-bundle of (TTM, π_{TM}, TM) as well as of $(TTM, T\pi_M, TM)$, its sections can be regarded as sections of these two tangent bundles. Thus, using the canonical embedding $i : J^2(M) \to TTM$, X can be identified with Y, that is, $Y = i \circ X$.

In natural local coordinates a second order vector field can be represented as

$$Y = \sum_{j=1}^{m} \left[\dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i(q,\, \dot{q}) \frac{\partial}{\partial \dot{q}^i} \right]. \tag{2.6}$$

A ***Birkhoffian vector field*** associated to a Birkhoffian ω of M (see [8]) is a smooth second order vector field on TM, $Y = i \circ X$, with $X : TM \to J^2(M)$, such that $ImX \subset D(\omega)$, that is

$$X^* \omega = 0. \tag{2.7}$$

In the natural local coordinate system, a Birkhoffian vector field is given by (2.6), such that $Q_j(q, \dot{q}, \ddot{q}(q, \dot{q})) = 0$.

A Birkhoffian ω is ***regular*** if and only if

$$\det \left[\frac{\partial Q_j}{\partial \ddot{q}^i}(q,\, \dot{q},\, \ddot{q}) \right]_{i,j=1,\ldots,m} \neq 0 \tag{2.8}$$

for all $(q,\, \dot{q},\, \ddot{q})$, and for each $(q,\, \dot{q})$, there exists $\ddot{q}$ such that $Q_j(q,\, \dot{q},\, \ddot{q}) = 0$, $j = 1, ..., m$.

If a Birkhoffian ω of M is regular, then it satisfies *the principle of determinism*, that is, there exists an unique Birkhoffian vector field $Y = i \circ X$ associated to ω such that $Im\, X = D(\omega)$ (see [8]).

A Birkhoffian ω of M is called ***conservative*** if and only if there exists a smooth function $E_\omega : TM \to \mathbf{R}$ such that

$$(X^* \omega) Y = dE_\omega(Y) \tag{2.9}$$

for all second order vector fields $Y = i \circ X$ (see [8]).

Equation (2.9) is equivalent, in the natural local coordinate system, to the identity (see [3], p. 16, eq. 4)

$$\sum_{j=1}^{m} Q_j(q\ \dot{q},\, \ddot{q}) \dot{q}^j = \sum_{j=1}^{m} \left[\frac{\partial E_\omega}{\partial q^j} \dot{q}^j + \frac{\partial E_\omega}{\partial \dot{q}^j} \ddot{q}^j \right]. \tag{2.10}$$

E_ω is constant on TM if and only if $dE_\omega(Y) = 0$ for all second order vector fields Y on TM (see [8]).

If ω is conservative and Y is a Birkhoffian vector field, then (2.9) becomes

$$dE_\omega(Y) = 0. \tag{2.11}$$

This means that E_ω is constant along the trajectories of Y.

It is also possible to introduce, in a natural manner, the notion of constrained Birkhoff system (see [8], §4).

Let $(M,\, \omega)$ be a Birkhoff system and $\mathfrak{S}$ a smooth constant rank affine subbundle of the affine bundle $\pi_J : J^2(M) \longrightarrow TM$. Locally, the submanifold $\mathfrak{S}$ of codimension N, is described by the vanishing of N independent affine functions

$$\phi^\nu(q, \dot{q}, \ddot{q}) = \sum_{i=1}^{m} b_i^\nu(q, \dot{q}) \ddot{q}^i + a^\nu(q, \dot{q}), \quad \nu = \overline{1, \mathrm{N}}. \tag{2.12}$$

A triple $(M,\, \omega,\, \mathfrak{S})$ is called ***constrained Birkhoff system***.

The ***constrained differential system associated to*** the constrained Birkhoff system $(M,\, \omega,\, \mathfrak{S})$ is the set

$$D(\omega, \mathfrak{S}) = \{z \in \mathfrak{S} |\, \omega(z) = 0\}. \tag{2.13}$$

Let us now generalize these concepts to time-dependent dynamical systems.

For the usual formulation of Lagrangian and Hamiltonian time-dependent mechanics (see for example [1], §5.1, [11] §4.1, §4.6), the configuration space has the form $\mathbf{R} \times M$, the phase space has the form $\mathbf{R} \times T^*M$, and the velocity space has the form $\mathbf{R} \times TM$, with some manifold M. If (t, q) is a coordinate system on $\mathbf{R} \times M$, then $(t, q, \dot{q})$ is a coordinate system on $\mathbf{R} \times TM$. Thus, $\mathbf{R} \times TM$ can be interpreted as a submanifold of $T(\mathbf{R} \times M)$ given by

$$\dot{t} = 1. \tag{2.14}$$

From the physical point of view, this means that a reference frame has been chosen. This is not the case for relativistic mechanics. The reference system provides a splitting between the time and the state coordinates of a mechanical system. Within the Birkhoffian framework, we follow the usual non-relativistic lines. Thus, for the time-dependent system, we have in addition the equation

$$\ddot{t} = 0. \tag{2.15}$$

In view of (2.14), (2.15), we choose in the study of time-dependent dynamical systems the extended bundle $\mathbf{R} \times J^2(M)$.

A ***time-dependent Birkhoffian*** is a smooth family of 1-forms ω_t on $J^2(M)$ defined by

$$\omega_t = \sum_{j=1}^{m} Q_j(t, q, \dot{q}, \ddot{q}) dq^j. \tag{2.16}$$

where $(t, q, \dot{q}, \ddot{q})$ is the natural local coordinate system on $\mathbf{R} \times J^2(M)$. Thus, our time-dependent Birkhoffian is obtained by merely freezing t and constructing the Birkhoffian for any fixed value of t as before.

A *time-dependent second order vector field* (see [11]) on $\mathbf{R} \times TM$ has the following representation in the natural local coordinate system

$$Y_t = \frac{\partial}{\partial t} + \sum_{j=1}^{m} \left[\dot{q}^j \frac{\partial}{\partial q^j} + \ddot{q}^j(t, q, \dot{q}) \frac{\partial}{\partial \dot{q}^j} \right]. \tag{2.17}$$

Thus, for a time-dependent system, a ***time-dependent Birkhoffian vector field*** on $\mathbf{R} \times T(M)$ has the representation (2.17), where $Q_j(t, q, \dot{q}, \ddot{q}(t, q, \dot{q})) = 0$.

A time-dependent Birkhoffian ω_t is ***regular*** if and only if

$$\det \left[\frac{\partial Q_j}{\partial \ddot{q}^i}(t, q,\, \dot{q},\, \ddot{q}) \right]_{i,j=1,\ldots,m} \neq 0 \tag{2.18}$$

for all $(t, q,\, \dot{q},\, \ddot{q})$, and for each $(t, q,\, \dot{q})$, there exists $\ddot{q}$ such that $Q_j(t, q,\, \dot{q},\, \ddot{q}) = 0,\, j = 1, ..., m$.

A time-dependent Birkhoffian ω_t is called ***conservative*** if and only if there exists a smooth family of functions $E_{\omega_t} : TM \longrightarrow \mathbf{R}$ such that, everywhere,

$$\sum_{j=1}^{m} Q_j(t, q\ \dot{q},\ \ddot{q})\dot{q}^j = \sum_{j=1}^{m} \left[\frac{\partial E_{\omega_t}}{\partial q^j}\dot{q}^j + \frac{\partial E_{\omega_t}}{\partial \dot{q}^j}\ddot{q}^j \right]. \tag{2.19}$$

If ω_t is conservative and Y_t is a time-dependent Birkhoffian vector field then, from (2.19), we obtain the ***generalized balance law***

$$\frac{dE_{\omega_t}}{dt} = \frac{\partial E_{\omega_t}}{\partial t} \tag{2.20}$$

along trajectories of Y_t.

3. LC circuit dynamics

A simple electrical circuit provides us with an *oriented connected* graph, that is, a collection of points, called nodes, and a set of connecting lines or arcs, called branches, such that in each branch is given a direction and there is at least one path between any two nodes. A path is a sequence of branches such that the origin of the next branch coincides with the end of the previous one. The graph will be assumed to be *planar*, that is, it can be drawn in a plane without branches crossing. For the graph theoretic terminology, see, for example [7].

Let b be the total number of branches in the graph, n be one less than the number of nodes and m be the cardinality of a selection of loops that cover the whole graph. Here, a loop is a path such that the first and last node coincide and that does not use the same branch more than once. By Euler's polyhedron formula, $b = m + n$.

A *cutset* in a connected graph, is a minimal set of branches whose removal from the graph, renders the graph disconnected. For example, the set branches tied to a node is a cutset.

We choose a reference node and a current direction in each l-branch of the graph, $l = 1, ..., b$. We also consider a covering of the graph with m loops, and a current direction in each j-loop, $j = 1, ..., m$. We assume that the associated graph has at least one loop, meaning that $m > 0$.

A graph can be described by matrices: a (bn)-matrix $B \in \mathfrak{M}_{bn}(\mathbf{R})$, $\mathrm{rank}(B) = n$, called *incidence matrix* and a (bm)-matrix $A \in \mathfrak{M}_{bm}(\mathbf{R})$, $\mathrm{rank}(A) = m$, called *loop matrix*. These matrices contain only 0,1, -1. An element of the matrix B is 0 if a branch b is not incident with a node n, 1 if branch b enters node n and -1 if branch b leaves node n, respectively. An element of the matrix A is 0 if a branch b does not belong to a loop m, 1 if branch b belongs to loop m and their directions agree and -1 if branch b belongs to loop m and their directions oppose, respectively. For the fundamentals of electrical circuit theory, see, for example [6].

The states of the circuit have two components, the currents through the branches, denoted by $\imath \in \mathbf{R}^b$, and the voltages across the branches, denoted by $v \in \mathbf{R}^b$.

Using the matrices A and B, Kirchhoff's current law and Kirchhoff's voltage law can be expressed by the equations

$$B^T \mathrm{I} = 0 \quad (KCL) \tag{3.1}$$

$$A^T v = 0 \quad (KVL) \tag{3.2}$$

Tellegen's theorem establishes a relation between the matrices A^T and B^T: *the kernel of the matrix B^T is orthogonal to the kernel of the matrix A^T* (see e.g., [5] page 5).

The next step is to introduce the branch elements in a simple electrical circuit. The branches of the graph associated to an LC electrical circuit, can be classified into two categories: inductor branches and capacitor branches. A capacitor loop will contain only capacitor branches and an inductor cutset will contain only inductor branches. Let k denote the number of inductor branches and p the number of capacitor branches, respectively. We assume that just one electrical device is associated to each branch, then, we have $b = k + p$. Thus, we can write $(\mathrm{I}_a, \mathrm{I}_\alpha) \in \mathbf{R}^r \times \mathbf{R}^p \simeq \mathbf{R}^b$, where I_a, I_α are the currents through the inductors, the capacitors, respectively, and $v = (v_a, v_\alpha) \in \mathbf{R}^k \times \mathbf{R}^p \simeq \mathbf{R}^b$, where v_a, v_α describe the voltage drops across the the inductors, the capacitors, respectively.

To exemplify, let us now write the matrices B and A, for a circuit which contains four inductors, three capacitors and which has the following oriented connected graph

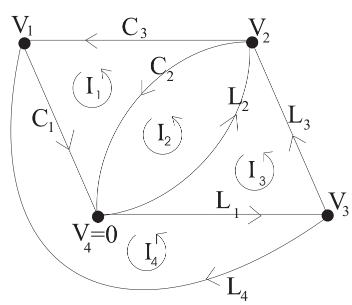

Figure 1

We have $k = 4$, $p = 3$, $n = 3$, $m = 4$, $b = 7$. We choose the reference node to be V_4 and the current directions as indicated in Figure 1. We cover the graph with the loops I_1, I_2, I_3, I_4. Let $V = (V_1, V_2, V_3) \in \mathbf{R}^3$ be the vector of node voltage values, $\mathrm{I} = (\mathrm{I}_a, \mathrm{I}_\alpha) \in \mathbf{R}^4 \times \mathbf{R}^3$ be the vector of branch current values and $v = (v_a, v_\alpha) \in \mathbf{R}^4 \times \mathbf{R}^3$ be the vector of branch voltage values.

The branches in Figure 1 are labelled as follows: the first, the second, the third and the fourth branch are the inductor branches L_1, L_2, L_3, L_4 and the last three branches are the capacitor branches C_1, C_2, C_3. The incidence and loop matrices,

$B \in \mathfrak{M}_{73}(\mathbf{R})$ and $A \in \mathfrak{M}_{74}(\mathbf{R})$, write as

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.3}$$

For another choice of the covering loops and of the current directions in the loops we obtain a different matrix A and for another choice of the reference node and of the current directions in the branches we obtain a different matrix B. □

Each capacitor is supposed to be charge-controlled. For the nonlinear capacitors we assume

$$v_\alpha = C_\alpha(\mathrm{Q}_\alpha), \quad \alpha = 1, ..., p \tag{3.4}$$

where the functions $C_\alpha : \mathbf{R} \longrightarrow \mathbf{R}\backslash\{0\}$ are smooth and invertible, and the Q_α's denote the charges of the capacitors. The current through a capacitor is given by the time-derivative of the corresponding charge

$$\mathrm{I}_\alpha = \frac{d\mathrm{Q}_\alpha}{dt}, \quad \alpha = 1, ..., p \tag{3.5}$$

t being the time variable.

Each inductor is supposed to be current-controlled. For the nonlinear inductors we assume

$$v_a = L_a(\mathrm{I}_a)\frac{d\mathrm{I}_a}{dt}, \quad a = 1, ..., k \tag{3.6}$$

where $L_a : \mathbf{R} \longrightarrow \mathbf{R}\backslash\{0\}$ are smooth invertible functions.

If the capacitors and the inductors are linear then the relations (3.4) and (3.6) become, respectively,

$$v_\alpha = \frac{\mathrm{Q}_\alpha}{\mathrm{C}_\alpha}, \quad v_a = \mathrm{L}_a\frac{d\mathrm{I}_a}{dt} \tag{3.7}$$

where $\mathrm{C}_\alpha \neq 0$ and $\mathrm{L}_a \neq 0$ are distinct constants.

Taking into account (3.4), (3.5), (3.6), the equations (3.1), (3.2), become

$$\begin{cases} B^T \begin{pmatrix} \mathrm{I}_a \\ \frac{d\mathrm{Q}_\alpha}{dt} \end{pmatrix} = 0 \\ \\ A^T \begin{pmatrix} L_a(\mathrm{I}_a)\,\frac{d\mathrm{I}_a}{dt} \\ C_\alpha(\mathrm{Q}_\alpha) \end{pmatrix} = 0 \end{cases}. \tag{3.8}$$

In the following we give a Birkhoffian formulation for the network described by the system of equations (3.8). Using the first set of equations (3.8), we are going to define a family of m-dimensional affine-linear configuration spaces $M_c \subset \mathbf{R}^b$ parameterized by a constant vector c in $\mathbf{R}^n$. This vector is related to the initial *values of the* Q*-variables at some instant of time. At this point we notice that actually already the initial values corresponding to the* Q*-variables associated to capacitors, together with those of m distinguished branch currents denoted by $\dot{q}^j$ below, parameterize the whole solution set of the equations in (3.8). A Birkhoffian ω_c of the configuration space M_c arises from a linear combination of the second set of equations (3.8). Thus, (M_c, ω_c) will be a family of Birkhoff systems that describe the LC circuit considered.*

We notice that the first set of equations (3.8) remains exactly the same for linear and nonlinear electrical devices. Thus, for obtaining the configuration space, it is not important whether the devices are linear or nonlinear. We shall see below that the only difference is that one ends up with a nonlinear configuration space or rather configuration manifold when one regularizes the resulting Birkhoffian system in the case of nonlinear networks.

Let $H : \mathbf{R}^b \longrightarrow \mathbf{R}^n$ be a linear map that, with respect to a coordinate system $(x^1, ..., x^b)$ on $\mathbf{R}^b$, is given by

$$H(x^1, ..., x^b) = B^T \begin{pmatrix} x^1 \\ \vdots \\ x^b \end{pmatrix}. \tag{3.9}$$

Then, $H^{-1}(c)$, with c a constant vector in $\mathbf{R}^n$, is an affine-linear subspace in $\mathbf{R}^b$. Its dimension is $m = b - n$, because $\operatorname{rank}(B) = n$.

We define M_c as

$$M_c := H^{-1}(c). \tag{3.10}$$

We denote a coordinate system on M_c by $q = (q^1, ..., q^m)$. Then, the natural coordinate system on the 2-jet bundle $J^2(M_c)$ is $(q, \dot{q}, \ddot{q})$.

Let us now represent the Birkhoffian in a specific coordinate system on M_c:

In the vector space $\mathbf{R}^k$, we identify points and vectors

$$\mathrm{I}_a := \frac{d\mathrm{Q}_{(a)}}{dt}, \tag{3.11}$$

where $(\mathrm{Q}_{(a)})_{a=1,...,k}$ is a coordinate system on $\mathbf{R}^k$. Taking into account (3.11) and the fact that the matrix B^T is a constant matrix, we integrate the first set of equations (3.8) to arrive at

$$B^T \begin{pmatrix} \mathrm{Q}_{(a)} \\ \mathrm{Q}_{\alpha} \end{pmatrix} = c \tag{3.12}$$

with c a constant vector in $\mathbf{R}^n$.

Likewise consider coordinates on $\mathbf{R}^b \simeq \mathbf{R}^k \times \mathbf{R}^p$ defined by

$$x^1 := \mathrm{Q}_{(1)},\ ...,\ x^k := \mathrm{Q}_{(k)},\ x^{k+1} := \mathrm{Q}_1,\ ...,\ x^b := \mathrm{Q}_p. \tag{3.13}$$

From (3.9), (3.10), we see that we can define coordinates on M_c by solving the equations in (3.12) in terms of an appropriate set of m of the Q-variables, say

$q = (q^1, ..., q^m)$. In other words, we express any of the x-variables as a function of $q = (q^1, ..., q^m)$, namely,

$$x^a = \sum_{j=1}^{m} \mathcal{N}_j^a q^j + const,\ a = 1, ..., k,$$
$$x^\alpha = \sum_{j=1}^{m} \mathcal{N}_j^\alpha q^j + const,\ \alpha = k+1, ..., b \tag{3.14}$$

with certain constants $\mathcal{N}_j^a$, and $\mathcal{N}_j^\alpha$. Here we can think of the constants *const* as being initial values of the x-variables at some instant of time.

From (3.5), (3.11), (3.13) and differentiating (3.14) we get

$$\mathrm{I} = \mathcal{N}\dot{q} \tag{3.15}$$

with the matrix of constants $\mathcal{N} \in \mathfrak{M}_{bm}(\mathbf{R})$, for some $\dot{q} \in \mathbf{R}^m$.

Using Tellegen's theorem and a fundamental theorem of linear algebra, we now find a relation between the matrices $\mathcal{N}$ and A. By a fundamental theorem of linear algebra we have

$$(Ker(A^T))^\perp = Im(A) \tag{3.16}$$

where $A \in \mathfrak{M}_{bm}(\mathbf{R})$, $Ker(A^T) := \{x \in \mathbf{R}^b \,|\, A^T x = 0\}$ is the kernel of A^T, $Im(A) := \{x \in \mathbf{R}^b \,|\, Ay = x, \text{ for some } y \in \mathbf{R}^m\}$ is the image of A and $^\perp$ denotes the orthogonal complement in $\mathbf{R}^b$ of the respective vector subspace.

For the incidence matrix $B \in \mathfrak{M}_{bn}(\mathbf{R})$ and the loop matrix $A \in \mathfrak{M}_{bm}(\mathbf{R})$, which satisfy Kirchhoff's law (3.1), (3.2), Tellegen's theorem writes as

$$Ker(B^T) = (Ker(A^T))^\perp. \tag{3.17}$$

From the first set of equations in (3.8), and by constraction of the matrix $\mathcal{N}$ in (3.15), we have

$$Ker(B^T) = Im(\mathcal{N}). \tag{3.18}$$

Therefore, using (3.16), (3.17), (3.18), we obtain $Im(A) = Im(\mathcal{N})$. Then, another application of (3.16) yields

$$Ker(A^T) = Ker(\mathcal{N}^T). \tag{3.19}$$

Taking into account (3.19), we see that there exists a nonsingular matrix $\mathcal{C} \in \mathfrak{M}_{mm}(\mathbf{R})$ satisfing

$$\mathcal{C}A^T = \mathcal{N}^T. \tag{3.20}$$

The matrix $\mathcal{C}$ provides a relation between the vector of the m independent loop currents and the coordinate vector q introduced on M_c.

Taking into account (3.19), we define the Birkhoffian ω_c of M_c such that the differential system (2.5) is the linear combination of the second set of equations in (3.8) obtained by replacing A^T with the matrix $\mathcal{N}^T$. Thus, in terms of q-coordinates as chosen before, the expressions of the components $Q_j(q, \dot{q}, \ddot{q})$ from (2.3) are

$$Q_j(q, \dot{q}, \ddot{q}) = F_j(\dot{q})\ddot{q} + G_j(q), \quad j = 1, ..., m \tag{3.21}$$

where

$$F_j(\dot q)\ddot q = \sum_{a=1}^{k} \mathcal{N}_j^{\,a} L_a \left(\sum_{l=1}^{m} \mathcal{N}_l^a \dot q^l\right) \sum_{i=1}^{m} \mathcal{N}_i^a \ddot q^i = \sum_{i=1}^{m}\left(\sum_{a=1}^{k} \mathcal{N}_j^{\,a} \mathcal{N}_i^a \widetilde{L}_a(\dot q)\right)\ddot q^i \quad (3.22)$$

$$G_j(q) = \sum_{\alpha=k+1}^{b} \mathcal{N}_j^{\,\alpha} C_{\alpha-k}\left(\sum_{l=1}^{m} \mathcal{N}_l^\alpha q^l + const\right) = \sum_{\alpha=k+1}^{b} \mathcal{N}_j^{\,\alpha} \widetilde{C}_{\alpha-k}(q)\,. \quad (3.23)$$

We claim that the Birkhoffian (3.21) is a **conservative** *one.*

Indeed, for our problem, the relation (2.10) becomes

$$\sum_{j=1}^{m}\left[\left(\sum_{i=1}^{m}\sum_{a=1}^{k} \mathcal{N}_j^{\,a}\mathcal{N}_i^a \widetilde{L}_a(\dot q)\ddot q^i\right)\dot q^j + G_j(q)\dot q^j\right] = \sum_{j=1}^{m}\left[\frac{\partial E_{\omega_c}}{\partial q^j}\dot q^j + \frac{\partial E_{\omega_c}}{\partial \dot q^j}\ddot q^j\right] \quad (3.24)$$

or (changing the indices of summation)

$$\sum_{j=1}^{m}\left[\left(\sum_{i=1}^{m}\sum_{a=1}^{k} \mathcal{N}_i^{\,a}\mathcal{N}_j^a \widetilde{L}_a(\dot q)\,\dot q^i\right)\ddot q^j + G_j(q)\dot q^j\right] = \sum_{j=1}^{m}\left[\frac{\partial E_{\omega_c}}{\partial q^j}\dot q^j + \frac{\partial E_{\omega_c}}{\partial \dot q^j}\ddot q^j\right]. \quad (3.25)$$

Because of the special form of the terms on the left hand side of (3.25), we can look for the required function $E_{\omega_c}(q,\dot q)$ as a sum of a function depending only on q, and a function depending only on $\dot q$. From the theory of total differentials, a necessary condition for the existence of such functions is the fulfilment of the following relations

$$\begin{cases} \frac{\partial G_j(q)}{\partial q^l} - \frac{\partial G_l(q)}{\partial q^j} = 0 \\ \frac{\partial \mathcal{F}_j(\dot q)}{\partial \dot q^l} - \frac{\partial \mathcal{F}_l(\dot q)}{\partial \dot q^j} = 0 \end{cases}. \quad (3.26)$$

for any $j,l = 1,...,m$, where

$$\mathcal{F}_j(\dot q) := \sum_{i=1}^{m}\sum_{a=1}^{k} \mathcal{N}_i^{\,a}\mathcal{N}_j^a \widetilde{L}_a(\dot q)\,\dot q^i. \quad (3.27)$$

In view of (3.23), (3.27) we get:

$$\frac{\partial G_j(q)}{\partial q^l} = \sum_{\alpha=k+1}^{b} \mathcal{N}_j^\alpha \mathcal{N}_l^\alpha \widetilde{C}'_{\alpha-k}(q) \quad (3.28)$$

$$\frac{\partial \mathcal{F}_j(\dot q)}{\partial \dot q^l} = \sum_{a=1}^{k} \mathcal{N}_l^a\mathcal{N}_j^a \widetilde{L}_a(\dot q) + \sum_{i=1}^{m}\sum_{a=1}^{k} \mathcal{N}_i^a\mathcal{N}_j^a\mathcal{N}_l^a \widetilde{L}'_a(\dot q)\dot q^i \quad (3.29)$$

where $\widetilde{C}'_\alpha := \frac{d\widetilde{C}_\alpha(\eta)}{d\eta}$, $\widetilde{L}'_a := \frac{d\widetilde{L}_a(\eta)}{d\eta}$. Therefore, the left hand side of (3.26) become

$$\sum_{\alpha=k+1}^{b} (\mathcal{N}_j^\alpha\mathcal{N}_l^\alpha - \mathcal{N}_l^\alpha\mathcal{N}_j^\alpha)\widetilde{C}'_{\alpha-k}(q) \quad (3.30)$$

$$\sum_{a=1}^{k}\left(\mathcal{N}_l^a\mathcal{N}_j^a-\mathcal{N}_j^a\mathcal{N}_l^a\right)\left(\widetilde{L}_a(\dot q)-\widetilde{L}'_a(\dot q)(\sum_{i=1}^{m}\mathcal{N}_i^a\dot q^i)\right). \tag{3.31}$$

We now easily see that the expressions in (3.30), (3.31) are zero and (3.26) are satisfied. Thus, we proved the existence of a function $E_{\omega_c}(q,\dot q)$ such that (3.25) is fulfilled.

Let us now look for the expression of this function. For linear devices, taking into account (3.7), we have

$$\widetilde{L}_a(\dot q)=\mathrm{L}_a,\quad \widetilde{C}_{\alpha-k}(q)=\frac{\sum_{i=1}^{m}\mathcal{N}_i^\alpha q^i}{\mathrm{C}_{\alpha-k}}+const \tag{3.32}$$

with L_a, C_α being real constants. Therefore, the functions $\mathcal{F}_j(\dot q)$ and $G_j(q)$ from (3.27), (3.23) become

$$\mathcal{F}_j(\dot q):=\sum_{i=1}^{m}\sum_{a=1}^{k}\mathrm{L}_a{\mathcal{N}_i}^a\mathcal{N}_j^a\dot q^i \tag{3.33}$$

$$G_j(q):=\sum_{\alpha=k+1}^{b}\sum_{i=1}^{m}\frac{{\mathcal{N}_j}^\alpha{\mathcal{N}_i}^\alpha}{\mathrm{C}_{\alpha-k}}q^i+(const)_j. \tag{3.34}$$

Thus, in the linear case, it is not difficult to find the function $E_{\omega_c}(q,\dot q)$ such that (3.25) is satisfied. This is

$$E_{\omega_c}(q,\dot q)=\frac{1}{2}\sum_{a=1}^{k}\sum_{i,j=1}^{m}\mathrm{L}_a\mathcal{N}_i^a\mathcal{N}_j^a\dot q^i\dot q^j+\frac{1}{2}\sum_{\alpha=k+1}^{b}\sum_{i,j=1}^{m}\frac{\mathcal{N}_i^\alpha\mathcal{N}_j^\alpha}{\mathrm{C}_{\alpha-k}}q^iq^j+\sum_{j=1}^{m}(const)_jq^j. \tag{3.35}$$

In order to derive such a function for nonlinear devices, we start with the equations

$$\begin{cases}\frac{\partial E_{\omega_c}}{\partial\dot q^1}=\mathcal{F}_1(\dot q)=\sum_{i=1}^{m}\sum_{a=1}^{k}{\mathcal{N}_i}^a\mathcal{N}_1^a\widetilde{L}_a(\dot q)\,\dot q^i\\ \frac{\partial E_{\omega_c}}{\partial q^1}=G_1(q)=\sum_{\alpha=k+1}^{b}{\mathcal{N}_1}^\alpha\widetilde{C}_{\alpha-k}(q)\end{cases}. \tag{3.36}$$

Integrating with respect to q^1 and $\dot q^1$, respectively, we get

$$E_{\omega_c}(q^1,...,q^m,\dot q^1,...,\dot q^m)=\sum_{a=1}^{k}\int\widetilde{L}_a(\dot q)\,{\mathcal{N}_i}^a\dot q^i\mathcal{N}_1^a d\dot q^1+f_1(\dot q^2,...,\dot q^m)+\sum_{\alpha=k+1}^{b}\int\widetilde{C}_{\alpha-k}(q)\,{\mathcal{N}_1}^\alpha dq^1+g_1(q^2,...,q^m) \tag{3.37}$$

f_1 depends only on $\dot q^2,...,\dot q^m$ and g_1 depends only on $q^2,...,q^m$. For $j=2$, we have

$$\begin{cases}\frac{\partial E_\omega}{\partial\dot q^2}=\mathcal{F}_2(\dot q)=\sum_{i=1}^{m}\sum_{a=1}^{k}{\mathcal{N}_i}^a\mathcal{N}_2^a\widetilde{L}_a(\dot q)\,\dot q^i\\ \frac{\partial E_\omega}{\partial q^1}=G_2(q)=\sum_{\alpha=k+1}^{b}{\mathcal{N}_2}^\alpha\widetilde{C}_{\alpha-k}(q)\end{cases} \tag{3.38}$$

and taking into account (3.37), we obtain

$$\begin{aligned} E_\omega(q^1,...,q^m,\dot q^1,...,\dot q^m) = & \sum_{a=1}^{k}\Big[\int \widetilde{L}_a\left(\dot q\right)\mathcal{N}_i^a\dot q^i\mathcal{N}_1^a d\dot q^1 + \int \widetilde{L}_a\left(\dot q\right)\mathcal{N}_i^a\dot q^i\mathcal{N}_2^a d\dot q^2 \\ & - \int\int \widetilde{L}'_a\left(\dot q\right)\mathcal{N}_i^a\dot q^i\mathcal{N}_1^a\mathcal{N}_2^a d\dot q^1 d\dot q^2 \\ & - \int\int \widetilde{L}_a\left(\dot q\right)\mathcal{N}_1^a\mathcal{N}_2^a d\dot q^1 d\dot q^2\Big] + f_2(\dot q^3,...,\dot q^m) \\ & + \sum_{\alpha=k+1}^{b}\Big[\int \widetilde{C}_{\alpha-k}\left(q\right)\mathcal{N}_1^\alpha dq^1 + \int \widetilde{C}_{\alpha-k}\left(q\right)\mathcal{N}_2^\alpha dq^2 \\ & - \int\int \widetilde{C}'_{\alpha-k}\left(q\right)\mathcal{N}_1^\alpha\mathcal{N}_2^\alpha dq^1 dq^2\Big] + g_2(q^3,...,q^m) \end{aligned} \quad (3.39)$$

which can be written in the form

$$\begin{aligned} & E_\omega(q,\dot q) \\ = & \sum_{a=1}^{k}\sum_{l=1}^{2}\sum_{i_1<...<i_l=1}^{2}(-1)^{l+1}\underbrace{\int ... \int}_{l}\Big[\widetilde{L}_a^{(l-1)}(\dot q)\mathcal{N}_i^a\dot q^i + (l-1)\widetilde{L}_a^{(l-2)}(\dot q)\Big]\mathcal{N}_{i_1}^a...\mathcal{N}_{i_l}^a d\dot q^{i_1}...d\dot q^{i_l} \\ & + \sum_{\alpha=k+1}^{b}\sum_{l=1}^{2}\sum_{i_1<...<i_l=1}^{2}(-1)^{l+1}\underbrace{\int ... \int}_{l}\widetilde{C}_{\alpha-k}^{(l-1)}(q)\mathcal{N}_{i_1}^\alpha...\mathcal{N}_{i_l}^\alpha dq^{i_1}...dq^{i_l} \\ & + f_2(\dot q^3,...,\dot q^m) + g_2(q^3,...,q^m) \end{aligned} \quad (3.40)$$

where $\widetilde{C}_\alpha^{(l)} := \frac{d^l\widetilde{C}_\alpha(\eta)}{d\eta^l}$, $\widetilde{L}_a^{(l)} := \frac{d^l\widetilde{L}_a(\eta)}{d\eta^l}$.

Repeating this procedure for j = 3, ... m, finally in the $m-th$ and last step, we obtain

$$\begin{aligned} & E_\omega(q,\dot q) \\ = & \sum_{a=1}^{k}\sum_{l=1}^{m}\sum_{i_1<...<i_l=1}^{m}(-1)^{l+1}\underbrace{\int ... \int}_{l}\Big[\widetilde{L}_a^{(l-1)}(\dot q)\mathcal{N}_i^a\dot q^i + (l-1)\widetilde{L}_a^{(l-2)}(\dot q)\Big]\mathcal{N}_{i_1}^a...\mathcal{N}_{i_l}^a d\dot q^{i_1}...d\dot q^{i_l} \\ & + \sum_{\alpha=k+1}^{b}\sum_{l=1}^{m}\sum_{i_1<...<i_l=1}^{m}(-1)^{l+1}\underbrace{\int ... \int}_{l}\widetilde{C}_{\alpha-k}^{(l-1)}(q)\mathcal{N}_{i_1}^\alpha...\mathcal{N}_{i_l}^\alpha dq^{i_1}...dq^{i_l}. \quad \square \end{aligned} \quad (3.41)$$

Let us now discuss the question, what to do when the Birkhoffian given by (3.21) is not regular in the sense of definition (2.8).

If there exists at least one loop in an LC circuit that contains only capacitors, then the Birkhoffian associated to the network is ***never regular****.*

Indeed, for the l-loop which contains only capacitors, on the column l of the matrix A we have $A_l^a = 0$ for any $a = 1, ..., k$. Without loss of generality, we will assume that $l = 1$, that is

$$A_1^a = 0, \quad \text{for any } a = 1, ..., k. \tag{3.42}$$

For the Birkhoffian (3.21), the determinant in (2.8) becomes

$$\det\left[\frac{\partial Q_j}{\partial \ddot{q}^i}(q,\, \dot{q},\, \ddot{q})\right]_{i,j=1,\ldots,m} = \det\left[\sum_{a=1}^{k} \mathcal{N}_j^{\,a}\mathcal{N}_i^a \widetilde{L}_a\,(\dot{q})\right]_{i,j=1,\ldots,m}. \tag{3.43}$$

From (3.20), we get $\mathcal{N}_j^a = \sum_{i_1=1}^{m} \mathcal{C}_j^{i_1} A_{i_1}^a$ for any $a = 1, ..., k$, $j = 1, ..., m$. Then, taking into account (3.42), we have, for example, in the case $m = 2$

$$\sum_{a=1}^{k} \mathcal{N}_j^{\,a}\mathcal{N}_i^a \widetilde{L}_a\,(\dot{q}) = \mathcal{C}_j^2\mathcal{C}_i^2\left[(A_2^1)^2\widetilde{L}_1\,(\dot{q}) + (A_2^2)^2\widetilde{L}_2\,(\dot{q}) + ... + (A_2^k)^2\widetilde{L}_k\,(\dot{q})\right]. \tag{3.44}$$

Then,

$$\det\left[\sum_{a=1}^{k} \mathcal{N}_j^{\,a}\mathcal{N}_i^a \widetilde{L}_a\,(\dot{q})\right]_{j,i=1,2} = \left[\sum_{a=1}^{k}(A_2^a)^2\widetilde{L}_a\,(\dot{q})\right]^2 \begin{vmatrix} \mathcal{C}_1^2\mathcal{C}_1^2 & \mathcal{C}_1^2\mathcal{C}_2^2 \\ \mathcal{C}_1^2\mathcal{C}_2^2 & \mathcal{C}_2^2\mathcal{C}_2^2 \end{vmatrix} = 0 \tag{3.45}$$

since the second factor obviously vanishes. In the case $m = 3$, we obtain

$$\begin{aligned}\sum_{a=1}^{k} \mathcal{N}_j^{\,a}\mathcal{N}_i^a \widetilde{L}_a\,(\dot{q}) \;=\; &\mathcal{C}_j^2\mathcal{C}_i^2\left[\sum_{a=1}^{k}(A_2^a)^2\widetilde{L}_a\,(\dot{q})\right] + \left(\mathcal{C}_j^2\mathcal{C}_i^3 + \mathcal{C}_i^2\mathcal{C}_j^3\right)\left[\sum_{a=1}^{k} A_2^a A_3^a\widetilde{L}_a\,(\dot{q})\right] \\ &+\mathcal{C}_j^3\mathcal{C}_i^3\left[\sum_{a=1}^{k}(A_3^a)^2\widetilde{L}_a\,(\dot{q})\right].\end{aligned} \tag{3.46}$$

Using basic calculus, the determinant of the matrix with elements (3.46) can be rearranged as a linear combination of determinants having the columns of the form $\begin{pmatrix} \mathcal{C}_1^{i_1}\mathcal{C}_1^{j_1} \\ \mathcal{C}_1^{i_1}\mathcal{C}_2^{j_1} \\ \mathcal{C}_1^{i_1}\mathcal{C}_3^{j_1} \end{pmatrix}$, $\begin{pmatrix} \mathcal{C}_2^{i_1}\mathcal{C}_1^{j_1} \\ \mathcal{C}_2^{i_1}\mathcal{C}_2^{j_1} \\ \mathcal{C}_2^{i_1}\mathcal{C}_3^{j_1} \end{pmatrix}$, $\begin{pmatrix} \mathcal{C}_3^{i_1}\mathcal{C}_1^{j_1} \\ \mathcal{C}_3^{i_1}\mathcal{C}_2^{j_1} \\ \mathcal{C}_3^{i_1}\mathcal{C}_3^{j_1} \end{pmatrix}$, respectively, with $i_1, j_1 = 2$ or 3 in each case. Hence, each of those determinants contain at least two linearly dependent columns, that is, they vanish, and this shows that the determinant is zero in the case $m = 3$ as well. Similarly, for an arbitrary m, the determinant of the matrix with the elements

$$\begin{aligned}\sum_{a=1}^{k} \mathcal{N}_j^{\,a}\mathcal{N}_i^a \widetilde{L}_a\,(\dot{q}) \;=\; &\sum_{i_1=2}^{m} \mathcal{C}_j^{i_1}\mathcal{C}_i^{i_1}\left[\sum_{a=1}^{k}(A_{i_1}^a)^2\widetilde{L}_a\,(\dot{q})\right] + \\ &\sum_{2<i_1<j_1}^{m}\left(\mathcal{C}_j^{i_1}\mathcal{C}_i^{j_1} + \mathcal{C}_i^{i_1}\mathcal{C}_j^{j_1}\right)\left[\sum_{a=1}^{k} A_{i_1}^a A_{j_1}^a\widetilde{L}_a\,(\dot{q})\right]\end{aligned} \tag{3.47}$$

is zero. □

If there exists in the network $m' < m$ loops which contain only capacitors, all the other loops containing at least an inductor, we can ***regularize*** *the Birkhoffian (3.21) via* ***reduction of the configuration space****. The reduced configuration space $\bar{M}_c$ of dimension $m-m'$, is a linear subspace of M_c or a manifold, depending on whether the capacitors are linear or nonlinear. We claim that the Birkhoffian $\bar{\omega}_c$ of the reduced configuration space $\bar{M}_c$ is still a* ***conservative Birkhoffian****. Under certain conditions on the functions L_a, $a = 1, ..., k$, which characterize the inductors, the reduced Birkhoffian $\bar{\omega}_c$ will be a* ***regular Birkhoffian****.*

Without loss of generality, we can assume that there is one loop in the network that contains only capacitors and in the coordinate system we have chosen

$$\mathcal{N}_1^a = 0,\ a = 1..., k. \tag{3.48}$$

Thus, the Birkhoffian components (3.21), with (3.22), (3.23), are given by, $j = 2, ..., m$,

$$\begin{aligned} Q_1(q,\dot{q},\ddot{q}) &= \sum_{\alpha=k+1}^{b} \mathcal{N}_1^\alpha \widetilde{C}_{\alpha-k}(q) \\ Q_j(q,\dot{q},\ddot{q}) &= \sum_{i=2}^{m}\sum_{a=1}^{k} \mathcal{N}_j^a \mathcal{N}_i^a \widetilde{L}_a\,(\dot{q})\,\ddot{q}^i + \sum_{\alpha=k+1}^{b} \mathcal{N}_j^\alpha \widetilde{C}_{\alpha-k}(q). \end{aligned} \tag{3.49}$$

We note that, according to (3.48), $\dot{q}^1$ does not appear in any function $\widetilde{L}_a(\dot{q})$ and the terms $\widetilde{L}_a(\dot{q})\ddot{q}^1$ do not appear in any of the Birkhoffian components $Q_2(q,\dot{q},\ddot{q}), ...,$ $Q_m(q,\dot{q},\ddot{q})$.

If the capacitors in this loop are linear devices, Q_1 is a linear combination of q's and we can use this relation to reduce the configuration space M_c, to an affine-linear subspace $\bar{M}_c$ of dimension $m-1$. If the capacitors in this loop are nonlinear devices, Q_1 depends nonlinearly on the q's. We define the $(m-1)$-dimensional manifold $\bar{M}_c \subset M_c$ by

$$\bar{M}_c = \left\{ q \in M_c \mid \sum_{\alpha=k+1}^{b} \mathcal{N}_1^\alpha \widetilde{C}_{\alpha-k}(q) = 0 \right\}. \tag{3.50}$$

By the implicit function theorem, we obtain a local coordinate system on the reduced configuration space $\bar{M}_c$. Taking $\bar{q}^1 := q^2$,..., $\bar{q}^{m-1} := q^m$, the Birkhoffian has the form $\bar{\omega}_c = \sum_{j=1}^{m-1} \bar{Q}_j d\bar{q}^j$,

$$\bar{Q}_j(\bar{q},\dot{\bar{q}},\ddot{\bar{q}}) = \bar{F}_j(\dot{\bar{q}})\ddot{\bar{q}} + \bar{G}_j(\bar{q}), \quad \text{where} \tag{3.51}$$

$$\bar{F}_j(\dot{\bar{q}})\ddot{\bar{q}} := \sum_{i=1}^{m-1}\sum_{a=1}^{k} \mathcal{N}_{(j+1)}^a \mathcal{N}_{(i+1)}^a L_a \left(\sum_{l=1}^{m-1} \mathcal{N}_{(l+1)}^a \dot{\bar{q}}^l \right) \ddot{\bar{q}}^i \tag{3.52}$$

$$\bar{G}_j(\bar{q}) := \sum_{\alpha=k+1}^{b} \mathcal{N}^{\alpha}_{(j+1)} C_{\alpha-k}\left(\mathcal{N}^{\alpha}_1 f(\bar{q}^1,...,\bar{q}^{m-1}) + \sum_{l=1}^{m-1}\mathcal{N}^{\alpha}_{(l+1)}\bar{q}^l + const\right). \tag{3.53}$$

$f : U \subset \mathbf{R}^{m-1} \longrightarrow \mathbf{R}$ being the unique function such that $f(\bar{q}_0) = q^1_0$, $q^1_0 \in \mathbf{R}$, and

$$\sum_{\alpha=k+1}^{b} \mathcal{N}^{\alpha}_1 C_{\alpha-k}\left(\mathcal{N}^{\alpha}_1 f(\bar{q}^1,...,\bar{q}^{m-1}) + \sum_{l=1}^{m-1}\mathcal{N}^{\alpha}_{(l+1)}\bar{q}^l + const\right) = 0 \tag{3.54}$$

for all $\bar{q} = (\bar{q}^1,...,\bar{q}^{m-1}) \in U$, with U a neighborhood of $\bar{q}_0 = (\bar{q}^1_0,...,\bar{q}^{m-1}_0)$.

We will now prove that the Birkhoffian (3.51) is conservative. In order to do so, we will show that there exists a function $\bar{E}_\omega(\bar{q},\dot{\bar{q}})$ satisfying

$$\sum_{j=1}^{m-1}\bar{Q}_j(\bar{q}\ \dot{\bar{q}},\ \ddot{\bar{q}})\dot{\bar{q}}^j = \sum_{j=1}^{m-1}\left[\frac{\partial \bar{E}_\omega}{\partial \bar{q}^j}\dot{\bar{q}}^j + \frac{\partial \bar{E}_\omega}{\partial \dot{\bar{q}}^j}\ddot{\bar{q}}^j\right]. \tag{3.55}$$

Because of the special form of the terms on the left side of (3.55), we may assume that $\bar{E}_\omega(\bar{q},\dot{\bar{q}})$ is a sum of a function depending only on $\bar{q}$, and a function depending only on $\dot{\bar{q}}$. From the theory of total differentials, a necessary condition for the existence of such functions is the fulfillment of the following relations

$$\begin{cases} \frac{\partial \bar{\mathcal{F}}_j(\dot{\bar{q}})}{\partial \dot{\bar{q}}^l} - \frac{\partial \bar{\mathcal{F}}_l(\dot{\bar{q}})}{\partial \dot{\bar{q}}^j} = 0 \\ \frac{\partial \bar{G}_j(\bar{q})}{\partial \bar{q}^l} - \frac{\partial \bar{G}_l(\bar{q})}{\partial \bar{q}^j} = 0 \end{cases}. \tag{3.56}$$

for any $j,l = 1,...,m-1$, where

$$\bar{\mathcal{F}}_j(\dot{\bar{q}}) := \sum_{i=1}^{m-1}\sum_{a=r+1}^{k}\mathcal{N}^a_{(j+1)}\mathcal{N}^a_{(i+1)}L_a\left(\sum_{l=1}^{m-1}\mathcal{N}^a_{(l+1)}\dot{\bar{q}}^l\right)\ddot{\bar{q}}^i. \tag{3.57}$$

We check in the same way as for the functions $\mathcal{F}_j(\dot{q})$ in (3.27), that the first relation in (3.56) is fulfilled. From (3.53), the second relation in (3.56) reads as

$$\sum_{\alpha=k+1}^{b}\left\{\mathcal{N}^{\alpha}_{(j+1)}\widetilde{C}'_{\alpha-k}(\bar{q})\left[\mathcal{N}^{\alpha}_1\frac{\partial f(\bar{q})}{\partial \bar{q}^l} + \mathcal{N}^{\alpha}_{(l+1)}\right]\right.$$
$$\left. -\mathcal{N}^{\alpha}_{(l+1)}\widetilde{C}'_{\alpha-k}(\bar{q})\left[\mathcal{N}^{\alpha}_1\frac{\partial f(\bar{q})}{\partial \bar{q}^j} + \mathcal{N}^{\alpha}_{(j+1)}\right]\right\} = 0 \tag{3.58}$$

where $\widetilde{C}'_{\alpha-k} := \frac{d\widetilde{C}_{\alpha-k}(\eta)}{d\eta}$. The relation (3.58) reduces to

$$\sum_{\alpha=k+1}^{b}\mathcal{N}^{\alpha}_{(j+1)}\widetilde{C}'_{\alpha-k}(\bar{q})\mathcal{N}^{\alpha}_1\frac{\partial f(\bar{q})}{\partial \bar{q}^l} - \mathcal{N}^{\alpha}_{(l+1)}\widetilde{C}'_{\alpha-k}(\bar{q})\mathcal{N}^{\alpha}_1\frac{\partial f(\bar{q})}{\partial \bar{q}^j} = 0. \tag{3.59}$$

Taking into account (3.54), the above relation is fulfilled, for any $j,l = 1,...,m-1$. Indeed, taking the derivatives with respect to $\bar{q}^j$ and also to $\bar{q}^l$, in the equation

(3.54), we obtain, respectively,

$$\sum_{\alpha=k+1}^{b} \mathcal{N}_1^\alpha \widetilde{C}'_{\alpha-k}(\bar{q}) \left[\mathcal{N}_1^\alpha \frac{\partial f(\bar{q})}{\partial \bar{q}^j} + \mathcal{N}_{(j+1)}^\alpha\right] = 0$$
$$\sum_{\alpha=k+1}^{b} \mathcal{N}_1^\alpha \widetilde{C}'_{\alpha-k}(\bar{q}) \left[\mathcal{N}_1^\alpha \frac{\partial f(\bar{q})}{\partial \bar{q}^l} + \mathcal{N}_{(l+1)}^\alpha\right] = 0. \tag{3.60}$$

Now we multiply in (3.60) the first equation with $\frac{\partial f(\bar{q})}{\partial \bar{q}^l}$, the second equation with $-\frac{\partial f(\bar{q})}{\partial \bar{q}^j}$ and we add the resulting equations to obtain the equation (3.59).

Thus, we proved the existence of a function $\bar{E}_\omega(\bar{q}, \dot{\bar{q}})$ such that (3.55) is fulfilled.

For any other loop which contains only capacitors, we just repeat this procedure. Thus, we finally arrive at a configuration space $\bar{M}_c$ of dimension $m - m'$, where m' denotes the total number of loops of that type. □

In case the network has loops which contain only inductors the Birkhoffian can be a regular one but we can further reduce the configuration space. Inductor loops can be considered as some conserved quantities of the network.

If there exists in the network $m'' < m$ loops which contain only linear inductors, all the other loops containing at least a capacitor, we can further reduce the configuration space. The reduced configuration space $\hat{M}_c$ of dimension $m - m''$, is a linear subspace of M_c. We claim that the Birkhoffian $\hat{\omega}_c$ of the reduced configuration space $\hat{M}_c$ is a ***conservative Birkhoffian****. Under certain conditions on the functions L_a, $a = 1, ..., k$, which characterize the inductors, the reduced Birkhoffian $\hat{\omega}_c$ will be a* ***regular Birkhoffian****.*

Without loss of generality, we can assume that there is one loop in the network that contains only inductors and in the coordinate system we have chosen

$$\mathcal{N}_1^\alpha = 0,\ \alpha = 1..., p. \tag{3.61}$$

Thus, the Birkhoffian components (3.21), with (3.22), (3.23), are given by, $j = 2, ..., m$,

$$Q_1(q, \dot{q}, \ddot{q}) = \sum_{i=1}^{m}\sum_{a=1}^{k} \mathcal{N}_1^a \mathcal{N}_i^a \widetilde{L}_a(\dot{q})\, \ddot{q}^i$$
$$Q_j(q, \dot{q}, \ddot{q}) = \sum_{i=1}^{m}\sum_{a=1}^{k} \mathcal{N}_j^a \mathcal{N}_i^a \widetilde{L}_a(\dot{q})\, \ddot{q}^i + \sum_{\alpha=k+1}^{b} \mathcal{N}_j^\alpha \widetilde{C}_{\alpha-k}(q). \tag{3.62}$$

We note that, according to (3.61), q^1 does not appear in any function $\widetilde{C}_{\alpha-k}(q)$.

If the inductors in this loop are linear devices, Q_1 is a linear combination of $\ddot{q}$'s. We can integrate this relation to obtain an affine-linear relation between q's (see the first example in Section 5). We can use this relation to reduce the configuration space M_c, to an affine-linear subspace $\hat{M}_c$ of dimension $m-1$. Taking $\hat{q}^1 := q^2$,..., $\hat{q}^{m-1} := q^m$, one can write the Birkhoffian components of $\hat{\omega}_c$ and one can prove, using the same ideas as in the previous reduction case, the existence of the function $\hat{E}_\omega$ such that this Birkhoffian is conservative.

For any other loop which contains only linear inductors, we just repeat this procedure. Thus, we finally arrive at a configuration space $\hat{M}_c$ of dimension $m - m''$, where m'' denotes the total number of loops of that type. □

If the devices in the $m'' < m$ inductor loops are nonlinear devices, then, $Q_1(q, \dot{q}, \ddot{q})$,..., $Q_{m''}(q, \dot{q}, \ddot{q})$, are nonlinear functions depending on $\dot{q}$'s and $\ddot{q}$'s. Using these relations we can define a smooth constant rank affine sub-bundle $\mathfrak{S}_c$ of the affine bundle $\pi_J : J^2(M_c) \longrightarrow TM_c$, on which we define the constrained Birkhoffian system $(M_c, \omega_c, \mathfrak{S}_c)$. The submanifold $\mathfrak{S}_c$ has codimension m''.

4. LC electric circuits with independent current/voltage sources

Let us now consider an electric circuit containing s_I independent current sources and s_V independent voltage sources, in addition to k inductors and p capacitors. Then $b = k+p+\mathrm{s}_I+\mathrm{s}_V = m+n$, where b, m, n have the same meaning as in Section 3. We suppose that $m - \mathrm{s}_I > 0$, $n - \mathrm{s}_V > 0$. The branches of the oriented connected graph associated to this circuit are labelled as follows: L_a, $a = 1, ..., k$, the inductor branches, C_α, $\alpha = 1, ..., p$, the capacitor branches, S_{I_i}, $i = 1, ..., \mathrm{s}_I$, the current source branches, and S_{V_j}, $j = 1, ..., \mathrm{s}_V$, the voltage source branches.

Let the basic equations governing the circuit be now written in the form

$$\begin{cases} \mathcal{B}_1^T \begin{pmatrix} \mathrm{I}_a \\ \frac{d\mathrm{Q}_\alpha}{dt} \end{pmatrix} + \mathcal{B}_2^T \left(\mathrm{I}_{s_I}(t) \right) = 0 \\ \\ \mathcal{A}_1^T \begin{pmatrix} L_a(\mathrm{I}_a) \frac{d\mathrm{I}_a}{dt} \\ C_\alpha(\mathrm{Q}_\alpha) \end{pmatrix} + \mathcal{A}_2^T \left(v_{s_V}(t) \right) = 0 \end{cases} \tag{4.1}$$

where $\mathcal{A}_1^T \in \mathfrak{M}_{(m-\mathrm{S}_I)(k+p)}(\mathbf{R})$, $\mathcal{A}_2^T \in \mathfrak{M}_{(m-\mathrm{S}_I)\mathrm{S}_V}(\mathbf{R})$, $\mathcal{B}_1^T \in \mathfrak{M}_{(n-\mathrm{S}_V)(k+p)}(\mathbf{R})$, $\mathcal{B}_2^T \in \mathfrak{M}_{(n-\mathrm{S}_V)\mathrm{S}_I}(\mathbf{R})$. We also assume that $\mathrm{rank}(\mathcal{A}_1^T) = m - \mathrm{s}_I$, $\mathrm{rank}(\mathcal{B}_1^T) = n - \mathrm{s}_V$. The functions $\mathrm{I}_{s_I}(t)$ and $v_{s_V}(t)$ are given vector functions of time. They describe the independent current sources and independent voltage sources, respectively. The other quantities in (4.1) are defined as in Section 3.

In the following we give a Birkhoffian formulation for the network described by the system of equations (4.1), using the same procedure as in Section 3. That is, using the first set of equations (4.1), we are going to define a family of $(m - s_I)$-dimensional affine-linear configuration spaces $M_c \subset \mathbf{R}^b$ parameterized by a constant vector c in $\mathbf{R}^{n-S_V}$. A Birkhoffian ω_{t_c} on the configuration space M_c arises from a linear combination of the second set of equations (4.1). Thus, (M_c, ω_{t_c}) will be a family of Birkhoff systems that describe the LC circuit with independent current/voltage sources considered.

Let $\mathfrak{H} : \mathbf{R}^{k+p} \longrightarrow \mathbf{R}^{n-\mathrm{S}_V}$ be the linear map that, with respect to a coordinate system $(x^1, ..., x^{k+p})$ on $\mathbf{R}^{k+p}$, is given by

$$\mathfrak{H}(x^1, ..., x^{k+p}) = \mathcal{B}_1^T \begin{pmatrix} x^1 \\ \vdots \\ x^{k+p} \end{pmatrix} + \mathcal{B}_2^T \left(\mathrm{I}_{s_I}(t) \right). \tag{4.2}$$

We define

$$M_c := \mathfrak{H}^{-1}(c) \tag{4.3}$$

c being a constant vector in $\mathbf{R}^{n-\mathrm{S}_V}$. M_c is a time-dependent affine linear subspace in $\mathbf{R}^{k+p}$. From $\mathrm{rank}(\mathcal{B}_1^T) = n - \mathrm{s}_V$, its dimension is $k + p + \mathrm{s}_V - n = m - \mathrm{s}_I$.

Let us figure out the relation between I_a, $\frac{d\mathrm{Q}_\alpha}{dt}$, and coordinates on M_c. As in the case without sources, taking into account (3.11) and the fact that the matrix $\mathcal{B}_1^T$ is a constant matrix, we integrate the first set of equations (4.1) to arrive at

$$\mathcal{B}_1^T \begin{pmatrix} \mathrm{Q}_{(a)} \\ \mathrm{Q}_\alpha \end{pmatrix} + \mathcal{I}(t) = c \tag{4.4}$$

with c a constant vector in $\mathbf{R}^{n-\mathrm{S}_V}$ and $\mathcal{I}(t)$ a primitive of $\mathcal{B}_2^T \left(\mathrm{I}_{s_I}(t) \right)$.

Likewise consider coordinates in $\mathbf{R}^{k+p}$

$$x^1 := \mathrm{Q}_{(1)},\ ..,\ x^k := \mathrm{Q}_{(k)},\ x^{k+1} := \mathrm{Q}_1,\ ..,\ x^{k+p} := \mathrm{Q}_p. \tag{4.5}$$

We can define coordinates on M_c by solving the equations (4.4) in terms of an appropriate set of $(m - \mathrm{s}_I)$ of the Q-variables, say $q = (q^1, ..., q^{m-\mathrm{S}_I})$. In other words, we express any of the x-variables as a function of $q = (q^1, ..., q^{m-\mathrm{S}_I})$, namely,

$$\begin{aligned} x^a &= \sum_{j=1}^{m-\mathrm{S}_I} \mathfrak{N}_j^a q^j + f^a(t) + const,\ a = 1, ..., k, \\ x^\alpha &= \sum_{j=1}^{m-\mathrm{S}_I} \mathfrak{N}_j^\alpha q^j + f^\alpha(t) + const,\ \alpha = k+1, ..., k+p \end{aligned} \tag{4.6}$$

with certain constants $\mathfrak{N}_j^a$, $\mathfrak{N}_j^\alpha$ and certain functions of t, $f^a(t)$, $f^\alpha(t)$.

The constant matrix $\mathfrak{N} = \begin{pmatrix} \mathfrak{N}_j^a \\ \mathfrak{N}_j^\alpha \end{pmatrix}_{\substack{a=1,...,k,\ \alpha=k+1,...,k+p \\ j=1,...,m-\mathrm{S}_I}}$ has rank $m - \mathrm{s}_I$, and there exists a nonsingular matrix $\mathcal{C} \in \mathfrak{M}_{(m-\mathrm{S}_I)(m-\mathrm{S}_I)}(\mathbf{R})$ such that

$$\mathcal{C}\mathcal{A}_1^T = \mathfrak{N}^T. \tag{4.7}$$

We define the Birkhoffian ω_{t_c} of M_c such that the differential system (2.5) is a linear combination of the second set of equations in (4.1), which is obtained multiplying the second set of equations in (4.1) by the matrix $\mathcal{C}$. Taking into *account (4.7), in terms of q-coordinates as chosen before, the expressions of the components* $Q_j(t, q, \dot{q}, \ddot{q})$, $j = 1, ..., m - s_I$ *are*

$$Q_j(t, q, \dot{q}, \ddot{q}) = F_j(t, \dot{q})\ddot{q} + G_j(t, q) + \mathcal{V}_j(t) \tag{4.8}$$

where

$$\begin{aligned} F_j(t, \dot{q})\ddot{q} &= \sum_{a=1}^{k} \mathfrak{N}_j^a \widetilde{L}_a(t, \dot{q}) \left(\sum_{i=1}^{m-\mathrm{S}_I} \mathfrak{N}_i^a \ddot{q}^i + \frac{d^2 f^a(t)}{dt^2} \right) \\ &= \sum_{i=1}^{m-\mathrm{S}_I} \left(\sum_{a=1}^{k} \mathfrak{N}_j^a \mathfrak{N}_i^a \widetilde{L}_a(t, \dot{q}) \right) \ddot{q}^i + \sum_{a=1}^{k} \mathfrak{N}_j^a \widetilde{L}_a(t, \dot{q}) \frac{d^2 f^a(t)}{dt^2} \end{aligned} \tag{4.9}$$

$$\begin{aligned} G_j(t, q) &= \sum_{\alpha=k+1}^{k+p} \mathfrak{N}_j^\alpha C_{\alpha-k} \left(\sum_{j=1}^{m-\mathrm{S}_I} \mathfrak{N}_j^\alpha q^j + f^\alpha(t) + const \right) \\ &= \sum_{\alpha=k+1}^{k+p} \mathfrak{N}_j^\alpha \widetilde{C}_{\alpha-k}(t, q) \end{aligned} \tag{4.10}$$

$$\mathcal{V}_j(t) = \sum_{s_V = k+p+\mathrm{S}_I+1}^{\mathrm{b}} (\mathcal{C}^T \mathcal{A}_2^T)_{j s_V} v_{s_V - k - p - S_I}(t). \tag{4.11}$$

If there exist in the network $m' < m - s_I$ *loops which contain only capacitors or capacitors and independent voltage sources, then the Birkhoffian associated to the network is* ***never*** **regular**.

Indeed, in this case the functions Q_j corresponding to such loops depend only on q's and t. Using the same procedure as in Section 3, the reduced configuration space $\bar{M}_c$ of dimension $(m - \mathrm{s}_I) - m'$, is a linear subspace of M_c or a manifold, depending on whether the devices in the loops are linear or nonlinear.

Let us finally discuss the question whether the Birkhoffian (4.8) is conservative or not.

For a linear LC circuit with independent current/voltage sources we claim that the Birkhoffian (4.8) is conservative. A nonlinear LC circuit with independent current/voltage sources is conservative if and only if it does not contain cutsets of inductors and independent current sources.

In order to show that the Birkhoffian (4.8) is conservative, we are looking for a smooth function $E_{\omega_t}(t, q, \dot{q})$ such that the relation (2.19) is fulfilled. For the Birkhoffian (4.8), this relation becomes

$$\left(\sum_{i=1}^{m-\mathrm{S}_I} \sum_{a=1}^{k} \mathfrak{N}_i^a \mathfrak{N}_j^a \widetilde{L}_a(t, \dot{q}) \, \dot{q}^i \right) \ddot{q}^j + \left(\sum_{a=1}^{k} \mathfrak{N}_j^a \widetilde{L}_a(t, \dot{q}) \frac{d^2 f^a(t)}{dt^2} + G_j(t, q) + \mathcal{V}_j(t) \right) \dot{q}^j$$

$$= \frac{\partial E_{\omega_t}}{\partial q^j}\dot{q}^j + \frac{\partial E_{\omega_t}}{\partial \dot{q}^j}\ddot{q}^j. \tag{4.12}$$

If the inductors and the capacitors in the network are linear devices, taking into account (3.7), we easily find the function

$$\begin{aligned} E_{\omega_t}(t,q,\dot{q}) \; &= \frac{1}{2}\sum_{a=1}^{k}\sum_{i,j=1}^{m-\mathrm{S}_I} \mathrm{L}_a \mathcal{N}_i^a \mathcal{N}_j^a \dot{q}^i \dot{q}^j + \frac{1}{2}\sum_{\alpha=k+1}^{\mathrm{B}}\sum_{i,j=1}^{m-\mathrm{S}_I} \frac{\mathcal{N}_i^\alpha \mathcal{N}_j^\alpha}{\mathrm{C}_{\alpha-k}} q^i q^j \\ &+ \sum_{j=1}^{m-\mathrm{S}_I} [\mathfrak{N}_j^a \mathrm{L}_a \frac{d^2 f^a(t)}{dt^2} + \mathcal{V}_j(t) + const_j] q^j \end{aligned} \tag{4.13}$$

which satisfies (4.12).

If the inductors and the capacitors in the network are nonlinear devices, the existence of the function $E_{\omega_t}(t,q,\dot{q})$ which satisfies (4.12), depends on the appearance of the term $\sum_{a=1}^{k} \mathfrak{N}_j^a \widetilde{L}_a\,(t,\dot{q})\,\frac{d^2 f^a(t)}{dt^2}$ in (4.12). For the networks which do not contain cutsets of inductors and independent current sources, this term does not appear at all in (4.12). In this case the proof of the existence of the function E_{ω_t} is the same as in the case without sources. If the term $\sum_{a=1}^{k} \mathfrak{N}_j^a \widetilde{L}_a\,(t,\dot{q})\,\frac{d^2 f^a(t)}{dt^2}$ is different from zero in (4.12), then, $\frac{\partial^2 E_{\omega_t}}{\partial q^j \partial \dot{q}^j} \neq \frac{\partial^2 E_{\omega_t}}{\partial \dot{q}^j \partial q^j}$. Therefore, the Birkhoffian (4.8) is not conservative in the sense of definition (2.19). □

5. Examples

The first example that we present is the example from the paper ([9]), in which we have interchanged the capacitor C_3 and the inductor L_1 to emphasize that networks which contain capacitor loops and inductor cutsets fit into the formalism presented in Section 3. The directed connected graph associated to this circuit is presented in Figure 1, page 182.

We first suppose that all devices are linear, that is, they are described by the relations (3.7). Then, taking into account the values of the matrices A, B given by (3.3), the equations (3.8) which govern the network have the form

$$\begin{cases} \mathrm{I}_4 - \frac{d\mathrm{Q}_1}{dt} + \frac{d\mathrm{Q}_3}{dt} = 0 \\ \mathrm{I}_2 + \mathrm{I}_3 - \frac{d\mathrm{Q}_2}{dt} - \frac{d\mathrm{Q}_3}{dt} = 0 \\ \mathrm{I}_1 - \mathrm{I}_3 - \mathrm{I}_4 = 0 \\ \\ \frac{\mathrm{Q}_1}{\mathrm{C}_1} - \frac{\mathrm{Q}_2}{\mathrm{C}_2} + \frac{\mathrm{Q}_3}{\mathrm{C}_3} = 0 \\ \mathrm{L}_2\frac{d\mathrm{I}_2}{dt} + \frac{\mathrm{Q}_2}{\mathrm{C}_2} = 0 \\ \mathrm{L}_1\frac{d\mathrm{I}_1}{dt} - \mathrm{L}_2\frac{d\mathrm{I}_2}{dt} + \mathrm{L}_3\frac{d\mathrm{I}_3}{dt} = 0 \\ -\mathrm{L}_1\frac{d\mathrm{I}_1}{dt} - \mathrm{L}_4\frac{d\mathrm{I}_4}{dt} - \frac{\mathrm{Q}_1}{\mathrm{C}_1} = 0 \end{cases} \tag{5.1}$$

where $\mathrm{C}_\alpha \neq 0$, $\alpha = 1,2,3$ and $\mathrm{L}_a \neq 0$, $a = 1,2,3,4$, are distinct constants. The relations (3.11), (3.13) read as follows for this example

$$\mathrm{I}_a := \frac{d\mathrm{Q}_{(a)}}{dt}, \quad a = 1,2,3,4 \tag{5.2}$$

$$x^1 := \mathrm{Q}_{(1)},\ ...,\ x^4 := \mathrm{Q}_{(4)},\ x^5 := \mathrm{Q}_1, ..., x^7 := \mathrm{Q}_3. \tag{5.3}$$

Using the first set of equations (5.1), we define the 4-dimensional affine-linear configuration space M_c. We solve the corresponding equations (3.12) in terms of 4 variables. In view of the notations (5.2), (5.3), we obtain, for example,

$$\begin{aligned} x^1 &= x^3 + x^4 + const \\ x^5 &= x^4 + x^7 + const \\ x^6 &= x^2 + x^3 - x^7 + const. \end{aligned} \tag{5.4}$$

Thus, a coordinate system on M_c is given by

$$q^1 := x^7, q^2 := x^2, q^3 := x^3, q^4 := x^4. \tag{5.5}$$

The matrices of constants $\mathcal{N} = \begin{pmatrix} \mathcal{N}^a_j \\ \mathcal{N}^\alpha_j \end{pmatrix}_{\substack{a=1,2,3,4,\ \alpha=5,6,7 \\ j=1,2,3,4}}$ and $\mathcal{C}$ in (3.14), (3.20) attain the form

$$\mathcal{N} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{5.6}$$

Note that, if we define the Birkhoffian ω_c of M_c using the second set of equations (5.1) and not a linear combination of them, that is, the matrix A^T instead of $\mathcal{C}A^T = \mathcal{N}^T$, then, in terms of the q-coordinates introduced in (5.5), we obtain $\omega_c = \sum_{j=1}^4 Q_j(q,\dot{q},\ddot{q})dq^j$, with

$$\begin{aligned} Q_1(q,\dot{q},\ddot{q}) &= \left(\frac{1}{\mathrm{C}_1} + \frac{1}{\mathrm{C}_2} + \frac{1}{\mathrm{C}_3}\right) q^1 - \frac{q^2}{\mathrm{C}_2} - \frac{q^3}{\mathrm{C}_2} + \frac{q^4}{\mathrm{C}_1} + const \\ Q_2(q,\dot{q},\ddot{q}) &= \mathrm{L}_2\ddot{q}^2 - \frac{q^1}{\mathrm{C}_2} + \frac{q^2}{\mathrm{C}_2} + \frac{q^3}{\mathrm{C}_2} + const \\ Q_3(q,\dot{q},\ddot{q}) &= -\mathrm{L}_2\ddot{q}^2 + (\mathrm{L}_1 + \mathrm{L}_3)\ddot{q}^3 + \mathrm{L}_1\ddot{q}^4 \\ Q_4(q,\dot{q},\ddot{q}) &= -\mathrm{L}_1\ddot{q}^3 - (\mathrm{L}_1 + \mathrm{L}_4)\ddot{q}^4 - \frac{q^1}{\mathrm{C}_1} - \frac{q^4}{\mathrm{C}_1} + const. \end{aligned} \tag{5.7}$$

The Birkhoffian (5.7) is ***not*** conservative. Indeed, for the Birkhoffian (5.7) two of the necessary conditions for the existence of the function $E_\omega : TM \to \mathbf{R}$ such that (2.10) is fulfilled, are

$$\begin{cases} \frac{\partial E_\omega}{\partial \dot{q}^2} = \mathrm{L}_2\dot{q}^2 - \mathrm{L}_2\dot{q}^3 \\ \\ \frac{\partial E_\omega}{\partial \dot{q}^3} = (\mathrm{L}_1 + \mathrm{L}_3)\dot{q}^3 + \mathrm{L}_1\dot{q}^4 \end{cases}. \tag{5.8}$$

Because $\mathrm{L}_2 \neq 0$, we see that $\frac{\partial^2 E_\omega}{\partial \dot{q}^3 \dot{q}^2} \neq \frac{\partial^2 E_\omega}{\partial \dot{q}^2 \dot{q}^3}$. Therefore, there does not exist a function E_ω such that (2.10) is fulfilled.

However, proceeding as suggested in Section 3, the functions $Q_j(q, \dot{q}, \ddot{q})$, $j = 1, 2, 3, 4$ are given by (3.21), (3.22), and (3.23), that is,

$$\begin{aligned} Q_1(q,\dot{q},\ddot{q}) &= \left(\frac{1}{\mathrm{C}_1} + \frac{1}{\mathrm{C}_2} + \frac{1}{\mathrm{C}_3}\right) q^1 - \frac{q^2}{\mathrm{C}_2} - \frac{q^3}{\mathrm{C}_2} + \frac{q^4}{\mathrm{C}_1} + const \\ Q_2(q,\dot{q},\ddot{q}) &= \mathrm{L}_2\ddot{q}^2 - \frac{q^1}{\mathrm{C}_2} + \frac{q^2}{\mathrm{C}_2} + \frac{q^3}{\mathrm{C}_2} + const \\ Q_3(q,\dot{q},\ddot{q}) &= (\mathrm{L}_1 + \mathrm{L}_3)\ddot{q}^3 + \mathrm{L}_1\ddot{q}^4 - \frac{q^1}{\mathrm{C}_2} + \frac{q^2}{\mathrm{C}_2} + \frac{q^3}{\mathrm{C}_2} + const \\ Q_4(q,\dot{q},\ddot{q}) &= \mathrm{L}_1\ddot{q}^3 + (\mathrm{L}_1 + \mathrm{L}_4)\ddot{q}^4 + \frac{q^1}{\mathrm{C}_1} + \frac{q^4}{\mathrm{C}_1} + const. \end{aligned} \tag{5.9}$$

The Birkhoffian (5.9) is **conservative**. The function $E_\omega(q, \dot{q})$ is given by (3.35), that is,

$$\begin{aligned} E_\omega(q,\dot{q}) \;=\; & \frac{1}{2}\mathrm{L}_1(\dot{q}^3 + \dot{q}^4)^2 + \frac{1}{2}\mathrm{L}_2(\dot{q}^2)^2 + \frac{1}{2}\mathrm{L}_3(\dot{q}^3)^2 + \frac{1}{2}\mathrm{L}_4(\dot{q}^4)^2 + \frac{1}{2\mathrm{C}_1}(q^1 + q^4)^2 + \\ & \frac{1}{2\mathrm{C}_2}(-q^1 + q^2 + q^3)^2 + \frac{1}{2\mathrm{C}_3}(q^1)^2 + \sum_{j=1}^{4}(const)_j q^j. \end{aligned} \tag{5.10}$$

Because we are in a situation where the network has one loop which contains only capacitors, the Birkhoffian corresponding to (5.9) is ***not* regular**. Indeed, the first row of the matrix $\left[\frac{\partial Q_j}{\partial \ddot{q}^i}\right]_{i,j=1,2,3,4}$ contains only zeros, therefore $\det\left[\frac{\partial Q_j}{\partial \ddot{q}^i}\right]_{i,j=1,2,3,4} = 0$.

As we have stated in Section 3, we can reduce the configuration space from dimension 4 to dimension 3. Using the first equation in (5.9) we define $\bar{M}_c \subset M_c$ by

$$\begin{aligned} \bar{M}_c = \Big\{ q = (q^1, q^2, q^3, q^4) \in M_c / \; & \left(\frac{1}{\mathrm{C}_1} + \frac{1}{\mathrm{C}_2} + \frac{1}{\mathrm{C}_3}\right) q^1 \\ & - \frac{q^2}{\mathrm{C}_2} - \frac{q^3}{\mathrm{C}_2} + \frac{q^4}{\mathrm{C}_1} + const = 0. \Big\} \end{aligned} \tag{5.11}$$

On the reduced configuration space $\bar{M}_c$, in the coordinate system $\bar{q}^1 := q^2$, $\bar{q}^2 :=$

q^3, $\bar{q}^3 := q^4$, the Birkhoffian has the form $\bar{\omega}_c = \sum_{j=1}^{3} \bar{Q}_j d\bar{q}^j$,

$$\begin{aligned}
\bar{Q}_1(\bar{q}, \dot{\bar{q}}, \ddot{\bar{q}}) &= \mathrm{L}_2\ddot{\bar{q}}^1 + \mathfrak{C}_1\bar{q}^1 + \mathfrak{C}_1\bar{q}^2 + \mathfrak{C}_2\bar{q}^3 + const \\
\bar{Q}_2(\bar{q}, \dot{\bar{q}}, \ddot{\bar{q}}) &= (\mathrm{L}_1 + \mathrm{L}_3)\ddot{\bar{q}}^2 + \mathrm{L}_1\ddot{\bar{q}}^3 + \mathfrak{C}_1\bar{q}^1 + \mathfrak{C}_1\bar{q}^2 + \mathfrak{C}_2\bar{q}^3 + const \\
\bar{Q}_3(\bar{q}, \dot{\bar{q}}, \ddot{\bar{q}}) &= \mathrm{L}_1\ddot{\bar{q}}^2 + (\mathrm{L}_1 + \mathrm{L}_4)\ddot{\bar{q}}^3 + \mathfrak{C}_2\bar{q}^1 + \mathfrak{C}_2\bar{q}^2 + \mathfrak{C}_3\bar{q}^3 + const.
\end{aligned} \tag{5.12}$$

where we have introduced the notation $\mathfrak{C}_1 := \frac{1}{\mathrm{C}_2}\left(1 - \frac{1}{\mathrm{C}_2}\left(\frac{1}{\mathrm{C}_1} + \frac{1}{\mathrm{C}_2} + \frac{1}{\mathrm{C}_3}\right)^{-1}\right)$,
$\mathfrak{C}_2 := \frac{1}{\mathrm{C}_2\mathrm{C}_1}\left(\frac{1}{\mathrm{C}_1} + \frac{1}{\mathrm{C}_2} + \frac{1}{\mathrm{C}_3}\right)^{-1}$, $\mathfrak{C}_3 := \frac{1}{\mathrm{C}_1}\left(1 - \frac{1}{\mathrm{C}_1}\left(\frac{1}{\mathrm{C}_1} + \frac{1}{\mathrm{C}_2} + \frac{1}{\mathrm{C}_3}\right)^{-1}\right)$.

Let us now see whether the Birkhoffian (5.12) is **regular** and/or **conservative**. We calculate

$$\det\left[\frac{\partial \bar{Q}_j}{\partial \ddot{\bar{q}}^i}\right]_{i,j=1,2,3} = \begin{vmatrix} \mathrm{L}_2 & 0 & 0 \\ 0 & \mathrm{L}_1 + \mathrm{L}_3 & \mathrm{L}_1 \\ 0 & \mathrm{L}_1 & \mathrm{L}_1 + \mathrm{L}_4 \end{vmatrix}. \tag{5.13}$$

Thus, if $\mathrm{L}_2\left[\mathrm{L}_1\mathrm{L}_4 + \mathrm{L}_3(\mathrm{L}_1 + \mathrm{L}_4)\right] \neq 0$, then the Birkhoffian (5.12) is **regular**.

The corresponding Birkhoffian vector field (see Section 2), is given by:

$$\begin{aligned}
Y = \;& \dot{\bar{q}}^1\frac{\partial}{\partial \bar{q}^1} + \dot{\bar{q}}^2\frac{\partial}{\partial \bar{q}^2} + \dot{\bar{q}}^3\frac{\partial}{\partial \bar{q}^3} - \frac{1}{\mathrm{L}_2}\left[\mathfrak{C}_1\bar{q}^1 + \mathfrak{C}_1\bar{q}^2 + \mathfrak{C}_2\bar{q}^3\right]\frac{\partial}{\partial \dot{\bar{q}}^1} \\
&+ \frac{1}{(\mathrm{L}_1 + \mathrm{L}_3)\mathrm{L}_4 + \mathrm{L}_1\mathrm{L}_3}\left[\left(-\mathfrak{C}_1(\mathrm{L}_1 + \mathrm{L}_4) + \mathfrak{C}_2\mathrm{L}_1\right)\bar{q}^1 + \left(-\mathfrak{C}_1(\mathrm{L}_1 + \mathrm{L}_4) + \mathfrak{C}_2\mathrm{L}_1\right)\bar{q}^2\right. \\
&\left. + \left(-\mathfrak{C}_2(\mathrm{L}_1 + \mathrm{L}_4) + \mathfrak{C}_3\mathrm{L}_1\right)\bar{q}^3 + const\right]\frac{\partial}{\partial \dot{\bar{q}}^2} \\
&+ \frac{1}{(\mathrm{L}_1 + \mathrm{L}_4)\mathrm{L}_3 + \mathrm{L}_1\mathrm{L}_4}\left[\left(-\mathfrak{C}_2(\mathrm{L}_1 + \mathrm{L}_3) + \mathfrak{C}_1\mathrm{L}_1\right)\bar{q}^1 + \left(-\mathfrak{C}_2(\mathrm{L}_1 + \mathrm{L}_3) + \mathfrak{C}_1\mathrm{L}_1\right)\bar{q}^2\right. \\
&\left. + \left(-\mathfrak{C}_3(\mathrm{L}_1 + \mathrm{L}_3) + \mathfrak{C}_2\mathrm{L}_1\right)\bar{q}^3 + const\right]\frac{\partial}{\partial \dot{\bar{q}}^3}.
\end{aligned} \tag{5.14}$$

Also, the Birkhoffian (5.12) is **conservative** with the function $\bar{E}_{\bar{\omega}}(\bar{q}, \dot{\bar{q}})$ given by

$$\begin{aligned}
\bar{E}_{\bar{\omega}}(\bar{q}, \dot{\bar{q}}) =\;& \frac{1}{2}\mathrm{L}_1(\dot{\bar{q}}^2 + \dot{\bar{q}}^3)^2 + \frac{1}{2}\mathrm{L}_2(\dot{\bar{q}}^1)^2 + \frac{1}{2}\mathrm{L}_3(\dot{\bar{q}}^2)^2 + \frac{1}{2}\mathrm{L}_4(\dot{\bar{q}}^3)^2 + \frac{1}{2}\mathfrak{C}_1(\bar{q}^1 + \bar{q}^2)^2 \\
&+ \mathfrak{C}_2(\bar{q}^1\bar{q}^3 + \bar{q}^2\bar{q}^3) + \frac{1}{2}\mathfrak{C}_3(\bar{q}^3)^2 + \sum_{j=1}^{3}(const)_j\bar{q}^j.
\end{aligned} \tag{5.15}$$

As we have pointed out in Section 3, because the network has one loop which contains only inductors, this is the loop I_3, we can further reduce the dimension of the configuration space by one. The equation that we use for doing this is Kirchhoff's voltage law equation for this loop, that is, the sixth equation in (5.1). For the chosen q-coordinate system (5.5) and after the transformation $\mathcal{C}A^T$, the sixth equation in (5.1) added with the fifth equation in (5.1) and it appeared in (5.9) by the function $Q_3(q, \dot{q}, \ddot{q})$. The Birkhoffian formulation was presented in a coordinate free fashion. In order to have a coordinate system in which the sixth equation in (5.1) appears in the initial form, we change the $\bar{q}$-coordinate system by the following relations

$$\begin{aligned}\bar{q}^1 &= \check{q}^1 - \check{q}^2\\ \bar{q}^2 &= \check{q}^2\\ \bar{q}^3 &= \check{q}^3.\end{aligned}\tag{5.16}$$

In terms of $\check{q}$-coordinates on $\bar{M}_c$, the Birkhoffian $\bar{\omega}_c = \sum_{j=1}^{3} \check{Q}_j d\check{q}^j$, where

$$\begin{aligned}\check{Q}_1(\check{q},\dot{\check{q}},\ddot{\check{q}}) &= \mathrm{L}_2\ddot{\check{q}}^1 - \mathrm{L}_2\ddot{\check{q}}^2 + \mathfrak{C}_1\check{q}^1 + \mathfrak{C}_2\check{q}^3 + const\\ \check{Q}_2(\check{q},\dot{\check{q}},\ddot{\check{q}}) &= -\mathrm{L}_2\ddot{\check{q}}^1 + (\mathrm{L}_1+\mathrm{L}_2+\mathrm{L}_3)\ddot{\check{q}}^2 + \mathrm{L}_1\ddot{\check{q}}^3\\ \check{Q}_3(\check{q},\dot{\check{q}},\ddot{\check{q}}) &= \mathrm{L}_1\ddot{\check{q}}^2 + (\mathrm{L}_1+\mathrm{L}_4)\ddot{\check{q}}^3 + \mathfrak{C}_2\check{q}^1 + \mathfrak{C}_3\check{q}^3 + const.\end{aligned}\tag{5.17}$$

Using the second equation (5.17), we define $\hat{M}_c \subset \bar{M}_c$ by

$$\hat{M}_c = \{\check{q} = (\check{q}^1,\check{q}^2,\check{q}^3) \in \bar{M}_c / -\mathrm{L}_2\check{q}^1 + (\mathrm{L}_1+\mathrm{L}_2+\mathrm{L}_3)\check{q}^2 + \mathrm{L}_1\check{q}^3 + g(t) + const = 0\}\tag{5.18}$$

with a certain function $g(t)$ depending on t.

On the reduced configuration space $\hat{M}_c$, in the coordinate system $\hat{q}^1 := \check{q}^1$, $\hat{q}^2 := \check{q}^3$, the Birkhoffian has the form $\hat{\omega} = \hat{Q}_1 d\hat{q}^1 + \hat{Q}_2 d\hat{q}^2$ with

$$\begin{aligned}\hat{Q}_1(\hat{q},\dot{\hat{q}},\ddot{\hat{q}}) &= \frac{\mathrm{L}_2(\mathrm{L}_1+\mathrm{L}_3)}{\mathrm{L}_1+\mathrm{L}_2+\mathrm{L}_3}\ddot{\hat{q}}^1 + \frac{\mathrm{L}_1\mathrm{L}_2}{\mathrm{L}_1+\mathrm{L}_2+\mathrm{L}_3}\ddot{\hat{q}}^2 + \mathfrak{C}_1\hat{q}^1 + \mathfrak{C}_2\hat{q}^2 + const\\ \hat{Q}_2(\hat{q},\dot{\hat{q}},\ddot{\hat{q}}) &= \frac{\mathrm{L}_1\mathrm{L}_2}{\mathrm{L}_1+\mathrm{L}_2+\mathrm{L}_3}\ddot{\hat{q}}^1 + \frac{(\mathrm{L}_1+\mathrm{L}_4)(\mathrm{L}_2+\mathrm{L}_3)+\mathrm{L}_1\mathrm{L}_4}{\mathrm{L}_1+\mathrm{L}_2+\mathrm{L}_3}\ddot{\hat{q}}^2 + \mathfrak{C}_2\hat{q}^1 + \mathfrak{C}_3\hat{q}^2 + const.\end{aligned}\tag{5.19}$$

Because $\mathrm{L}_i \neq 0$, $i = 1,...,4$, the determinant $\frac{\mathrm{L}_2}{(\mathrm{L}_1+\mathrm{L}_2+\mathrm{L}_3)^2}\begin{vmatrix}\mathrm{L}_1+\mathrm{L}_3 & \mathrm{L}_1\\ \mathrm{L}_1\mathrm{L}_2 & (\mathrm{L}_1+\mathrm{L}_4)(\mathrm{L}_2+\mathrm{L}_3)+\mathrm{L}_1\mathrm{L}_4\end{vmatrix} \neq 0$, then, the Birkhoffian (5.19) is **regular**.

Moreover, the Birkhoffian (5.19) is **conservative**, and

$$\begin{aligned}\hat{E}_\omega(\hat{q},\dot{\hat{q}}) \;=\; &\frac{\mathrm{L}_2(\mathrm{L}_1+\mathrm{L}_3)}{2(\mathrm{L}_1+\mathrm{L}_2+\mathrm{L}_3)}(\dot{\hat{q}}^1)^2 + \frac{\mathrm{L}_1\mathrm{L}_2}{\mathrm{L}_1+\mathrm{L}_2+\mathrm{L}_3}\dot{\hat{q}}^1\dot{\hat{q}}^2\\ &+\frac{(\mathrm{L}_1+\mathrm{L}_4)(\mathrm{L}_2+\mathrm{L}_3)+\mathrm{L}_1\mathrm{L}_4}{2(\mathrm{L}_1+\mathrm{L}_2+\mathrm{L}_3)}(\dot{\hat{q}}^2)^2\\ &+\frac{1}{2}\mathfrak{C}_1(\hat{q}^1)^2 + \mathfrak{C}_2\hat{q}^1\hat{q}^2 + \frac{1}{2}\mathfrak{C}_3(\hat{q}^2)^2 + \sum_{j=1}^{2}(const)_j\hat{q}^j.\end{aligned}\tag{5.20}$$

Let us now suppose that the inductors and the capacitors in the network are nonlinear devices, their constitutive relations being of the form (3.4), (3.6). The equations (3.8) which govern the network have now the form

$$\begin{cases} \mathrm{I}_4 - \frac{d\mathrm{Q}_1}{dt} + \frac{d\mathrm{Q}_3}{dt} = 0 \\ \mathrm{I}_2 + \mathrm{I}_3 - \frac{d\mathrm{Q}_2}{dt} - \frac{d\mathrm{Q}_3}{dt} = 0 \\ \mathrm{I}_1 - \mathrm{I}_3 - \mathrm{I}_4 = 0 \\ \\ C_1(\mathrm{Q}_1) - C_2(\mathrm{Q}_2) + C_3(\mathrm{Q}_3) = 0 \\ L_2(\mathrm{I}_2)\frac{d\mathrm{I}_2}{dt} + C_2(\mathrm{Q}_2) = 0 \\ L_1(\mathrm{I}_1)\frac{d\mathrm{I}_1}{dt} - L_2(\mathrm{I}_2)\frac{d\mathrm{I}_2}{dt} + L_3(\mathrm{I}_3)\frac{d\mathrm{I}_3}{dt} = 0 \\ -L_1(\mathrm{I}_1)\frac{d\mathrm{I}_1}{dt} - L_4(\mathrm{I}_4)\frac{d\mathrm{I}_4}{dt} - C_1(\mathrm{Q}_1) = 0 \end{cases} \tag{5.21}$$

where $L_a : \mathbf{R} \longrightarrow \mathbf{R}\backslash\{0\}$, $C_\alpha : \mathbf{R} \longrightarrow \mathbf{R}\backslash\{0\}$ are smooth invertible functions.

As we have pointed out in Section 3, the first set of equations in (5.21) is the same as in the linear case, therefore the configuration space M_c is the same, too. For the coordinate system on M_c given by (5.5), the matrices $\mathcal{N}$, $\mathcal{C}$ have the same expressions (5.6) as before. Thus, in the nonlinear case, the Birkhoffian becomes $\omega_c = \sum_{j=1}^{4} Q_j(q,\dot{q},\ddot{q})dq^j$ with the functions Q_j given by (3.21), (3.22), (3.23), that is,

$$\begin{aligned} Q_1(q,\dot{q},\ddot{q}) &= C_3(q^1) + C_1(q^1+q^4+const) - C_2(-q^1+q^2+q^3+const) \\ Q_2(q,\dot{q},\ddot{q}) &= L_2(\dot{q}^2)\ddot{q}^2 + C_2(-q^1+q^2+q^3+const) \\ Q_3(q,\dot{q},\ddot{q}) &= \left(L_3(\dot{q}^3) + L_1(\dot{q}^3+\dot{q}^4)\right)\ddot{q}^3 + L_1(\dot{q}^3+\dot{q}^4)\ddot{q}^4 \\ &\quad + C_2(-q^1+q^2+q^3+const) \\ Q_4(q,\dot{q},\ddot{q}) &= L_1(\dot{q}^3+\dot{q}^4)\ddot{q}^3 + \left(L_4(\dot{q}^4) + L_1(\dot{q}^3+\dot{q}^4)\right)\ddot{q}^4 + C_1(q^1+q^4+const). \end{aligned} \tag{5.22}$$

The Birkhoffian (5.22) is **conservative** with the function $E_\omega(q,\dot{q})$ given by (3.41), that is,

$$\begin{aligned} E_\omega(q,\dot{q}) = &\int \widetilde{L}_1(\dot{q})(\dot{q}^3+\dot{q}^4)(d\dot{q}^3+d\dot{q}^4) + \int L_2(\dot{q}^2)\dot{q}^2 d\dot{q}^2 + \int L_3(\dot{q}^3)\dot{q}^3 d\dot{q}^3 \\ &+ \int L_4(\dot{q}^4)\dot{q}^4 d\dot{q}^4 - \int\int \widetilde{L}_1'(\dot{q})(\dot{q}^3+\dot{q}^4)d\dot{q}^3 d\dot{q}^4 - \int\int \widetilde{L}_1(\dot{q})d\dot{q}^3 d\dot{q}^4 \\ &+ \int \widetilde{C}_1(q)(dq^1+dq^4) + \int \widetilde{C}_2(q)(-dq^1+dq^2+dq^3) + \int C_3(q^1)dq^1 \\ &- \int\int \widetilde{C}_1'(q)dq^1 dq^4 - \int\int \widetilde{C}_2'(q)(-dq^1dq^2 - dq^1dq^3 + dq^2dq^3) \\ &- \int\int\int \widetilde{C}_2''(q)dq^1dq^2dq^3. \end{aligned} \tag{5.23}$$

The Birkhoffian (5.22) is ***not* regular**, since the first row of the matrix $\left[\frac{\partial Q_j}{\partial \ddot{q}^i}\right]_{i,j=1,2,3,4}$ contains only zeros. But just as in the linear case, we can reduce the configuration space from dimension 4 to dimension 3. Different from the linear case, the reduced configuration space will ***not*** be a linear subspace of M_c.

If the functions C_1, C_2, C_3 are such that the Jacobian matrix for the first equation in (5.22) has rank one, we define the 3-dimensional manifold $\bar{M}_c \subset M_c$ by

$$\begin{aligned}\bar{M}_c = \{q = (q^1, q^2, q^3, q^4) \in M_c / \, C_3(q^1) + C_1(q^1 + q^4 + const) \\ -C_2(-q^1 + q^2 + q^3 + const) = 0\}.\end{aligned} \tag{5.24}$$

By the implicit function theorem, we obtain a local coordinate system on the reduced configuration space $\bar{M}_c$. Taking $\bar{q}^1 := q^2$, $\bar{q}^2 := q^3$, $\bar{q}^3 := q^4$, the Birkhoffian has the form $\bar{\omega}_c = \sum_{j=1}^{3} \bar{Q}_j d\bar{q}^j$, with

$$\begin{aligned}\bar{Q}_1(\bar{q}, \dot{\bar{q}}, \ddot{\bar{q}}) &= L_2(\dot{\bar{q}}^1)\ddot{\bar{q}}^1 + C_2(-f(\bar{q}^1, \bar{q}^2, \bar{q}^3) + \bar{q}^1 + \bar{q}^2 + const) \\ \bar{Q}_2(\bar{q}, \dot{\bar{q}}, \ddot{\bar{q}}) &= \left(L_3(\dot{\bar{q}}^2) + L_1(\dot{\bar{q}}^2 + \dot{\bar{q}}^3)\right)\ddot{\bar{q}}^2 + L_1(\dot{\bar{q}}^2 + \dot{\bar{q}}^3)\ddot{\bar{q}}^3 \\ &\quad + C_2(-f(\bar{q}^1, \bar{q}^2, \bar{q}^3) + \bar{q}^1 + \bar{q}^2 + const) \\ \bar{Q}_3(\bar{q}, \dot{\bar{q}}, \ddot{\bar{q}}) &= L_1(\dot{\bar{q}}^2 + \dot{\bar{q}}^3)\ddot{\bar{q}}^2 + \left(L_4(\dot{\bar{q}}^3) + L_1(\dot{\bar{q}}^2 + \dot{\bar{q}}^3)\right)\ddot{\bar{q}}^3 \\ &\quad + C_1(f(\bar{q}^1, \bar{q}^2, \bar{q}^3) + \bar{q}^3 + const)\end{aligned} \tag{5.25}$$

$f : U \subset \mathbf{R}^3 \longrightarrow \mathbf{R}^1$ being an unique function such that $f(\bar{q}_0) = q_0^1$, $q_0^1 \in \mathbf{R}$, and $C_3(f(\bar{q})) + C_1(f(\bar{q}) + \bar{q}^3 + const) - C_2(-f(\bar{q}) + \bar{q}^1 + \bar{q}^2 + const) = 0$, $\forall \bar{q} = (\bar{q}^1, \bar{q}^2, \bar{q}^3) \in U$, with U a neighborhood of $\bar{q}_0 = (\bar{q}_0^1, \bar{q}_0^2, \bar{q}_0^3)$. On account of $L_1, L_2, L_3, L_4 : \mathbf{R} \longrightarrow \mathbf{R}\backslash\{0\}$, we have

$$\begin{vmatrix} L_2(\dot{\bar{q}}^1) & 0 & 0 \\ 0 & L_3(\dot{\bar{q}}^2) + L_1(\dot{\bar{q}}^2 + \dot{\bar{q}}^3) & L_1(\dot{\bar{q}}^2 + \dot{\bar{q}}^3) \\ 0 & L_1(\dot{\bar{q}}^2 + \dot{\bar{q}}^3) & L_4(\dot{\bar{q}}^3) + L_1(\dot{\bar{q}}^2 + \dot{\bar{q}}^3) \end{vmatrix} \neq 0 \tag{5.26}$$

then, the Birkhoffian (5.25) is **regular**.

Because the network has one loop which contains only inductors, let us perform a further reduction of the dimension of the configuration space by one, just as we have done in the linear case. In the coordinate system $\check{q}$ defined in (5.16), the Birkoffian $\bar{\omega}_c = \sum_{j=1}^{3} \check{Q}_j d\check{q}^j$, where

$$\begin{aligned}\check{Q}_1(\check{q}, \dot{\check{q}}, \ddot{\check{q}}) &= L_2(\dot{\check{q}}^1 - \dot{\check{q}}^2)\ddot{\check{q}}^1 - L_2(\dot{\check{q}}^1 - \dot{\check{q}}^2)\ddot{\check{q}}^2 + C_2(\check{q}^1, \check{q}^2, \check{q}^3) \\ \check{Q}_2(\check{q}, \dot{\check{q}}, \ddot{\check{q}}) &= -L_2(\dot{\check{q}}^1 - \dot{\check{q}}^2)\ddot{\check{q}}^1 + \left[L_1(\dot{\check{q}}^2 + \dot{\check{q}}^3) + L_2(\dot{\check{q}}^1 - \dot{\check{q}}^2) + L_3(\dot{\check{q}}^2)\right]\ddot{\check{q}}^2 \\ &\quad + L_1(\dot{\check{q}}^2 + \dot{\check{q}}^3)\ddot{\check{q}}^3 \\ \check{Q}_3(\check{q}, \dot{\check{q}}, \ddot{\check{q}}) &= L_1(\dot{\check{q}}^2 + \dot{\check{q}}^3)\ddot{\check{q}}^2 + \left[L_1(\dot{\check{q}}^2 + \dot{\check{q}}^3) + L_4(\dot{\check{q}}^3)\right]\ddot{\check{q}}^3 + C_1(\check{q}^1, \check{q}^2, \check{q}^3)\end{aligned} \tag{5.27}$$

Using the second equation in (5.27), we can define a smooth constant rank affine sub-bundle $\mathfrak{S}_c$ of the affine bundle $\pi_J : J^2(\bar{M}_c) \longrightarrow T\bar{M}_c$ via

$$\begin{aligned}\mathfrak{S}_c = \{(\check{q}, \dot{\check{q}}, \ddot{\check{q}}) \in J^2(\bar{M}_c) / -L_2(\dot{\check{q}}^1 - \dot{\check{q}}^2)\ddot{\check{q}}^1 + \left[L_1(\dot{\check{q}}^2 + \dot{\check{q}}^3) + L_2(\dot{\check{q}}^1 - \dot{\check{q}}^2)\right. \\ \left. + L_3(\dot{\check{q}}^2)\right]\ddot{\check{q}}^2 + L_1(\dot{\check{q}}^2 + \dot{\check{q}}^3)\ddot{\check{q}}^3 = 0\}.\end{aligned} \tag{5.28}$$

The constraint $\mathfrak{S}_c$ is integrable, in the sense that we have the foliation

$$\mathfrak{F}_{const} := \{(\check{q}, \dot{\check{q}}) \in T(\bar{M}_c) / -\mathcal{L}_2(\dot{\check{q}}^1 - \dot{\check{q}}^2) + \mathcal{L}_1(\dot{\check{q}}^2 + \dot{\check{q}}^3) + \mathcal{L}_3(\dot{\check{q}}^2) = const\}. \tag{5.29}$$

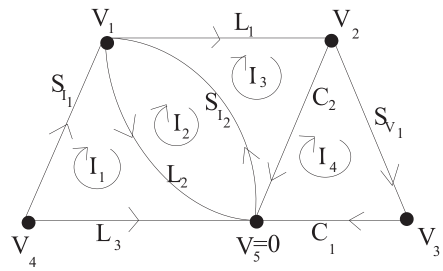

Figure 2

Thus, in the nonlinear case, we draw the conclusion that we can further reduce the configuration space only if it is possible to find from (5.29) new configuration coordinates $\hat{q}^1$, $\hat{q}^2$, that is, when the constraint (5.29) is holonomic.

In order to underline that, depending on the topology of the networks with independent sources, the associated Birkhoffian is conservative or not, we consider the circuit shown in Figure 2 above. This circuit contains a loop formed by capacitors and independent voltage sources and a cutset formed by inductors and independent current sources. We shall see that the Birkhoffian associated to such a circuit is ***not* regular** and ***not* even conservative** for nonlinear inductors and capacitors.

We have $k = 3$, $p = 2$, $\mathrm{s}_I = 2$, $\mathrm{s}_V = 1$, $n = 4$, $m = 4$, $b = 8$. We choose the reference node to be V_5 and the current directions as indicated in Figure 2. We cover the associated graph with the loops I_1, I_2, I_3, I_4. Let $V = (V_1, V_2, V_3, V_4) \in \mathbf{R}^4$ be the vector of node voltage values, $\mathrm{I} = (\mathrm{I}_a, \mathrm{I}_\alpha, \mathrm{I}_{\mathrm{S}_I}, \mathrm{I}_{\mathrm{S}_V}) \in \mathbf{R}^3 \times \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^1$ be the vector of branch current values and $v = (v_a, v_\alpha, v_{\mathrm{S}_I}, \mathrm{S}_V) \in \mathbf{R}^3 \times \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^1$ be the vector of branch voltage values.

The branches in Figure 2 are labelled as follows: the first, the second, and the third branch are the inductor branches L_1, L_2, L_3, the forth and the fifth branch are the capacitor branches C_1, C_2, the next two branches are the current source branches $\mathrm{S}_{\mathrm{I}_1}$, $\mathrm{S}_{\mathrm{I}_2}$, and the last branch is the voltage source branch $\mathrm{S}_{\mathrm{V}_1}$. The incidence and loop matrices, $B \in \mathfrak{M}_{84}(\mathbf{R})$ and $A \in \mathfrak{M}_{84}(\mathbf{R})$, write as

$$B = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.30}$$

For linear inductors and capacitors, the governing equations have the form:

$$\begin{cases} -\mathrm{I}_1 - \mathrm{I}_2 + \mathrm{I}_{s_{I_1}}(t) + \mathrm{I}_{s_{I_2}}(t) = 0 \\ \mathrm{I}_1 - \frac{d\mathrm{Q}_2}{dt} - \mathrm{I}_{s_{V_1}} = 0 \\ -\frac{d\mathrm{Q}_1}{dt} + \mathrm{I}_{s_{V_1}} = 0 \\ -\mathrm{I}_3 - \mathrm{I}_{s_{I_1}}(t) = 0 \\ \\ \mathrm{L}_2\frac{d\mathrm{I}_2}{dt} - \mathrm{L}_3\frac{d\mathrm{I}_3}{dt} + v_{s_{I_1}} = 0 \\ -\mathrm{L}_2\frac{d\mathrm{I}_2}{dt} - v_{s_{I_2}} = 0 \\ \mathrm{L}_1\frac{d\mathrm{I}_1}{dt} + \frac{\mathrm{Q}_2}{\mathrm{C}_2} + v_{s_{I_2}} = 0 \\ \frac{\mathrm{Q}_1}{\mathrm{C}_1} - \frac{\mathrm{Q}_2}{\mathrm{C}_2} + v_{s_{V_1}}(t) = 0 \end{cases} \tag{5.31}$$

where $\mathrm{C}_\alpha \neq 0$, $\alpha = 1, 2$ and $\mathrm{L}_a \neq 0$, $a = 1, 2, 3$, are distinct constants.

Note that $\mathrm{I}_{s_{I_1}}$, $\mathrm{I}_{s_{I_2}}$, $v_{s_{V_1}}$ are given functions of time which describe the currents associated to the independent current sources S_{I_1}, S_{I_2} and the voltage associated to the independent voltage source S_{V_1}, respectively.

Once we know the unknowns I_1, I_2, I_3, Q_1, Q_2, we can determine all the other circuit variables.

From the first set of equations (5.31), we have

$$\mathrm{I}_{s_{V_1}} = \frac{d\mathrm{Q}_1}{dt} \tag{5.32}$$

and from the second set of equations (5.31), we conclude

$$\begin{aligned} v_{s_{I_1}} &= -\mathrm{L}_2\frac{d\mathrm{I}_2}{dt} + \mathrm{L}_3\frac{d\mathrm{I}_3}{dt} \\ v_{s_{I_2}} &= -\mathrm{L}_2\frac{d\mathrm{I}_2}{dt}. \end{aligned} \tag{5.33}$$

Therefore, the system (4.1) has now the form

$$\begin{cases} -\mathrm{I}_1 - \mathrm{I}_2 + \mathrm{I}_{s_{I_1}}(t) + \mathrm{I}_{s_{I_2}}(t) = 0 \\ \mathrm{I}_1 - \frac{d\mathrm{Q}_2}{dt} - \frac{d\mathrm{Q}_1}{dt} = 0 \\ -\mathrm{I}_3 - \mathrm{I}_{s_{I_1}}(t) = 0 \\ \\ \mathrm{L}_1\frac{d\mathrm{I}_1}{dt} + \frac{\mathrm{Q}_2}{\mathrm{C}_2} - \mathrm{L}_2\frac{d\mathrm{I}_2}{dt} = 0 \\ \frac{\mathrm{Q}_1}{\mathrm{C}_1} - \frac{\mathrm{Q}_2}{\mathrm{C}_2} + v_{s_{V_1}}(t) = 0 \end{cases} \tag{5.34}$$

with

$$\mathcal{B}_1^T = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix} \quad \mathcal{B}_2^T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ -1 & 0 \end{pmatrix} \tag{5.35}$$

$$\mathcal{A}_1^T = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad \mathcal{A}_2^T = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5.36}$$

The relations (3.11), (4.5), read as follows for this example

$$\mathrm{I}_a := \frac{d\mathrm{Q}_{(a)}}{dt},\ a = 1,2,3 \tag{5.37}$$

$$x^1 := \mathrm{Q}_{(1)},\ x^2 := \mathrm{Q}_{(2)},\ x^3 := \mathrm{Q}_{(3)},\ ,x^4 := \mathrm{Q}_1,\ x^5 := \mathrm{Q}_2. \tag{5.38}$$

Using the first set of equations (5.34), we define the 2-dimensional affine-linear configuration space M_c. We solve the corresponding equations (4.4) in terms of 2 variables. In view of the notations (5.37), (5.38), we obtain, for example,

$$\begin{aligned} x^2 &= -x^1 + f^2(t) + const \\ x^3 &= f^3(t) + const \\ x^5 &= x^1 - x^4 + const \end{aligned} \tag{5.39}$$

with $f^2(t) = \int (\mathrm{I}_{s_1}(t) + \mathrm{I}_{s_2}(t))\,\mathrm{d}t$, $f^3(t) = -\int \mathrm{I}_{s_1}(t)\mathrm{d}t$ and the other components of f in (4.6) being zero. Thus a coordinate system on M_c is given by

$$q^1 := x^1, q^2 := x^4. \tag{5.40}$$

and the matrices of constants $\mathfrak{N} = \begin{pmatrix} \mathfrak{N}^a_j \\ \mathfrak{N}^\alpha_j \end{pmatrix}_{\substack{a=1,2,\ \alpha=3,4,5 \\ j=1,2}}$ and $\mathcal{C}$ are

$$\mathfrak{N} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.41}$$

In terms of the coordinates (5.40), we define the Birkhoffian $\omega_{t_c} = Q_1 dq^1 + Q_2 dq^2$, as in (4.8)-(4.11), that is,

$$\begin{aligned} Q_1(t,q,\dot{q},\ddot{q}) &= (\mathrm{L}_1 + \mathrm{L}_2)\ddot{q}^1 - \mathrm{L}_2 \frac{d^2 f^2(t)}{dt^2} + \frac{q^1}{\mathrm{C}_2} - \frac{q^2}{\mathrm{C}_2} + const \\ Q_2(t,q,\dot{q},\ddot{q}) &= -\frac{q^1}{\mathrm{C}_2} + \left(\frac{1}{\mathrm{C}_1} + \frac{1}{\mathrm{C}_2}\right) q^2 + v_{s_{\mathrm{V}_1}}(t) + const. \end{aligned} \tag{5.42}$$

Because there exists a loop which contains only capacitors and independent voltage sources, the Birkhoffian (5.42) is ***not* regular**. Indeed, the second row of the matrix $\left[\frac{\partial Q_j}{\partial \ddot{q}^i}\right]_{i,j=1,2}$ contains only zeros, therefore, det $\left[\frac{\partial Q_j}{\partial \ddot{q}^i}\right]_{i,j=1,2} = 0$.

Though there exists in the network a cutset formed by inductors and independent current sources, the Birkhoffian (5.42) is **conservative** in the sense of definition (2.19). The function $E_{\omega_t}(t,q,\dot{q})$ is given by (4.13), that is,

$$\begin{aligned} E_{\omega_t}(t,q,\dot{q}) \;=\; & \frac{1}{2}\mathrm{L}_1(\dot{q}^1)^2 + \frac{1}{2}\mathrm{L}_2(\dot{q}^1)^2 + \frac{1}{2\mathrm{C}_1}(q^2)^2 + \frac{1}{2\mathrm{C}_2}(q^1 - q^2)^2 \\ & + \left(-\mathrm{L}_2 \frac{d^2 f^2(t)}{dt^2} + const_1\right) q^1 + \left(v_{s_{\mathrm{V}_1}}(t) + const_2\right) q^2. \end{aligned} \tag{5.43}$$

In order to obtain a regular Birkhoffian we could use the second equation from (5.42) and reduce the configuration space M_c to a vector space $\bar{M}_c$ of dimension 1. The procedure is the same as in the first example with linear devices.

For nonlinear inductors and capacitors, in the coordinate system (5.40) on the configuration space M_c of dimension 2, the Birkhoffian $\omega_{t_c} = Q_1 dq^1 + Q_2 dq^2$, where

$$\begin{aligned} Q_1(t,q,\dot{q},\ddot{q}) \; &= \left[L_1(\dot{q}^1) + L_2\left(-\dot{q}^1 + \frac{df^2(t)}{dt}\right)\right]\ddot{q}^1 - L_2\left(-\dot{q}^1 + \frac{df^2(t)}{dt}\right)\frac{d^2f^2(t)}{dt^2} \\ &\quad + C_2(q^1 - q^2 + const) \\ Q_2(t,q,\dot{q},\ddot{q}) \; &= C_1(q^2) - C_2(q^1 - q^2 + const) + v_{s_{V_1}}(t). \end{aligned} \tag{5.44}$$

The Birkhoffian (5.44) is ***not* regular** and ***not* conservative**. Indeed, two of the necessary conditions for the existence of the function $E_{\omega_t} : TM \to \mathbf{R}$ such that $\sum_{j=1}^{2} Q_j(t,q,\dot{q},\ddot{q})dq^j = \sum_{j=1}^{2} \frac{\partial E_{\omega_t}}{\partial q^j}\dot{q}^j + \frac{\partial E_{\omega_t}}{\partial \dot{q}^j}\ddot{q}^j$, are

$$\begin{cases} \frac{\partial E_{\omega_t}}{\partial \dot{q}^1} = L_1(\dot{q}^1) + L_2\left(-\dot{q}^1 + \frac{df^2(t)}{dt^2}\right) \\ \\ \frac{\partial E_{\omega_t}}{\partial q^1} = -L_2\left(-\dot{q}^1 + \frac{df^2(t)}{dt^2}\right)\frac{d^2f^2(t)}{dt^2} + C_2(q^1 - q^2 + const) \end{cases} . \tag{5.45}$$

We easily see that for almost all values of the parameters, $\frac{\partial^2 E_{\omega_t}}{\partial \dot{q}^1 q^1} \neq \frac{\partial^2 E_{\omega_t}}{\partial q^1 \dot{q}^1} = 0$.

Let us now consider a network that has the oriented connected graph as in Figure 2 in which we interchanged the inductor branch L_3 with the capacitor branch C_1 and the inductor branch L_2 with the capacitor branch C_2. We will see that the Birkhoffian associated to this network is ***conservative*** even if the inductors and the capacitors in the network are nonlinear devices.

The system (4.1) has now the form

$$\begin{cases} -\mathrm{I}_1 - \frac{d\mathrm{Q}_2}{dt} + \mathrm{I}_{s_{I_1}}(t) + \mathrm{I}_{s_{I_2}}(t) = 0 \\ \mathrm{I}_1 - \mathrm{I}_2 - \mathrm{I}_3 = 0 \\ -\frac{d\mathrm{Q}_1}{dt} - \mathrm{I}_{s_{I_1}}(t) = 0 \\ \\ L_1(\mathrm{I}_1)\frac{d\mathrm{I}_1}{dt} + L_2(\mathrm{I}_2)\frac{d\mathrm{I}_2}{dt} - C_2(\mathrm{Q}_2) = 0 \\ -L_2(\mathrm{I}_2)\frac{d\mathrm{I}_2}{dt} + L_3(\mathrm{I}_3)\frac{d\mathrm{I}_3}{dt} + v_{s_{V_1}}(t) = 0 \end{cases} \tag{5.46}$$

and

$$\mathrm{I}_{s_{V_1}} = \mathrm{I}_3 \tag{5.47}$$

$$\begin{aligned} v_{s_{I_1}} &= C_1(\mathrm{Q}_1) - C_2(\mathrm{Q}_2) \\ v_{s_{I_2}} &= -C_2(\mathrm{Q}_2). \end{aligned} \tag{5.48}$$

Using the same procedure as above we get the configuration space $\mathcal{M}_c$ of dimension 2. In view of the notations (5.37), (5.38), a coordinate system on $\mathcal{M}_c$ is given by

$$q^1 := x^1, q^2 := x^3. \tag{5.49}$$

and the Birkhoffian $\omega_{t_c} = Q_1 dq^1 + Q_2 dq^2$, where

$$\begin{aligned} Q_1(t,q,\dot{q},\ddot{q}) &= \left[L_1(\dot{q}^1) + L_2\left(\dot{q}^1 - \dot{q}^2\right)\right]\ddot{q}^1 - L_2\left(\dot{q}^1 - \dot{q}^2\right)\ddot{q}^2 \\ &\quad -C_2(-q^1 + f^5(t) + const) \\ Q_2(t,q,\dot{q},\ddot{q}) &= -L_2\left(\dot{q}^1 - \dot{q}^2\right)\ddot{q}^1 + \left[L_2\left(\dot{q}^1 - \dot{q}^2\right) + L_3(\dot{q}^2)\ddot{q}^2\right]\ddot{q}^2 + v_{s_{V_1}}(t) \end{aligned} \tag{5.50}$$

with $f^5(t) = \int (\mathrm{I}_{s_1}(t) + \mathrm{I}_{s_2}(t))\,\mathrm{d}t$ and the other components of f in (4.6) being zero. The Birkhoffian (5.50) is conservative in the sense of definition (2.19), with the function $E_{\omega_t}(t,q,\dot{q})$ given by

$$\begin{aligned} E_{\omega_t}(t,q,\dot{q}) &= \int L_1(\dot{q}^1)\dot{q}^1 d\dot{q}^1 + \int \widetilde{L}_2(\dot{q})(\dot{q}^1 - \dot{q}^2)(d\dot{q}^1 - d\dot{q}^2) + \int L_3(\dot{q}^2)\dot{q}^2 d\dot{q}^2 \\ &\quad + \int\int \widetilde{L}_2'(\dot{q})(\dot{q}^1 - \dot{q}^2) d\dot{q}^1 d\dot{q}^2 + \int\int \widetilde{L}_2(\dot{q}) d\dot{q}^1 d\dot{q}^2 \\ &\quad - \int C_2(-q^1 + f^5(t) + const) dq^1 + v_{s_{V_1}}(t) q^2 \end{aligned} \tag{5.51}$$

where $\widetilde{L}_2' := \frac{d\widetilde{L}_2(\eta)}{d\eta^l}$. □

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References

[1] R. Abraham and J. Marsden, *Foundations of Mechanics*. Benjamin/Cummings Publishing Company, Inc. 1978.

[2] G. M. Bernstein and M. A. Lieberman, A method for obtaining a canonical Hamiltonian for nonlinear LC circuits. *IEEE Trans. Circuits and Systems* **36** (3) (1989), 411–420.

[3] G. D. Birkhoff, *Dynamical Systems*. American Mathematical Society Colloquium Publications, vol. IX, New York 1927.

[4] A. M. Bloch and P. E. Crouch, Representations of Dirac structures on vector spaces and nonlinear LC circuits. *Differential Geometry and Control*, volume **64** of *Proceedings of Symposia in Pure Mathematics*, pages 103–117. American Mathematical Society, 1999.

[5] R.K. Brayton and J.K. Moser, A theory of nonlinear networks I, II. *Quarterly of Applied Mathematics* **22** (1964), 1–33, 81–104.

[6] L. O. Chua and J. D. McPherson, Explicit topological formulation of Lagrangian and Hamiltonian equations for nonlinear networks. *IEEE Trans. Circuits and Systems* **21** (2) (1974), 277–286.

[7] L. R. Foulds, *Graph Theory Applications*. Springer-Verlag New York, Inc. 1992.

[8] M. H. Kobayashi and W. M. Oliva, On the Birkhoff approach to classical mechanics. *Resenhas IME-USP* **6** (1) (2003), 1–71.

[9] B. M. Maschke, A. J. van der Schaft and P. C. Breedveld, An intrinsic Hamiltonian formulation of the dynamics of LC circuits. *IEEE Trans. Circuits and Systems*, **42** (2) (1995), 73–82.

[10] B. M. Maschke and A. J. van der Schaft, The Hamiltonian formulation of energy conserving physical systems with external ports. *Archiv für Elektronik und Übertragungstechnik* **49** (1995), 362–371.

[11] D. J. Saunders, *The Geometry of Jet Bundles*. London Mathematical Society Lecture Note Series, **142**, Cambridge University Press, 1989.

[12] A. J. van der Schaft, Implicit Hamiltonian systems with symmetry. *Rep. Math. Phys.* **41** (1998), 203–221.

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