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CAN THE NOTION OF A HOMOGENEOUS GRAVITATIONAL FIELD BE TRANSFERRED FROM CLASSICAL MECHANICS TO THE RELATIVISTIC THEORY OF GRAVITY?

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Bogorodskii generalized the classical mechanical concept of a homogeneous gravitational field to the case of Einstein's general relativity. We seek such a generalization to the case of the relativistic theory of gravity. The corresponding solutions in these two theories differ substantially. The solution obtained in accordance with the relativistic theory of gravity does not satisfy the causality principle in that theory. The problem of constructing a generalization of the classical notion of a homogeneous gravitational field in the framework of the relativistic theory of gravity therefore remains open.

1. Introduction

The homogeneous gravitational field in Newton's classical mechanics is a field with the same potential gradient at all points. Such a field can be generated by an infinite material plane with constant surface mass density (see Sec. 3).

Can the classical notion of a homogeneous gravitational field be preserved in the relativistic theory of gravity (RTG)? The starting point for discussing this problem is the monograph by Bogorodskii (see Sec. 17 in [1]), where the author addressed the problem of finding the gravitational field generated by a system of mass homogeneously distributed on a plane in the framework of Einstein's general relativity (GR). In Sec. 4, we describe what Bogorodskii understood by a homogeneous gravitational field in GR. We demonstrate that its solution contains an irremovable singularity, which has no physical explanation.

The problem of a homogeneous gravitational field in the RTG was briefly addressed in [2]. In Sec. 5, we analyze this problem in detail. The solution of the complete system of RTG equations for the problem under investigation differs from that obtained by Bogorodskii. Although our solution is regular in the entire domain of definition, it is unacceptable as a physically meaningful gravitational field because it does not satisfy the causality principle in the RTG. The problem of finding such a field in the RTG therefore remains *open*.

In Sec. 6, we show that if the constructed solution is used, then the speed of some free test particles in this field can exceed the speed of light in a vacuum.

2. Equations of the RTG and the causality principle in the RTG

The RTG was constructed by Logunov and collaborators (see [3], [4]) as a field theory of gravitation based on special relativity (SR). The Minkowski space–time is the fundamental space, which contains all physical fields including the gravitational field. The element of length in this space is

$$d\sigma^2 = \gamma_{mn}(x)\, dx^m\, dx^n, \tag{2.1}$$

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where x^m, $m = 1, 2, 3, 4$, are admissible coordinates in the Minkowski space–time and $\gamma_{mn}(x)$ are the components of the Minkowski metric in the chosen coordinate system.

The gravitational field is described by the symmetric second-rank tensor $\phi^{mn}(x)$, which determines the effective Riemannian space–time. A basic RTG assumption is that the behavior of matter in the Minkowski space–time with the metric $\gamma_{mn}(x)$ under the action of the gravitational field $\phi^{mn}(x)$ is equivalent to its behavior in the effective Riemannian space–time with the metric $g_{mn}(x)$ determined by the relation

$$\tilde{g}^{mn} = \sqrt{-g}g^{mn} = \sqrt{-\gamma}\gamma^{mn} + \sqrt{-\gamma}\phi^{mn}, \qquad g = \det(g_{mn}), \qquad \gamma = \det(\gamma_{mn}). \tag{2.2}$$

Such an interaction between the gravitational field and matter was called the *geometrization principle of the RTG.*

The RTG dynamics of the gravitational field is governed by the differential equations

$$R^m_n - \frac{1}{2}\delta^m_n R + \frac{m_{\rm g}^2}{2}\left(\delta^m_n + g^{mk}\gamma_{kn} - \frac{1}{2}\delta^m_n g^{kl}\gamma_{kl}\right) = 8\pi T^m_n, \tag{2.3}$$

$$D_m\tilde{g}^{mn} = 0, \quad m, n, k, l = 1, 2, 3, 4, \tag{2.4}$$

where R^m_n is the Ricci tensor corresponding to the metric g_{mn}, $R = R^m_m$ is the scalar curvature, δ^m_n is the Kronecker symbol, $m_{\rm g}$ is the graviton mass, and T^m_n is the energy–momentum tensor of the gravitational field sources. In Eq. (2.4), D_m is the operator of covariant differentiation w.r.t. the metric γ_{mn}. Equations (2.3) and (2.4) are covariant under arbitrary coordinate transformations with a nonzero Jacobian. All the field variables in the RTG depend on universal space–time coordinates of the Minkowski space. The presence of mass terms in Eq. (2.3) allows unambiguously determining the space–time geometry and the density of the gravitational field energy–momentum tensor in the absence of matter. It follows from Eq. (2.4) that the gravitational field only has states with the spins 0 and 2. In [4], this equation, which determines the field polarization, was obtained based on the gravitational field source being a universal conserved density of the energy–momentum tensor of the total matter including the gravitational field. The graviton mass essentially affects the Universe's evolution and changes the character of the gravitational collapse.

In the present paper, because the graviton mass is negligibly small ($m_{\rm g} \simeq 10^{-66}$ g), we analyze the problem of finding the homogeneous gravitational field in the RTG omitting the mass terms in Eq. (2.3). Hereafter, we use the relativistic system of units. Equation (2.4) can be written in the form (see Appendix 1 in [3])

$$D_m\tilde{g}^{mn} = \tilde{g}^{mn}{}_{,m} + \gamma^n_{mp}\tilde{g}^{mp} = 0, \tag{2.5}$$

where γ^n_{mp} are the components of the metric connection generated by the metric γ_{mn} and the comma in (2.5) denotes differentiation w.r.t. the corresponding coordinate.

The causality principle in the RTG was proposed and analyzed by Logunov in Chap. 6 in [4]. According to this principle, each motion of a pointlike test body must occur inside the causality light cone in the Minkowski space–time. Following Logunov, the causality principle holds if and only if for any isotropic Minkowski vector u^m, i.e., for any vector satisfying the condition

$$\gamma_{mn}u^m u^n = 0, \tag{2.6}$$

the metric of the effective Riemannian space–time satisfies the restriction

$$g_{mn}u^m u^n \leq 0. \tag{2.7}$$

Following the causality principle in the RTG, only those solutions of system (2.3), (2.4) that satisfy this restriction are physically meaningful.

We stress that the above causality principle can be formulated only in the RTG because it is only in RTG that the space–time is the Minkowski space–time and the gravitational field is described by the field of the symmetric second-rank tensor $\phi_{mn}(x)$, where x^m are the admissible coordinates in the Minkowski space–time, x^1, x^2, and x^3 are the spacelike variables, and x^4 is the timelike variable.

3. A homogeneous gravitational field in classical mechanics

A gravitational field is said to be homogeneous in classical mechanics if its intensity is constant or piecewise constant. Such a field can be generated by a system of masses homogeneously distributed over a plane. The relation between the surface mass density σ and the acceleration $\mathcal{G}$ caused by the gravitational field is

$$\mathcal{G} = 2\pi\sigma k > 0, \tag{3.1}$$

where k is the Newton gravitational constant.

Choosing the x and y axes of the rectangular system of coordinates in the plane of the mass distribution and the z axis orthogonal to this plane, we find that the motion of a free test particle in this gravitational field is governed by the equations

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = 0, \tag{3.2}$$

$$\frac{d^2z}{dt^2} + \mathcal{G} = 0 \quad \text{for } z > 0, \qquad \frac{d^2z}{dt^2} - \mathcal{G} = 0 \quad \text{for } z < 0, \tag{3.3}$$

where t is the Newtonian time. In a noninertial reference frame that propagates with the constant proper acceleration $\mathcal{G}$, the equations of motion of a free test particle are

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = 0, \qquad \frac{d^2z}{dt^2} + \mathcal{G} = 0 \quad \text{for any } z, \tag{3.4}$$

where x, y, and z are particle coordinates in the noninertial reference frame.

Although system of equations (3.2), (3.3) is formally similar to system of equations (3.4), these two systems are not equivalent. To see this, it suffices to compare Eqs. (3.3) and (3.4). Whereas laws (3.4) can be transformed into the form

$$\frac{d^2X}{dt^2} = 0, \qquad \frac{d^2Y}{dt^2} = 0, \qquad \frac{d^2Z}{dt^2} = 0 \quad \text{for any } Z \tag{3.5}$$

by a coordinate transformation, this is impossible for laws (3.2), (3.3). The two cases are nonequivalent because we deal with a real gravitational field in the first case, but we have only an inertial force in the second case. This example demonstrates that the difference between a real gravitational force caused by a mass distribution and an inertia force acting in a noninertial reference frame exists even in classical mechanics.

4. A homogeneous gravitational force in GR

In his monograph, Bogorodskii (see Chap. 17 in [1]) tried to answer the following questions: Does a homogeneous gravitational field exist in GR? In GR, which Riemannian metric corresponds to a gravitational field generated by an infinite material plane with a constant surface mass density? We present his answers to these questions below.

Using classic results, Bogorodskii sought a solution of the Einstein equations in the form

$$ds^2 = -A\,dx^2 - A\,dy^2 - C\,dz^2 + D\,dt^2, \tag{4.1}$$

where A, C, and D are positive-definite functions depending only on z. The energy–momentum tensor of sources homogeneously distributed over the plane $z = 0$ is

$$T^{mn} \equiv 0 \quad \text{for all } z \neq 0. \tag{4.2}$$

Bogorodskii concluded that the Einstein field equations for metric (4.1) with energy–momentum tensor (4.2) are satisfied for arbitrary functions A, C, and D that satisfy the equations

$$\begin{aligned} &2\left(\frac{A'}{A}\right)' - \frac{A'C'}{AC} + \frac{A'}{A}\left(\frac{2A'}{A} + \frac{D'}{D}\right) = 0, \\ &2\left(\frac{D'}{D}\right)' - \frac{C'D'}{CD} + \frac{D'}{D}\left(\frac{2A'}{A} + \frac{D'}{D}\right) = 0, \\ &\frac{A'}{A}\left(\frac{A'}{A} + \frac{2D'}{D}\right) = 0, \end{aligned} \tag{4.3}$$

where the prime denotes differentiation w.r.t. the z coordinate. In accordance with the last equation, we have two possibilities:

$$A' = 0 \qquad \text{or} \qquad \left(\frac{A'}{A} + \frac{2D'}{D}\right) = 0. \tag{4.4}$$

In the first case, we can set $A = 1$ because the functions A, C, and D are determined up to a constant. In this case, the first equation of system (4.3) is obviously satisfied, and the second equation in (4.3) becomes

$$\left(\frac{D'}{D}\right)' - \frac{1}{2}\frac{C'D'}{CD} + \frac{1}{2}\left(\frac{D'}{D}\right)^2 = 0. \tag{4.5}$$

From (4.5), we obtain $C = aD^{-1}D'^{\,2}$, where a is the real constant of integration. In this case, the solution of the Einstein equations has the form

$$A = 1, \qquad C = aD^{-1}D'^{\,2}, \tag{4.6}$$

where D is an arbitrary function of z.

For the second case in (4.4), we can choose $A = D^{-2}$, and the first two equations of system (4.3) then give

$$\left(\frac{D'}{D}\right)' - \frac{1}{2}\frac{C'D'}{CD} - \frac{3}{2}\left(\frac{D'}{D}\right)^2 = 0. \tag{4.7}$$

From (4.7), we obtain $C = bD^{-5}D'^{\,2}$, where b is the constant of integration. The solution of the Einstein equations in the second case is therefore

$$A = D^{-2}, \qquad C = bD^{-5}D'^{\,2}, \tag{4.8}$$

where D is an arbitrary function of z.

To determine $D(z)$, Bogorodskii then found that the motion of the free test particle in the obtained gravitational field is governed by the geodesic equations

$$\frac{d^2x}{dt^2} + \left(\frac{A'}{A} - \frac{D'}{D}\right)\frac{dx}{dt}\frac{dz}{dt} = 0, \qquad \frac{d^2y}{dt^2} + \left(\frac{A'}{A} - \frac{D'}{D}\right)\frac{dy}{dt}\frac{dz}{dt} = 0, \tag{4.9}$$

$$\frac{d^2z}{dt^2} - \frac{A'}{2C}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] + \left(\frac{C'}{2C} - \frac{D'}{D}\right)\left(\frac{dz}{dt}\right)^2 + \frac{D'}{2C} = 0. \tag{4.10}$$

For this system, we find that vertical motion with the initial speed zero is described by the system of equations

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = 0, \tag{4.11}$$

$$\frac{d^2z}{dt^2} + \left(\frac{C'}{2C} - \frac{D'}{D}\right)\left(\frac{dz}{dt}\right)^2 + \frac{D'}{2C} = 0. \tag{4.12}$$

The term containing the speed must be dropped in the case of slow motion; Eq. (4.12) then becomes

$$\frac{d^2z}{dt^2} + \frac{D'}{2C} = 0. \tag{4.13}$$

Comparing the system of equations (4.11) and (4.13) with the classical system, Bogorodskii imposed the restriction

$$\frac{D'}{2C} = \mathcal{G}. \tag{4.14}$$

We discuss this restriction below.

Eventually, setting the constants a and b equal to $1/(4\mathcal{G}^2)$ and using relation (4.14), Bogorodskii obtained the following two solutions from (4.6) and (4.8):

$$A = 1, \qquad C = e^{2\mathcal{G}z}, \qquad D = e^{2\mathcal{G}z} \tag{4.15}$$

and

$$A = (1 - 8\mathcal{G}z)^{1/2}, \qquad C = (1 - 8\mathcal{G}z)^{-5/4}, \qquad D = (1 - 8\mathcal{G}z)^{-1/4}. \tag{4.16}$$

The Riemann–Christoffel curvature tensor vanishes for the first solution (Eqs. (4.15)). Bogorodskii therefore concluded that solution (4.15) does not correspond to a real gravitational field. Using the transformations

$$X = x, \qquad Y = y, \qquad Z = \frac{1}{\mathcal{G}}\big[e^{\mathcal{G}z}\cosh(\mathcal{G}t) - 1\big], \qquad T = \frac{1}{\mathcal{G}}e^{\mathcal{G}z}\sinh(\mathcal{G}t), \tag{4.17}$$

it is easy to see that fundamental invariant (4.1) becomes the Minkowski metric element

$$d\sigma^2 = -dX^2 - dY^2 - dZ^2 + dT^2. \tag{4.18}$$

(A detailed description of properties and singularities of a noninertial system described by relations (4.17) is given in Chap. 15 in [5].)

We thus find that the first solution (Eq. (4.15)) corresponds to a noninertial system of reference whose origin moves with the constant proper acceleration $\mathcal{G}$ in the positive direction of the Z axis of the inertial system. The Riemann–Christoffel curvature tensor corresponding to the second solution (Eq. (4.16)) is nonzero. According to Bogorodskii, this solution describes a real homogeneous gravitational field generated in GR by the mass distribution under consideration.

We first note that solution (4.16) has a singularity at $z = 1/(8\mathcal{G})$ that is difficult to explain. We now turn to Bogorodskii's condition (4.14). We noted in Sec. 3 that the motion of a free test particle in an actual gravitational field is described by the system of equations (3.2) and (3.3) in classical mechanics, not by system (3.4). Bogorodskii's condition (4.14) must therefore be replaced with the relation

$$\frac{D'}{2C} = \begin{cases} \mathcal{G}, & z > 0, \\ -\mathcal{G}, & z < 0. \end{cases} \tag{4.19}$$

Bogorodskii's solution (4.16) must then be replaced with

$$A = (1 \mp 8\mathcal{G}z)^{1/2}, \qquad C = (1 \mp 8\mathcal{G}z)^{-5/4}, \qquad D = (1 \mp 8\mathcal{G}z)^{-1/4}, \tag{4.20}$$

where the plus and minus signs correspond to the respective cases $z < 0$ and $z > 0$. Solution (4.20) can exist only in the range $-1/(8\mathcal{G}) < z < 1/(8\mathcal{G})$, and the singularities at $z = \pm 1/(8\mathcal{G})$ have no physical explanation.

5. A homogeneous gravitational field in the RTG

Solving a problem in the RTG framework relates to solving Eqs. (2.3) and (2.4) w.r.t. the coordinates of the underlying Minkowski space–time. Only solutions that satisfy the causality principle can correspond to physically meaningful gravitational fields. We keep the same starting point as Bogorodskii and seek the solution for a homogeneous gravitational field in the RTG in form (4.1). We can then proceed in two ways; we consider both alternatives below.

We first use already obtained solutions (4.15) and (4.20) satisfying Eqs. (2.3) (without the mass terms). We now check whether these solutions satisfy Eqs. (2.4). The components of the initial Minkowski metric and the components of the effective Riemannian metric coincide for solution (4.15) because this solution appears when passing from an inertial reference frame to an accelerated reference frame in the Minkowski space–time. Equations (2.4) are therefore obviously satisfied, and D_m is the operator of the covariant differentiation w.r.t. the Minkowski metric.

For solution (4.20), we must first find the reference frame in the underlying Minkowski space–time. This reference frame can be obtained under the assumption of a vanishing gravitational field. In the above reference frame, metric (4.1) for $\sigma = 0$ and hence for $\mathcal{G}$=0 then becomes

$$d\sigma^2 = -dx^2 - dy^2 - dz^2 + dt^2. \tag{5.1}$$

In the coordinate system thus chosen, the components of the metric connection γ^n_{mp} vanish, and Eq. (2.5) becomes merely

$$\tilde{g}^{mn}{}_{,m} = 0. \tag{5.2}$$

Taking relations (2.2), (4.1), and (4.20) into account, we find that Eq. (5.2) is not satisfied. Solution (4.20) is therefore not an RTG admissible solution. To find an RTG admissible solution, we use the procedure in Chap. 13 in [3], namely, we seek a coordinate system $\{\eta^i\} = \{X, Y, Z, T\}$ in which Eq. (2.3) (without mass terms) is satisfied and Eq. (2.4) establishes a one-to-one correspondence between the sets of coordinates $\{\eta^i\}$ and $\{\xi^i\} = \{x, y, z, t\}$ in the Minkowski space–time. This coordinate replacement is such that when the gravitational field switched off, we obtain the Minkowski space–time with the metric

$$d\sigma^2 = -dX^2 - dY^2 - dZ^2 + dT^2. \tag{5.3}$$

The components of the tensor γ^n_{mp} must therefore be identically zero in this coordinate system. Because the components of metric (4.1) depend only on z, we can pass from the variables $\{\xi^i\}$ to the variables $\{\eta^i\}$ assuming that

$$X = x, \qquad Y = y, \qquad Z = Z(z), \qquad T = t. \tag{5.4}$$

In this reference frame, we can write Eqs. (2.4) in the form (see also relations (13.17) and (13.22) in [3])

$$\frac{\partial}{\partial \xi^m}\left(\sqrt{-g(\xi)}g^{mn}(\xi)\frac{\partial \eta^p}{\partial \xi^n}\right) = 0. \tag{5.5}$$

By virtue of relations (4.1) and (4.20), Eq. (5.5) for transformations (5.4) becomes

$$\frac{d}{dz}\left((1 \mp 8\mathcal{G}z)\frac{dZ}{dz}\right) = 0. \tag{5.6}$$

Integrating this equation and choosing the integration constant such that the variable Z tends to z as $\mathcal{G}$ tends to zero, we obtain

$$Z = \mp\frac{1}{8\mathcal{G}}\log(1 \mp 8\mathcal{G}z). \tag{5.7}$$

We can find components of metric (4.1) in coordinate system (5.4), (5.7) using the transformation law for tensors. The result is

$$\begin{aligned} &A = e^{-4\mathcal{G}Z}, \qquad C = e^{-6\mathcal{G}Z}, \qquad D = e^{2\mathcal{G}Z} \quad \text{for } Z > 0, \\ &A = e^{4\mathcal{G}Z}, \qquad C = e^{6\mathcal{G}Z}, \qquad D = e^{-2\mathcal{G}Z} \quad \text{for } Z < 0. \end{aligned} \tag{5.8}$$

Solution (5.8) satisfies the total system comprising Eqs. (2.3) (without mass terms) and (2.4). This solution is regular for all $Z \neq 0$ and is nondifferentiable at $Z = 0$; the derivatives of the functions A, C, and D have finite jumps when passing through this plane. This singularity concentrated in the plane $Z = 0$ obviously results from the source of the actual gravitational field being the system of mass distributed over this plane. We also see that condition (4.19) can be satisfied only approximately because, for example, it follows from (5.8) that

$$\frac{D'}{2C} = \mathcal{G}e^{8\mathcal{G}Z} \quad \text{for any } Z > 0. \tag{5.9}$$

At the same time, we note that this approximation is sufficiently correct. Indeed, using the standard system of units, we obtain

$$\frac{D'}{2C} = \frac{\mathcal{G}}{c^2}e^{8\mathcal{G}Z/c^2} \quad \text{for any } Z > 0, \tag{5.10}$$

where c is the speed of light in a vacuum in the inertial system. Therefore, for positive Z, the relation $D'/2C$ can be considered approximately constant if

$$Z \ll \frac{c^2}{\mathcal{G}}. \tag{5.11}$$

The RTG analysis is still incomplete because we must verify that the obtained solutions satisfy the causality principle. First solution (4.15) obviously satisfies the causality principle because the Minkowski metric and the Riemannian metric coincide in this case, as already mentioned. Solution (4.15) is therefore RTG admissible; its physical meaning was already clarified.

Taking formula (5.3) for the initial Minkowski space–time into account, we find that the vector $u = (1,0,0,1)$ is an isotropic vector of the Minkowski space for the second solution (Eq. (5.8)). Condition (2.7) is therefore satisfied if

$$e^{2\mathcal{G}Z} \le e^{-4\mathcal{G}Z} \quad \text{for } Z>0 \qquad \text{and} \qquad e^{-2\mathcal{G}Z} \le e^{4\mathcal{G}Z} \quad \text{for } Z<0. \tag{5.12}$$

These conditions are not satisfied for any $Z > 0$ or $Z < 0$ if the acceleration $\mathcal{G}$ is nonzero, and we conclude that a generalization of a homogeneous gravitational field in Bogorodskii's sense cannot be in agreement with the RTG.

We can find an RTG solution of the problem under consideration using an alternative approach. From Eq. (2.3) (without mass terms) in the case where $T_n^m \equiv 0$ for $z \neq 0$, we obtain solutions (4.6) and (4.8). We stress that the function $D(z)$ is arbitrary in (4.6) and (4.8). This clearly demonstrates that the Einstein field equations are insufficient for finding a unique gravitational field generated by the mass distribution under consideration. We can determine the unknown function $D(z)$ using Eqs. (2.4).

We now consider the Galilean coordinates x, y, z, and t of the inertial system. Equations (2.4) then become merely (5.2). Taking (2.2), (4.1), (4.6), and (4.8) into account, we obtain

$$D(z) = pe^{qz} \tag{5.13}$$

from (5.2), where p and q are real constants. Therefore, substituting (5.13) in (4.6), we obtain the first RTG solution

$$A = 1, \qquad C = ape^{qz}, \qquad D = pe^{qz}. \tag{5.14}$$

From (4.8), we find the second RTG solution

$$A = p^{-2}e^{-2qz}, \qquad C = bp^{-3}e^{-3qz}, \qquad D = pe^{qz}. \tag{5.15}$$

The Riemann–Christoffel curvature tensor is identically zero for solution (5.14) and is nonzero for solution (5.15).

The constants a, b, p, and q must be determined from the correspondence principle: when the gravitational field is switched off, the space curvature must vanish, and we obtain the Minkowski space–time in the chosen reference frame. The equations of motion then become classical in the chosen reference frame. The geometrization principle in the RTG then claims that the equations of motion in the gravitational field under consideration are given by Eqs. (4.9) and (4.10). In the case of slow vertical motion, this system of equations becomes system (4.11), (4.13). It thus follows from the correspondence principle that we must obtain relation (4.14) for solution (5.14) and relation (4.19) for solution (5.15).

Following the same principle, the metric must tend to the Galilean metric as $\mathcal{G} \to 0$. We thus obtain solution (4.15) in the first case and solution (5.8) in the second case. As mentioned above, relation (4.19) is satisfied only approximately for the second solution.

We have thus obtained the same result using two different approaches: solution (4.15) represents inertial forces, whereas solution (5.8) is inadmissible as an actual gravitational field generated by the mass distribution under consideration because it fails to satisfy the causality principle (see (5.12)).

6. The importance of the causality principle in the RTG

The example in this section clearly demonstrates that the causality principle is crucial when we conclude that solution (5.8) is RTG inadmissible. We show that there are free test particles in the constructed space–time that move faster than light in a vacuum.

We choose a free test particle, which is situated at the distance $h > 0$ from the mass distribution plane $z = 0$ at the initial moment $t = 0$. For simplicity, we consider the problem in the plane xOz. At the initial instant $t = 0$, when the particle occupies the initial position,

$$x(0) = 0, \tag{6.1}$$

$$z(0) = h > 0, \tag{6.2}$$

we release it. The geometrization principle claims that if we want to study the behavior of this particle under the action of the homogeneous gravitational field under consideration, we can study its motion in the effective Riemannian space–time (see (5.8))

$$ds^2 = -e^{-4\mathcal{G}z}\, dx^2 - e^{-6\mathcal{G}z}\, dz^2 + e^{2\mathcal{G}z}\, dt^2. \tag{6.3}$$

By virtue of geodesic equations (4.9) and (4.10) and Riemannian metric (6.3), we find that the particle trajectory is described by the equations

$$\frac{d^2x}{dt^2} - 6\mathcal{G}\frac{dx}{dt}\frac{dz}{dt} = 0, \tag{6.4}$$

$$\frac{d^2z}{dt^2} + 2\mathcal{G}e^{2\mathcal{G}z}\left(\frac{dx}{dt}\right)^2 - 5\mathcal{G}\left(\frac{dz}{dt}\right)^2 + \mathcal{G}e^{8\mathcal{G}z} = 0. \tag{6.5}$$

We also assume that at the initial instant $t = 0$,

$$\dot{x}(0) = a > 0, \tag{6.6}$$

$$\dot{z}(0) = b > 0, \tag{6.7}$$

where a and b are real constants.

Solving Eq. (6.4) and taking relations (6.2) and (6.6) into account, we obtain

$$\dot{x}(t) = ae^{6\mathcal{G}(z-h)}. \tag{6.8}$$

Also solving Eq. (6.5) and taking Eqs. (6.2), (6.6), and (6.7) into account, we obtain

$$\dot{z}(t) = e^{4\mathcal{G}z}\sqrt{1 + Le^{2\mathcal{G}z} - a^2e^{-12\mathcal{G}h}e^{6\mathcal{G}z}}, \tag{6.9}$$

where L is the real constant

$$L = b^2e^{-10\mathcal{G}h} + a^2e^{-8\mathcal{G}h} - e^{-2\mathcal{G}h}. \tag{6.10}$$

We now segregate the timelike and spacelike components in (6.3),

$$ds^2 = d\sigma^2 - dl^2, \tag{6.11}$$

where

$$d\sigma^2 = e^{2\mathcal{G}z}\, dt^2, \tag{6.12}$$

$$dl^2 = e^{-4\mathcal{G}z}\, dx^2 + e^{-6\mathcal{G}z}\, dz^2. \tag{6.13}$$

Therefore, if one local event has the coordinates $(x, 0, z, t)$ and another local event has the coordinates $(x+dx, 0, z+dz, t+dt)$, then an observer situated at the point $(x, 0, z, t)$ and moving with the four-velocity $(dx/ds, 0, dz/ds, dt/ds)$ measures the space interval dl and the proper time interval $d\sigma$ between the two events.

The particle velocity $v(v^1 = dx/dt, 0, v^3 = dz/dt)$ in the effective Riemannian space–time under consideration has the absolute value

$$v^2 = \frac{dl^2}{dt^2} = e^{-4\mathcal{G}z}\dot{x}^2 + e^{-6\mathcal{G}z}\dot{z}^2. \tag{6.14}$$

It is natural to require the initial speed of the test particle to be less than the speed of light in a vacuum. We therefore obtain the inequality

$$e^{-4\mathcal{G}h}a^2 + e^{-6\mathcal{G}h}b^2 < 1 \tag{6.15}$$

from (6.2), (6.6), (6.7), and (6.14). Taking expressions (6.10) and (6.15) into account, we find the restriction on the constant L:

$$L < e^{-4\mathcal{G}h} - e^{-2\mathcal{G}h}. \tag{6.16}$$

Because we assume that $h > 0$, formula (6.16) implies

$$L < 0. \tag{6.17}$$

We now demonstrate that in the given case, there exist certain coordinates z where the particle speed exceeds the speed of light in a vacuum, i.e., that we can find a value of z such that

$$v^2(z) > 1. \tag{6.18}$$

Substituting (6.8) and (6.9) in (6.14), we find that inequality (6.18) is equivalent to the inequality

$$e^{2\mathcal{G}z} + Le^{4\mathcal{G}z} > 1. \tag{6.19}$$

Taking (6.17) into account, we find that for

$$-\frac{1}{4} < L < 0, \tag{6.20}$$

i.e., under some conditions imposed on the initial particle velocity, inequality (6.19) is satisfied for any z satisfying the inequality

$$\frac{1}{2\mathcal{G}}\log\left(\frac{-1+\sqrt{1+4L}}{2L}\right) < z < \frac{1}{2\mathcal{G}}\log\left(\frac{-1-\sqrt{1+4L}}{2L}\right). \tag{6.21}$$

For example, restrictions (6.20) hold for

$$a = \rho e^{2\mathcal{G}h}\cos\theta, \qquad b = \rho e^{3\mathcal{G}h}\sin\theta$$

$$\text{for } \theta \in \left(0, \frac{\pi}{2}\right), \quad 0 < \rho < 1, \quad \rho^2 > 1 - \frac{(e^{2\mathcal{G}h} - 2)^2}{4}.$$

Hence, if we consider a free test particle moving in the effective Riemannian space–time (5.8) from position (6.1), (6.2) with the velocity (6.6), (6.7) at the initial instant, then it follows from (6.10) that the real constants a and b satisfy restrictions (6.20), and this particle moves faster than light in a vacuum.

7. Conclusion

As we have seen, in classical mechanics, GR, and the RTG, if a reference frame propagates with a constant acceleration w.r.t. an inertial reference frame, then the inertial field generated by the inertial force is a constant field. The expressions for Minkowski length element (4.15) coincide in GR and the RTG. In the classical mechanics approach, this constant field is said to be indistinguishable from a homogeneous gravitational field generated by an infinite material plane. But expressions (3.2), (3.3), and (3.4) imply that these two fields differ, even in the classical mechanics approach. This is because the gravitational force is an attraction force. The difference between a constant field generated by an inertial force and a homogeneous gravitational field generated by masses is essential in both GR and the RTG. As mentioned, Bogorodskii's solution (4.16) for a homogeneous gravitational field in GR contains an irremovable singularity. The RTG solution has form (5.8). Unfortunately, this solution fails to satisfy the RTG causality principle and must be rejected. Indeed, in obtained space–time (5.8), the speed of a free test particle can exceed the speed of light in a vacuum. To conclude our analysis, we note that the interesting problem of finding the gravitational field generated in the RTG by homogeneously distributed masses on an infinite plane remains *open*.

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REFERENCES

1. A. F. Bogorodskii, *Gravity* [in Russian], Naukova Dumka, Kiev (1971).
2. D. Ionescu and E. Soós, "Consequences of the causality principle in the relativistic theory of gravitation," in: *Proc. 23rd Intl. Workshop on High Energy Physics and Field Theory* (Protvino, June 21–23, 2000) (to be published).
3. A. A. Logunov and M. Mestvirishvili, *The Relativistic Theory of Gravitation*, Mir, Moscow (1989).
4. A. A. Logunov, *Relativistic Theory of Gravity and the Mach Principle* [in Russian], Dubna (1996).
5. A. I. Zhukov, *Introduction to General Relativity* [in Russian], Fizmatgiz, Moscow (1961).