QUARTERLY OF APPLIED MATHEMATICS
VOLUME , NUMBER 0
XXXX XXXX, PAGES 000–000
S 0033-569X(XX)0000-0

A NEW TWO-COMPONENT SYSTEM MODELLING SHALLOW-WATER WAVES

By

DELIA IONESCU-KRUSE

Simion Stoilow Institute of Mathematics of the Romanian Academy, Research Unit No. 6, P.O. Box 1-764, 014700 Bucharest, Romania

Abstract.
For propagation of surface shallow-water waves on irrotational flows, we derive a new two-component system. The system is obtained by a variational approach in the Lagrangian formalism. The system has a non-canonical Hamiltonian formulation. We also find its exact solitary-wave solutions.

1. Introduction. In this paper we obtain the following system of nonlinear partial differential equations

$$\left\{\begin{array}{l} u_t + 3uu_x + HH_x = \left[H^2\left(uu_{xx} + u_{xt} - \frac{u_x^2}{2}\right)\right]_x \\ \\ H_t + (Hu)_x = 0, \end{array}\right. \tag{1.1}$$

with $x \in \mathbf{R}$, $t \in \mathbf{R}$, $u(x,t) \in \mathbf{R}$, $H(x,t) \in \mathbf{R}$.
We start from a general dimensionless version of the two-dimensional irrotational water-wave problem with a free surface and a flat bottom. We focus on the motion of shallow-water waves, waves whose length is still large compared with the depth of the water in which they propagate. In this shallow-water regime, many two-component systems have already been derived and studied. One of them is the well-known Green-Nagdhi system [16]

$$\left\{\begin{array}{l} u_t + uu_x + H_x = \frac{1}{3H}\left[H^3(uu_{xx} + u_{xt} - u_x^2)\right]_x \\ \\ H_t + (Hu)_x = 0, \end{array}\right. \tag{1.2}$$

which models shallow-water waves whose amplitude (in the dimensionless version, the amplitude parameter, that is, the ratio of the wave amplitude to the depth of the water)

2000 *Mathematics Subject Classification.* 35Q35, 76B15, 76M30, 37K05, 76B25.
Key words and phrases. shallow-water waves, variational methods, Hamiltonian structures, solitary waves.
E-mail address: Delia.Ionescu@imar.ro

is not necessarily small. $u(x,t)$ represents the horizontal velocity or the depth-averaged[1] horizontal velocity and $H(x,t)$ is the free upper surface. The Green-Naghdi equations are mathematically well-posed in the sense that they admit solutions over the relevant time scale for any initial data that are reasonably smooth (see [26], [3]). The solution of the Green-Naghdi equations provides a good approximation of the solution of the full water-wave problem (see [26], [4]). The Green-Naghdi equations have nice structural properties that facilitate the derivation of simplified model equations in the shallow water regime. For example, the celebrated Korteweg-de Vries, Benjamin-Bona-Mahoney, Camassa-Holm and Degasperis-Procesi equations arise as approximations to the Green-Naghdi equations cf. the discussion in [13].

Actually, Green and Naghdi considered in [16] the three-dimensional water-wave problem with a free surface and a variable bottom, and no assumption of an irrotational flow was made a priori. The equations were derived by imposing the condition that the horizontal velocity is independent of the vertical coordinate z, the condition that the vertical velocity has only a linear dependence on z and by using the mass conservation equation and the energy equation in integral form plus invariance under rigid-body translation. For one horizontal x-coordinate and for a flat bottom, the equations have the form (1.2). In the two-dimensional case (only one horizontal dimension) and for a domain with a flat bottom, the system (1.2) was originally derived in 1953 by Serre [31], and independently rediscovered by Su and Gardner [33] in 1969. Serre ([31], Sect. V.) integrated the Euler equations over z on the interval $[0, H(x,t)]$ and made the assumption that the horizontal component of fluid velocity is equal to its depth-averaged value. Su and Gardner [33] obtained the system (1.2) by depth-averaging the two-dimensional irrotational water-wave problem and by using a long-wave asymptotic expansion. In the literature, the equations (1.2) are sometimes referred to as the Serre equations, or the Su-Gardner equations but usually they are called the Green-Naghdi equations. Very recently, Ionescu-Kruse [21] obtained, by a variational approach in the Lagrangian formalism, the system (1.2) for the propagation of arbitrary amplitude shallow-water waves on two-dimensional irrotational flows.

In Section 3 of the present paper, we derive, by the same approach as in [21], the system (1.1). We are in the shallow-water regime and we consider surface waves of arbitrary amplitude. We are looking for a higher-order correction to the classical shallow-water equations (2.14). The second equation of the system (2.14) is a transport equation, the free surface is advected or Lie transported (in the geometry literature), by the fluid flow. In the system (1.1), we keep this equation as it is. We obtain the first equation of the system (1.1) by calculating the critical points of an action functional in the space of paths with fixed endpoints, within the Lagrangian formalism. We arrive at this action functional as follows. Within the Eulerian formalism, we consider the Lagrangian function integrated over time in the action functional to have the traditional form, that is, the kinetic energy minus the potential energy. According to a velocity field with a horizontal component (2.9) independent of the vertical coordinate z and a vertical component (2.10) having only a linear dependence on z, we take for the kinetic energy at the free surface

[1]The depth-averaged value of a quantity $q(x,z,t)$ is defined by $\bar{q}(x,t) := \frac{1}{H(x,t)} \int_0^{H(x,t)} q(x,z,t)dz$.

of the water the expression (3.12) and for the potential energy calculated with respect to the undisturbed water level the expression (3.13). Then, we transport the Lagrangian function (3.15) from the Eulerian picture to the tangent bundle which represents the velocity phase space in the Lagrangian formalism, this transport being made taking into account the second equation of the system (1.1) too. Thus, we get the Lagrangian function (3.16). We point out that the Lagrangian (3.15) as well as (3.16) are not metrics; the pursuit of an advanced geometrical approach is not necessarily dependent upon the existence of a metric, as illustrated in the recent papers [14] and [15] too.
The type of considerations made in the present paper proved also very useful (in similar contexts) to qualitative studies of some model equations. For example, in the derivation of criteria for global existence and blow-up of solutions as well as in studies of the propagation speed for some model equations for shallow water waves, see e.g. the papers [8], [11], [17], [9], [18].

The Green-Naghdi equations (1.2) have the following Hamiltonian formulation (see [19], [7])

$$\begin{pmatrix} m_t \\ \\ H_t \end{pmatrix} = -\begin{pmatrix} \partial_x m + m\partial_x & H\partial_x \\ \\ \partial_x H & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta \mathcal{H}_{GN}}{\delta m} \\ \\ \frac{\delta \mathcal{H}_{GN}}{\delta H} \end{pmatrix}, \tag{1.3}$$

where $\mathcal{H}_{GN}$ is the total energy (kinetic plus potential) given by

$$\frac{1}{2}\int_{-\infty}^{\infty} \left(Hu^2 + \frac{1}{3}H^3 u_x^2 + (H-1)^2 \right) dx, \tag{1.4}$$

and m is the momentum density defined by

$$m := \frac{\delta \mathcal{H}_{GN}}{\delta u} = Hu - \frac{1}{3}\left(H^3 u_x\right)_x, \tag{1.5}$$

$\frac{\delta \mathcal{H}_{GN}}{\delta u}$, $\frac{\delta \mathcal{H}_{GN}}{\delta m}$ and $\frac{\delta \mathcal{H}_{GN}}{\delta H}$ being the variational derivatives of $\mathcal{H}_{GN}$ with respect to u, m and H, respectively.
In Section 4 of the present paper, we show that the the system (1.1) has the Hamiltonian formulation (1.3), with a different total energy $\mathcal{H}_N$ given by (4.2) and a different momentum density m given by (4.3).

The solitary-wave solution of the Green-Naghdi equations (1.2) has the form (see [31], pag. 863-864 and [33], pag 539)

$$\begin{aligned} H(x,t) &= 1 + (c^2-1)\,\mathrm{sech}^2\left[\tfrac{\sqrt{3}}{2}\tfrac{\sqrt{c^2-1}}{c}(x-ct)\right] \\ u(x,t) &= c\left(1 - \tfrac{1}{H(x,t)}\right), \end{aligned} \tag{1.6}$$

with c the speed of the traveling wave. These waves exist for all c such that the following condition:

$$c^2 > 1 \tag{1.7}$$

is satisfied. In [24], [25], the eigenvalue problem obtained from linearizing the equations about solitary-wave solutions is investigated and it is established that small-amplitude solitary-wave solutions of the Green-Naghdi equations are linearly stable.

In Section 5 of the present paper, we find the solitary-wave solution of the system (1.1). Its expression (5.19)-(5.20) is different from (1.6). The speed c of the traveling wave has to satisfy the condition (5.9), that is, the condition (1.7).

2. Preliminaries. We recall the classical water-wave problem for gravity waves propagating at the free surface of a two-dimensional inviscid incompressible fluid. The fluid occupies the domain:

$$-\infty < x < \infty, \quad 0 \le z \le h_0 + \eta(x,t), \tag{2.1}$$

where the constant $h_0 > 0$ is the undisturbed depth of the water and $\eta(x,t)$ is the displacement of the free surface from the undisturbed state. (x,z) are the Cartesian coordinates, the x-axis being in the direction of wave propagation and the z-axis pointing vertically upwards. The governing equations are Euler's equations and the continuity equation with appropriate surface and bottom boundary conditions (see, for example, [10]):

$$\begin{array}{ll} u_t + uu_x + vu_z = -p_x & \\ v_t + uv_x + vv_z = -p_z - g & \\ u_x + v_z = 0 & \\ v = \eta_t + u\eta_x & \text{on } z = h_0 + \eta(x,t) \\ p = p_0 & \text{on } z = h_0 + \eta(x,t) \\ v = 0 & \text{on } z = 0. \end{array} \tag{2.2}$$

Here $(u(x,z,t), v(x,z,t))$ is the velocity field of the water - no motion takes place in the y-direction, $p(x,z,t)$ denotes the pressure, p_0 being the constant atmospheric pressure and g is the acceleration due to gravity. We set the constant density $\rho = 1$.
The water flow is assumed to be irrotational, that is, in addition to the system (2.2) we also have the equation

$$u_z - v_x = 0. \tag{2.3}$$

We introduce the following dimenionless variables (see, for example, [23]):

$$\begin{gathered} \bar{x} = \tfrac{x}{\lambda}, \quad \bar{z} = \tfrac{z}{h_0}, \quad \bar{t} = \tfrac{\sqrt{gh_0}}{\lambda}t, \quad \bar{\eta} = \tfrac{\eta}{a}, \\ \bar{u} = \tfrac{1}{\sqrt{gh_0}}u, \quad \bar{v} = \tfrac{1}{h_0}\tfrac{\lambda}{\sqrt{gh_0}}v, \\ \bar{p} = \tfrac{1}{gh_0}[p - p_0 - g(h_0 - z)], \end{gathered} \tag{2.4}$$

where a represents a measure of the amplitude of the waves and λ the typical wavelength for the considered waves. The dimensionless variables considered above are good choices for showing the magnitude of the different terms that appear in the equations. Substituting (2.4) in the system (2.2)-(2.3), one finds that the equations of motion depend upon two parameters ϵ and δ defined as follows:

$$\epsilon := \frac{a}{h_0}, \quad \delta := \frac{h_0}{\lambda}. \tag{2.5}$$

The amplitude parameter ϵ is associated with the nonlinearity of the wave, and the long-wave parameter δ is associated with the dispersion of the wave. Omitting the bars for the sake of clarity, the dimensionless form of the system (2.2)-(2.3) is:

$$\begin{aligned} u_t + uu_x + vu_z &= -p_x \\ \delta^2(v_t + uv_x + vv_z) &= -p_z \\ u_x + v_z &= 0 \\ u_z - \delta^2 v_x &= 0 \\ v = \epsilon(\eta_t + u\eta_x) & \quad \text{on } z = 1 + \epsilon\eta(x,t) \\ p = \epsilon\eta & \quad \text{on } z = 1 + \epsilon\eta(x,t) \\ v = 0 & \quad \text{on } z = 0. \end{aligned} \tag{2.6}$$

Making smallness hypotheses on the parameters ϵ and δ, one reduces the problem to different physical regimes. Our analysis is concerned with the shallow-water regime, that is,

$$\delta << 1. \tag{2.7}$$

The amplitude of waves is governed by ϵ. We consider relatively large amplitude surface waves, meaning that no smallness assumption is made on ϵ. For $\delta = 0$, the leading-order system becomes:

$$\begin{aligned} u_t + uu_x + vu_z &= -p_x \\ p_z &= 0 \\ u_x + v_z &= 0 \\ u_z &= 0 \\ v = \epsilon(\eta_t + u\eta_x) & \quad \text{on } z = 1 + \epsilon\eta(x,t) \\ p = \epsilon\eta(x,t) & \quad \text{on } z = 1 + \epsilon\eta(x,t) \\ v = 0 & \quad \text{on } z = 0. \end{aligned} \tag{2.8}$$

The system of equations (2.8) reduces to

$$u = u(x,t), \tag{2.9}$$

$$v = -zu_x, \tag{2.10}$$

$$p = \epsilon\eta(x,t) \tag{2.11}$$

and

$$\begin{cases} u_t + uu_x + \epsilon\eta_x = 0 \\ \epsilon\eta_t + [(1+\epsilon\eta)u]_x = 0. \end{cases} \tag{2.12}$$

Let us denote by

$$H(x,t) := 1 + \epsilon\eta(x,t). \tag{2.13}$$

Then, the system of equations (2.12) becomes:

$$\begin{cases} u_t + uu_x + H_x = 0 \\ H_t + (Hu)_x = 0, \end{cases} \tag{2.14}$$

that is, the classical shallow-water equations (see, for example, [32]). These equations possess an infinite number of integrals of motion (the conserved quantities) due to Benney [5] and can be written in Hamiltonian form relative to a symplectic structure introduced by Manin [27]. The second Hamiltonian structure for the system (2.14) was obtained by Cavalcante and McKean [6]. In fact, the system (2.14) is Hamiltonian with respect to three distinct Hamiltonian structures [29]. These Hamiltonian structures are compatible and thus, the system of equations (2.14) is completely integrable [30]. For a rigorous analysis of the system (2.14) as an approximate model of the water-wave problem see [4].

3. The variational derivation of a new two-component shallow-water system. In what follows we consider ϵ arbitrary but fixed, there is no smallness assumption on the wave amplitude. We are looking for a higher-order correction to the classical shallow-water equations (2.12), or (2.14) in view of the notation (2.13). We observe that the second equation in (2.14) is exactly the second equation of the new two-component shallow-water system (1.1). The first equation of the system (1.1) we will derive directly from a variational principle in the Lagrangian formalism.

We introduce now the following map

$$\gamma : \mathbf{R} \times [0, T] \mapsto \mathbf{R}, \quad \gamma(X, t) = x, \tag{3.1}$$

such that, for a fixed t, $\gamma(\cdot, t)$ is an invertible C^1-mapping, that is,

$$\gamma(\cdot, t) \in \mathrm{Diff}(\mathbf{R}), \tag{3.2}$$

and such that

$$u(x, t) = \gamma_t(X, t), \quad \text{that is,} \quad u(\cdot, t) = \gamma_t \circ \gamma^{-1}. \tag{3.3}$$

This map reminds us of the flow map used in the Lagrangian description of the fluid which maps a fluid particle labeled by its initial location X to its later Eulerian position x. In the Lagrangian description of the fluid, the Lagrangian velocity $\gamma_t(X, t)$ represents the velocity of the fluid particle labeled X, while the Eulerian velocity $\gamma_t(\gamma^{-1}(x), t)$ represents the velocity of the particle passing the location x at time t.
In the Eulerian formalism for our problem, for a fixed t, $u(x, t)$ can be regarded as a vector field on $\mathbf{R}$, that is, it belongs to the Lie algebra of $\mathrm{Diff}(\mathbf{R})$. In the Lagrangian formalism for our problem, the velocity phase space is the tangent bundle $T\mathrm{Diff}(\mathbf{R})$. For the configuration space $\mathrm{Diff}(\mathbf{R})$, we add the technical assumption that the smooth functions defined on $\mathbf{R}$ with value in $\mathbf{R}$ vanish rapidly at $\pm\infty$ together with as many derivatives as necessary.

The other unknown of our problem is $H(x, t)$, which for a fixed t can be regarded as a real function on $\mathbf{R}$, $H(\cdot, t) \in \mathcal{F}(\mathbf{R})$. We settle that the evolution equation of $H(x, t)$ is the second equation in (2.14). This equation is an advection equation. In the language of geometry, this equation expresses the fact that the 1-form

$$\mathrm{H}(x, t) := H(x, t)dx \tag{3.4}$$

is Lie transported by the vector field

$$\mathrm{u}(x, t) := u(x, t)\partial_x, \tag{3.5}$$

that is,

$$\frac{\partial \mathrm{H}}{\partial t} + \mathrm{L}_{\mathrm{u}}\mathrm{H} = 0, \tag{3.6}$$

where L_{u} denotes the Lie derivative with respect to the vector field u (see, for example, [1] Section 2.2.). The equation (3.6) is an equation written in the Eulerian formalism.

With the aid of the pull back map γ^*, in the Lagrangian formalism this becomes:

$$\gamma^* \left(\frac{\partial \mathrm{H}}{\partial t} + \mathrm{L_u H} \right) = 0. \tag{3.7}$$

By interpreting the Lie derivative of a time-dependent 1-form along a time-dependent vector field in terms of the flow of the vector field (see, for example, [1], Section 2), we get that

$$\frac{d}{dt} [\gamma^*(\mathrm{H})] = \gamma^*(\mathrm{L_u H}) + \gamma^* \left(\frac{\partial \mathrm{H}}{\partial t} \right) \overset{(3.7)}{=} 0, \tag{3.8}$$

that is, we get the following time invariant 1-form

$$\mathrm{H}_0 := \gamma^*(\mathrm{H}), \quad \mathrm{H}_0(X,t) = \mathrm{H}_0(X,0). \tag{3.9}$$

By the definition of the pull back map (see, for example, [1], Section 2), we get between the components of the 1-forms $\mathrm{H}_0(X,t) := H_0(X,t)dX$ and $\mathrm{H}(x,t) := H(x,t)dx$ the following relation:

$$H_0 = (H \circ \gamma) J_\gamma, \tag{3.10}$$

where $J_\gamma := \frac{\partial \gamma}{\partial X}$ is the Jacobian of γ, or,

$$H = (H_0 \circ \gamma^{-1}) J_{\gamma^{-1}}. \tag{3.11}$$

Our goal is to show that the first equation of the system (1.1) yields the critical points of an appropriate action functional which is completely determined by a scalar function called Lagrangian. We take the traditional form of the Lagrangian, that is, the kinetic energy minus the potential energy. In the Eulerian formalism, taking into account the components (2.9) and (2.10) of the velocity field, the kinetic energy has at the free surface $z = 1 + \epsilon\eta(x,t)$ the expression

$$\begin{aligned} E_c(u,\eta) &= \frac{1}{2} \int_{-\infty}^{\infty} [u^2 + (1+\epsilon\eta)^2 u_x^2] dx \\ &\overset{(2.13)}{=} \frac{1}{2} \int_{-\infty}^{\infty} [u^2 + H^2 u_x^2] dx =: E_c(u,H). \end{aligned} \tag{3.12}$$

In non-dimensional variables, with ρ and g settled at 1, we define the gravitational potential energy at the free surface $z = 1 + \epsilon\eta(x,t)$, gained by the fluid parcel when it is vertically displaced from its undisturbed position with $\epsilon\eta(x,t)$, by

$$\begin{aligned} E_p(\eta) &= \int_{-\infty}^{\infty} \left(\int_0^{1+\epsilon\eta} (z-1)\, dz \right) dx = \frac{1}{2} \int_{-\infty}^{\infty} (\epsilon\eta)^2 dx \\ &\overset{(2.13)}{=} \frac{1}{2} \int_{-\infty}^{\infty} (H-1)^2 dx =: E_p(H). \end{aligned} \tag{3.13}$$

We require in (3.12) and (3.13) that at any instant t,

$$u \to 0, \quad u_x \to 0 \text{ and } H \to 1 \text{ as } x \to \pm\infty. \tag{3.14}$$

Thus, in the Eulerian formalism, the Lagrangian function has the form

$$\mathfrak{L}(u,H) = E_c(u,H) - E_p(H) = \frac{1}{2} \int_{-\infty}^{\infty} [u^2 + H^2 u_x^2 - (H-1)^2] dx. \tag{3.15}$$

Within the Lagrangian formalism, the Lagrangian for our problem will be obtained by transporting the Lagrangian (3.15) from the Eulerian formalism, to all tangent spaces $T\mathrm{Diff}(\mathbf{R})$, this transport being made taking into account (3.3) and (3.11).
For each function $H_0 \in \mathcal{F}(\mathbf{R})$ independent of time, we define the Lagrangian $\mathcal{L}_{H_0}$: $T\mathrm{Diff}(\mathbf{R}) \to \mathbf{R}$ by

$$\begin{aligned}\mathcal{L}_{H_0}(\gamma,\gamma_t) \quad := \quad & \frac{1}{2}\int_{-\infty}^{\infty}\{(\gamma_t\circ\gamma^{-1})^2 + [(H_0\circ\gamma^{-1})J_{\gamma^{-1}}]^2[\partial_x(\gamma_t\circ\gamma^{-1})]^2 - \\ & -[(H_0\circ\gamma^{-1})J_{\gamma^{-1}}-1]^2\}dx. \end{aligned} \tag{3.16}$$

The Lagrangian $\mathcal{L}_{H_0}$ depends smoothly on H_0 and it is right invariant under the action of the subgroup

$$\mathrm{Diff}(\mathbf{R})_{H_0} = \{\psi\in\mathrm{Diff}(\mathbf{R}) | (H_0\circ\psi^{-1})J_{\psi^{-1}} = H_0\}, \tag{3.17}$$

that is, if we replace the path $\gamma(t,\cdot)$ by $\gamma(t,\cdot)\circ\psi(\cdot)$, for a fixed time-independent ψ in $\mathrm{Diff}(\mathbf{R})_{H_0}$, then $\mathcal{L}_{H_0}$ is unchanged.
The action on a path $\gamma(t,\cdot)$, $t\in[0,T]$, in $\mathrm{Diff}(\mathbf{R})$ is

$$\mathfrak{a}(\gamma) := \int_0^T \mathcal{L}_{H_0}(\gamma,\gamma_t)dt. \tag{3.18}$$

The critical points of the action (3.18) in the space of paths with fixed endpoints, satisfy

$$\frac{d}{d\varepsilon}\mathfrak{a}(\gamma+\varepsilon\varphi)\Big|_{\varepsilon=0} = 0, \tag{3.19}$$

for every path $\varphi(t,\cdot)$, $t\in[0,T]$, in $\mathrm{Diff}(\mathbf{R})$ with endpoints at zero, that is,

$$\varphi(0,\cdot) = 0 = \varphi(T,\cdot), \tag{3.20}$$

and such that $\gamma+\varepsilon\varphi$ is a small variation of γ on $\mathrm{Diff}(\mathbf{R})$. With (3.16) and (3.18) in view, the condition (3.19) becomes

$$\begin{aligned}\int_0^T\int_{-\infty}^{\infty} \quad & \Big\{\left(\gamma_t\circ\gamma^{-1}\right)\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[(\gamma_t+\varepsilon\varphi_t)\circ(\gamma+\varepsilon\varphi)^{-1}\right] \\ & +(H_0\circ\gamma^{-1})J_{\gamma^{-1}}^2\left[\partial_x(\gamma_t\circ\gamma^{-1})\right]^2\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}[H_0\circ(\gamma+\varepsilon\varphi)^{-1}] \\ & +(H_0\circ\gamma^{-1})^2J_{\gamma^{-1}}\left[\partial_x(\gamma_t\circ\gamma^{-1})\right]^2\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}[J_{(\gamma+\varepsilon\varphi)^{-1}}] \\ & +[(H_0\circ\gamma^{-1})J_{\gamma^{-1}}]^2\partial_x(\gamma_t\circ\gamma^{-1})\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[\partial_x\left((\gamma_t+\varepsilon\varphi_t)\circ(\gamma+\varepsilon\varphi)^{-1}\right)\right] \\ & -(H_0\circ\gamma^{-1})J_{\gamma^{-1}}^2\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}[H_0\circ(\gamma+\varepsilon\varphi)^{-1}] \\ & -(H_0\circ\gamma^{-1})^2J_{\gamma^{-1}}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}[J_{(\gamma+\varepsilon\varphi)^{-1}}] \\ & +(J_{\gamma^{-1}})\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}[H_0\circ(\gamma+\varepsilon\varphi)^{-1}] \\ & +(H_0\circ\gamma^{-1})\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}[J_{(\gamma+\varepsilon\varphi)^{-1}}]\Big\}dxdt = 0. \end{aligned} \tag{3.21}$$

After calculation (for more details see, for example, [22]), we get

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[(\gamma_t+\varepsilon\varphi_t)\circ(\gamma+\varepsilon\varphi)^{-1}\right] = \partial_t(\varphi\circ\gamma^{-1})+(\gamma_t\circ\gamma^{-1})\partial_x(\varphi\circ\gamma^{-1}) -(\varphi\circ\gamma^{-1})\partial_x(\gamma_t\circ\gamma^{-1}), \tag{3.22}$$

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}[H_0\circ(\gamma+\varepsilon\varphi)^{-1}] = -(\varphi\circ\gamma^{-1})\partial_x(H_0\circ\gamma^{-1}) \tag{3.23}$$

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}[J_{(\gamma+\varepsilon\varphi)^{-1}}] = -(J_{\gamma^{-1}})\partial_x(\varphi\circ\gamma^{-1})-\partial_x(J_{\gamma^{-1}})(\varphi\circ\gamma^{-1}), \tag{3.24}$$

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[\partial_x\left((\gamma_t+\varepsilon\varphi_t)\circ(\gamma+\varepsilon\varphi)^{-1}\right)\right] = \partial_{tx}(\varphi\circ\gamma^{-1})+(\gamma_t\circ\gamma^{-1})\partial_x^2(\varphi\circ\gamma^{-1}) -[\partial_x^2(\gamma_t\circ\gamma^{-1})](\varphi\circ\gamma^{-1}). \tag{3.25}$$

Thus, from (3.22)-(3.25), the condition (3.21) becomes:

$$\begin{aligned}\int_0^T\int_{-\infty}^{\infty} &\Big\{u\left[\partial_t(\varphi\circ\gamma^{-1})+u\partial_x(\varphi\circ\gamma^{-1})-(\varphi\circ\gamma^{-1})u_x\right]\\ &-HH_xu_x^2(\varphi\circ\gamma^{-1})-H^2u_x^2\partial_x(\varphi\circ\gamma^{-1})\\ &+H^2u_x\left[\partial_{tx}(\varphi\circ\gamma^{-1})+u\partial_x^2(\varphi\circ\gamma^{-1})-(\varphi\circ\gamma^{-1})u_{xx}\right]\\ &+HH_x(\varphi\circ\gamma^{-1})+H^2\partial_x(\varphi\circ\gamma^{-1})\\ &-H_x(\varphi\circ\gamma^{-1})-H\partial_x(\varphi\circ\gamma^{-1})\Big\}\,dxdt=0,\end{aligned} \tag{3.26}$$

where $u=\gamma_t\circ\gamma^{-1}$ and $H=(H_0\circ\gamma^{-1})J_{\gamma^{-1}}$. In the above formula, we integrate by parts with respect to t and x, we take into account (3.14) and (3.20), and we get

$$\begin{aligned}-\int_0^T\int_{-\infty}^{\infty} &(\varphi\circ\gamma^{-1})\left[u_t+3uu_x-HH_xu_x^2-H^2u_xu_{xx}\right.\\ &\left.-(H^2u_x)_{tx}-(H^2uu_x)_{xx}+HH_x\right]dxdt=0\end{aligned} \tag{3.27}$$

With H satisfying the second equation in (1.1), the condition (3.27) becomes:

$$\begin{aligned}-\int_0^T\int_{-\infty}^{\infty} &(\varphi\circ\gamma^{-1})\Big\{u_t+3uu_x+HH_x-\\ &-\left[H^2\left(u_{xt}+uu_{xx}-\frac{u_x^2}{2}\right)\right]_x\Big\}\,dxdt=0.\end{aligned} \tag{3.28}$$

Therefore, we proved:

Theorem 3.1. For an irrotational shallow-water flow, the non-dimensional horizontal velocity of the water $u(x,t)$ and the non-dimensional free upper surface $H(x,t)=1+\epsilon\eta(x,t)$, for ϵ arbitrary fixed, satisfy the system (1.1).

We emphasize that for our considerations we do not require any hypothesis of small amplitude. Under the additional assumption of a small or moderate amplitude regime, similar considerations lead to a variational derivation of the celebrated Korteweg-de Vries and Camassa-Holm model equations (see [20] and [12]).

4. The Hamiltonian structure for the shallow-water system (1.1). The use of a variational principle in fluid dynamics, beside the aesthetic attraction in condensing the equations by extremizing a scalar quantity, retains the Hamiltonian structure with consequent energy conservation. We present below the Hamiltonian structure of the two-component shallow-water system (1.1).

THEOREM 4.1. The shallow-water system (1.1) has the following Hamiltonian form:

$$\begin{pmatrix} m_t \\ \\ H_t \end{pmatrix} = - \begin{pmatrix} \partial_x m + m\partial_x & H\partial_x \\ \\ \partial_x H & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta \mathcal{H}_N}{\delta m} \\ \\ \frac{\delta \mathcal{H}_N}{\delta H} \end{pmatrix}, \tag{4.1}$$

where $\mathcal{H}_N$ is the total energy , that is,

$$\mathcal{H}_N(u,H) := E_c(u,H) + E_p(H) = \frac{1}{2}\int_{-\infty}^{\infty} \left[u^2 + H^2 u_x^2 + (H-1)^2\right] dx, \tag{4.2}$$

and m is the momentum density defined by

$$m := \frac{\delta \mathcal{H}_N}{\delta u} = u - (H^2 u_x)_x. \tag{4.3}$$

Proof. $\frac{\delta \mathcal{H}_N}{\delta u}$ is the variational derivative of $\mathcal{H}_N$ with respect to u, that is,

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{H}_N(u+\epsilon\delta u, H) = \int_{-\infty}^{\infty} \frac{\delta \mathcal{H}_N}{\delta u} \delta u \, dx. \tag{4.4}$$

From the expression (4.2) of $\mathcal{H}_N$ we have

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{H}_N(u+\epsilon\delta u, H) = \int_{-\infty}^{\infty} \left[u\delta u + H^2 u_x (\delta u)_x\right] dx. \tag{4.5}$$

Integrating by parts and taking into account (3.14), we get

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{H}_N(u+\epsilon\delta u, H) = \int_{-\infty}^{\infty} \left[u - (H^2 u_x)_x\right] \delta u \, dx. \tag{4.6}$$

Therefore, m has the expression (4.3).
In order to calculate $\frac{\delta \mathcal{H}_N}{\delta m}$ and $\frac{\delta \mathcal{H}_N}{\delta H}$, that is, the variational derivatives of $\mathcal{H}_N$ with respect to m and H, respectively, we write the total energy $\mathcal{H}_N$ in terms of m and H. Integrating by parts the second term in the right-hand integral (4.2) and taking into account (3.14) we obtain

$$\mathcal{H}_N = \frac{1}{2}\int_{-\infty}^{\infty} \left[mu + (H-1)^2\right] dx. \tag{4.7}$$

We can regard (4.3) as an operator equation, that is,

$$m = u - (H^2 u_x)_x =: \mathcal{T}_H u. \tag{4.8}$$

$\mathcal{T}_H$ is a linear operator defined on the space of real functions u satisfying (3.14), with the inner product defined by

$$< \mathcal{T}_H u, v > := \int_{-\infty}^{\infty} \left[uv - (H^2 u_x)_x v\right] dx. \tag{4.9}$$

For two functions u and v satisfying (3.14), integrating by parts the second term in the right-hand integral (4.9) we obtain

$$< \mathcal{T}_H u, v >=< u, \mathcal{T}_H v > \tag{4.10}$$

that is, $\mathcal{T}_H$ is a self-adjoint operator.

$$< \mathcal{T}_H u, u >= \int_{-\infty}^{\infty} \left[u^2 + H^2 u_x^2\right] dx, \tag{4.11}$$

thus, the operator $\mathcal{T}_H$ is positive definite too.
The operator equation (4.8) may be inverted to determine u as a continuous function of m,

$$u = \mathcal{T}_H^{-1} m, \tag{4.12}$$

$\mathcal{T}_H^{-1}$ being the inverse operator.
Then, (4.7) becomes

$$\mathcal{H}_N(m, H) = \frac{1}{2} \int_{-\infty}^{\infty} \left[m(\mathcal{T}_H^{-1} m) + (H-1)^2\right] dx. \tag{4.13}$$

Let us calculate now $\frac{\delta \mathcal{H}_N}{\delta m}$ and $\frac{\delta \mathcal{H}_N}{\delta H}$, where

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{H}_N(m + \epsilon \delta m, H) = \int_{-\infty}^{\infty} \frac{\delta \mathcal{H}_N}{\delta m} \delta m \, dx \tag{4.14}$$

and

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{H}_N(m, H + \epsilon \delta H) = \int_{-\infty}^{\infty} \frac{\delta \mathcal{H}_N}{\delta H} \delta H \, dx. \tag{4.15}$$

From (4.13), taking into account that $\mathcal{T}_H^{-1}$ is a linear and self-adjoint operator too, we obtain

$$\begin{aligned} \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{H}_N(m + \epsilon \delta m, H) &= \frac{1}{2} \int_{-\infty}^{\infty} \left[\delta m(\mathcal{T}_H^{-1} m) + m(\mathcal{T}_H^{-1} \delta m)\right] dx \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \left[\delta m(\mathcal{T}_H^{-1} m) + (\mathcal{T}_H^{-1} m)\delta m\right] dx \\ &= \int_{-\infty}^{\infty} (\mathcal{T}_H^{-1} m) \delta m \, dx. \end{aligned} \tag{4.16}$$

Therefore,

$$\frac{\delta \mathcal{H}_N}{\delta m} = \mathcal{T}_H^{-1} m = u. \tag{4.17}$$

From (4.13), we also have

$$\begin{aligned} \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{H}_N(m, H + \epsilon \delta H) &= \frac{1}{2} \int_{-\infty}^{\infty} \left(m \frac{d}{d\varepsilon}\Big|_{\epsilon=0} \mathcal{T}_{(H+\epsilon\delta H)}^{-1} m \right) dx \\ &+ \int_{-\infty}^{\infty} (H-1) \delta H \, dx. \end{aligned} \tag{4.18}$$

Differentiating with respect to ε the identity

$$\mathcal{T}_{(H+\epsilon\delta H)} \circ \mathcal{T}_{(H+\epsilon\delta H)}^{-1} = Id \tag{4.19}$$

we get

$$\frac{d}{d\varepsilon}\Big|_{\epsilon=0}\mathcal{T}^{-1}_{(H+\epsilon\delta H)}m = -\mathcal{T}^{-1}_{H}\left(\frac{d}{d\varepsilon}\Big|_{\epsilon=0}\mathcal{T}_{(H+\epsilon\delta H)}u\right). \tag{4.20}$$

Thus, taking into account that $\mathcal{T}_H^{-1}$ is a self-adjoint operator and the relation (4.20), the first term in the right-hand side of (4.18) becomes

$$\frac{1}{2}\int_{-\infty}^{\infty}\left(m\frac{d}{d\varepsilon}\Big|_{\epsilon=0}\mathcal{T}^{-1}_{(H+\epsilon\delta H)}m\right)dx = -\frac{1}{2}\int_{-\infty}^{\infty}\left(u\frac{d}{d\varepsilon}\Big|_{\epsilon=0}\mathcal{T}_{(H+\epsilon\delta H)}u\right)dx. \tag{4.21}$$

From the expression (4.8) of the linear operator $\mathcal{T}_H$,

$$\frac{d}{d\varepsilon}\Big|_{\epsilon=0}\mathcal{T}_{(H+\epsilon\delta H)}u = -2(u_xH_x + Hu_{xx})\delta H - 2Hu_x(\delta H)_x. \tag{4.22}$$

We replace (4.22) in the right-hand side of (4.21), we integrate by parts, for a function u satisfying (3.14), and finally we obtain

$$\frac{1}{2}\int_{-\infty}^{\infty}\left(m\frac{d}{d\varepsilon}\Big|_{\epsilon=0}\mathcal{T}^{-1}_{(H+\epsilon\delta H)}m\right)dx = \int_{-\infty}^{\infty}\left(-Hu_x^2\right)\delta H\,dx. \tag{4.23}$$

Substituting (4.23) into (4.18) yields

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\mathcal{H}_N(m, H+\epsilon\delta H) = \int_{-\infty}^{\infty}\left(-Hu_x^2 + H - 1\right)\delta H\,dx, \tag{4.24}$$

that is,

$$\frac{\delta\mathcal{H}_N}{\delta H} = -Hu_x^2 + H - 1. \tag{4.25}$$

It remains to check now that the system

$$\begin{pmatrix} m_t \\ \\ H_t \end{pmatrix} = -\begin{pmatrix} \partial_x m + m\partial_x & H\partial_x \\ \\ \partial_x H & 0 \end{pmatrix}\begin{pmatrix} u \\ -Hu_x^2 + H - 1 \end{pmatrix} \tag{4.26}$$

is the shallow-water system (1.1). It is clear that the second equation of the system (4.26) is the second equation of the system (1.1). A straightforward calculation, with H satisfying the second equation in (1.1), shows that the first equation of the two systems coincide too.

What is left is to show that the operator

$$-\begin{pmatrix} \partial_x m + m\partial_x & H\partial_x \\ \\ \partial_x H & 0 \end{pmatrix} \tag{4.27}$$

is skew-symmetric and satisfies Jacobi's identity. The verification of Jacobi's identity can be done directly (see, for example, [7]) or with the assistance of the Lie-Poisson structure (see, for example, [28]). This completes the proof. □

REMARK 4.2. The Lagrangian (3.15) does not depend on time and on the space coordinate x, that is, it is invariant (symmetric) under the time and space translations. Noether's theorem implies for each invariance a unique conservation law (see, for example, [2]). Thus, we get for the system of equations (1.1) the conservation of the total energy (4.2) and the conservation of the momentum density (4.3), respectively. The local conservation law for the momentum density has the form

$$\begin{aligned} m_t &= -\partial_x(mu) - m\partial_x(u) - H\partial_x\left(-Hu_x^2 + H - 1\right) \\ &= -\partial_x\left(mu + \frac{u^2}{2} + \frac{H^2}{2} + \frac{3H^2u_x^2}{2}\right). \end{aligned} \tag{4.28}$$

5. Solitary waves for the shallow-water system (1.1). We are now interested in finding the solitary-wave solution of the nonlinear system (1.1). For a solution

$$u(x,t) = u(x-ct), \quad H(x,t) = H(x-ct), \tag{5.1}$$

travelling with speed $c > 0$, the system (1.1) takes the form

$$\begin{cases} -cu' + 3uu' + HH' = \left[H^2(u-c)u'' - H^2\frac{(u')^2}{2}\right]' \\ (-cH + Hu)' = 0. \end{cases} \tag{5.2}$$

We require that, at any instant t,

$$u \to 0, \quad u' \to 0 \quad u'' \to 0 \text{ and } H \to 1 \text{ as } x \to \pm\infty. \tag{5.3}$$

Integrating each equation of the system (5.2) and taking into account the asymptotic limits (5.3), we get

$$\begin{cases} -cu + \frac{3}{2}u^2 + \frac{H^2}{2} = H^2(u-c)u'' - H^2\frac{(u')^2}{2} + \frac{1}{2} \\ u = c\left(1 - \frac{1}{H}\right). \end{cases} \tag{5.4}$$

Plugging the expression of u into the first equation of the system (5.4) yields an ordinary differential equation for H:

$$\frac{c^2}{2} - \frac{2c^2}{H} + \frac{3c^2}{2}\frac{1}{H^2} + \frac{H^2}{2} = -c^2\frac{H''}{H} + \frac{3c^2}{2}\frac{(H')^2}{H^2} + \frac{1}{2}. \tag{5.5}$$

We multiply the above equation by $2\frac{H'}{H^2}$, we integrate, we take into account the asymptotic limits (5.3) and we obtain

$$-\frac{c^2}{H} + \frac{2c^2}{H^2} - \frac{c^2}{H^3} + H = -c^2\frac{(H')^2}{H^3} - \frac{1}{H} + 2. \tag{5.6}$$

Now (5.6) becomes

$$c^2(H')^2 = (H-1)^2(c^2 - H^2). \tag{5.7}$$

From (5.7) it follows that

$$c^2 > H^2, \tag{5.8}$$

which according to the asymptotic behavior (5.3) of H, yields the following condition for c:

$$c^2 > 1. \tag{5.9}$$

The solution of the separable differential equation (5.7) is obtained by integration. We denote

$$H - 1 =: \frac{1}{K}. \tag{5.10}$$

Then, we get the integral

$$I := \int \frac{c\,dH}{(H-1)\sqrt{c^2-H^2}} = -\int \frac{c\,dK}{\sqrt{(c^2-1)K^2-2K-1}}. \tag{5.11}$$

With the condition (5.9) in view, we denote

$$\sqrt{(c^2-1)K^2-2K-1} =: w - \sqrt{c^2-1}\,K. \tag{5.12}$$

In this way,

$$K = \frac{w^2+1}{2(w\sqrt{c^2-1}-1)} \tag{5.13}$$

and the integral (5.11) becomes

$$I = -\int \frac{c\,dw}{w\sqrt{c^2-1}-1} = -\frac{c}{\sqrt{c^2-1}}\log(w\sqrt{c^2-1}-1). \tag{5.14}$$

From the notations (5.12) and (5.10), we conclude that

$$I = -\frac{c}{\sqrt{c^2-1}}\log\left[\frac{\sqrt{c^2-1}\sqrt{c^2-H^2}+c^2-H}{H-1}\right]. \tag{5.15}$$

Therefore, the solution of the differential equation (5.7) has the following implicit form

$$\frac{\sqrt{c^2-1}\sqrt{c^2-H^2}+c^2-H}{H-1} = \exp\left[-\frac{\sqrt{c^2-1}}{c}(x-ct)\right]. \tag{5.16}$$

We add 1 to both sides of the above equation, we divide by $\sqrt{c^2-1}$, we raise to the second power and we get

$$\frac{2c^2-H^2-1+2\sqrt{c^2-1}\sqrt{c^2-H^2}}{(H-1)^2} = \left(\frac{\exp\left[-\frac{\sqrt{c^2-1}}{c}(x-ct)\right]+1}{\sqrt{c^2-1}}\right)^2. \tag{5.17}$$

By adding again 1 to both sides of the above equation

$$\left(\frac{2}{H-1}\right)\frac{\sqrt{c^2-1}\sqrt{c^2-H^2}+c^2-H}{H-1} = \left(\frac{\exp\left[-\frac{\sqrt{c^2-1}}{c}(x-ct)\right]+1}{\sqrt{c^2-1}}\right)^2+1,$$

and by (5.16), we finally obtain

$$\frac{2}{H-1}\exp\left[-\frac{\sqrt{c^2-1}}{c}(x-ct)\right] = \left(\frac{\exp\left[-\frac{\sqrt{c^2-1}}{c}(x-ct)\right]+1}{\sqrt{c^2-1}}\right)^2+1. \tag{5.18}$$

Thus, we have:

THEOREM 5.1. The solitary-wave solution of the shallow-water system (1.1) has the form:

$$\begin{aligned} H(x,t) &= 1 + \frac{2(c^2-1)\exp\left[\frac{\sqrt{c^2-1}}{c}(x-ct)\right]}{c^2\exp\left[2\frac{\sqrt{c^2-1}}{c}(x-ct)\right] + 2\exp\left[\frac{\sqrt{c^2-1}}{c}(x-ct)\right] + 1} \\ &= 1 + \frac{c^2-1}{1 + \frac{c^2+1}{2}\cosh\left[\frac{\sqrt{c^2-1}}{c}(x-ct)\right] + \frac{c^2-1}{2}\sinh\left[\frac{\sqrt{c^2-1}}{c}(x-ct)\right]} \end{aligned} \tag{5.19}$$

and

$$u(x,t) = c\left(1 - \frac{1}{H(x,t)}\right). \tag{5.20}$$

References

[1] R. Abraham and J. E. Marsden, *Foundations of Mechanics*, Benjamin-Cummings, London ISBN 0-8053-0102-X, 1978.

[2] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Second Edition, Springer-Verlag, New York, 1989.

[3] B. Alvarez-Samaniego and D. Lannes, *A Nash-Moser theorem for singular evolution equations. Application to the Serre and Green-Naghdi equations*, Indiana Univ. Math. J. **57** (2008), 97–131.

[4] B. Alvarez-Samaniego and D. Lannes, *Large time existence for 3D water-waves and asymptotics*, Invent. Math. **171** (2008), 485–541.

[5] D. J. Benney, *Some properties of long non-linear waves*, Studies Appl. Math. **52** (1973) 45–50.

[6] J. Cavalcante and H. P. McKean, *The Classical Shallow Water Equations: Symplectic Geometry*, Physica 4D(1982), 253–260.

[7] A. Constantin, *The Hamiltonian structure of the Camassa-Holm equation*, Expo. Math **15** (1997), 053–085.

[8] A. Constantin, *Existence of permanent and breaking waves for a shallow water equation: a geometric approach*, Ann. Inst. Fourier (Grenoble) **50** (2000), no. 2, 321–362.

[9] A. Constantin, *Finite propagation speed for the Camassa-Holm equation*, J. Math. Phys. **46** (2005), 023506.

[10] A. Constantin, *Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis*, CBMS-NSF Conference Series in Applied Mathematics, Vol. 81, SIAM, Philadelphia, 2011.

[11] A. Constantin and J. Escher, *On the blow-up rate and the blow-up set of breaking waves for a shallow water equation*, Math. Z. **233** (2000), no. 1, 75–91.

[12] A. Constantin, T. Kappeler, B. Kolev and P. Topalov, *On geodesic exponential maps of the Virasoro group*, Ann. Global Anal. Geom. **31** (2007), 155-180.

[13] A. Constantin and D. Lannes, *The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations*, Arch. Ration. Mech. Anal. **192** (2009), 165–186.

[14] J. Escher, *Non-metric two-component Euler equations on the circle*, Monatsh. Math. **167** (2012), no. 3–4, 449–459.

[15] J. Escher and B. Kolev, *The Degasperis-Procesi equation as a non-metric Euler equation*, Math. Z. **269** (2011), no. 3-4, 1137–1153.

[16] A. Green and P. Naghdi, *A derivation of equations for wave propagation in water of variable depth*, J. Fluid Mech. **78** (1976), 237–246.

[17] G. Gui and Y. Liu, *On the Cauchy problem for the Degasperis-Procesi equation*, Quart. Appl. Math. **69** (2011), no. 3, 445-464.

[18] D. Henry, *Infinite propagation speed for the Degasperis-Procesi equation*, J. Math. Anal. Appl. **311** (2005), no. 2, 755–759.

[19] D. D. Holm, *Hamiltonian structure for two-dimensional hydrodynamics with nonlinear dispersion*, Phys. Fluids **31** (1988), 2371–2373.

[20] D. Ionescu-Kruse, *Variational derivation of the Camassa-Holm shallow water equation*, J. Nonlinear Math. Phys. **14** (2007), 303–312.

[21] D. Ionescu-Kruse, *Variational derivation of the Green-Naghdi shallow-water equations*, J. Nonlinear Math. Phys. **19** (2012), id. 1240001.

[22] D. Ionescu-Kruse, *Variational derivation of two-component Camassa-Holm shallow water system*, Appl. Anal. 2012, DOI:10.1080/00036811.2012.667082.

[23] R. S. Johnson, *A Modern Introduction to the Mathematical Theory of Water Waves*, Cambridge Univeristy Press, 1997.

[24] Y. A. Li, *Linear stability of solitary waves of the Green-Naghdi equations*, Commun. Pure Appl. Math. **54** (2001), 501–536.

[25] Y. A. Li, *Hamiltonian structure and linear stability of solitary waves of the Green-Naghdi equations*, J. Nonlinear Math. Phys. **9** (2002), 99–105.

[26] Y. A. Li, *A shallow-water approximation to the full water wave problem*, Commun. Pure Appl. Math. **59** (2006), 1225–1285.

[27] Yu. I. Manin, *Algebraic Aspects of Nonlinear Differential Equations*, Sov. Prob. Mat. **11** (1978), 5–152.

[28] P. J. Morrison, *Mathematical Methods in Hydrodynamics and Integrability of Dynamical Systems*, edited by M. Tabor and Y. M. Treve (American Institute of Physics, New York, 1982), vol. 88 of AIP Conference Proceedings, 13-46.

[29] Y. Nutku, *On a new class of completely integrable nonlinear wave equations. II. Multi-Hamiltonian structure*, J. Math. Phys. **28** (1987), 2579–2585.

[30] P.J. Olver and Y. Nutku, *Hamiltonian structures for systems of hyperbolic conservation laws*, J. Math. Phys. **29** (1988), 1610.

[31] F. Serre, *Contribution à l'étude des écoulements permanents et variables dans les canaux*, La Houille Blanche **3** (1953), 374–388, and **6** (1953), 830–872.

[32] J. J. Stoker, *Water Waves: The Mathematical Theory with Applications*, Wiley-Interscience New-York, 1992.

[33] C. H. Su and C. S. Gardner, *Korteweg-de Vries Equation and Generalizations. III. Derivation of the Korteweg-de Vries Equation and Burgers Equation*, J. Math. Phys. **10** (1969), 536–539.