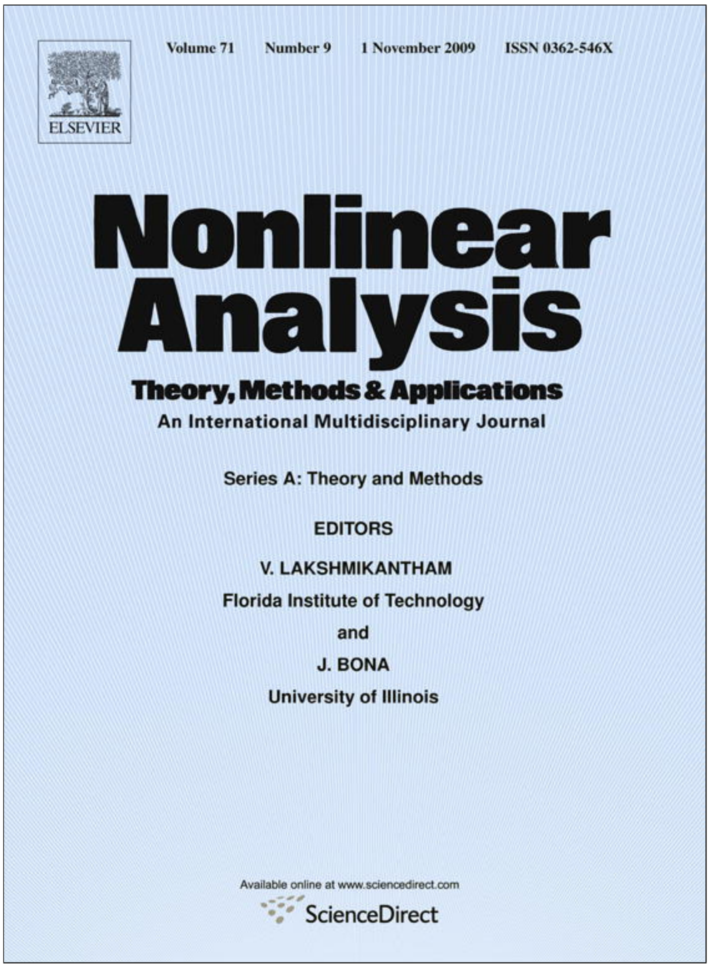
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Particle trajectories beneath small amplitude shallow water waves in constant vorticity flows

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ABSTRACT

We investigate the particle trajectories in a constant vorticity shallow water flow over a flat bed as periodic waves propagate on the water's free surface. Within the framework of small amplitude waves, we find the solutions of the nonlinear differential equations system which describes the particle motion in the considered case, and we describe the possible particle trajectories. Depending on the relation between the initial data and the constant vorticity, some particle trajectories are undulating curves to the right, or to the left, others are loops with forward drift, or with backward drift, others can follow some peculiar shapes.

1. Introduction

The motion of water particles under regular waves which propagate on the water's free surface is a very old problem. It was widely believed that the particle trajectories are closed. After the linearization of the governing equations for water waves, analysing the first approximation of the nonlinear ordinary differential equations system which describes the particle motion, one obtained that all water particles trace closed, circular or elliptic, orbits (see, for example, [1], [2–5], [6,7] — a conclusion apparently supported by photographs with a long exposure [1,6,7]).

While in this first approximation all particle paths appear to be closed, in [8] it is shown, using phase-plane considerations for the nonlinear system describing the particle motion, that in linear periodic gravity water waves no particles trajectory is actually closed, unless the free surface is flat. Each particle trajectory involves per period a backward/forward movement, and the path is an elliptical arc with a forward drift; on the flat bed the particle path degenerates to a backward/forward motion. Similar results hold for the particle trajectories in deep-water, that is, the trajectories are not closed existing a forward drift over a period, which decreases with greater depth (see [9]). These conclusions are in agreement with Stokes' observation [10]: "There is one result of a second approximation which may be of possible importance. It appears that the forward motion of the particles is not altogether compensated by their backward motion; so that, in addition to their motion of oscillation, the particles have a progressive motion in the direction of the propagation of the waves. In the case in which the depth of the fluid is very great, this progressive motion decreases rapidly as the depth of the particle considered increases".

For linearized irrotational shallow water waves, obtained very recently in [11] were the exact solutions of the nonlinear differential equations system which describes the particle motion. Beside the phase-plane analysis, the exact solutions allow a better understanding of the dynamics. In [11] it is shown that depending on the strength of underlying uniform current,

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beneath the irrotational shallow water waves some particle trajectories are undulating paths to the right or to the left, some are looping curves with a drift to the right.

The steady (traveling) linear water waves with constant vorticity were studied recently in [12,13]. Here linearity means that the waves are small perturbations of shear flows. The linear system obtained in this way is solvable. Further, making a phase portrait study in steady variables for the nonlinear differential equations system which describes the particle paths, it is found that for positive vorticity, the steady wave resembles that of the irrotational situation, though for large enough vorticity the particles trace closed orbits within the fluid domain. For negative vorticity all the fluid particles display a forward drift.

Using the same approach as in [11] within the framework of linear water waves theory, we investigate in this paper the particle trajectories beneath shallow water waves in a constant vorticity flow. The obtained results and the way in which the paper is organized are presented below.

Let us now give some references on the results obtained for the governing equations without linearization. Analyzing a free boundary problem for harmonic functions in a planar domain, in [14] it is shown that there are no closed orbits for Stokes waves of small or large amplitude propagating at the surface of water over a flat bed; for an extension of the investigation in [14] to deep-water Stokes waves see [15]. Within a period each particle experiences a backward/forward motion with a slight forward drift. In a very recent preprint [16], the results in [14] are recovered by a simpler approach and there are also described all possible particle trajectories beneath a Stokes wave. The particle trajectories change considerably according to whether the Stokes waves enter a still region of water or whether they interact with a favorable or adverse uniform current. Some particle trajectories are closed orbits, some are undulating paths and most are looping orbits that drift either to the right or to the left, depending on the underlying current.

Analyzing a free boundary problem for harmonic functions in an infinite planar domain, in [17] it is shown that under a solitary wave, each particle is transported in the wave direction but slower than the wave speed. As the solitary wave propagates, all particles located ahead of the wave crest are lifted while those behind have a downward motion.

Notice that there are only a few explicit solutions to the nonlinear governing equations: Gerstner's wave (see [18] and the discussion in [19]), the edge wave solution related to it (see [20]), and the capillary waves in water of infinite or finite depth (see [21,22]). These solutions are peculiar and their special features (a specific vorticity for Gerstner's wave and its edge wave correspondent, and complete neglect of gravity in the capillary case) are not deemed relevant to sea waves.

The present paper is organized as follows. In Section 2 we recall the governing equations for gravity water waves. In Section 3 we present their nondimensionalization and scaling. It is natural to start the investigation for shallow water waves by simplifying the governing equations via linearization. The linearized problems for an irrotational shallow water flow and for a constant vorticity shallow water flow are written in Section 4. We also obtain here the general solutions of these two linear problems. In the next section we find the solutions of the nonlinear differential equations systems which describe the particle motion in the two cases, and we describe the possible particle trajectories beneath shallow water waves. We see that these particle trajectories are not closed. Section 5.1 contains the irrotational case. Depending on the strength of the underlying uniform current, the particle trajectories are undulating path to the right or to left, are looping curves with a drift to the right, and, if there is no underlying current or the underlying current is moving in the same direction as the irrotational shallow water wave with the strength of the current smaller than 2, then, the particle trajectories obtained are not physically acceptable (Theorem 1). In dealing with the linearized problem not with the full governing equations, we expect solutions to appear which are not physically acceptable. Section 5.2 contains the case of a constant vorticity flow. Also in this case the particle trajectories are not closed. Depending on the relation between the initial data (x_0, z_0) and the constant vorticity ω_0, some particle trajectories are undulating curves to the right, or to the left, others are loops with forward drift, or with backward drift while others can follow some peculiar shapes (Theorems 2 and 3).

2. The governing equations for gravity water waves

We consider a two-dimensional inviscid incompressible fluid in a constant gravitational field. For gravity water waves these are physically reasonable assumptions (see [2] and [5]). Thus, the motion of water is given by Euler's equations

$$\begin{aligned} u_t + uu_x + vu_z &= -\frac{1}{\rho}p_x \\ v_t + uv_x + vv_z &= -\frac{1}{\rho}p_z - g. \end{aligned} \tag{1}$$

Here (x, z) are the space coordinates, $(u(x, z, t), v(x, z, t))$ is the velocity field of the water, $p(x, z, t)$ denotes the pressure, g is the constant gravitational acceleration in the negative z direction and ρ is the constant density. The assumption of incompressibility implies the equation of mass conservation

$$u_x + v_z = 0. \tag{2}$$

Let $h_0 > 0$ be the undisturbed depth of the fluid and let $z = h_0 + \eta(x, t)$ represent the free upper surface of the fluid (see Fig. 1). The boundary conditions at the free surface are a constant pressure

$$p = p_0 \quad \text{on } z = h_0 + \eta(x, t), \tag{3}$$

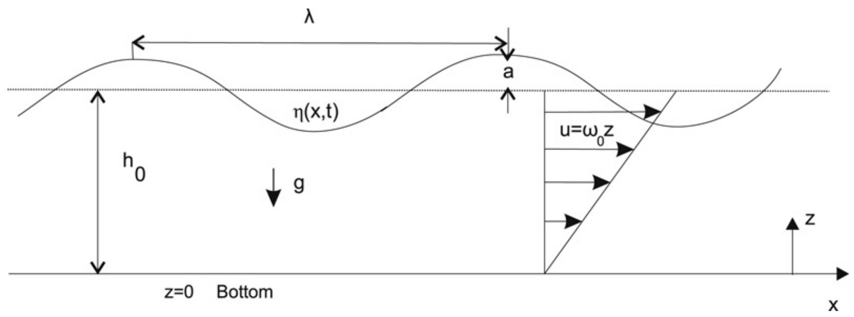

Fig. 1. Linear shear flow.

p_0 being the constant atmospheric pressure, and the continuity of fluid velocity and surface velocity

$$v = \eta_t + u\eta_x \quad \text{on } z = h_0 + \eta(x,t). \tag{4}$$

On the flat bottom $z = 0$, only one condition is required for an inviscid fluid, that is,

$$v = 0 \quad \text{on } z = 0. \tag{5}$$

Summing up, the exact solution for the water–wave problem is given by the system (1)–(5). In respect of the well-posedness for the initial-value problem for (1)–(5) there has been significant recent progress, see [23] and the references therein.

A key quantity in fluid dynamics is the *curl* of the velocity field, called vorticity. For two-dimensional flows with the velocity field $(u(x,z,t), v(x,z,t))$, we denote the scalar vorticity of the flow by

$$\omega(x,z) = u_z - v_x. \tag{6}$$

Vorticity is adequate for the specification of a flow: a flow which is uniform with depth is described by a zero vorticity (irrotational case), a constant non-zero vorticity corresponds to a linear shear flow and a non-constant vorticity indicates highly sheared flows. See in the Fig. 1 an example of a linear shear flow with a constant vorticity $\omega = \text{const} := \omega_0 > 0$.

The full Euler equations (1)–(5) are often too complicated to analyze directly. One can pursue for example a mathematical study of their periodic steady solutions in the irrotational case (see see [24,25]) or a study of their periodic steady solutions in the case of non-zero vorticity (see [26], [27]). But in order to obtain detailed information about qualitative features of water waves, it is useful to derive approximate models which are more amenable to an in-depth analysis.

3. Nondimensionalisation and scaling

In order to develop a systematic approximation procedure, we need to characterize the water–wave problem (1)–(5) in terms of the sizes of various fundamental parameters. These parameters are introduced by defining a set of non-dimensional variables.

First we introduce the appropriate length scales: the undisturbed depth of water h_0, as the vertical scale and a typical wavelength λ (see Fig. 1), as the horizontal scale. In order to define a time scale we require a suitable velocity scale. An appropriate choice for the scale of the horizontal component of the velocity is $\sqrt{gh_0}$. Then, the corresponding time scale is $\frac{\lambda}{\sqrt{gh_0}}$ and the scale for the vertical component of the velocity is $h_0\frac{\sqrt{gh_0}}{\lambda}$. The surface wave itself leads to the introduction of a typical amplitude of the wave a (see Fig. 1). For more details see [2]. Thus, we define the set of non-dimensional variables

$$x \mapsto \lambda x, \qquad z \mapsto h_0 z, \qquad \eta \mapsto a\eta, \qquad t \mapsto \frac{\lambda}{\sqrt{gh_0}}t, u \mapsto \sqrt{gh_0}u, \qquad v \mapsto h_0\frac{\sqrt{gh_0}}{\lambda}v \tag{7}$$

where, to avoid new notations, we have used the same symbols for the non-dimensional variables x, z, η, t, u, v, on the right-hand side. The partial derivatives will be replaced by

$$\begin{aligned} &u_t \mapsto \frac{gh_0}{\lambda}u_t, \qquad u_x \mapsto \frac{\sqrt{gh_0}}{\lambda}u_x, \qquad u_z \mapsto \frac{\sqrt{gh_0}}{h_0}u_z, \\ &v_t \mapsto \frac{gh_0^2}{\lambda^2}v_t, \qquad v_x \mapsto h_0\frac{\sqrt{gh_0}}{\lambda^2}v_x, \qquad v_z \mapsto \frac{\sqrt{gh_0}}{\lambda}v_z. \end{aligned} \tag{8}$$

Let us now define the non-dimensional pressure. If the water would be stationary, that is, $u \equiv v \equiv 0$, from the Eqs. (1) and (3) with $\eta = 0$, we get for a non-dimensionalized z, the hydrostatic pressure $p_0 + \rho g h_0(1-z)$. Thus, the non-dimensional pressure is defined by

$$p \mapsto p_0 + \rho g h_0(1-z) + \rho g h_0 p \tag{9}$$

therefore

$$p_x \mapsto \rho\frac{gh_0}{\lambda}p_x, \qquad p_z \mapsto -\rho g + \rho g p_z. \tag{10}$$

Taking into account (7)–(10), the water–wave problem (1)–(5) writes in non-dimensional variables, as

$$\begin{aligned} & u_t + uu_x + vu_z = -p_x \\ & \delta^2(v_t + uv_x + vv_z) = -p_z \\ & u_x + v_z = 0 \\ & v = \epsilon(\eta_t + u\eta_x) \quad \text{and} \quad p = \epsilon\eta \quad \text{on } z = 1 + \epsilon\eta(x, t) \\ & v = 0 \quad \text{on } z = 0 \end{aligned} \tag{11}$$

where we have introduced the amplitude parameter $\epsilon = \frac{a}{h_0}$ and the shallowness parameter $\delta = \frac{h_0}{\lambda}$. In view of (8), the vorticity equation (6) writes in non-dimensional variables as

$$u_z = \delta^2 v_x + \frac{\sqrt{gh_0}}{g}\omega(x, z). \tag{12}$$

For zero vorticity flows (irrotational flows) this equation is written as

$$u_z = \delta^2 v_x. \tag{13}$$

For constant non-zero vorticity flows, that is, $\omega(x, z) = \text{const} := \omega_0$, the Eq. (12) becomes

$$u_z = \delta^2 v_x + \frac{\sqrt{gh_0}}{g}\omega_0. \tag{14}$$

After the nondimensionalization of the system (1)–(5) let us now proceed with the scaling transformation. First we observe that, on $z = 1 + \epsilon\eta$, both v and p are proportional to ϵ. This is consistent with the fact that as $\epsilon \to 0$ we must have $v \to 0$ and $p \to 0$, and it leads to the following scaling of the non-dimensional variables

$$p \mapsto \epsilon p, \qquad (u, v) \mapsto \epsilon(u, v) \tag{15}$$

where we avoided again the introduction of a new notation. The problem (11) becomes

$$\begin{aligned} & u_t + \epsilon(uu_x + vu_z) = -p_x \\ & \delta^2[v_t + \epsilon(uv_x + vv_z)] = -p_z \\ & u_x + v_z = 0 \\ & v = \eta_t + \epsilon u\eta_x \quad \text{and} \quad p = \eta \quad \text{on } z = 1 + \epsilon\eta(x, t) \\ & v = 0 \quad \text{on } z = 0 \end{aligned} \tag{16}$$

and the Eq. (12) keeps the same form.

In what follows we will consider in turn the cases of an irrotational flow and a constant vorticity flow. The system which describes our problem in the irrotational case is given by

$$\begin{aligned} & u_t + \epsilon(uu_x + vu_z) = -p_x \\ & \delta^2[v_t + \epsilon(uv_x + vv_z)] = -p_z \\ & u_x + v_z = 0 \\ & u_z = \delta^2 v_x \\ & v = \eta_t + \epsilon u\eta_x \quad \text{and} \quad p = \eta \quad \text{on } z = 1 + \epsilon\eta(x, t) \\ & v = 0 \quad \text{on } z = 0. \end{aligned} \tag{17}$$

In the constant vorticity case, the problem is described by the following system

$$\begin{aligned} & u_t + \epsilon(uu_x + vu_z) = -p_x \\ & \delta^2[v_t + \epsilon(uv_x + vv_z)] = -p_z \\ & u_x + v_z = 0 \\ & u_z = \delta^2 v_x + \frac{\sqrt{gh_0}}{g}\omega_0 \\ & v = \eta_t + \epsilon u\eta_x \quad \text{and} \quad p = \eta \quad \text{on } z = 1 + \epsilon\eta(x, t) \\ & v = 0 \quad \text{on } z = 0. \end{aligned} \tag{18}$$

4. The linearized problem

The two important parameters ϵ and δ that arise in water–waves theories, are used to define various approximations of the governing equations and the boundary conditions. The scaled versions (17) and (18) of the equations for our problem, allow immediately the identification of the linearized problem, by letting $\epsilon \to 0$, for arbitrary δ. The linearized problem in the shallow water regime is obtained by letting further $\delta \to 0$. Thus, we get the following linear systems, in the irrotational case

$$\begin{aligned} & u_t + p_x = 0 \\ & p_z = 0 \\ & u_x + v_z = 0 \\ & u_z = 0 \\ & v = \eta_t \quad \text{and} \quad p = \eta \quad \text{on } z = 1 \\ & v = 0, \quad \text{on } z = 0 \end{aligned} \tag{19}$$

and in the constant vorticity case

$$\begin{aligned} & u_t + p_x = 0 \\ & p_z = 0 \\ & u_x + v_z = 0 \\ & u_z = \frac{\sqrt{gh_0}}{g}\omega_0 \\ & v = \eta_t \quad \text{and} \quad p = \eta \quad \text{on } z = 1 \\ & v = 0 \quad \text{on } z = 0. \end{aligned} \tag{20}$$

From the second equation in (19), respectively (20), we get in the both cases that p does not depend on z. Because $p = \eta(x, t)$ on $z = 1$, we have

$$p = \eta(x, t) \quad \text{for any } 0 \le z \le 1. \tag{21}$$

Therefore, using the first equation and the fourth equation in (19), respectively (20), we obtain, in the irrotational case

$$u = -\int_0^t \eta_x(x, s)\mathrm{d}s + \mathcal{F}(x). \tag{22}$$

and in the constant vorticity case

$$u = -\int_0^t \eta_x(x, s)\mathrm{d}s + \mathcal{F}(x) + \frac{\omega_0\sqrt{gh_0}}{g}z \tag{23}$$

where $\mathcal{F}$ is an arbitrary function such that

$$\mathcal{F}(x) = u(x, 0, 0). \tag{24}$$

Differentiating (22), respectively (23), with respect to x and using the third equation in (19), respectively (20), we get, after an integration against z, the same equation in the both cases

$$v = -zu_x = z\left(\int_0^t \eta_{xx}(x, s)\mathrm{d}s - \mathcal{F}'(x)\right). \tag{25}$$

In view of the fifth equation in (19) and (20), we get after a differentiation with respect to t, that η has to satisfy the equation

$$\eta_{tt} - \eta_{xx} = 0. \tag{26}$$

The general solution of this equation is $\eta(x, t) = f(x - t) + g(x + t)$, where f and g are differentiable functions. It is convenient first to restrict ourselves to waves which propagate in only one direction, thus, we choose

$$\eta(x, t) = f(x - t). \tag{27}$$

From (25) and (27) and the condition $v = \eta_t$ on $z = 1$, we obtain

$$\mathcal{F}(x) = f(x) + c_0 \tag{28}$$

where c_0 is constant.

Therefore, in the irrotational case, taking into account (21), (22), (25), (27) and (28), the solution of the linear system (19) is given by

$$\begin{aligned} & \eta(x, t) = f(x - t) \\ & p(x, t) = f(x - t) \\ & u(x, z, t) = f(x - t) + c_0 \\ & v(x, z, t) = -zf'(x - t) = -zu_x. \end{aligned} \tag{29}$$

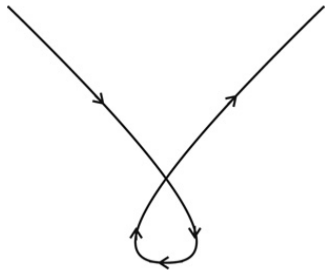

Fig. 2. Particle trajectory, in the irrotational case, for $c_0 = 0$.

In the case of a constant vorticity flow, from (21), (23), (25), (27) and (28), the solution of the linear system (20) is given by

$$\begin{aligned} \eta(x,t) &= f(x-t)\\ p(x,t) &= f(x-t)\\ u(x,z,t) &= f(x-t) + \frac{\omega_0\sqrt{gh_0}}{g}z + c_0\\ v(x,z,t) &= -zf'(x-t) = -zu_x. \end{aligned} \tag{30}$$

5. Particle trajectories beneath linearized shallow water waves

Let $(x(t), z(t))$ be the path of a particle in the fluid domain, with location $(x(0), z(0))$ at time $t = 0$. The motion of the particle is described by the differential system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = u(x,z,t)\\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = v(x,z,t) \end{cases} \tag{31}$$

with the initial data $(x(0), z(0)) := (x_0, z_0)$.

5.1. *The case of an irrotational flow*

This case was investigated in [11]. We summarize below the results presented in detail in reference [11]. Some new observations concerning the solutions shown in Figs. 2 and 6 are added.

Making the *Ansatz*

$$f(x-t) = \cos(2\pi(x-t)) \tag{32}$$

from (29), the differential system (31) becomes

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \cos(2\pi(x-t)) + c_0\\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = 2\pi z\sin(2\pi(x-t)). \end{cases} \tag{33}$$

Notice that the constant c_0 is the average of the horizontal fluid velocity over any horizontal segment of length 1, that is,

$$c_0 = \frac{1}{1}\int_x^{x+1} u(s,z,t)\mathrm{d}s, \tag{34}$$

representing therefore the strength of the underlying uniform current. Thus, $c_0 = 0$ will correspond to a region of still water with no underlying current, $c_0 > 0$ will characterize a favorable uniform current and $c_0 < 0$ will characterize an adverse uniform current.

The right-hand side of the differential system (33) is smooth and bounded, therefore, the unique solution of the Cauchy problem with initial data (x_0, z_0) is defined globally in time.

To study the exact solution of the system (33) it is more convenient to re-write it in the following moving frame

$$X = 2\pi(x-t), \qquad Z = z. \tag{35}$$

This transformation yields

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = 2\pi\cos(X) + 2\pi(c_0 - 1)\\ \dfrac{\mathrm{d}Z}{\mathrm{d}t} = 2\pi Z\sin(X). \end{cases} \tag{36}$$

For $\mathbf{c_0} = \mathbf{0}$, we get the following exact solution of the system (36):

$$X(t) = 2\text{arccot}(2\pi t + a), \quad a = \text{constant} \tag{37}$$

$$Z(t) = Z(0)\exp\left(\int_0^t 2\pi \sin(X(s))\,\mathrm{d}s\right) = Z(0)\exp\left(\ln\left[\frac{1+(2\pi t+a)^2}{1+a^2}\right]\right). \tag{38}$$

Taking into account (35), (37) and (38), we obtain the solution of the system (33) with the initial data (x_0, z_0):

$$\begin{cases} x(t) = t + \dfrac{1}{\pi}\text{arccot}(2\pi t + a) \\ z(t) = \dfrac{z_0}{1+a^2}[1+(2\pi t+a)^2]. \end{cases} \tag{39}$$

From the initial conditions, we have $a := \cot(\pi x_0)$.

Studying the derivatives of $x(t), z(t)$ with respect to t and the limits of $x(t), z(t), \frac{z(t)}{x(t)}$ for $t \to \pm\infty$, we draw in Fig. 2 the graph of the parametric curve (39).

As is shown in Fig. 2, the coordinate z increases indefinitely in time, thus, the solution (39) is *not* physically acceptable. In dealing with the linearized problem not with the full governing equations, we expect solutions to appear which are not physically acceptable.

For $\mathbf{c_0}(\mathbf{c_0} - \mathbf{2}) > \mathbf{0}$, the exact solution of the system (36) has the following expression:

$$X(t) = 2\text{arccot}\Big[\mathfrak{C}_0 \tan(\alpha(t))\Big], \tag{40}$$

$$Z(t) = Z(0)\exp\left(\int_0^t \frac{4\pi\mathfrak{C}_0 \tan(\alpha(s))}{1+\Big[\mathfrak{C}_0\tan(\alpha(s))\Big]^2}\mathrm{d}s\right) \tag{41}$$

where

$$\mathfrak{C}_0 := \sqrt{\frac{c_0 - 2}{c_0}} \tag{42}$$

$$\alpha(t) := -\frac{c_0\mathfrak{C}_0}{2}(2\pi t + a). \tag{43}$$

From (35), (40) and (41), we obtain the solution of the system (33) with the initial data (x_0, z_0), $z_0 > 0$,

$$\begin{cases} x(t) = t + \dfrac{1}{\pi}\text{arccot}\Big[\mathfrak{C}_0\tan(\alpha(t))\Big] \\ z(t) = z_0\exp\left(\displaystyle\int_0^t \frac{4\pi\mathfrak{C}_0\tan(\alpha(s))}{1+\Big[\mathfrak{C}_0\tan(\alpha(s))\Big]^2}\mathrm{d}s\right). \end{cases} \tag{44}$$

Studying the derivatives of $x(t)$ and $z(t)$ with respect to t, we get the graphs presented in Figs. 3–5.

For $\mathbf{c_0} \in (\mathbf{0}, \mathbf{2}]$, the exact solution of the system (36) has for $|\cot\left(\frac{X}{2}\right)| > \mathfrak{K}_0$ the following expression:

$$X(t) = 2\text{arccot}\Big[\mathfrak{K}_0\coth(\beta(t))\Big], \qquad Z(t) = Z(0)\exp\left(\int_0^t \frac{4\pi\mathfrak{K}_0\coth(\beta(s))}{1+\Big[\mathfrak{K}_0\coth(\beta(s))\Big]^2}\mathrm{d}s\right), \tag{45}$$

and for $|\cot\left(\frac{X}{2}\right)| < \mathfrak{K}_0$ the following expression:

$$X(t) = 2\text{arccot}\Big[\mathfrak{K}_0\tanh(\beta(t))\Big], \qquad Z(t) = Z(0)\exp\left(\int_0^t \frac{4\pi\mathfrak{K}_0\tanh(\beta(s))}{1+\Big[\mathfrak{K}_0\tanh(\beta(s))\Big]^2}\mathrm{d}s\right). \tag{46}$$

In the above formulas we have denoted by

$$\mathfrak{K}_0 := \sqrt{\frac{2 - c_0}{c_0}} \tag{47}$$

$$\beta(t) := \frac{c_0\mathfrak{K}_0}{2}(2\pi t + a). \tag{48}$$

Fig. 3. Particle trajectory, in the irrotational case, for $c_0 < -1$.

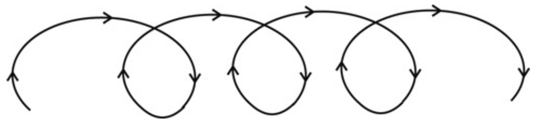

Fig. 4. Particle trajectory, in the irrotational case, for $-1 \leq c_0 < 0$.

Fig. 5. Particle trajectory, in the irrotational case, for $c_0 > 2$.

From (35), (45) and (46), we obtain the solution of the system (33) with the initial data (x_0, z_0), $z_0 > 0$,

$$\begin{cases} x(t) = t + \dfrac{1}{\pi}\operatorname{arccot}\Big[\mathfrak{K}_0 \coth\left(\beta(t)\right)\Big] \\ z(t) = z_0 \exp\left(\displaystyle\int_0^t \frac{4\pi\mathfrak{K}_0 \coth\left(\beta(s)\right)}{1+\Big[\mathfrak{K}_0 \coth\left(\beta(s)\right)\Big]^2} \mathrm{d}s\right) \end{cases} \tag{49}$$

or

$$\begin{cases} x(t) = t + \dfrac{1}{\pi}\operatorname{arccot}\Big[\mathfrak{K}_0 \tanh\left(\beta(t)\right)\Big] \\ z(t) = z_0 \exp\left(\displaystyle\int_0^t \frac{4\pi\mathfrak{K}_0 \tanh\left(\beta(s)\right)}{1+\Big[\mathfrak{K}_0 \tanh\left(\beta(s)\right)\Big]^2} \mathrm{d}s\right). \end{cases} \tag{50}$$

The graphs of the parametric curves in (49) and (50) are drawn in Fig. 6

As is shown in Fig. 6, the coordinate z increases indefinitely in time, hence, the solutions (49) and (50), are *not* physically acceptable.

Thus, one gets the following theorem:

Theorem 1. *In the case that the underlying uniform current is moving in the same direction as an irrotational shallow water wave and the strength of the current is bigger than 2, then the particle trajectories beneath the wave are undulating paths to the right (see Fig. 5).*

In the case that the underlying uniform current is moving in the opposite direction as an irrotational shallow water wave and the strength of the current is smaller than -1, then the particle trajectories beneath the wave are undulating paths to the left (see Fig. 3). If the strength of the adverse current is bigger than -1, then the particle trajectories are loops with forward drift (see Fig. 4).

In the case of no underlying current and in the case that the underlying uniform current is moving in the same direction as an irrotational shallow water wave with the strength of the current smaller than 2, the particle trajectories obtained (see Figs. 2 and 6) are not physically acceptable. In these cases it seems necessary to study the full nonlinear problem.

5.2. The case of a constant vorticity flow

Making the *Ansatz*

$$f(x-t) = \cos(2\pi(x-t)) \tag{51}$$

from (30), the differential system (31) becomes

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \cos(2\pi(x-t)) + \dfrac{\omega_0\sqrt{gh_0}}{g} z + c_0 \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = 2\pi z \sin(2\pi(x-t)). \end{cases} \tag{52}$$

Notice that the constant c_0 is the average of the horizontal fluid velocity on the bottom over any horizontal segment of length 1, that is,

$$c_0 = \frac{1}{1}\int_x^{x+1} u(s, 0, t)\mathrm{d}s. \tag{53}$$

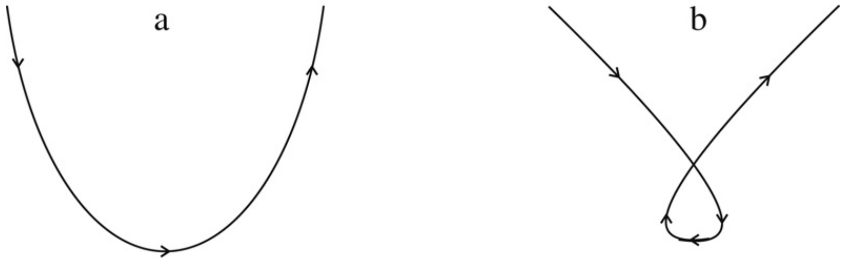

Fig. 6. Particle trajectory, in the irrotational case, for $0 < c_0 \leq 2$.

The right-hand side of the differential system (52) is smooth and bounded, therefore, the unique solution of the Cauchy problem with initial data (x_0, z_0) is defined globally in time.

To study the exact solution of the system (52) it is more convenient to re-write it in the following moving frame

$$X = 2\pi(x - t), \qquad Z = z. \tag{54}$$

This transformation yields

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = 2\pi \cos(X) + 2\pi \dfrac{\omega_0\sqrt{gh_0}}{g} Z + 2\pi(c_0 - 1) \\ \dfrac{\mathrm{d}Z}{\mathrm{d}t} = 2\pi Z \sin(X). \end{cases} \tag{55}$$

Let us now investigate the differential system (55).

Differentiating with respect to t the first equation in (55) and taking into account (55), this first equation becomes

$$\frac{\mathrm{d}^2 X}{\mathrm{d}t^2} = 4\pi^2 \sin(X)\left[1 - c_0 - \cos(X)\right]. \tag{56}$$

Like in the irrotational case (see [11]), we use the following substitution (see [28], I.76, page 308)

$$\cot\left(\frac{X}{2}\right) = y, \qquad \sin(X) = \frac{2y}{y^2+1}, \qquad \cos(X) = \frac{y^2-1}{y^2+1}, \qquad \mathrm{d}X = -\frac{2}{y^2+1}\mathrm{d}y. \tag{57}$$

In the new variable, the Eq. (56) takes the form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{2y}{y^2+1}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \frac{8\pi^2 y}{y^2+1} - 4\pi^2 c_0 y = 0. \tag{58}$$

The solution of the system (55) has then the following expression:

$$X(t) = 2\mathrm{arccot}(y(t)), \tag{59}$$

$$Z(t) = Z(0)\exp\left(\int_0^t \frac{4\pi y(s)}{y^2(s)+1}\,\mathrm{d}s\right) \tag{60}$$

with $y(t)$ satisfying the ordinary differential equation (58).

From (54), (59) and (60), we obtain the solution of the system (52) with the initial data (x_0, z_0):

$$\begin{cases} x(t) = t + \dfrac{1}{\pi}\mathrm{arccot}(y(t)) \\ z(t) = z_0 \exp\left(\displaystyle\int_0^t \frac{4\pi y(s)}{y^2(s)+1}\,\mathrm{d}s\right) \end{cases} \tag{61}$$

where $y(t)$ satisfies the ordinary differential equation (58) with the following initial conditions:

$$\begin{aligned} & y(0) = \cot(\pi x_0) \\ & \frac{\mathrm{d}y}{\mathrm{d}t}(0) = \pi\left[2 - \left(\frac{\omega_0\sqrt{gh_0}}{g}z_0 + c_0\right)\left(\cot^2(\pi x_0) + 1\right)\right] \end{aligned} \tag{62}$$

obtained from (57) and the first equation in (55).

5.2.1. The case $\mathbf{c_0 = 0}$

In this case the Eq. (58) writes as

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{2y}{y^2+1}\left[\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - 4\pi^2\right]. \tag{63}$$

For

$$\frac{\mathrm{d}y}{\mathrm{d}t} \neq \pm 2\pi \iff y \neq \pm 2\pi t + a, \quad \text{with } a \text{ constant,} \tag{64}$$

we put the Eq. (63) into the following form

$$\frac{y'}{(y')^2 - 4\pi^2} y'' = \frac{2y}{y^2+1} y', \tag{65}$$

where $y' := \frac{\mathrm{d}y}{\mathrm{d}t}$, $y'' := \frac{d^2y}{dt^2}$.

We integrate in (65) and we get

$$(y')^2 - 4\pi^2 = A^2(y^2+1)^2 \iff (y')^2 = A^2(y^2+1)^2 + 4\pi^2. \tag{66}$$

A being an integration constant, which, taking into account (62), can be expressed as a function of the initial data (x_0, z_0) and of the constant vorticity ω_0.

The solution of the Eq. (66) involves an elliptic integral of first kind, that is,

$$\pm \int \frac{\mathrm{d}y}{\sqrt{A^2(y^2+1)^2 + 4\pi^2}} = t. \tag{67}$$

The elliptic integral of first kind from (67) may by reduced to Legendre's normal form. In order to do this we first consider the substitution

$$y^2 = w. \tag{68}$$

The left hand side in (67) becomes

$$\pm \int \frac{\mathrm{d}y}{\sqrt{A^2(y^2+1)^2 + 4\pi^2}} = \pm \frac{1}{2|A|} \int \frac{\mathrm{d}w}{\sqrt{w\left(w^2 + 2w + 1 + \frac{4\pi^2}{A^2}\right)}}.$$

We introduce now the variable φ by (see [29] Ch. VI, §4, page 603)

$$w = \sqrt{1 + \frac{4\pi^2}{A^2}} \tan^2 \frac{\varphi}{2} \tag{69}$$

and we get

$$w\left(w^2 + 2w + 1 + \frac{4\pi^2}{A^2}\right) = \left(\sqrt{1 + \frac{4\pi^2}{A^2}}\right)^3 \left(1 - k^2 \sin^2 \varphi\right) \frac{\tan^2 \frac{\varphi}{2}}{\cos^4 \frac{\varphi}{2}}$$

$$\mathrm{d}w = \sqrt{1 + \frac{4\pi^2}{A^2}} \frac{\tan \frac{\varphi}{2}}{\cos^2 \frac{\varphi}{2}} \mathrm{d}\varphi$$

where the constant $0 < k^2 < 1$ is given by

$$k^2 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1 + \frac{4\pi^2}{A^2}}}\right).$$

Therefore we obtain the Legendre normal form of the integral in (67), that is,

$$\pm \frac{1}{2|A|\left(1 + \frac{4\pi^2}{A^2}\right)} \int \frac{\mathrm{d}\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}. \tag{70}$$

Taking into account (66), the derivatives of $x(t)$ and $z(t)$ from (61) with respect to t have the expressions

$$\begin{aligned} x'(t) &= \frac{\pi(y^2+1) \mp \sqrt{A^2(y^2+1)^2 + 4\pi^2}}{\pi(y^2+1)} \\ z'(t) &= \frac{4\pi z_0 y}{y^2+1} \exp\left(\int_0^t \frac{4\pi y(s)}{y^2(s)+1} \mathrm{d}s\right) \end{aligned} \tag{71}$$

where $y = y(t)$ is given implicitly by (67).

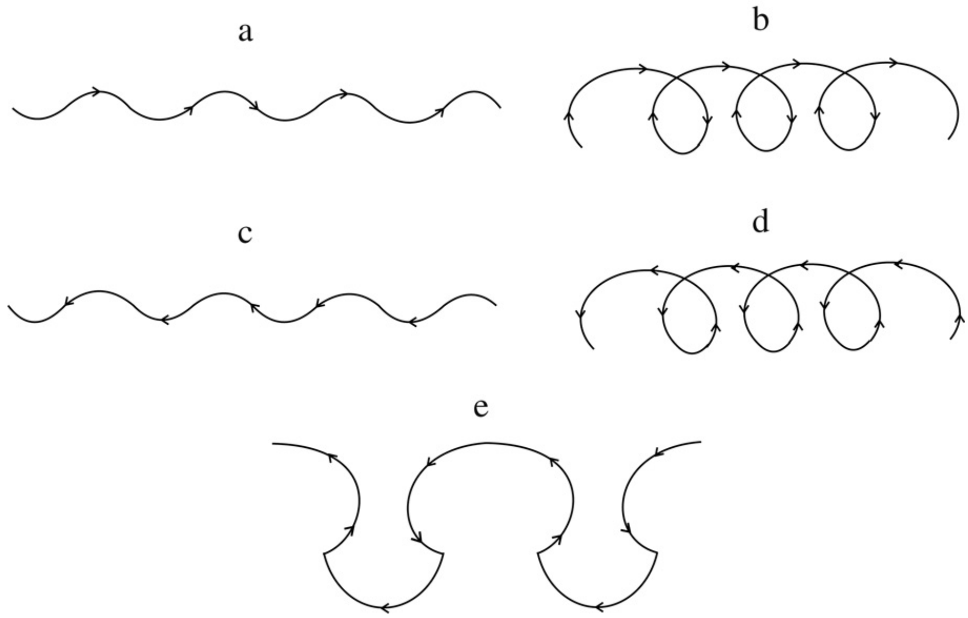

Fig. 7. Particle trajectory in the constant vorticity case.

For the alternative with "+" in the expression (71) of $x'(t)$, we obtain that $x'(t) > 0$, for all t. The sign of $z'(t)$ depends on $y(t)$. For $y(t) < 0$ we get $z'(t) < 0$, and for $y(t) > 0$ we get $z'(t) > 0$. Concerning the zeros of the function $y(t)$ we can tell more by looking at the Legendre normal form of the integral in (67), that is, at the expression (70). The inverse of this integral is, excepting a constant factor, the Jacobian elliptic function $\mathrm{sn}(t) := \sin\varphi$ (see, for example, [30]). Thus, taking into account (68) and (69) and the formula $\tan\frac{\varphi}{2} = \frac{\sin\varphi}{1+\cos\varphi}$, we get that the zeros of the function $y(t)$ are the zeros of the periodic Jacobian elliptic function $\mathrm{sn}(t)$ (for the graph of $\mathrm{sn}(t)$ see, for example, [30], Figure 10, page 26). Therefore, the particle trajectory looks like the one in Fig. 7a.

Let us consider now the alternative with "−" in the expression (71) of $x'(t)$.

If

$$|A| > \pi \tag{72}$$

then we have $x'(t) < 0$, for all t. The sign of $z'(t)$ depends on $y(t)$: for $y(t) < 0$ we have $z'(t) < 0$, and for $y(t) > 0$ we have $z'(t) > 0$. The function $y(t)$ being periodic and having multiple zeros, the particle trajectory looks like the one in Fig. 7c.

If

$$|A| < \pi \tag{73}$$

then the zeros of $x'(t)$ are obtained for

$$y(t) = \pm\sqrt{\frac{2\pi}{\sqrt{\pi^2 - A^2}} - 1}. \tag{74}$$

The sign of $z'(t)$ depends on $y(t)$ as above, that is, $z'(t) < 0$ for $y(t) < 0$, and $z'(t) > 0$ for $y(t) > 0$. Summing up,

for $y(t) \in \left(-\infty, -\sqrt{\frac{2\pi}{\sqrt{\pi^2-A^2}} - 1}\right)$ we have $x'(t) > 0, z'(t) < 0$,

for $y(t) \in \left(-\sqrt{\frac{2\pi}{\sqrt{\pi^2-A^2}} - 1}, 0\right)$ we have $x'(t) < 0, z'(t) < 0$,

for $y(t) \in \left(0, \sqrt{\frac{2\pi}{\sqrt{\pi^2-A^2}} - 1}\right)$ we have $x'(t) < 0, z'(t) > 0$,

for $y(t) \in \left(\sqrt{\frac{2\pi}{\sqrt{\pi^2-A^2}} - 1}, \infty\right)$ we have $x'(t) > 0, z'(t) > 0$.

If the periodic function $y(t)$ has values in each interval from above, then the particle trajectory looks like in Fig. 7b.

Theorem 2. *As periodic waves propagate on the water's free surface of a constant vorticity shallow water flow over a flat bed, with the average of the horizontal fluid velocity on the bottom over any horizontal segment of length* 1 *equals zero, a single pattern for all the water particles does not exist. The particle paths are not closed, and depending on the relation between the initial data* (x_0, z_0) *and the constant vorticity* ω_0*, some particle trajectories are undulating curves to the right (see* Fig. 7a*), or to the left (see* Fig. 7c*), others are loops with forward drift (see* Fig. 7b*).*

5.2.2. The case $\mathbf{c_0 \neq 0}$

In this case we transform the Eq. (58), by (see [28], 6.45, page 551)

$$w(y) := \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2, \tag{75}$$

into the following equation

$$\frac{1}{2}\frac{\mathrm{d}w}{\mathrm{d}y} - \frac{2y}{y^2+1}w(y) + \frac{8\pi^2 y}{y^2+1} - 4\pi^2 c_0 y = 0. \tag{76}$$

The homogeneous equation:

$$\frac{1}{2}\frac{\mathrm{d}w}{\mathrm{d}y} - \frac{2y}{y^2+1}w(y) = 0 \tag{77}$$

has the solution

$$w_h(y) = B(y^2+1)^2 \tag{78}$$

where B is an integration constant. By the method of variation of constants, the general solution of the non-homogeneous equation (76) is given by

$$w(y) = B(y)(y^2+1)^2 \tag{79}$$

where $B(y)$ is a continuous function which satisfies the equation

$$\frac{\mathrm{d}B}{\mathrm{d}y} = \frac{8\pi^2 c_0 y}{(y^2+1)^2} - \frac{16\pi^2 y}{(y^2+1)^3}. \tag{80}$$

The solution of the Eq. (80) is

$$B(y) = -\frac{4\pi^2 c_0}{y^2+1} + \frac{4\pi^2}{(y^2+1)^2} + \mathcal{C} \tag{81}$$

where $\mathcal{C}$ is constant. Therefore, the solution of the non-homogeneous equation (76) has the expression

$$w(y) = \mathcal{C}(y^2+1)^2 - 4\pi^2 c_0(y^2+1) + 4\pi^2. \tag{82}$$

Taking into account (75), we now obtain, instead of the Eq. (66), the following equation

$$(y')^2 = \mathcal{C}(y^2+1)^2 - 4\pi^2 c_0(y^2+1) + 4\pi^2 \tag{83}$$

where $y' := \frac{\mathrm{d}y}{\mathrm{d}t}$, and the following condition has to be satisfied

$$\pi^2 c_0^2 < \mathcal{C} \tag{84}$$

in order to have the right hand side in (83) bigger then zero for any y.

The solution of the Eq. (83) involves an elliptic integral of first kind

$$\pm\int \frac{\mathrm{d}y}{\sqrt{\mathcal{C}(y^2+1)^2 - 4\pi^2 c_0(y^2+1) + 4\pi^2}} = t. \tag{85}$$

The elliptic integral of first kind from (85) may by reduced to Legendre's normal form. In order to do this we first consider the substitution

$$y^2 = w.$$

The left hand side in (85) becomes

$$\pm\int \frac{\mathrm{d}y}{\sqrt{\mathcal{C}(y^2+1)^2 - 4\pi^2 c_0(y^2+1) + 4\pi^2}} = \pm\frac{1}{2}\int \frac{\mathrm{d}w}{\sqrt{\mathcal{C}w\left[w^2 + w\left(2 - \frac{4\pi^2 c_0}{\mathcal{C}}\right) + 1 + \frac{4\pi^2}{\mathcal{C}} - \frac{4\pi^2 c_0}{\mathcal{C}}\right]}}.$$

We introduce now the variable φ by (see [29] Ch. VI, Section 4, page 603)

$$w = \sqrt{1 + \frac{4\pi^2}{\mathcal{C}} - \frac{4\pi^2 c_0}{\mathcal{C}}}\,\tan^2\frac{\varphi}{2}$$

and we get

$$w\left[w^2+w\left(2-\frac{4\pi^2c_0}{\mathcal{C}}\right)+1+\frac{4\pi^2}{\mathcal{C}}-\frac{4\pi^2c_0}{\mathcal{C}}\right]=\left(\sqrt{1+\frac{4\pi^2}{\mathcal{C}}-\frac{4\pi^2c_0}{\mathcal{C}}}\right)^3\left(1-k^2\sin^2\varphi\right)\frac{\tan^2\frac{\varphi}{2}}{\cos^4\frac{\varphi}{2}}$$

$$\mathrm{d}w=\sqrt{1+\frac{4\pi^2}{\mathcal{C}}-\frac{4\pi^2c_0}{\mathcal{C}}}\,\frac{\tan\frac{\varphi}{2}}{\cos^2\frac{\varphi}{2}}\mathrm{d}\varphi$$

where the constant $0<k^2<1$ is given by

$$k^2=\frac{1}{2}\left(1-\frac{\mathcal{C}-2\pi^2c_0}{\mathcal{C}\sqrt{1+\frac{4\pi^2}{\mathcal{C}}-\frac{4\pi^2c_0}{\mathcal{C}}}}\right).$$

Therefore we obtain the Legendre normal form of the integral in (85), that is,

$$\pm\frac{1}{2\sqrt{\mathcal{C}}\left(1+\frac{4\pi^2}{\mathcal{C}}-\frac{4\pi^2c_0}{\mathcal{C}}\right)}\int\frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}}. \tag{86}$$

Taking into account (83), the derivatives of $x(t)$ and $z(t)$ from (61) with respect to t, have the expressions

$$\begin{aligned} x'(t)&=\frac{\pi(y^2+1)\mp\sqrt{\mathcal{C}(y^2+1)^2-4\pi^2c_0(y^2+1)+4\pi^2}}{\pi(y^2+1)}\\ z'(t)&=\frac{4\pi z_0y}{y^2+1}\exp\left(\int_0^t\frac{4\pi y(s)}{y^2(s)+1}\mathrm{d}s\right)\end{aligned} \tag{87}$$

where $y=y(t)$ is given implicitly by (85).

For the alternative with "+" in the expression (87) of $x'(t)$, we obtain that $x'(t)>0$, for all t. The sign of $z'(t)$ depends on $y(t)$. For $y(t)<0$ we get $z'(t)<0$, and for $y(t)>0$ we get $z'(t)>0$. Taking into account the Legendre normal form of the integral in (85), that is, the expression in (86), and using the same arguments as in the case $c_0=0$ on the page 11, we get that the zeros of $y(t)$ are the zeros of the periodic Jacobian elliptic function $\mathrm{sn}(t)$. Therefore, in this case the particle trajectory looks like the one in Fig. 7a.

Let us consider now the alternative with "−" in the expression (87) of $x'(t)$. The zeros of $x'(t)$ are obtained by solving the equation

$$(\pi^2-\mathcal{C})(y^2+1)^2+4\pi^2c_0(y^2+1)-4\pi^2=0. \tag{88}$$

The discriminant of the quadratic equation in W

$$(\pi^2-\mathcal{C})W^2+4\pi^2c_0W-4\pi^2=0 \tag{89}$$

is

$$\Delta=16\pi^2(\pi^2c_0^2+\pi^2-\mathcal{C}). \tag{90}$$

(I) If

$$\mathcal{C}>\pi^2c_0^2+\pi^2 \tag{91}$$

then $\Delta<0$ and $\pi^2-\mathcal{C}<0$. This yields $x'(t)<0$, for all t. The sign of $z'(t)$ depends on $y(t)$ as was mentioned above. The function $y(t)$ being periodic and having multiple zeros, the particle trajectory looks like the one in Fig. 7c.

(II) If

$$\mathcal{C}<\pi^2c_0^2+\pi^2 \tag{92}$$

then $\Delta>0$ and the Eq. (89) has two real solutions.

II (a) If in addition to the condition (92) we have

$$\pi^2-\mathcal{C}>0 \tag{93}$$

then, one of the solutions of the Eq. (89) is positive and the other one is negative. We denote the positive solution of the Eq. (89) by

$$W_1:=\frac{-2\pi^2c_0+2\pi\sqrt{\pi^2c_0^2+\pi^2-\mathcal{C}}}{\pi^2-\mathcal{C}}. \tag{94}$$

Further, if $W_1 - 1 < 0$, then the sign of $x'(t)$ is given by $\pi^2 - \mathcal{C}$, that is, in this case $x'(t) > 0$, for all t. Taking into account that $z'(t) > 0$ for $y(t) > 0$, and $z'(t) < 0$ for $y(t) < 0$, the particle trajectory looks now like in Fig. 7a.

If $W_1 - 1 > 0$, then the zeros of $x'(t)$ are obtained for

$$y(t) = \pm\sqrt{W_1 - 1}. \tag{95}$$

The sign of $z'(t)$ depends on $y(t)$ as above. Thus,

for $y(t) \in (-\infty, -\sqrt{W_1 - 1})$ we have $x'(t) > 0, z'(t) < 0$,

for $y(t) \in (-\sqrt{W_1 - 1}, 0)$ we have $x'(t) < 0, z'(t) < 0$,

for $y(t) \in (0, \sqrt{W_1 - 1})$ we have $x'(t) < 0, z'(t) > 0$,

for $y(t) \in (\sqrt{W_1 - 1}, \infty)$ we have $x'(t) > 0, z'(t) > 0$.

If the periodic function $y(t)$ has values in each interval from above, then the particle trajectory looks like the one in Fig. 7b.

II (b) If in addition to the condition (92) we have

$$\pi^2 - \mathcal{C} < 0 \tag{96}$$

then, the solutions of the Eq. (89) are both negative or both positive.

If

$$c_0 < 0 \tag{97}$$

then both solutions of (89) are negative. Therefore, the sign of $x'(t)$ is given by $\pi^2 - \mathcal{C}$, that is, in this case $x'(t) < 0$, for all t. How $z'(t) < 0$ for $y(t) < 0$ and $z'(t) > 0$ for $y(t) > 0$, the particle trajectory looks like the one in Fig. 7c.

If

$$c_0 > 0 \tag{98}$$

then both solutions of (89) are positive. We denote these positive solutions by

$$W_1 := \frac{-2\pi^2 c_0 + 2\pi\sqrt{\pi^2 c_0^2 + \pi^2 - \mathcal{C}}}{\pi^2 - \mathcal{C}},$$
$$W_2 := \frac{-2\pi^2 c_0 - 2\pi\sqrt{\pi^2 c_0^2 + \pi^2 - \mathcal{C}}}{\pi^2 - \mathcal{C}}. \tag{99}$$

We observe that $W_1 < W_2$.

Further, if $W_1 - 1 < 0$ and $W_2 - 1 < 0$ then the sign of $x'(t)$ is given by $\pi^2 - \mathcal{C}$, that is, in this case $x'(t) < 0$, for all t. Therefore, the particle trajectory looks like the one in Fig. 7c.

If $W_1 - 1 < 0$ and $W_2 - 1 > 0$ then the zeros of $x'(t)$ are obtained for

$$y(t) = \pm\sqrt{W_2 - 1}. \tag{100}$$

The sign of $z'(t)$ depends on $y(t)$ as above. Thus, taking into account (96),

for $y(t) \in (-\infty, -\sqrt{W_2 - 1})$ we get $x'(t) < 0, z'(t) < 0$,

for $y(t) \in (-\sqrt{W_2 - 1}, 0)$ we get $x'(t) > 0, z'(t) < 0$,

for $y(t) \in (0, \sqrt{W_2 - 1})$ we get $x'(t) > 0, z'(t) > 0$,

for $y(t) \in (\sqrt{W_2 - 1}, \infty)$ we get $x'(t) < 0, z'(t) > 0$.

If the periodic function y(t) has values in each interval from above, then the particle trajectory looks like the one in Fig. 7d.

If $W_1 - 1 > 0$ and $W_2 - 1 > 0$ then the zeros of $x'(t)$ are obtained for

$$y(t) = \pm\sqrt{W_1 - 1}, \qquad y = \pm\sqrt{W_2 - 1}. \tag{101}$$

The sign of $z'(t)$ depends on $y(t)$ as above. Thus, taking into account (96),

for $y(t) \in (-\infty, -\sqrt{W_2 - 1})$ we have $x'(t) < 0, z'(t) < 0$,

for $y(t) \in (-\sqrt{W_2 - 1}, -\sqrt{W_1 - 1})$ we have $x'(t) > 0, z'(t) < 0$,

for $y(t) \in (-\sqrt{W_1 - 1}, 0)$ we have $x'(t) < 0, z'(t) < 0$,

for $y(t) \in (0, \sqrt{W_1 - 1})$ we have $x'(t) < 0, z'(t) > 0$,

for $y(t) \in (\sqrt{W_1 - 1}, \sqrt{W_2 - 1})$ we have $x'(t) > 0, z'(t) > 0$,

for $y(t) \in (\sqrt{W_2 - 1}, \infty)$ we have $x'(t) < 0, z'(t) > 0$.

If the periodic function $y(t)$ has values in each interval from above, then the particle trajectory has the peculiar shape as in Fig. 7e.

Theorem 3. *As periodic waves propagate on the water's free surface of a constant vorticity shallow water flow over a flat bed, with the average of the horizontal fluid velocity on the bottom over any horizontal segment of length* 1 *different from zero, a single pattern for all the water particles does not exist. The particle paths are not closed, and depending on the relation between the initial data* (x_0, z_0) *and the constant vorticity* ω_0*, some particle trajectories are undulating curves to the right (see Fig. 7*a*), or to the left (see Fig. 7*c*), others are loops with forward drift (see Fig. 7*b*), or with backward drift (see Fig. 7*d*), others can follow peculiar shapes (see Fig. 7*e*).*

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