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Variational derivation of the Camassa-Holm shallow water equation

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Abstract

We describe the physical hypothesis in which an approximate model of water waves is obtained. For an irrotational unidirectional shallow water flow, we derive the Camassa-Holm equation by a variational approach in the Lagrangian formalism.

1 Introduction

The Camassa-Holm equation reads

$$U_t + 2\kappa U_x + 3UU_x - U_{txx} = 2U_xU_{xx} + UU_{xxx} \tag{1.1}$$

with $x \in \mathbf{R}$, $t \in \mathbf{R}$, $U(x,t) \in \mathbf{R}$. Subscripts here, and later, denote partial derivatives. The constant κ is related to the critical shallow water speed. For κ=0 the equation (1.1) possesses peaked soliton solutions ([3]). The physical derivation of (1.1) as a model for the evolution of a shallow water layer under the influence of gravity, is due to Camassa and Holm [3]. See also Refs. [5], [16] for alternative derivations within the shallow water regime. In [19] it was shown that the equation (1.1) describes a geodesic flow on the one dimensional central extension of the group of diffeomorphisms of the circle (for the case κ=0, a geodesic flow on the diffeomorphism group of the circle, see also [9]). It should be mentioned that, prior to Camassa and Holm, Fokas and Fuchssteiner [12] obtained formally, by the method of recursion operators in the context of hereditary symmetries, families of integrable equations containing (1.1). These equations are bi-Hamiltonian generalizations of the KdV equation and possess infinitely many conserved quantities in involution ([12]). In order to see how the equation (1.1) relates with one of equations in the families introduced in [12], see [14], §2.2. Also, the equation (5.3) in [13] is the equation (1.1) but with errors in the coefficients.

The Camassa-Holm equation attracted a lot of interest, due to its complete integrability [10] (for the periodic case), [8] and the citations therein (for the integrability on the line), the existence of waves of permanent form and of breaking waves [7] and the presence of peakon solutions [3] (for a more rigorous study on the peakons see for example [2]). An important question to be answered is whether solutions of the water wave problem can really be approximated by solutions of the Camassa-Holm equation. In [18] it is shown that suitable solutions of the water wave problem and solutions of the Camassa-Holm equation stay close together for long times, in the case $\epsilon = \delta^2$, ϵ being the amplitude parameter and δ the shallowness parameter. In [18] it is also shown that the peakon equation cannot strictly be derived from the Euler equation and hence it is at most a phenomenological model.

The present paper is concerned with the physical hypothesis in which an approximate model of water waves is obtained, with the derivation of the Camassa-Holm equation by a variational approach in the Lagrangian formalism, and with the role of this equation within the shallow water problem.

2 The governing equations for water waves

We consider water moving in a domain with a free upper surface at $z = h_0 + \eta(x,t)$, for a constant $h_0 > 0$, and a flat bottom at $z = 0$. The undisturbed water surface is $z = h_0$. Let $(u(x,z,t), v(x,z,t))$ be the velocity of the water - no motion take place in the y-direction. The fluid is acted on only by the acceleration of gravity g, the effects of surface tension are ignored. For the gravity water waves, the appropriate equations of motion are Euler's equations (EE)(see [15]). Another realistic assumption for gravity water wave problem is the incompressibility (constant density ρ) (see [15]), which implies the equation of mass conservation (MC). The boundary conditions for the water wave problem are the kinematic boundary conditions as well as the dynamic boundary condition. The kinematic boundary conditions (KBC) express the fact that the same particles always form the free water surface and that the fluid is assumed to be bounded below by a hard horizontal bed $z = 0$. The dynamic boundary condition (DBC) express the fact that on the free surface the pressure is equal with the constant atmospheric pressure denoted p_0. Summing up, the exact solution for the water-wave problem is given by the system

$$
\begin{array}{ll}
\begin{array}{c} u_t + uu_x + vu_z = -\frac{1}{\rho}p_x \\ v_t + uv_x + vv_z = -\frac{1}{\rho}p_z - g \end{array} & \text{(EE)} \\[2ex]
u_x + v_z = 0 & \text{(MC)} \\[2ex]
\begin{array}{c} v = \eta_t + u\eta_x \quad \text{on} \quad z = h_0 + \eta(x,t) \\ v = 0 \quad \text{on } z = 0 \end{array} & \text{(KBC)} \\[2ex]
p = p_0, \quad \text{on } z = h_0 + \eta(x,t) & \text{(DBC)}
\end{array}
\tag{2.1}
$$

where $p(x,z,t)$ denotes the pressure.

We non-dimensionalise this set of equations and boundary conditions using the undisturbed depth of water, h_0, as the vertical scale, a typical wavelength λ, as the horizontal scale, and a typical amplitude of the surface wave a (for more details see [15], [16]). An appropriate choice for the scale of the horizontal component of the velocity is $\sqrt{gh_0}$. Then, the corresponding time scale is $\frac{\lambda}{\sqrt{gh_0}}$ and the scale for the vertical component of the velocity is $h_0\frac{\sqrt{gh_0}}{\lambda}$. Thus, we define the set of non-dimensional variables

$$\begin{aligned} x\mapsto \lambda x, \quad z\mapsto h_0 z, \quad \eta\mapsto a\eta, \quad t\mapsto \frac{\lambda}{\sqrt{gh_0}}t, \\ u\mapsto \sqrt{gh_0}u, \quad v\mapsto h_0\frac{\sqrt{gh_0}}{\lambda}v \end{aligned} \tag{2.2}$$

where, to avoid new notations, we have used the same symbols for the non-dimensional variables x, z, η, t, u, v, in the right-hand side. The partial derivatives will be replaced by

$$\begin{aligned} u_t\mapsto \frac{gh_0}{\lambda}u_t, \quad u_x\mapsto \frac{\sqrt{gh_0}}{\lambda}u_x, \quad u_z\mapsto \frac{\sqrt{gh_0}}{h_0}u_z, \\ v_t\mapsto \frac{gh_02}{\lambda^2}v_t, \quad v_x\mapsto h_0\frac{\sqrt{gh_0}}{\lambda^2}v_x, \quad v_z\mapsto \frac{\sqrt{gh_0}}{\lambda}v_z, \end{aligned} \tag{2.3}$$

Let us now define the non-dimensional pressure. If the water would be stationary, that is, $u\equiv v\equiv 0$, from the first two equations and the last condition with $\eta=0$, of the system (2.1), we get for a non-dimensionalised z, the hydrostatic pressure $p_0+\rho g h_0(1-z)$. Thus, the non-dimensional pressure is defined by

$$p\mapsto p_0+\rho g h_0(1-z)+\rho g h_0 p \tag{2.4}$$

and

$$p_x\mapsto \rho\frac{gh_0}{\lambda}p_x, \quad p_z\mapsto -\rho g+\rho g p_z \tag{2.5}$$

Taking into account (2.2), (2.3), (2.4), and (2.5), the water-wave problem (2.1) writes in non-dimensional variables, as

$$\begin{aligned} u_t+uu_x+vu_z=-p_x \\ \delta^2(v_t+uv_x+vv_z)=-p_z \\ u_x+v_z=0 \\ v=\epsilon(\eta_t+u\eta_x) \ \text{ and } \ p=\epsilon\eta \ \text{ on } \ z=1+\epsilon\eta(x,t) \\ v=0 \ \text{ on } z=0 \end{aligned} \tag{2.6}$$

where we have introduced the amplitude parameter $\epsilon=\frac{a}{h_0}$ and the shallowness parameter $\delta=\frac{h_0}{\lambda}$. The small-amplitude shallow water is obtained in the limits $\epsilon\to 0$, $\delta\to 0$. We observe that, on $z=1+\epsilon\eta$, both v and p are proportional to ϵ. This is consistent with the fact that as $\epsilon\to 0$ we must have $v\to 0$ and $p\to 0$, and it leads to the following scaling of the non-dimensional variables

$$p\mapsto \epsilon p, \quad (u,v)\mapsto \epsilon(u,v) \tag{2.7}$$

where we avoided again the introduction of a new notation. The problem (2.6) becomes

$$\begin{aligned} u_t+\epsilon(uu_x+vu_z)=-p_x \\ \delta^2[v_t+\epsilon(uv_x+vv_z)]=-p_z \\ u_x+v_z=0 \\ v=\eta_t+\epsilon u\eta_x \ \text{ and } \ p=\eta \ \text{ on } \ z=1+\epsilon\eta(x,t) \\ v=0 \ \text{ on } z=0 \end{aligned} \tag{2.8}$$

Further, the parameter δ can be removed from the system (2.8) (see [16]), this being equivalent to use only h_0 as the length scale of the problem. In order to do this, the non-dimensional variables x, t and v from (2.2) are replaced by

$$x \mapsto \frac{\sqrt{\epsilon}}{\delta} x, \quad t \mapsto \frac{\sqrt{\epsilon}}{\delta} t, \quad v \mapsto \frac{\delta}{\sqrt{\epsilon}} v \tag{2.9}$$

Therefore the equations in the system (2.8) are recovered, but with δ^2 replaced by ϵ in the second equation of the system, that is, this equation writes as

$$\epsilon[v_t + \epsilon(uv_x + vv_z)] = -p_z \tag{2.10}$$

3 The Camassa-Holm equation

The non-dimensionalisation and scaling presented above will be useful in obtaining a scheme of approximation of the governing water-wave problem (2.1).

In the limit $\epsilon \to 0$, that is for small-amplitude waves, the system (2.8) with the second equation given by (2.10) becomes

$$\begin{aligned} u_t + p_x &= 0 \\ p_z &= 0 \\ u_x + v_z &= 0 \\ v = \eta_t \text{ and } p &= \eta \text{ on } z = 1 \\ v &= 0 \text{ on } z = 0 \end{aligned} \tag{3.1}$$

From the second equation in (3.1), we get that p does not depend on z. Because $p = \eta(x, t)$ on $z = 1$, we have

$$p = \eta(x, t) \qquad \text{for any} \quad 0 \le z \le 1 \tag{3.2}$$

Therefore, using the first equation in (3.1), we obtain

$$u = -\int \eta_x(x, t)dt + \mathcal{F}(x, z) \tag{3.3}$$

where $\mathcal{F}$ is an arbitrary function. Differentiating (3.3) with respect to x and using the third equation in (3.1), we get, after an integration against z,

$$v = z \int \eta_{xx}(x, t)dt - \mathcal{G}(x, z) + \mathcal{G}(x, 0) \tag{3.4}$$

where $\mathcal{G}_z(x, z) = \mathcal{F}_x(x, z)$ and we have also taken into account the last condition in the system (3.1). Making $z = 1$ in (3.4), and taking into account that $v = \eta_t$ on $z = 1$, we get after a differentiation with respect to t, that η has to satisfy the equation

$$\eta_{tt} - \eta_{xx} = 0 \tag{3.5}$$

The general solution of this equation is $\eta(x, t) = f(x - t) + g(x + t)$, where f and g are differentiable functions. It is convenient first to restrict ourselves to waves which propagate in only one direction, thus, we choose

$$\eta(x, t) = f(x - t) \tag{3.6}$$

Therefore, for u and v in (3.3), (3.4) we get

$$u = \eta + \mathcal{F}(x,z), \quad v = -z\eta_x - \mathcal{G}(x,z) + \mathcal{G}(x,0) \tag{3.7}$$

with $\mathcal{G}_z(x,z) = \mathcal{F}_x(x,z)$, $\mathcal{G}(x,1) = \mathcal{G}(x,0)$, arbitrary functions. Thus, the solutions to the shallow water problem are determined by the evolution of the function $\eta(t,x)$, which represents the displacement of the free surface from the undisturbed (flat) state.

Under the assumption that the fluid is irrotational, we get

$$\mathcal{F}(x,z) = \text{ const} := c_0 \quad \text{and} \quad \mathcal{G}(x,z) = 0 \tag{3.8}$$

Indeed, if the fluid is irrotational the vorticity is zero, that is, in addition to the system (2.1), we also have the equation

$$u_z - v_x = 0 \tag{3.9}$$

For a discussion of the role of vorticity in water wave flows see for example [11], [17].
In the equation (3.9), the velocity components u and v are written in the physical (dimensional) variables. If we non-dimensionalise this equation using (2.2), (2.3), we obtain

$$u_z = \delta^2 v_x \tag{3.10}$$

After scaling (2.7) and transformation (2.9), the equation (3.10) writes as

$$u_z = \epsilon v_x \tag{3.11}$$

Therefore, in the limit $\epsilon \to 0$, we get in addition to the system (3.1), the equation

$$u_z = 0 \tag{3.12}$$

The relation (3.2) remains the same but instead of (3.3) we have now

$$u = -\int \eta_x(x,t)dt + \tilde{\mathcal{F}}(x) \tag{3.13}$$

where $\tilde{\mathcal{F}}$ is an arbitrary function. Using the third equation in (3.1), we get now

$$v = -zu_x = z\left(\int \eta_{xx}(x,t)dt - \tilde{\mathcal{F}}'(x)\right) \tag{3.14}$$

where we have taken into account the last condition in the system (3.1). Making $z = 1$ in (3.14), and taking into account that $v = \eta_t$ on $z = 1$, we get after a differentiation with respect to t, that η has to satisfy the equation (3.5). We consider the solution of (3.5) into the form the solution (3.6). Therefore, for u and v in (3.13), (3.14), we have $u = \eta + \tilde{\mathcal{F}}(x)$, $v = -z\left(\eta_x + \tilde{\mathcal{F}}'(x)\right)$. The condition $v = \eta_t$ on $z = 1$, yields $\tilde{\mathcal{F}}(x) = \text{const} := c_0$. Thus, for the irrotational case the solution of the system (3.1) plus the equation (3.12), can be written into the form

$$\eta(x,t) = f(x-t), \quad u = \eta + c_0, \quad v = -z\eta_x \tag{3.15}$$

We underline the fact that in our approximation the vertical velocity component maintains a dependence on the z-variable. In analyzing the motion of the fluid particles, this means that the particles below the surface may perform a vertical motion. This is in agreement with a recent general result obtained in [6] for the Stokes waves, i.e. particular waves in an irrotational flow which are solutions of the full Euler equations.

By consistently neglecting the ϵ contribution, we will derive using variational methods in the Lagrangian formalism (see [5]), the equation (1.1) governing unidirectional propagation of shallow water waves.

In the Lagrangian formulation of a fluid, the flow pattern is obtained by describing the path of each individual water particle. Consider the ambient space M whose points are supposed to represent the fluid particles at $t=0$. A diffeomorphism of M represents the rearrangement of the particles with respect to their initial positions. The motion of the fluid is described by a time-dependent family of orientation-preserving diffeomorphisms $\gamma(t,\cdot)\in \mathrm{Diff}(M)$. A point x in M follows the trajectory $\gamma(t,x)$ through M. For our problem, since a particle on the water's free surface will always stay on the surface and describes progressive plane wave (no motion take place in the y direction), we may regard the motion at that of a one-dimensional membrane. For the one-dimensional periodic motion, $M=\mathbf{S}^1$ the unit circle. We can allow $M=\mathbf{R}$ and add the technical assumption that the smooth functions defined on $\mathbf{R}$ with value in $\mathbf{R}$ vanish rapidly at $\pm\infty$ together with as many derivatives as necessary (see [4] for a possible choice of weighted Sobolev spaces). In what follows we focus on the latter situation.

For a fluid particle initially located at x, the velocity at time t is $\gamma_t(t,x)$, this being the material velocity used in the Lagrangian description. The spatial velocity, used in the Eulerian description, is the flow velocity $w(t,X)=\gamma_t(t,x)$ at the location $X=\gamma(t,x)$ at time t, that is, $w(t,\cdot)=\gamma_t\circ\gamma^{-1}$. In Lagrangian description, the equation of motion is the equation satisfied by a critical point of a certain functional (called the action) defined on all paths $\{\gamma(t,\cdot),\ t\in[0,T]\}$ in $\mathrm{Diff}(\mathbf{R})$, having fixed endpoints. Following Arnold's approach to Euler equations on diffeomorphism groups ([1]), the action for our problem will be obtained by transporting the kinetic energy to all tangent spaces of $\mathrm{Diff}(\mathbf{R})$ by means of right translations. For small surface elevation, the potential energy is negligible compared to the kinetic energy. Taking into account (3.15), the kinetic energy on the surface is

$$K=\frac{1}{2}\int_{-\infty}^{\infty}(u^2+v^2)dx=\frac{1}{2}\int_{-\infty}^{\infty}\left[u^2+(1+\epsilon\eta)^2u_x^2\right]dx\approx\frac{1}{2}\int_{-\infty}^{\infty}\left(u^2+u_x^2\right)dx \quad (3.16)$$

to the order of our approximation (see [5]). We observe that if we replace the path $\gamma(t,\cdot)$ by $\gamma(t,\cdot)\circ\psi(\cdot)$, for a fixed time-independent ψ in $\mathrm{Diff}(\mathbf{R})$, then the spatial velocity is unchanged $\gamma_t\circ\gamma^{-1}$. Transforming K to a right-invariant Lagrangian, the action on a path $\gamma(t,\cdot)$, $t\in[0,T]$, in $\mathrm{Diff}(\mathbf{R})$ is

$$\mathfrak{a}(\gamma)=\frac{1}{2}\int_0^T\int_{-\infty}^{\infty}\{(\gamma_t\circ\gamma^{-1})^2+[\partial_x(\gamma_t\circ\gamma^{-1})]^2\}dxdt \quad (3.17)$$

The critical points of the action (3.17) in the space of paths with fixed endpoints, verify

$$\frac{d}{d\varepsilon}\mathfrak{a}(\gamma+\varepsilon\varphi)\Big|_{\varepsilon=0}=0, \quad (3.18)$$

for every path $\varphi(t,\cdot)$, $t\in[0,T]$, in Diff($\mathbf{R}$) with endpoints at zero, that is, $\varphi(0,\cdot)=0=\varphi(T,\cdot)$ and such that $\gamma+\varepsilon\varphi$ is a small variation of γ on Diff($\mathbf{R}$). Taking into account (3.17), the condition (3.18) becomes

$$\begin{aligned}\int_0^T\int_{-\infty}^{\infty} &\left\{(\gamma_t\circ\gamma^{-1})\,\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[(\gamma_t+\varepsilon\varphi_t)\circ(\gamma+\varepsilon\varphi)^{-1}\right]\right.\\ &\left.+\partial_x(\gamma_t\circ\gamma^{-1})\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[\partial_x\left((\gamma_t+\varepsilon\varphi_t)\circ(\gamma+\varepsilon\varphi)^{-1}\right)\right]\right\}dxdt=0 \qquad (3.19)\end{aligned}$$

After calculation (see [5]), we get

$$\begin{aligned}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[(\gamma_t+\varepsilon\varphi_t)\circ(\gamma+\varepsilon\varphi)^{-1}\right] &= \varphi_t\circ\gamma^{-1}-(\varphi\circ\gamma^{-1})\partial_x(\gamma_t\circ\gamma^{-1})\\ &= \partial_t(\varphi\circ\gamma^{-1})+(\gamma_t\circ\gamma^{-1})\partial_x(\varphi\circ\gamma^{-1})\\ &\quad-(\varphi\circ\gamma^{-1})\partial_x(\gamma_t\circ\gamma^{-1}) \qquad (3.20)\end{aligned}$$

and

$$\begin{aligned}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[\partial_x\left((\gamma_t+\varepsilon\varphi)\circ(\gamma+\varepsilon\varphi)^{-1}\right)\right] &= \partial_x(\varphi_t\circ\gamma^{-1})-\partial_x(\gamma_t\circ\gamma^{-1})\partial_x(\varphi\circ\gamma^{-1})\\ &\quad-(\varphi\circ\gamma^{-1})\partial_x^2(\gamma_t\circ\gamma^{-1})\\ &= \partial_{tx}(\varphi\circ\gamma^{-1})+(\gamma_t\circ\gamma^{-1})\partial_x^2(\varphi\circ\gamma^{-1})\\ &\quad-(\varphi\circ\gamma^{-1})\partial_x^2(\gamma_t\circ\gamma^{-1}) \qquad (3.21)\end{aligned}$$

where are used the formulas of the type

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[(\gamma+\varepsilon\varphi)^{-1}\right]=-\frac{\varphi\circ\gamma^{-1}}{\gamma_x\circ\gamma^{-1}} \qquad (3.22)$$

$$\partial_x(\gamma_t\circ\gamma^{-1})=\frac{\gamma_{tx}\circ\gamma^{-1}}{\gamma_x\circ\gamma^{-1}} \qquad (3.23)$$

$$\partial_t(\varphi\circ\gamma^{-1})=\varphi_t\circ\gamma^{-1}+\left(\varphi_x\circ\gamma^{-1}\right)\partial_t(\gamma^{-1})=\varphi_t\circ\gamma^{-1}-(\gamma_t\circ\gamma)\partial_x(\varphi\circ\gamma^{-1}) \quad (3.24)$$

Thus, denoting $\gamma_t\circ\gamma^{-1}=u$, from (3.20), (3.21), the condition (3.19) writes as

$$\begin{aligned}\int_0^T\int_{-\infty}^{\infty} &\left\{u\left[\partial_t(\varphi\circ\gamma^{-1})+u\partial_x(\varphi\circ\gamma^{-1})-(\varphi\circ\gamma^{-1})u_x\right]\right.\\ &\left.+u_x\left[\partial_{tx}(\varphi\circ\gamma^{-1})+u\partial_x^2(\varphi\circ\gamma^{-1})-(\varphi\circ\gamma^{-1})u_{xx}\right]\right\}dxdt=0 \qquad (3.25)\end{aligned}$$

We integrate by parts with respect to t and x in the above formula, we take into account that φ has endpoints at zero, the smooth functions defined on $\mathbf{R}$ with value in $\mathbf{R}$, together with as many derivatives as necessary, vanish rapidly at $\pm\infty$, and we obtain

$$-\int_0^T\int_{-\infty}^{\infty}(\varphi\circ\gamma^{-1})\left[u_t+3uu_x-u_{txx}-2u_xu_{xx}-uu_{xxx}\right]dxdt=0 \qquad (3.26)$$

Therefore, we get that for an irrotational unidirectional shallow water flow, the horizontal velocity component of the water $u(x,t)$ satisfies the Cammassa-Holm equation (1.1) for $\kappa=0$.

Let us see now which equation fulfill the displacement $\eta(x,t)$ of the free surface from the flat state. Taking into account (3.15), the kinetic energy on the surface is

$$K=\frac{1}{2}\int_{-\infty}^{\infty}(u^2+v^2)dx=\frac{1}{2}\int_{-\infty}^{\infty}\left[(\eta+c_0)^2+(1+\epsilon\eta)^2\eta_x^2\right]dx\approx\frac{1}{2}\int_{-\infty}^{\infty}\left[(\eta+c_0)^2+\eta_x^2\right]dx \tag{3.27}$$

to the order of our approximation. Transforming K to a right-invariant Lagrangian, the action on a path $\Gamma(t,\cdot)$, $t\in[0,T]$, in Diff($\mathbf{R}$) is

$$\mathfrak{a}(\Gamma)=\frac{1}{2}\int_0^T\int_{-\infty}^{\infty}\{(\Gamma_t\circ\Gamma^{-1}+c_0)^2+[\partial_x(\Gamma_t\circ\Gamma^{-1})]^2\}dxdt \tag{3.28}$$

The critical points of the action (3.28) in the space of paths with fixed endpoints, verify

$$\frac{d}{d\varepsilon}\mathfrak{a}(\Gamma+\varepsilon\Phi)\Big|_{\varepsilon=0}=0, \tag{3.29}$$

for every path $\Phi(t,\cdot)$, $t\in[0,T]$, in Diff($\mathbf{R}$) with endpoints at zero, that is, $\Phi(0,\cdot)=0=\Phi(T,\cdot)$ and such that $\Gamma+\varepsilon\Phi$ is a small variation of Γ on Diff($\mathbf{R}$). Taking into account (3.28), the condition (3.29) becomes

$$\begin{aligned}\int_0^T\int_{-\infty}^{\infty}\quad&\Big\{\left(\Gamma_t\circ\Gamma^{-1}+c_0\right)\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[(\Gamma_t+\varepsilon\Phi_t)\circ(\Gamma+\varepsilon\Phi)^{-1}\right]\\&+\partial_x(\Gamma_t\circ\Gamma^{-1})\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[\partial_x\left((\Gamma_t+\varepsilon\Phi_t)\circ(\Gamma+\varepsilon\Phi)^{-1}\right)\right]\Big\}dxdt=0\end{aligned} \tag{3.30}$$

After calculation, denoting $\Gamma_t\circ\Gamma^{-1}=\eta$, (3.30) writes as

$$\begin{aligned}\int_0^T\int_{-\infty}^{\infty}\quad&\left\{(\eta+c_0)\left[\partial_t(\Phi\circ\Gamma^{-1})+\eta\partial_x(\Phi\circ\Gamma^{-1})-(\Phi\circ\Gamma^{-1})\eta_x\right]\right.\\&\left.+\eta_x\left[\partial_{tx}(\Phi\circ\Gamma^{-1})+\eta\partial_x^2(\Phi\circ\Gamma^{-1})-(\Phi\circ\Gamma^{-1})\eta_{xx}\right]\right\}dxdt=0\end{aligned} \tag{3.31}$$

We integrate by parts with respect to t and x in the above formula, we take into account that Φ has endpoints at zero, the smooth functions defined on $\mathbf{R}$ with values in $\mathbf{R}$, together with as many derivatives as necessary, vanish rapidly at $\pm\infty$, and we obtain

$$-\int_0^T\int_{-\infty}^{\infty}(\Phi\circ\Gamma^{-1})\left[\eta_t+3\eta\eta_x+2c_0\eta_x-\eta_{txx}-2\eta_x\eta_{xx}-\eta\eta_{xxx}\right]dxdt=0 \tag{3.32}$$

Therefore, we get that for an irrotational unidirectional shallow water flow, the displacement $\eta(x,t)$ of the free surface from the flat state, satisfies the Camassa-Holm equation (1.1) for $\kappa=c_0$.

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