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Comparative analysis of the electrogravitational Kepler problem in GRT and RTG

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Abstract

In the framework of Einstein's General Relativity Theory and of the Relativistic Theory of Gravitation, the equations governing the trajectories of charged particles in the field created by a charged mass point are given. An analysis of the shape of the trajectories in both theories is presented. The first and the second order approximate solutions of the *electrogravitational Kepler problem* are found in the two theories and the results are compared with each other. I have pointed out the differences between the predictions in the two theories.

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1. Introduction

In this paper, I study the motion of a charged mass point P having mass m and electric charge q , in the electrogravitational field produced by a charged mass point S having mass M and electric charge Q . The *electrogravitational Kepler problem* constitutes an analogue to the problem of the motion of a planet about a fixed sun, under Newtonian attraction. This problem will be considered, in turn, in the framework of Newton's Classical Mechanics (CM), of Einstein's General Relativity Theory (GRT) and of the Relativistic Theory of Gravitation (RTG).

For the sake of comparison with the relativistic versions of the considered problem, in Section 2, I present, the description of the motion in the electrogravitational field according to CM.

The development of RTG and the differences between this and GRT are described in detail in [1,2]. A very important test for a theory of gravitation is to confirm the astronomical predictions. The predictions of RTG for the gravitational effects are unique and consistent with the available experimental data. If the accuracy of the astronomical measurements could be raised to a level at which the effects of order $velocity^4/c^4$, with $velocity \ll c$, come into play, it will be possible to verify the differences between the predictions of the two theories. Besides, in total contradiction with GRT, static spherically symmetric bodies in RTG cannot have dimensions less than Schwarzschild radius. Therefore, the absence of black holes and gravitational collapse in RTG has been confirmed (see, [1–3]).

In the framework of GRT, a spherically symmetric solution of the coupled system of Einstein's equations and Maxwell's equations is that of Nordström and Jeffrey (see Ref. [4, Section 56]). The gravitational radius of the source point S, as a function depending on Q^2 and M^2 , has a discontinuity in $Q^2 = kM^2$. In

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Section 3, I present equations of motion of the charged mass point P in the Nordström metric. In this section, I also present the analysis of the shape of the orbits in the equatorial plane. What happens in the vicinity of the gravitational radius is not presented in detail. For this, see [5] and for a complete bibliography of papers on the geodesics in the Nordström metric see [6]. In this section, an approximate solution of order velocity²/ c^2 , with velocity $\ll c$, for the considered problem is given.

The problem of finding the electrogravitational field produced by the charged mass point S in RTG, was first analyzed by Karabut and Chugreev [7], but assuming only that $kM^2 \geq Q^2$. Soós and I have reanalyzed the problem in RTG (see [8]) considering also the possibility $Q^2 > kM^2$. It is important to analyze this case because the variant is true for the electron. The analytical form of the solution we found, as well as its domain of definition, i.e. the gravitational radius r_g , depend essentially on the relation existing between Q^2 and kM^2 . But, in [14] it is shown that this solution does not fulfill the Causality Principle in RTG. Therefore, this solution cannot be an acceptable solution for this theory. In [14], I have determined the unique solution of electrogravitational field produced by a charged mass point according to RTG. The obtained solution has the same analytical form for all order relations between Q^2 and kM^2 . The gravitational radius depends on this relation, but it is a continuous function depending on Q^2 and M^2 . In Section 4, I present equations of motion of the charged mass point P in the electrogravitational metric according to RTG. I also analyze the shape of the orbits in the equatorial plane. I do not analyze in detail what happens in the vicinity of the gravitational radius. This problem will be treated in a future paper. In this section, I also write an approximate solution of order velocity²/ c^2 , with velocity $\ll c$, for the considered problem and I compare it with the one obtained in the framework of GRT in Section 3.

In Section 5, I write solutions to order velocity⁴/ c^4 , with velocity $\ll c$, in both theories and then I compare the predictions of the two theories.

2. Orbits in the electrogravitational field in CM

To study the problem of motion of the charged mass point P having mass m and electric charge q , in the field produced by the charged mass point S having

mass M and electric charge Q , we consider a system of coordinates centered in S. The position of P is denoted by the position vector $\mathbf{r} = \mathbf{SP}$.

When P moves in the field produced by S, it is acting on the electrostatic force \mathbf{F}_e due to Q and the gravitational force \mathbf{F}_g due to M . The motion of P is governed by

$$m\mathbf{a} = \mathbf{F}_e + \mathbf{F}_g, \quad (2.1)$$

\mathbf{a} representing the acceleration of P.

The expression of the electrostatic force \mathbf{F}_e is given by Coulomb's law:

$$\mathbf{F}_e = \frac{qQ}{r^3} \mathbf{r}, \quad (2.2)$$

where r denotes the length of \mathbf{r} . The fact that like charges repel and unlike charges attract each other is reflected by the direction of the force \mathbf{F}_e . When q, Q have the opposite signs, \mathbf{F}_e has the inverse direction with \mathbf{r} . When they have the same sign, \mathbf{F}_e has the same direction with \mathbf{r} .

The expression of the gravitational force \mathbf{F}_g is given by Newton's law:

$$\mathbf{F}_g = -k \frac{mM}{r^3} \mathbf{r}, \quad (2.3)$$

k being the gravitational constant with the empirical value $k=6,673 \times 10^{-8} \text{ g}^{-1} \text{ cm}^3 \text{ s}^{-2}$. The minus sign in (2.3) indicates that particles attract each other because of the gravitation.

Introducing (2.2) and (2.3) in (2.1), equation of the motion for P takes the form

$$m\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = -k \frac{mM}{r^3} \mathbf{r} + \frac{qQ}{r^3} \mathbf{r}, \quad (2.4)$$

\mathbf{v} representing the velocity of P. Then

$$\mathbf{r} \times m\mathbf{v} = \text{const} = \mathbf{C}. \quad (2.5)$$

Multiplying scalar (2.5) by the vector \mathbf{r} we obtain

$$\mathbf{r} \cdot \mathbf{C} = 0. \quad (2.6)$$

So the trajectory of P under the action of \mathbf{F}_e and \mathbf{F}_g is situated in a fixed plane which includes S.

We choose the trajectory plane Sxy . We can describe in this plane the motion of P using the polar coordinates r and θ , where $x = r \cos \theta$, $y = r \sin \theta$. For any position of P in the plane Sxy , there is a positive value r and an infinity of values θ which differ by an

integer multiple of 2π . If P coincides with S, then $r=0$ and θ is indefinite.

In the polar coordinates, Eq. (2.1) takes the form

$$m \left(\frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right) = -k \frac{mM}{r^2} + \frac{qQ}{r^2}, \quad (2.7)$$

$$m \left(2 \frac{dr}{dt} \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2} \right) = 0. \quad (2.8)$$

Eq. (2.8) shows that during the motion

$$r^2 \frac{d\theta}{dt} = \text{const} = J. \quad (2.9)$$

The value of the constant J , which denotes the angular momentum of P per unit mass, can be determined from the initial conditions. We denote by r_0, θ_0 the polar coordinates of P at the initial moment t_0 , by v_0 the magnitude of the initial velocity and by α the angle between \mathbf{r}_0 and \mathbf{v}_0 . Knowing the expression of the velocity in polar coordinates we can write

$$\frac{dr}{dt}(0) = v_0 \cos \alpha, \quad r_0 \frac{d\theta}{dt}(0) = v_0 \sin \alpha. \quad (2.10)$$

From (2.9) and (2.10), for the constant of the motion J , we get the value

$$J = r_0 v_0 \sin \alpha. \quad (2.11)$$

As in Kepler's classical problem (see for example [9, Chapter XV]), we can simplify matters by considering r as a function of θ instead of t . Any functional relation $r=r(\theta)$ defines a curve in the polar coordinates system.

Assume $J \neq 0$, so r and $d\theta/dt$ are never 0. Then, the substitution $u = 1/r$ transforms Eq. (2.7) into Binet's differential equation for the orbit of P:

$$\frac{d^2u}{d\theta^2} + u = \frac{kM}{J^2} - \frac{qQ}{mJ^2}. \quad (2.12)$$

This equation has the general solution:

$$u = \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) + \mathcal{A} \cos(\theta - \psi), \quad (2.13)$$

where \mathcal{A} and ψ are constants determined from the initial conditions. By rotating the coordinates we can make $\psi = 0$.

So, if $J \neq 0$, the orbit of P is the conic:

$$\frac{1}{r} = \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) + \mathcal{A} \cos \theta, \quad (2.14)$$

S being situated in a focus of this conic.

If the gravitational force \mathbf{F}_g and the electrostatic force \mathbf{F}_e have opposite directions and the same magnitude, $kmM - qQ = 0$, than, from (2.14), P moves along the line:

$$\frac{1}{r} = \mathcal{A} \cos \theta. \quad (2.15)$$

Now, we want to see how the nature of the conic (2.14) depends on the sign of the expression $kmM - qQ$ and on the initial conditions.

Multiplying Eq. (2.12) by $2(du/d\theta)$, then integrating, replacing $u = 1/r$ and taking into account (2.9), we obtain the energy equation:

$$\left(\frac{dr}{dt} \right)^2 = -\frac{J^2}{r^2} + \frac{2}{m} (kmM - qQ) \frac{1}{r} + 2E, \quad (2.16)$$

E being the total energy of P per unit mass. This constant of the motion is also determined from the initial conditions.

Allowing for (2.10) and (2.11), we obtain

$$2E = v_0^2 - \frac{2}{mr_0} (kmM - qQ). \quad (2.17)$$

Since $(dr/dt)^2 \geq 0$, (2.16) yields $E \geq \frac{1}{2}((J^2/r^2) - (2/m)(kmM - qQ)(1/r))$.

The sign of the expression $kmM - qQ$ and the value of E determine the range of r and implicitly the shape of the orbit described by P. We denote the right member of Eq. (2.16) by $F(1/r)$.

Case (I): $E < 0$. This case happens only when $kmM - qQ > 0$. Hence, Eq. $F(1/r) = 0$ has two positive roots, so r oscillates between finite endpoints. We get an ellipse as the trajectory of P. If $F(1/r) = 0$ has a double root, then we get a circular orbit. By virtue of (2.17), if

$$v_0^2 - \frac{2}{mr_0} (kmM - qQ) < 0, \text{ the orbit is an ellipse.}$$

Case (II): $E > 0$. If $kmM - qQ < 0$ then $E > 0$, but this can also happen if $kmM - qQ > 0$. Eq. $F(1/r) = 0$ has one positive root and one negative root. Hence,

$0 < 1/r \leq$ a positive root. We obtain a hyperbola as the trajectory of P. So, by virtue of (2.17), if

$$v_0^2 - \frac{2}{mr_0} (kmM - qQ) > 0, \text{ the orbit is a hyperbola.}$$

Case (III): $E = 0$. This case happens only when $kmM - qQ > 0$. Eq. $F(1/r) = 0$ has a root zero and the other root is positive. Hence, $0 \leq 1/r \leq$ positive root. We obtain a parabola as the trajectory of P. So, by virtue of (2.17), if

$$v_0^2 - \frac{2}{mr_0} (kmM - qQ) = 0, \text{ the orbit is a parabola.}$$

In conclusion, if the gravitational force F_g and the electrostatic force F_e have the same direction or they have opposite directions, but the magnitude of F_g is greater than the magnitude of F_e , then the orbit described by P in the field produced by S is an ellipse or a hyperbola or a parabola, all depending on the initial position and velocity of P. Finally, if F_g and F_e have opposite direction s and the magnitude of F_g is smaller than the magnitude of F_e , then P moves on a hyperbola.

3. Orbits in the electrogravitational field in GRT

Because the basic concepts of Einstein’s GRT are so different from those of Newton’s CM, we want to know more about the differences between the predictions of the two theories in the considered problem. The study of classical Kepler’s problem in GRT is well known (see for example [10, Chapter VI, Section 3] and [11, Chapter VII, Section 8]). In [12], Soós has revealed that this problem was one of the main questions taken into account by Einstein. The capacity of obtaining the correct value for the perihelion rotation of Mercury has represented a permanent test for the successively elaborated Einstein’s theories of gravitation during the period 1907–1915.

Let us study the *electrogravitational Kepler problem* in GRT. The electrogravitational field produced by S, having mass M and electric charge Q , is described by the following metric (see [4, Section 56]):

$$ds^2 = g_{ij} dx^i dx^j = \left(1 - \frac{2kM}{c^2 r} + \frac{kQ^2}{c^4 r^2} \right) (dx^4)^2$$

$$- \frac{1}{1 - (2kM/c^2 r) + (kQ^2/c^4 r^2)} dr^2 - r^2 d\varphi^2 - r^2 \sin^2 \varphi d\theta^2, \tag{3.1}$$

$c = 3 \times 10^{10} \text{ cm s}^{-1}$ being the velocity of light in vacuum.

The metric (3.1) is written in the system of coordinates $(x^i)_{i=1,4} = (r, \varphi, \theta, ct)$ centered in S. The coordinates (r, φ, θ) are the spherical coordinates of any point situated in this field. The domains of definition for these coordinates are: $0 \leq r_g < r < \infty$, $0 \leq \varphi \leq \pi$, $0 \leq \theta \leq 2\pi$, $-\infty < t < \infty$; r_g representing the gravitational radius of the point source S. According to GRT, the value of this gravitational radius depends on the relation between Q^2 and M^2 in the following manner (see [4, Section 56]):

$$r_g = \begin{cases} kM/c^2 + (1/c^2) & \text{for } Q^2 \leq kM^2, \\ \sqrt{k^2 M^2 - kQ^2} & \\ 0 & \text{for } Q^2 > kM^2. \end{cases} \tag{3.2}$$

The motion of P in the field created by S follows a timelike geodesic line $(x^j(s))_{j=1,4} = (r(s), \varphi(s), \theta(s), ct(s))$. The parameter s of this curve is such that ds^2 is given by (3.1). Equation of motion of the charged particle P of mass m and charge q , moving in the field of gravitation (g_{ij}) and electromagnetism (F_{ij}) , a field which is not influenced by the particle itself, is (see [13, Section 103]):

$$m \left(\frac{d^2 x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} \right) + \frac{q}{c^2} F_j^i \frac{dx^j}{ds} = 0, \tag{3.3}$$

$i = 1, 2, 3, 4.$

In (3.3), Γ_{jk}^i are the components of the metric connection generated by the metric (3.1): $\Gamma_{jk}^i = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk})$ and $F_j^i = g^{il} F_{lj}$ are the mixed components of the electromagnetic tensor (F_{ij}) . For our problem, the non-zero components of the electromagnetic tensor are (see [4, Section 56]):

$$F_{14} = -F_{41} = F^{41} = -F^{14} = \frac{Q}{r^2}. \tag{3.4}$$

Allowing for (3.1), the non-zero components of F_j^i are

$$F_1^4 = -f^2 \frac{Q}{r^2}, \quad F_4^1 = -\frac{1}{f^2} \frac{Q}{r^2}, \tag{3.5}$$

where

$$f^2 = \frac{1}{1 - (2kM/c^2r) + (kQ^2/c^4r^2)}. \quad (3.6)$$

Taking into account (3.1), the non-zero components $\Gamma_{jk}^2, \Gamma_{jk}^3, \Gamma_{jk}^4$ of the metric connection, which will be used in (3.3), are

$$\begin{aligned} \Gamma_{12}^2 &= \Gamma_{21}^2 = \Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{r}, & \Gamma_{33}^2 &= -\sin \varphi \cos \varphi, \\ \Gamma_{23}^3 &= \Gamma_{32}^3 = \cot \varphi, \\ \Gamma_{14}^4 &= \Gamma_{41}^4 = -\frac{1}{f} \frac{df}{dr}. \end{aligned} \quad (3.7)$$

Thus, allowing for (3.7) and setting $i = 2$ in (3.3), we get

$$\frac{d^2\varphi}{ds^2} + \frac{2}{r} \frac{d\varphi}{ds} \frac{dr}{ds} - \sin \varphi \cos \varphi \left(\frac{d\theta}{ds}\right)^2 = 0. \quad (3.8)$$

By an appropriate orientation of the axes, we can initially have $\varphi(s_0) = \pi/2$ and $d\varphi/ds(s_0) = 0$. Thus the solution of Eq. (3.8) is

$$\varphi(s) = \frac{\pi}{2}. \quad (3.9)$$

So we can see that as in the classical case, the orbit lies in a plane.

Considering $i = 3$ in (3.3) and taking into account (3.7) and (3.9) we obtain

$$\frac{d^2\theta}{ds^2} + \frac{2}{r} \frac{dr}{ds} \frac{d\theta}{ds} = 0. \quad (3.10)$$

Integrating this equation, we find

$$r^2 \frac{d\theta}{ds} = \text{const} = L. \quad (3.11)$$

This equation is similar to Eq. (2.9), hence we can call L the angular momentum of P per unit mass.

We set $i = 4$ in (3.3) and from (3.5) and (3.7), we get

$$\frac{d^2x^4}{ds^2} - 2 \frac{1}{f} \frac{df}{ds} \frac{dx^4}{ds} = f^2 \frac{qQ}{mc^2r^2} \frac{dr}{ds}. \quad (3.12)$$

Eq. (3.12) integrates to

$$\frac{dx^4}{ds} = \left(E - \frac{qQ}{mc^2} \frac{1}{r}\right) f^2, \quad (3.13)$$

E being a constant. To obtain dr/ds we can consider $i = 1$ in (3.3) but it is more convenient to divide the line element (3.1) by ds^2 . Allowing for (3.9), (3.11) and (3.13), we find equation

$$\begin{aligned} f^2 \left(\frac{dr}{ds}\right)^2 + \frac{L^2}{r^2} - f^2 \left(E - \frac{qQ}{mc^2} \frac{1}{r}\right)^2 + 1 \\ = 0, \end{aligned} \quad (3.14)$$

which is analogous to the classical energy Eq. (2.16).

As in the problem considered in the framework of CM, we consider r as a function of θ instead of s . Thus, taking into account (3.11), we have

$$\frac{dr}{ds} = \frac{dr}{d\theta} \frac{d\theta}{ds} = \frac{L}{r^2} \frac{dr}{d\theta}. \quad (3.15)$$

Putting

$$u = \frac{1}{r} \quad (3.16)$$

and considering the case when $L \neq 0$, Eq. (3.14) becomes

$$\begin{aligned} \left(\frac{du}{d\theta}\right)^2 &= -\frac{kQ^2}{c^4} u^4 + \frac{2kM}{c^2} u^3 \\ &\quad - \left[1 + \frac{Q^2}{L^2 m^2 c^4} (km^2 - q^2)\right] u^2 \\ &\quad + \left(\frac{2kM}{L^2 c^2} - E \frac{2qQ}{mL^2 c^2}\right) u - \frac{1 - E^2}{L^2}. \end{aligned} \quad (3.17)$$

Eq. (3.17) governs the geometry of the orbits described by P in the plane $\varphi = \pi/2$.

We denote the right member of Eq. (3.17) by $F(u)$. It is also clear that the disposition of the roots of equation $F(u) = 0$ will determine the shape of the orbit. By (3.17), $F(u) \geq 0$ throughout the orbit and $F(u)$ tends to $-\infty$ for very large values of u . It follows that $F(u) = 0$ has four real roots or two real roots and a complex conjugate pair. A root signifies a turning point where $du/d\theta$ changes the sign. A negative root has no physical meaning, so at least one positive root should occur.

The consideration of Eq. (3.17) is conveniently separated into the following parts: $E^2 < 1$, $E^2 > 1$, $E^2 = 1$. These distinctions determine whether the orbits are bound or unbound (i.e. whether along the orbit r

remains bounded or not). These classes of orbits are characterized by total energies (exclusive of the rest energy) which are negative, positive or zero.

We denote u_1, u_2, u_3, u_4 the zeros of $F(u)$, with $u_1 < u_2 < u_3 < u_4$ if they are all real. We have

$$u_1 + u_2 + u_3 + u_4 = \frac{2Mc^2}{Q^2} > 0, \tag{3.18}$$

$$u_1 u_2 u_3 u_4 = \frac{1 - E^2}{L^2} \frac{c^4}{kQ^2}. \tag{3.19}$$

Case (I): $E^2 < 1$. In this case, from (3.18) and (3.19), if there are only two real roots, they must be positive. If all the roots are real, then two of them or all four must be positive. We also have $F(u) < 0$ for u tends to 0. Thus, we get an orbit of elliptic type with u oscillating in the range $u_1 \leq u \leq u_2$ or $u_3 \leq u \leq u_4$ (u_1 or u_3 , corresponds to aphelion, u_2 or u_4 corresponds to perihelion).

Case (II): $E^2 > 1$. In this case, by virtue of (3.18) and (3.19), one (for example u_4) or three roots (for example u_2, u_3, u_4) must be positive. We also have $F(u) > 0$ for u tends to 0. Thus, we get an orbit of hyperbolic type restricted to the interval $0 < u \leq u_4$ or $0 < u \leq u_2$ or an orbit of elliptic type restricted to the interval $u_3 \leq u \leq u_4$.

Case (III): $E^2 = 1$. In this case, one of the solutions is zero. Thus, we get an orbit of parabolic type restricted to the interval $0 \leq u \leq u_2$ or an orbit of elliptic type restricted to the interval $u_3 \leq u \leq u_4$.

In the special case of double roots, we get a circular orbit.

From (3.2), if $Q^2 > kM^2$, then $r_g = 0$. The biggest positive solution of the equation $F(u) = 0$ can take any large value and the non-capture orbits occur. But if $Q^2 \leq kM^2$, then from (3.2), $r_g = (kM/c^2) + (1/c^2)\sqrt{k^2M^2 - kQ^2}$, so it will be possible that the biggest positive solution of $F(u) = 0$ overpasses $1/r_g$. From the viewpoint of GRT (for a complete bibliography of papers on the geodesics in the Nordström metric (3.1) see Sharp [6]; see also Chandrasekhar [5], Chapter 5), the particle will cross the horizon $r = r_g$ in this case only in the inside direction and its trajectory will formally terminate at this turning point. In addition, from (3.14), at this turning point $1 - (2kM/c^2r) + (kQ^2/c^4r^2)$ must be positive, so this turning point is in the interval $(0, (kM/c^2) - (1/c^2)\sqrt{k^2M^2 - kQ^2})$. For an external

observer, P will take an infinite time to reach the horizon $r = r_g$, but the falling observer with P will cross the horizon $r = r_g$ and reach the turning point in a finite time which is its own proper time.

Let us now explore Eq. (3.17) with the view to find a solution. The exact solution of this equation expresses the angle θ as an elliptic integral of $u = 1/r$ and conversely it gives u as an implicit function of θ . Unfortunately, this implicit form of the solution does not make evident the approximate classical form of the trajectory. To establish a closer connection with the classical problem, we differentiate Eq. (3.17) with respect to θ . One possible solution is obtained by setting the common factor $du/d\theta$ equal to zero. This yields $u = \text{const}$, so $r = \text{const}$. Thus the circular motion occurs also in GRT. Removing the common factor $2 du/d\theta$, we obtain

$$\begin{aligned} \frac{d^2u}{d\theta^2} + u = & -\frac{2kQ^2}{c^4} u^3 + \frac{3kM}{c^2} u^2 \\ & + \frac{Q^2}{L^2 c^4 m^2} (q^2 - km^2) u \\ & + \frac{kM}{L^2 c^2} - E \frac{qQ}{mL^2 c^2}. \end{aligned} \tag{3.20}$$

In the case of slow motion in weak gravitational fields, Eq. (3.20) must reduce to the classical Eq. (2.12). Indeed, for a slowly moving particle in a weak field, we have $dx^4/ds \simeq 1$ and $E \simeq 1 + E/c^2$, where E is the total energy of P per unit mass, given by (2.16). Thus, from (2.9) and (3.11), we get

$$\begin{aligned} \frac{1}{L^2 c^2} &= \frac{1}{r^2 (d\theta/ds)^2 c^2} = \frac{1}{r^2 (d\theta/dt)^2 (dt/ds)^2 c^2} \\ &= \frac{1}{J^2 (dx^4/ds)^2} \simeq \frac{1}{J^2}. \end{aligned} \tag{3.21}$$

and taking $E \simeq 1$, Eq. (3.20) reduces to Eq. (2.12) in the case of slow motion in weak gravitational fields.

We notice that relativistic Eq. (3.20) differs from the classical Eq. (2.12) through the addition of three terms containing u , and it has a slightly different constant term.

For the sake of simplicity, in the view of (3.21) and for $E \simeq 1$, let us now investigate the orbital equation:

$$\frac{d^2u}{d\theta^2} + u = \frac{kM}{J^2} - \frac{qQ}{mJ^2}$$

$$\begin{aligned}
 & + \frac{Q^2}{J^2 c^2 m^2} (q^2 - km^2)u + \frac{3kM}{c^2} u^2 \\
 & - \frac{2kQ^2}{c^4} u^3. \tag{3.22}
 \end{aligned}$$

Let us evaluate the order of magnitude of the three terms containing u from the right side of Eq. (3.22). The order of magnitude of u is $1/l$, where l is a length. The order of magnitude of kMu is v_1^2 , v_1 being considered a velocity much smaller than the velocity of light in vacuum. Thus the term of second order in u has the magnitude $(v_1^2/c^2)(1/l)$. The order of magnitude of $(Q^2/J^2 m^2)(q^2 - km^2)$ is also the square of a velocity v_2 , considered much smaller than the velocity of light in vacuum. Thus, the term of the first order in u has the order of magnitude $(v_2^2/c^2)(1/l)$. Finally, $kQ^2 u^2$ has the order of magnitude v_3^4 , where v_3 is considered a velocity much smaller than the velocity of light in vacuum. So, the term of the third order in u has the order of magnitude $(v_3^4/c^4)(1/l)$. The term of order v_3^4/c^4 is very small compared with the unity, so, if we want to find a solution of Eq. (3.22) to order velocity²/ c^2 , velocity $\ll c$, we neglect the last term in Eq. (3.22). We define the small dimensionless quantities:

$$\varepsilon = \frac{Q^2}{J^2 m^2} (q^2 - km^2) \frac{1}{c^2}, \quad \delta = \frac{k^2 M^2}{J^2 c^2}. \tag{3.23}$$

Thus, Eq. (3.22) becomes

$$\frac{d^2 u}{d\theta^2} + u(1 - \varepsilon) = \frac{kM}{J^2} - \frac{qQ}{mJ^2} + 3\delta \frac{J^2}{kM} u^2. \tag{3.24}$$

Let us find an approximate solution of this non-linear equation by a perturbation approach.

To solve this, we assume a solution of the form

$$\begin{aligned}
 u(\theta) = & u_0(\theta) + \varepsilon V(\theta) + \delta W(\theta) + O(\varepsilon^2) \\
 & + O(\delta^2) + O(\varepsilon\delta). \tag{3.25}
 \end{aligned}$$

Substituting this form for u in the differential Eq. (3.24) and keeping only the terms of order 0 and 1 in ε and δ , we find

$$\begin{aligned}
 & \frac{d^2 u_0}{d\theta^2} + \varepsilon \frac{d^2 V}{d\theta^2} + \delta \frac{d^2 W}{d\theta^2} + u_0 + \varepsilon V + \delta W - \varepsilon u_0 \\
 & = \frac{kM}{J^2} - \frac{qQ}{mJ^2} + 3\delta \frac{J^2}{kM} u_0^2. \tag{3.26}
 \end{aligned}$$

Equating the zeroth-order terms in ε and δ we get

$$\frac{d^2 u_0}{d\theta^2} + u_0 = \frac{kM}{J^2} - \frac{qQ}{mJ^2}, \tag{3.27}$$

which is the classical Eq. (2.12). The solution of this equation is given by (2.14).

Equating the first-order terms in ε and taking into account (2.14), we get

$$\frac{d^2 V}{d\theta^2} + V = \frac{kM}{J^2} - \frac{qQ}{mJ^2} + \mathcal{A} \cos \theta. \tag{3.28}$$

We need only a non-homogeneous solution to this equation, since the zeroth-order solution already contains a term $\mathcal{A} \cos \theta$, which is the general solution to the homogeneous equation. Thus, we find

$$V(\theta) = V_1(\theta) + V_2(\theta), \tag{3.29}$$

where

$$V_1(\theta) = \frac{kM}{J^2} - \frac{qQ}{mJ^2}, \quad V_2(\theta) = \frac{\mathcal{A}}{2} \theta \sin \theta. \tag{3.30}$$

Similarly, equating the first-order terms in δ and by virtue of (2.14), we obtain

$$\begin{aligned}
 \frac{d^2 W}{d\theta^2} + W = & 3 \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + \frac{3}{2} \frac{J^2}{kM} \mathcal{A}^2 \\
 & + 6 \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \mathcal{A} \cos \theta \\
 & + \frac{3}{2} \frac{J^2}{kM} \mathcal{A}^2 \cos 2\theta. \tag{3.31}
 \end{aligned}$$

With the non-homogeneous solution:

$$W(\theta) = W_1(\theta) + W_2(\theta) + W_3(\theta), \tag{3.32}$$

where

$$\begin{aligned}
 W_1(\theta) = & 3 \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + \frac{3}{2} \frac{J^2}{kM} \mathcal{A}^2, \\
 W_2(\theta) = & 3 \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \mathcal{A} \theta \sin \theta, \\
 W_3(\theta) = & -\frac{1}{2} \frac{J^2}{kM} \mathcal{A}^2 \cos 2\theta. \tag{3.33}
 \end{aligned}$$

Introducing (2.14), (3.29), (3.32) into (3.25), we find the solution for the orbit to first order in ε and δ :

$$u(\theta) = \left[\frac{kM}{J^2} - \frac{qQ}{mJ^2} + \varepsilon \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \right]$$

$$\begin{aligned}
 & + \delta \frac{3J^2}{kM} \left(\left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + \frac{\mathcal{A}^2}{2} \right) \\
 & + \left[\mathcal{A} \cos \theta - \delta \frac{J^2 \mathcal{A}^2}{2kM} \cos 2\theta \right] \\
 & + \left[\frac{\varepsilon}{2} + \delta \frac{3J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \right] \\
 & \times \mathcal{A} \theta \sin \theta. \tag{3.34}
 \end{aligned}$$

In the solution (3.34), only the last term is non-periodic. To clarify further the effect of this non-periodic term, we note that, to the first order in ε and δ ,

$$\begin{aligned}
 & \cos \left(\theta - \left(\frac{\varepsilon}{2} + \delta \frac{3J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \right) \theta \right) \\
 & = \cos \theta + \left(\frac{\varepsilon}{2} + \delta \frac{3J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \right) \\
 & \quad \times \theta \sin \theta, \tag{3.35}
 \end{aligned}$$

so the solution may be written as

$$\begin{aligned}
 u(\theta) & = \frac{kM}{J^2} - \frac{qQ}{mJ^2} + \mathcal{A} \cos(\theta) \\
 & - \left(\frac{\varepsilon}{2} + \delta \frac{3J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \right) \theta \\
 & + (\text{periodic terms of order } \varepsilon \text{ and } \delta). \tag{3.36}
 \end{aligned}$$

The effect of the last term is to introduce small periodic variations in the radial distance of P.

In the case of an orbit of elliptic type, the effect of the second term can be clarified. The small differences between relativistic orbit and the Newtonian ellipse $(kM/J^2) - (qQ/mJ^2) + \mathcal{A} \cos \theta$ are due to this term which influences the perihelion position of P. The perihelion of P is the point of closest approach to S. This occurs when u is maximum. From (3.36) we see that u is maximum when

$$\begin{aligned}
 & \theta \left(1 - \left(\frac{\varepsilon}{2} + \delta \frac{3J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \right) \right) \\
 & = 2\pi n. \tag{3.37}
 \end{aligned}$$

Keeping the terms to the first order in ε and δ , the interval between successive perihelia is

$$\begin{aligned}
 \Delta\theta & \\
 & = 2\pi \left(1 + \frac{\varepsilon}{2} + \delta \frac{3J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \right), \tag{3.38}
 \end{aligned}$$

instead of 2π like in periodic motion. So, according to the notations (3.23), the shift of the perihelion is approximately

$$\begin{aligned}
 \Delta\theta - 2\pi & = 6\pi \frac{k^2 M^2}{c^2 J^2} - 6\pi \frac{qQkM}{mJ^2 c^2} \\
 & + \pi \frac{Q^2(q^2 - km^2)}{J^2 m^2 c^2}. \tag{3.39}
 \end{aligned}$$

Therefore, the orbit of P is to be regarded as an ellipse which rotates slowly. We see that if Q, q are zero, we find the known formula for the advance of perihelion per revolution, which is one of the famous formulas of GRT. From (3.39), we see that when the charges of P and S are taken into account, if

$$6kmM(kmM - qQ) + Q^2(q^2 - km^2) > 0 \tag{3.40}$$

then the ellipse rotates in the direction in which it is described and if

$$6kmM(kmM - qQ) + Q^2(q^2 - km^2) < 0 \tag{3.41}$$

then the ellipse rotates in the opposite direction in which it is described.

4. Orbits in the electrogravitational field in RTG

GRT encounters serious difficulties with the evaluation of the physical characteristics of the gravitational field and the formulation of the energy–momentum conservation laws. Combining Poincaré’s idea of the gravitational field as a Faraday–Maxwell physical field with Einstein’s idea of a Riemannian space–time geometry, Logunov and his co-workers have elaborated a *new* relativistic theory of gravitation, named RTG (see [1,2]), in the framework of the Special Theory of Relativity (SRT). In this theory, the Minkowski space–time is a fundamental space that incorporates all physical fields, including gravitation. The gravitational field is described by a second-order symmetric tensor ϕ^{ij} , possessing energy–momentum density, rest mass and polarization states corresponding to spin 2 and 0. Owing to the action of this field, an effective Riemannian space–time g_{ij} arises. GRT characterizes the gravitational field by the metric tensor g_{ij} , whereas in RTG it is determined by the tensor value ϕ^{ij} , the effective Riemannian space–time being constructed

with the help of the field ϕ^{ij} and of the Minkowski metric tensor to fix the choice of the coordinate system. The construction rule is the following: $\tilde{g}^{ij} = \sqrt{-g}g^{ij} = \sqrt{-\gamma}\gamma^{ij} + \sqrt{-\gamma}\phi^{ij}$, $g = \det(g_{ij})$, $\gamma = \det(\gamma_{ij})$. Metric properties are determined by the effective Riemannian space–time tensor g_{ij} in the presence of a gravitational field and by the Minkowski space–time tensor γ_{ij} in the absence of this field. The interaction between tensor gravitational field and matter can be introduced as though it deformed the Minkowski space, changing the metric properties without affecting the causality. The Causality Principle in RTG affirms that the light cone in the effective Riemannian space–time does not go beyond the causality cone of the Minkowskian space–time. For the differential laws of RTG and the analytic formulation of the Causality Principle in RTG see [1,2].

The problem of finding the field of an electrically charged mass point having mass M and electric charge Q , was first analyzed by Karabut and Chugreev in [7], but only assuming $kM^2 \geq Q^2$. Soós and I have reanalyzed this problem in RTG in [8], considering also the possibility $Q^2 > kM^2$. It is important to analyze this case because the variant is true for the electron. The analytical form of the solution we found, as well as its domain of definition, i.e. the gravitational radius r_g , depend essentially on the relation existing between Q^2 and kM^2 . But in [14] I have shown that the solution obtained by us does not fulfill the Causality Principle in RTG. So, this solution cannot be an acceptable solution in this theory. I have determined in [14] the unique solution of electrogravitational field produced by a charged mass point according to RTG. The obtained solution has the same analytical form for all order relations between Q^2 and kM^2 . The gravitational radius depends on this relation but it is a continuous function depending on Q^2 and kM^2 .

Solving the coupled system of RTG’s and Maxwell’s equations, and taking into account the Causality Principle in RTG, we get the following effective Riemannian space–time due to the electrogravitational field produced by a charged mass point with mass M and charge Q :

$$ds^2 = g_{ij} dx^i dx^j = \left(1 - \frac{2kM}{c^2(r + (kM/c^2))} + \frac{kQ^2}{c^4(r + (kM/c^2))^2} \right) (dx^4)^2$$

$$- \frac{1}{1 - (2kM/c^2(r + (kM/c^2))) + (kQ^2/c^4(r + (kM/c^2))^2)} dr^2 - \left(r + \frac{kM}{c^2} \right)^2 (d\varphi^2 + \sin^2 \varphi d\theta^2). \tag{4.1}$$

The metric (4.1) is written in the Minkowskian system of coordinates $(x^i)_{i=\overline{1,4}} = (r, \varphi, \theta, ct)$ centered in S. The domains of definition for these coordinates are: $0 \leq r_g < r < \infty$, $0 \leq \varphi \leq \pi$, $0 \leq \theta \leq 2\pi$, $-\infty < t < \infty$; r_g representing the gravitational radius of the point source S. According to RTG, the value of this gravitational radius depends on the relation between Q^2 and kM^2 in the following manner (see [14]):

$$r_g = \begin{cases} 1/c^2 \sqrt{k^2 M^2 - kQ^2} & \text{for } Q^2 \leq kM^2, \\ 0 & \text{for } kM^2 < Q^2 < 2kM^2, \\ 1/2c^2 M(Q^2 - 2kM^2) & \text{for } Q^2 \geq 2kM^2. \end{cases} \tag{4.2}$$

We notice in (4.2), that the function r_g depending on Q^2 and kM^2 is a continuous one, which is not the case for GRT’s function r_g from (3.2).

The metric of the Minkowski space–time in which we happen to be when the gravitational field is switched off is

$$d\sigma^2 = \gamma_{ij} dx^i dx^j = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2. \tag{4.3}$$

According to the principles of RTG, the motion of matter under the action of a gravitational field in the Minkowski space–time is identical to its motion in the effective Riemannian space–time with the metric g_{ij} . Thus, the equation of motion of the charged particle P of mass m and charge q , moving in the field produced by S, is

$$m \left(\frac{d^2 x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} \right) + \frac{q}{c^2} F_j^i \frac{dx^j}{ds} = 0, \quad i = 1, 2, 3, 4. \tag{4.4}$$

Eqs. (4.4) are similar to Einstein’s Eqs. (3.3), the one important difference being that in RTG all field variables depend on the universal spatial–temporal coordinates in the Minkowski space–time. Γ_{jk}^i are the components of the metric connection generated by

the effective Riemannian metric (4.1) and $F_j^i = g^{il}F_{lj}$ are the mixed components of the electromagnetic tensor (F_{ij}). For our problem, the non-zero components of the electromagnetic tensor are (see [14])

$$F_{14} = -F_{41} = F^{41} = -F^{14} = \frac{Q}{(r + (kM/c^2))^2}. \quad (4.5)$$

Allowing for (4.1), the non-zero components of F_j^i are

$$F_1^4 = -f^2 \frac{Q}{(r + (kM/c^2))^2},$$

$$F_4^1 = -\frac{1}{f^2} \frac{Q}{(r + (kM/c^2))^2}, \quad (4.6)$$

where

$$f^2 = \frac{1}{1 - (2kM/c^2(r + (kM/c^2))) + (kQ^2/c^4(r + (kM/c^2))^2)}. \quad (4.7)$$

Taking into account (4.1), the non-zero components $\Gamma_{jk}^2, \Gamma_{jk}^3, \Gamma_{jk}^4$ of the metric connection, which will be used in (4.4), are

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{(r + (kM/c^2))},$$

$$\Gamma_{33}^2 = -\sin \varphi \cos \varphi,$$

$$\Gamma_{23}^3 = \Gamma_{32}^3 = \cot \varphi, \quad \Gamma_{14}^4 = \Gamma_{41}^4 = -\frac{1}{f} \frac{df}{dr}. \quad (4.8)$$

Using the same procedures as in Section 3, we find

$$\varphi(s) = \frac{\pi}{2}. \quad (4.9)$$

So we can see that like in CM and GRT, the orbit lies in a plane.

Integrating Eq. (4.4) for $i=3$ and taking into account (4.8) and (4.9) we obtain

$$\left(r + \frac{kM}{c^2}\right)^2 \frac{d\theta}{ds} = \text{const} = \mathcal{L}. \quad (4.10)$$

Allowing for (4.6) and (4.8), Eq. (4.4) for $i = 4$ integrates to

$$\frac{dx^4}{ds} = \left(\mathfrak{E} - \frac{qQ}{mc^2} \frac{1}{(r + (kM/c^2))}\right) f^2 \quad (4.11)$$

\mathfrak{E} being a constant.

Dividing the line element (4.1) by ds^2 and allowing for (4.9)–(4.11), we find the following equation, which is analogous to the classical energy Eq. (2.16), according to RTG:

$$f^2 \left(\frac{dr}{ds}\right)^2 + \frac{\mathcal{L}^2}{(r + (kM/c^2))^2} - f^2 \left(\mathfrak{E} - \frac{qQ}{mc^2} \frac{1}{(r + (kM/c^2))}\right)^2 + 1 = 0. \quad (4.12)$$

As in the problem considered in the framework of CM or GRT, we will consider r as a function of θ instead of s . Thus, taking into account (4.10):

$$\frac{dr}{ds} = \frac{dr}{d\theta} \frac{d\theta}{ds} = \frac{\mathcal{L}}{(r + (kM/c^2))^2} \frac{dr}{d\theta} \quad (4.13)$$

and putting

$$Z = \frac{1}{r + (kM/c^2)}, \quad (4.14)$$

for the case when $\mathcal{L} \neq 0$, Eq. (4.12) becomes

$$\left(\frac{dZ}{d\theta}\right)^2 = -\frac{kQ^2}{c^4} Z^4 + \frac{2kM}{c^2} Z^3 - \left[1 + \frac{Q^2}{\mathcal{L}^2 m^2 c^4} (km^2 - q^2)\right] Z^2 + \left(\frac{2kM}{\mathcal{L}^2 c^2} - \mathfrak{E} \frac{2qQ}{m\mathcal{L}^2 c^2}\right) Z - \frac{1 - \mathfrak{E}^2}{\mathcal{L}^2}. \quad (4.15)$$

Eq. (4.15) governs, according to RTG, the geometry of the orbits described by P in the plane $\varphi = \pi/2$. As in GRT, denoting by $\mathcal{F}(Z)$ the right member of this equation, the range for r and the shape of the orbit are determined by the disposition of the roots of equation $\mathcal{F}(Z) = 0$ and by the value of \mathfrak{E}^2 . Equation $\mathcal{F}(Z) = 0$ has the same roots as equation $F(u) = 0$ from (3.17), so the discussion concerning the disposition of these roots depending on the value of \mathfrak{E}^2 , rests the same as in Section 3. Thus, if $\mathfrak{E}^2 < 1$ we obtain elliptic type orbits, if $\mathfrak{E}^2 > 1$ we get hyperbolic or elliptic type orbits and if $\mathfrak{E}^2 = 1$ we get parabolic or elliptic type orbits. For these orbits, Z oscillates in the same range as u , so the range of r is moved back with kM/c^2 .

The substantial difference between the solution in GRT and RTG is established in the region close to the gravitational radius r_g , given by (4.2). This case must be treated as Logunov and Mestvirishvili have made in [3] for the case of Schwarzschild metric. For this purpose, we must consider the solution of the system of RTG's equations with a non-zero gravitation mass and Maxwell's equations. All details concerning this problem will be treated in a future paper. For now, the following remark can be made here. The particle cannot continue its trajectory beyond the horizon $r = r_g$. We introduce for P the co-moving variables $(\zeta^i)_{i=\overline{1,4}} = (R(r, t), \varphi, \theta, c\tau(r, t))$, where τ is the proper time of P, these coordinates forming, according to RTG, another coordinate system in Minkowski space–time. In the co-moving reference system ζ^i , the metric tensor γ_{ij} of the Minkowski space–time has not the form (4.3) it is determined from the tensor transformation law. In principle, the one-to-one transformation between the system of coordinates x^i and ζ^i can be established using the fact that the metric coefficients $g_{ij}(\zeta)$ must satisfy the general covariant equations of RTG which tell us that a gravitational field can have only spin states 0 and 2, i.e. equations $D_i \tilde{g}^{ij} = 0, i, j = \overline{1,4}$. The connection between the proper time interval $d\tau$ and dt is $d\tau = (1 - (2kM/c^2)(1/r) + (kQ^2/c^4)(1/r^2))dt$. Thus, P cannot continue the trajectory beyond the horizon $r = r_g$ because in this region, the expression $1 - (2kM/c^2)(1/r) + (kQ^2/c^4)(1/r^2)$ takes negative values and therefore the proper time τ cannot be a time which P measures on its own clock.

As in Section 3, let us now explore equation (4.15) with to find its solution.

Differentiating this equation with respect to θ and then removing the common factor $2 dZ/d\theta$, we obtain

$$\begin{aligned} \frac{d^2Z}{d\theta^2} + Z &= -\frac{2kQ^2}{c^4} Z^3 + \frac{3kM}{c^2} Z^2 \\ &+ \frac{Q^2}{\mathcal{L}^2 c^4 m^2} (q^2 - km^2) Z \\ &+ \frac{kM}{\mathcal{L}^2 c^2} - \mathfrak{E} \frac{qQ}{m \mathcal{L}^2 c^2}. \end{aligned} \tag{4.16}$$

Let us now find a solution of order velocity²/ c^2 for this relativistic equation.

Allowing for the notations (3.16) and (4.14), the expression of Z depending on u is

$$Z = \frac{u}{1 + u(kM/c^2)}. \tag{4.17}$$

As we have already discussed in the Section 3, we assume that the order of magnitude of kMu is v_1^2 , where v_1 is a velocity much smaller than the velocity of light in vacuum. Keeping only to the terms of order v_1^2/c^2 in (4.17), we get for Z :

$$Z = u - \frac{kM}{c^2} u^2. \tag{4.18}$$

From (4.18), we get to terms of order v_1^2/c^2 ,

$$\frac{d^2Z}{d\theta^2} = \left(1 - \frac{2kM}{c^2} u\right) \frac{d^2u}{d\theta^2} - \frac{2kM}{c^2} \left(\frac{du}{d\theta}\right)^2. \tag{4.19}$$

Thus, by virtue of (4.18) and (4.19), in an approximation of order velocity²/ c^2 , Eq. (4.16) becomes

$$\begin{aligned} \left(1 - \frac{2kM}{c^2} u\right) \frac{d^2u}{d\theta^2} + u &= \frac{kM}{\mathcal{L}^2 c^2} - \mathfrak{E} \frac{qQ}{m \mathcal{L}^2 c^2} + \frac{Q^2}{\mathcal{L}^2 c^4 m^2} (q^2 - km^2) u \\ &+ 4 \frac{kM}{c^2} u^2 + \frac{2kM}{c^2} \left(\frac{du}{d\theta}\right)^2. \end{aligned} \tag{4.20}$$

In the case of slow motion in weak gravitational fields, Eq. (4.20) must reduce to the classical Eq. (2.12). As in Section 3, this happens for $\mathfrak{E} \simeq 1$ and $1/\mathcal{L}^2 c^2 \simeq 1/J^2$. Taking also into account the notations (3.23) for the small dimensionless quantities, let us find an approximate solution to order velocity²/ c^2 for equation

$$\begin{aligned} \frac{d^2u}{d\theta^2} \left(1 - 2\delta \frac{J^2}{kM} u\right) + u(1 - \varepsilon) &= \frac{kM}{J^2} - \frac{qQ}{mJ^2} + 4\delta \frac{J^2}{kM} u^2 \\ &+ 2\delta \frac{J^2}{kM} \left(\frac{du}{d\theta}\right)^2. \end{aligned} \tag{4.21}$$

To solve this we assume a solution of the form:

$$\begin{aligned} u(\theta) &= u_0(\theta) + \varepsilon V(\theta) + \delta(\theta) \\ &+ O(\varepsilon^2) + O(\delta^2) + O(\varepsilon\delta). \end{aligned} \tag{4.22}$$

Substituting this form for u in the differential Eq. (4.21) and keeping only the terms of order 0 and 1 in ε and δ , we find

$$\begin{aligned} \frac{d^2 u_0}{d\theta^2} + \varepsilon \frac{d^2 V}{d\theta^2} + \delta \frac{d^2 \mathcal{W}}{d\theta^2} - 2\delta \frac{J^2}{kM} \frac{d^2 u_0}{d\theta^2} u_0 \\ + u_0 + \varepsilon V + \delta \mathcal{W} - \varepsilon u_0 = \frac{kM}{J^2} - \frac{qQ}{mJ^2} \\ + 4\delta \frac{J^2}{kM} u_0^2 + 2\delta \frac{J^2}{kM} \left(\frac{du_0}{d\theta} \right)^2. \end{aligned} \tag{4.23}$$

Equating the zeroth-order terms in ε and δ we get Eq. (2.12) with the solution (2.14). Equating the first-order terms in ε and taking into account (2.14), we get Eq. (3.28) with the non-homogeneous solutions (3.29) and (3.30).

Similarly, equating the first-order terms in δ and by virtue of (2.14), we obtain

$$\begin{aligned} \frac{d^2 \mathcal{W}}{d\theta^2} + \mathcal{W} = 4 \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + 2 \frac{J^2}{kM} \mathcal{A}^2 \\ + 6 \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \mathcal{A} \cos \theta \end{aligned} \tag{4.24}$$

with the non-homogeneous solution:

$$\mathcal{W}(\theta) = \mathcal{W}_1(\theta) + \mathcal{W}_2(\theta), \tag{4.25}$$

where

$$\begin{aligned} \mathcal{W}_1(\theta) = 4 \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + 2 \frac{J^2}{kM} \mathcal{A}^2 \\ \mathcal{W}_2(\theta) = 3 \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \mathcal{A} \theta \sin \theta. \end{aligned} \tag{4.26}$$

Introducing (2.14), (3.29), (4.25) into (4.22), we obtain the solution for the orbit to first order in ε and δ :

$$\begin{aligned} u(\theta) = \left[\frac{kM}{J^2} - \frac{qQ}{mJ^2} + \varepsilon \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \right. \\ \left. + \delta \frac{4J^2}{kM} \left(\left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + \frac{\mathcal{A}^2}{2} \right) \right] \\ + \mathcal{A} \cos \theta + \left[\frac{\varepsilon}{2} + \delta \frac{3J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \right] \\ \times \mathcal{A} \theta \sin \theta. \end{aligned} \tag{4.27}$$

As in the framework of GRT, in the solution (4.27) only the last term is non-periodic. In fact, this last term is exactly as in (3.34). Therefore, using (3.35) the solution (4.27) may be written in the form (3.36). The only small difference between GRT's solution and RTG's solution is in the periodic term of order δ . But the effect of these terms is to introduce small periodic variations in the radial distance of P and it is difficult to be detected. For an orbit of elliptic type, keeping the terms to first order in ε and δ , the interval between successive perihelia is given by (3.38).

In conclusion, the predictions of RTG in the considered problem are the same as in GRT in the approximation of order velocity²/ c^2 .

5. Second-order approximation of the solution in GRT and RTG

Let us see if the solutions in GRT and RTG remain the same if we consider them in an approximation of order velocity⁴/ c^4 .

Let us return to Section 3 at Eq. (3.22). If we want to find a solution of Eq. (3.22) to order velocity⁴/ c^4 , velocity $\ll c$, we will not neglect the last term from this equation. We define the small dimensionless quantity ε, δ as in (3.23) and in addition:

$$\zeta = \frac{k^3 Q^2 M^2}{J^4 c^4}. \tag{5.1}$$

Thus Eq. (3.22) becomes

$$\begin{aligned} \frac{d^2 u}{d\theta^2} + u(1 - \varepsilon) = \frac{kM}{J^2} - \frac{qQ}{mJ^2} + 3\delta \frac{J^2}{kM} u^2 \\ - 2\zeta \left(\frac{J^2}{kM} \right)^2 u^3. \end{aligned} \tag{5.2}$$

To find a solution of this non-linear equation to order velocity⁴/ c^4 , we assume a solution of the form:

$$\begin{aligned} u(\theta) = u_0(\theta) + \varepsilon V(\theta) + \delta W(\theta) + \zeta \Upsilon(\theta) \\ + \varepsilon^2 S(\theta) + \delta^2 X(\theta) + \varepsilon \delta Y(\theta) \\ + O(\varepsilon^3) + O(\delta^3) + O(\zeta^2) + O(\zeta \varepsilon) \\ + O(\zeta \delta) + O(\varepsilon^2 \delta) + O(\varepsilon \delta^2). \end{aligned} \tag{5.3}$$

Substituting this form for u in the differential Eq. (5.2) and keeping only the terms to order 2 in ε and δ and

to order 1 in ζ , we find

$$\begin{aligned} \frac{d^2 u_0}{d\theta^2} + \varepsilon \frac{d^2 V}{d\theta^2} + \delta \frac{d^2 W}{d\theta^2} + \zeta \frac{d^2 \Upsilon}{d\theta^2} + \varepsilon^2 \frac{d^2 S}{d\theta^2} \\ + \delta^2 \frac{d^2 X}{d\theta^2} + \varepsilon \delta \frac{d^2 Y}{d\theta^2} + u_0 + \varepsilon V + \delta W + \zeta \Upsilon \\ + \varepsilon^2 S + \delta^2 X + \varepsilon \delta Y - \varepsilon u_0 - \varepsilon^2 V - \varepsilon \delta W = \frac{kM}{J^2} \\ - \frac{qQ}{mJ^2} + 3\delta \frac{J^2}{kM} u_0^2 + 6\varepsilon\delta \frac{J^2}{kM} u_0 V \\ + 6\delta^2 \frac{J^2}{kM} u_0 W - 2\zeta \left(\frac{J^2}{kM}\right)^2 u_0^3. \end{aligned} \tag{5.4}$$

Equating the zeroth-order terms in ε , δ and ζ we get Eq. (2.12) with the solution (2.14). Equating the first-order terms in ε and δ , and taking into account (2.14), we get Eq. (3.28), respectively (3.31) with the non-homogeneous solution (3.29), (3.30), respectively (3.32), (3.33). Similarly, taking into account (2.14), for the terms of order ζ we obtain

$$\begin{aligned} \frac{d^2 \Upsilon}{d\theta^2} + \Upsilon = -2 \left(\frac{J^2}{kM}\right)^2 \left[\left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right)^3 \right. \\ \left. + \frac{3\mathcal{A}^2}{2} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right) \right] - 2 \left(\frac{J^2}{kM}\right)^2 \\ \times \left(3 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right)^2 + \frac{3\mathcal{A}^2}{4} \right) \mathcal{A} \cos \theta \\ - 3 \left(\frac{J^2}{kM}\right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right) \mathcal{A}^2 \cos 2\theta \\ - \frac{1}{2} \left(\frac{J^2}{kM}\right)^2 \mathcal{A}^3 \cos 3\theta \end{aligned} \tag{5.5}$$

with the non-homogeneous solution

$$\Upsilon(\theta) = \Upsilon_1(\theta) + \Upsilon_2(\theta) + \Upsilon_3(\theta) + \Upsilon_4(\theta), \tag{5.6}$$

where

$$\begin{aligned} \Upsilon_1(\theta) = -2 \left(\frac{J^2}{kM}\right)^2 \left[\left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right)^3 \right. \\ \left. + \frac{3\mathcal{A}^2}{2} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right) \right], \end{aligned}$$

$$\begin{aligned} \Upsilon_2(\theta) = - \left(\frac{J^2}{kM}\right)^2 \left(3 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right)^2 + \frac{3\mathcal{A}^2}{4} \right) \\ \times \mathcal{A} \theta \sin \theta, \end{aligned}$$

$$\Upsilon_3(\theta) = \left(\frac{J^2}{kM}\right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right) \mathcal{A}^2 \cos 2\theta,$$

$$\Upsilon_4(\theta) = \frac{1}{16} \left(\frac{J^2}{kM}\right)^2 \mathcal{A}^3 \cos 3\theta. \tag{5.7}$$

Identifying the second-order terms in ε^2 and allowing for (3.29) and (3.30), it results in:

$$\frac{d^2 S}{d\theta^2} + S = \frac{kM}{J^2} - \frac{qQ}{mJ^2} + \frac{\mathcal{A}}{2} \theta \sin \theta \tag{5.8}$$

with the non-homogeneous solution:

$$S(\theta) = S_1(\theta) + S_2(\theta), \tag{5.9}$$

where

$$S_1(\theta) = \frac{kM}{J^2} - \frac{qQ}{mJ^2},$$

$$S_2(\theta) = \frac{\mathcal{A}}{8} (\theta \sin \theta - \theta^2 \cos \theta). \tag{5.10}$$

Equating the second-order terms in δ^2 , by virtue of (2.14) and (3.32), (3.33), we get

$$\begin{aligned} \frac{d^2 X}{d\theta^2} + X = 18 \left(\frac{J^2}{kM}\right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right) \\ \times \left[\left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right)^2 + \frac{\mathcal{A}^2}{2} \right] \\ + 18 \left(\frac{J^2}{kM}\right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right)^2 \mathcal{A} \theta \sin \theta \\ + 18 \left(\frac{J^2}{kM}\right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right)^2 \\ \times \left(\frac{5}{12} \mathcal{A}^2\right) \mathcal{A} \cos \theta \\ + 9 \left(\frac{J^2}{kM}\right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right) \mathcal{A}^2 \theta \sin 2\theta \\ - 3 \left(\frac{J^2}{kM}\right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right) \mathcal{A}^2 \cos 2\theta \\ - \frac{3}{2} \left(\frac{J^2}{kM}\right)^2 \mathcal{A}^3 \cos 3\theta. \end{aligned} \tag{5.11}$$

The non-homogeneous solution of (5.11) can be easily checked:

$$X(\theta) = X_1(\theta) + X_2(\theta) + X_3(\theta) + X_4(\theta) + X_5(\theta) + X_6(\theta), \tag{5.12}$$

where

$$\begin{aligned} X_1(\theta) &= 18 \left(\frac{J^2}{kM} \right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \\ &\quad \times \left[\left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + \frac{\mathcal{A}^2}{2} \right], \\ X_2(\theta) &= \frac{9}{2} \left(\frac{J^2}{kM} \right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 \\ &\quad \times \mathcal{A}(\theta \sin \theta - \theta^2 \cos \theta), \\ X_3(\theta) &= 9 \left(\frac{J^2}{kM} \right)^2 \left(\left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + \frac{5}{12} \mathcal{A}^2 \right) \\ &\quad \times \mathcal{A} \theta \sin \theta, \\ X_4(\theta) &= \left(\frac{J^2}{kM} \right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \\ &\quad \times \mathcal{A}^2(-3\theta \sin 2\theta - 4 \cos 2\theta), \\ X_5(\theta) &= \left(\frac{J^2}{kM} \right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \mathcal{A}^2 \cos 2\theta, \\ X_6(\theta) &= \frac{3}{16} \left(\frac{J^2}{kM} \right)^2 \mathcal{A}^3 \cos 3\theta. \end{aligned} \tag{5.13}$$

Equating the terms in $\varepsilon\delta$ and by virtue of (2.14), (3.29), (3.30), (3.32), (3.33), we get

$$\begin{aligned} \frac{d^2 Y}{d\theta^2} + Y &= \frac{J^2}{kM} \left[9 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + \frac{3}{2} \mathcal{A}^2 \right] \\ &\quad + 6 \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \mathcal{A} \theta \sin \theta \\ &\quad - \frac{1}{2} \frac{J^2}{kM} \mathcal{A}^2 \cos 2\theta \\ &\quad + 6 \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \mathcal{A} \cos \theta \\ &\quad + \frac{3}{2} \frac{J^2}{kM} \mathcal{A}^2 \theta \sin 2\theta. \end{aligned} \tag{5.14}$$

Having the following non-homogeneous solution:

$$Y(\theta) = Y_1(\theta) + Y_2(\theta) + Y_3(\theta) + Y_4(\theta) + Y_5(\theta), \tag{5.15}$$

where

$$\begin{aligned} Y_1(\theta) &= \frac{J^2}{kM} \left[9 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + \frac{3}{2} \mathcal{A}^2 \right], \\ Y_2(\theta) &= \frac{3}{2} \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \\ &\quad \times \mathcal{A}(\theta \sin \theta - \theta^2 \cos \theta), \\ Y_3(\theta) &= \frac{1}{6} \frac{J^2}{kM} \mathcal{A}^2 \cos 2\theta, \\ Y_4(\theta) &= 3 \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \mathcal{A} \cos \theta, \\ Y_5(\theta) &= \frac{J^2}{kM} \mathcal{A}^2 \left(-\frac{1}{2} \theta \sin 2\theta - \frac{2}{3} \cos 2\theta \right). \end{aligned} \tag{5.16}$$

Introducing (2.14), (3.29), (3.32), (5.6), (5.9), (5.12), (5.15) into (5.4), we find the non-periodic term of the solution

$$\begin{aligned} &\left[\frac{\varepsilon}{2} + \delta \frac{3J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) - \zeta \left(\frac{J^2}{kM} \right)^2 \right. \\ &\quad \times \left. \left(3 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + \frac{3\mathcal{A}^2}{4} \right) \right] \mathcal{A} \theta \sin \theta + \left[\frac{\varepsilon^2}{8} \right. \\ &\quad \left. + \delta^2 \left(\frac{J^2}{kM} \right)^2 \left(\frac{27}{2} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + \frac{15}{4} \mathcal{A}^2 \right) \right. \\ &\quad \left. + \varepsilon\delta \frac{3}{2} \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \right] \mathcal{A} \theta \sin \theta \\ &\quad - \left[\frac{\varepsilon^2}{8} + \delta^2 \frac{9}{2} \left(\frac{J^2}{kM} \right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 \right. \\ &\quad \left. + \varepsilon\delta \frac{3}{2} \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \right] \mathcal{A} \theta^2 \cos \theta \\ &\quad - \left[3\delta^2 \left(\frac{J^2}{kM} \right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) + \varepsilon\delta \frac{1}{2} \frac{J^2}{kM} \right] \\ &\quad \times \mathcal{A}^2 \theta \sin 2\theta. \end{aligned} \tag{5.17}$$

We note that to the second order in ε and δ , and to the first order in ζ , the following expression:

$$\begin{aligned} & \cos\left(\theta - \theta \left[\frac{\varepsilon}{2} + \delta \frac{3J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \right. \right. \\ & - \zeta \left(\frac{J^2}{kM} \right)^2 \left(3 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + \frac{3\mathcal{A}^2}{4} \right) + \frac{\varepsilon^2}{8} \\ & + \delta^2 \left(\frac{J^2}{kM} \right)^2 \left(\frac{27}{2} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + \frac{15}{4} \mathcal{A}^2 \right) \\ & + \varepsilon\delta \frac{3}{2} \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) - \left(6\delta^2 \left(\frac{J^2}{kM} \right)^2 \right. \\ & \left. \left. \times \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) + \varepsilon\delta \frac{J^2}{kM} \right) \mathcal{A} \cos \theta \right], \quad (5.18) \end{aligned}$$

reduces to $\cos \theta$ plus the expression written in (5.17). Therefore, the solution is an approximation of order velocity⁴/ c^4 for the electrogravitational Kepler problem in the framework of GRT and may be written as

$$\begin{aligned} u(\theta) = & \frac{kM}{J^2} - \frac{qQ}{mJ^2} \\ & + \mathcal{A} \cos\left(\theta - \theta \left[\frac{\varepsilon}{2} + \delta \frac{3J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \right. \right. \\ & - \zeta \left(\frac{J^2}{kM} \right)^2 \left(3 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + \frac{3\mathcal{A}^2}{4} \right) \\ & + \frac{\varepsilon^2}{8} + \delta^2 \left(\frac{J^2}{kM} \right)^2 \left(\frac{27}{2} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 \right. \\ & + \frac{15}{4} \mathcal{A}^2 \left. \right) + \varepsilon\delta \frac{3}{2} \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \\ & - \left(6\delta^2 \left(\frac{J^2}{kM} \right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \right. \\ & \left. + \varepsilon\delta \frac{J^2}{kM} \right) \mathcal{A} \cos \theta \left. \right) \\ & + (\text{periodic terms of order } \varepsilon, \delta, \varepsilon^2, \delta^2, \zeta). \quad (5.19) \end{aligned}$$

Let us now see how this approximation of order velocity⁴/ c^4 for the electrogravitational Kepler problem looks in the framework of RTG.

We return to Eq. (4.16). Keeping in (4.17) to terms of order velocity⁴/ c^4 , we get for Z :

$$Z = u - \frac{kM}{c^2} u^2 + \frac{k^2 M^2}{c^4} u^3. \quad (5.20)$$

From (5.20) we get to terms of order velocity⁴/ c^4 :

$$\begin{aligned} \frac{d^2 Z}{d\theta^2} = & \left(1 - \frac{2kM}{c^2} u + \frac{3k^2 M^2}{c^4} u^2 \right) \frac{d^2 u}{d\theta^2} \\ & + \left(-\frac{2kM}{c^2} + \frac{6k^2 M^2}{c^4} u \right) \left(\frac{du}{d\theta} \right)^2. \quad (5.21) \end{aligned}$$

Thus, by virtue of (5.20), (5.21), for $\mathfrak{E} \simeq 1$ and $1/\mathcal{L}^2 c^2 \simeq \frac{1}{J^2}$, Eq. (4.16) becomes, in an approximation of order velocity⁴/ c^4 :

$$\begin{aligned} \frac{d^2 u}{d\theta^2} & \left(1 - 2\delta \frac{J^2}{kM} u + 3\delta^2 \left(\frac{J^2}{kM} \right)^2 u^2 \right) + u(1 - \varepsilon) \\ & = \frac{kM}{J^2} - \frac{qQ}{mJ^2} + 4\delta \frac{J^2}{kM} u^2 - \varepsilon\delta \frac{J^2}{kM} u^2 \\ & - 7\delta^2 \left(\frac{J^2}{kM} \right)^2 u^3 - 2\zeta \left(\frac{J^2}{kM} \right)^2 u^3 \\ & + \left(2\delta - 6\delta^2 \frac{J^2}{kM} u \right) \frac{J^2}{kM} \left(\frac{du}{d\theta} \right)^2. \quad (5.22) \end{aligned}$$

To find a solution of this non-linear equation to order velocity⁴/ c^4 , we assume a solution of the form:

$$\begin{aligned} u(\theta) = & u_0(\theta) + \varepsilon V(\theta) + \delta \mathcal{W}(\theta) + \zeta \Upsilon(\theta) + \varepsilon^2 S(\theta) \\ & + \delta^2 \mathcal{X}(\theta) + \varepsilon\delta \mathcal{Y}(\theta) + O(\varepsilon^3) + O(\delta^3) \\ & + O(\zeta^2) + O(\zeta\varepsilon) + O(\zeta\delta) \\ & + O(\varepsilon^2\delta) + O(\varepsilon\delta^2). \quad (5.23) \end{aligned}$$

Substituting this form for u in the differential Eq. (5.22) and keeping only the terms to the order 2 in ε and δ and to the order 1 in ζ , we find

$$\begin{aligned} \frac{d^2 u_0}{d\theta^2} + \varepsilon \frac{d^2 V}{d\theta^2} + \delta \frac{d^2 \mathcal{W}}{d\theta^2} + \zeta \frac{d^2 \Upsilon}{d\theta^2} + \varepsilon^2 \frac{d^2 S}{d\theta^2} \\ + \delta^2 \frac{d^2 \mathcal{X}}{d\theta^2} + \varepsilon\delta \frac{d^2 \mathcal{Y}}{d\theta^2} - \delta \frac{2J^2}{kM} u_0 \frac{d^2 u_0}{d\theta^2} \\ - \varepsilon\delta \frac{2J^2}{kM} V \frac{d^2 u_0}{d\theta^2} - \delta^2 \frac{2J^2}{kM} \mathcal{W} \frac{d^2 u_0}{d\theta^2} \end{aligned}$$

$$\begin{aligned}
 & -\varepsilon\delta \frac{2J^2}{kM} u_0 \frac{d^2V}{d\theta^2} - \delta^2 \frac{2J^2}{kM} u_0 \frac{d^2\mathcal{W}}{d\theta^2} \\
 & + 3\delta^2 \left(\frac{J^2}{kM}\right)^2 u_0^2 \frac{d^2u_0}{d\theta^2} + u_0 + \varepsilon V + \delta\mathcal{W} + \zeta\Upsilon \\
 & + \varepsilon^2 S + \delta^2 \mathcal{X} + \varepsilon\delta\mathcal{Y} - \varepsilon u_0 - \varepsilon^2 V - \varepsilon\delta\mathcal{W} \\
 & = \frac{kM}{J^2} - \frac{qQ}{mJ^2} + \delta \frac{4J^2}{kM} u_0^2 + \varepsilon\delta \frac{8J^2}{kM} u_0 V \\
 & + \delta^2 \frac{8J^2}{kM} u_0 \mathcal{W} - \varepsilon\delta \frac{J^2}{kM} u_0^3 \\
 & - 7\delta^2 \left(\frac{J^2}{kM}\right)^2 u_0^3 - 2\zeta \left(\frac{J^2}{kM}\right)^2 u_0^3 \\
 & + \delta \frac{2J^2}{kM} \left(\frac{du_0}{d\theta}\right)^2 + \varepsilon\delta \frac{4J^2}{kM} \frac{du_0}{d\theta} \frac{dV}{d\theta} \\
 & + \delta^2 \frac{4J^2}{kM} \frac{du_0}{d\theta} \frac{d\mathcal{W}}{d\theta} \\
 & - 6\delta^2 \left(\frac{J^2}{kM}\right)^2 u_0 \left(\frac{du_0}{d\theta}\right)^2. \tag{5.24}
 \end{aligned}$$

Equating the zeroth-order terms in ε , δ and ζ we get Eq. (2.12) with the solution (2.14). Equating the first-order terms in ε and δ , and taking into account (2.14), we get Eq. (3.28), respectively (4.24) with the non-homogeneous solution (3.29), (3.30), respectively (4.25), (4.26). For the terms of order ζ we obtain (5.5), with the non-homogeneous solution (5.6), (5.7). Identifying the second-order terms in ε^2 and allowing for (3.29), (3.30), it results Eq. (5.8), having the non-homogeneous solution (5.9), (5.10).

Equating the second-order terms in δ^2 , we get

$$\begin{aligned}
 & \frac{d^2\mathcal{X}}{d\theta^2} + \mathcal{X} \\
 & = \left(\frac{J^2}{kM}\right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right) \\
 & \times \left[25 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right)^2 + \frac{11}{2} \mathcal{A}^2 \right] \\
 & + 18 \left(\frac{J^2}{kM}\right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right)^2 \mathcal{A}\theta \sin \theta \\
 & + \left(\frac{J^2}{kM}\right)^2 \left(18 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right)^2 + \frac{15}{2} \mathcal{A}^2 \right) \mathcal{A} \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{15}{2} \left(\frac{J^2}{kM}\right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right) \mathcal{A}^2 \cos 2\theta \\
 & + \frac{1}{2} \left(\frac{J^2}{kM}\right)^2 \mathcal{A}^3 \cos 3\theta. \tag{5.25}
 \end{aligned}$$

The non-homogeneous solution of (5.25) can be easily checked:

$$\begin{aligned}
 \mathcal{X}(\theta) = & \mathcal{X}_1(\theta) + \mathcal{X}_2(\theta) + \mathcal{X}_3(\theta) \\
 & + \mathcal{X}_4(\theta) + \mathcal{X}_5(\theta), \tag{5.26}
 \end{aligned}$$

where

$$\begin{aligned}
 \mathcal{X}_1(\theta) = & \left(\frac{J^2}{kM}\right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right) \\
 & \times \left[25 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right)^2 + \frac{11}{2} \mathcal{A}^2 \right], \\
 \mathcal{X}_2(\theta) = & \frac{9}{2} \left(\frac{J^2}{kM}\right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right)^2 \\
 & \times \mathcal{A}(\theta \sin \theta - \theta^2 \cos \theta), \\
 \mathcal{X}_3(\theta) = & \left(\frac{J^2}{kM}\right)^2 \left(9 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right)^2 + \frac{15}{4} \mathcal{A}^2 \right) \\
 & \times \mathcal{A}\theta \sin \theta, \\
 \mathcal{X}_4(\theta) = & -\frac{5}{2} \left(\frac{J^2}{kM}\right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right) \mathcal{A}^2 \cos 2\theta, \\
 \mathcal{X}_5(\theta) = & -\frac{1}{16} \left(\frac{J^2}{kM}\right)^2 \mathcal{A}^3 \cos 3\theta. \tag{5.27}
 \end{aligned}$$

Equating the terms in $\varepsilon\delta$ and by virtue of (2.14), (3.29), (3.30), (4.25), (4.26), we get

$$\begin{aligned}
 \frac{d^2\mathcal{Y}}{d\theta^2} + \mathcal{Y} = & \frac{J^2}{kM} \left[11 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right)^2 + \frac{3}{2} \mathcal{A}^2 \right] \\
 & + 6 \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right) \mathcal{A}\theta \sin \theta \\
 & + \frac{J^2}{kM} \mathcal{A}^2 \cos 2\theta \\
 & + 6 \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2}\right) \mathcal{A} \cos \theta \\
 & + \frac{1}{2} \frac{J^2}{kM} \mathcal{A}^2 \theta \sin 2\theta. \tag{5.28}
 \end{aligned}$$

With the following non-homogeneous solution:

$$\mathcal{Y}(\theta) = \mathcal{Y}_1(\theta) + \mathcal{Y}_2(\theta) + \mathcal{Y}_3(\theta) + \mathcal{Y}_4(\theta) + \mathcal{Y}_5(\theta), \tag{5.29}$$

where

$$\mathcal{Y}_1(\theta) = \frac{J^2}{kM} \left[11 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + \frac{3}{2} \mathcal{A}^2 \right],$$

$$\mathcal{Y}_2(\theta) = \frac{3}{2} \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \times \mathcal{A}(\theta \sin \theta - \theta^2 \cos \theta),$$

$$\mathcal{Y}_3(\theta) = -\frac{1}{3} \frac{J^2}{kM} \mathcal{A}^2 \cos 2\theta,$$

$$\mathcal{Y}_4(\theta) = 3 \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \mathcal{A} \cos \theta,$$

$$\mathcal{Y}_5(\theta) = \frac{J^2}{kM} \mathcal{A}^2 \left(-\frac{1}{6} \theta \sin 2\theta - \frac{2}{9} \cos 2\theta \right). \tag{5.30}$$

Introducing (2.14), (3.29), (4.25), (5.6), (5.9), (5.26), (5.29) into (5.23), we find the non-periodic term of this solution:

$$\begin{aligned} & \left[\frac{\varepsilon}{2} + \delta \frac{3J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) - \zeta \left(\frac{J^2}{kM} \right)^2 \right. \\ & \left. \left(3 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + \frac{3\mathcal{A}^2}{4} \right) \right] \mathcal{A} \theta \sin \theta \\ & + \left[\frac{\varepsilon^2}{8} + \delta^2 \left(\frac{J^2}{kM} \right)^2 \left(\frac{27}{2} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 \right. \right. \\ & \left. \left. + \frac{15}{4} \mathcal{A}^2 \right) + \varepsilon \delta \frac{3}{2} \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \right] \mathcal{A} \theta \sin \theta \\ & - \left[\varepsilon^2 \frac{1}{8} + \delta^2 \frac{9}{2} \left(\frac{J^2}{kM} \right)^2 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 \right. \\ & \left. + \varepsilon \delta \frac{3}{2} \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \right] \mathcal{A} \theta^2 \cos \theta \\ & - \varepsilon \delta \frac{1}{6} \frac{J^2}{kM} \mathcal{A}^2 \theta \sin 2\theta. \tag{5.31} \end{aligned}$$

We note that to the second order in ε and δ , and to the first order in ζ , the following expression:

$$\begin{aligned} & \cos \left(\theta - \theta \left[\frac{\varepsilon}{2} + \delta \frac{3J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \right. \right. \\ & \left. \left. - \zeta \left(\frac{J^2}{kM} \right)^2 \left(3 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + \frac{3\mathcal{A}^2}{4} \right) + \frac{\varepsilon^2}{8} \right. \right. \\ & \left. \left. + \delta^2 \left(\frac{J^2}{kM} \right)^2 \left(\frac{27}{2} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + \frac{15}{4} \mathcal{A}^2 \right) \right. \right. \\ & \left. \left. + \varepsilon \delta \frac{3}{2} \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \right. \right. \\ & \left. \left. - \varepsilon \delta \frac{1}{3} \frac{J^2}{kM} \mathcal{A} \cos \theta \right) \right], \tag{5.32} \end{aligned}$$

reduces to $\cos \theta$ plus the expression written in (5.31). Therefore, the solution in an approximation of order velocity⁴/ c^4 for the electrogravitational Kepler problem and in the framework of RTG may be written as

$$\begin{aligned} u(\theta) &= \frac{kM}{J^2} - \frac{qQ}{mJ^2} \\ &+ \mathcal{A} \cos \left(\theta - \theta \left[\frac{\varepsilon}{2} + \delta \frac{3J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \right. \right. \\ & \left. \left. - \zeta \left(\frac{J^2}{kM} \right)^2 \left(3 \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + \frac{3\mathcal{A}^2}{4} \right) + \frac{\varepsilon^2}{8} \right. \right. \\ & \left. \left. + \delta^2 \left(\frac{J^2}{kM} \right)^2 \left(\frac{27}{2} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right)^2 + \frac{15}{4} \mathcal{A}^2 \right) \right. \right. \\ & \left. \left. + \varepsilon \delta \frac{3}{2} \frac{J^2}{kM} \left(\frac{kM}{J^2} - \frac{qQ}{mJ^2} \right) \right. \right. \\ & \left. \left. - \varepsilon \delta \frac{1}{3} \frac{J^2}{kM} \mathcal{A} \cos \theta \right) \right) \\ &+ (\text{periodic terms of order } \varepsilon, \delta, \varepsilon^2, \delta^2, \zeta). \tag{5.33} \end{aligned}$$

This solution must be compared with the solution (5.19) obtained in the framework of GRT. As we can see, considering approximate solutions to order velocity⁴/ c^4 , velocity $\ll c$, the shift of the perihelion has different values in the two theories.

6. Conclusions

We can conclude that the orbits described by the charged mass point with mass m and electric charge q , in the electrogravitational field produced by the charged mass point with M and electric charge Q , have the same shape in GRT and RTG. In RTG, the range of variable r is moved back, with respect to the one in GRT, with kM/c^2 .

The substantial difference between the solution in GRT and RTG is established in the region close to the gravitational radius, r_g . From the viewpoint of GRT, P crosses the horizon $r = r_g$ only in the inside direction. This trajectory does not reach the singularity $r = 0$; it ends at some point inside the region with r in the interval $(0, (kM/c^2) - (1/c^2)\sqrt{k^2M^2 - kQ^2})$. From the viewpoint of RTG, the trajectory of P cannot continue beyond the horizon $r = r_g$.

The orbits of elliptic type of P, rotate slowly in the same direction or in the opposite directions in which they are described.

In an approximation of the solution for the electrogravitational Kepler problem to the first order, the advance of perihelion per revolution is the same in GRT and RTG. In an approximation of the solution to the second order this advance of perihelion differs in the two theories.

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