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VARIATIONAL DERIVATION OF THE GREEN–NAGHDI SHALLOW-WATER EQUATIONS

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We consider the two-dimensional irrotational water-wave problem with a free surface and a flat bottom. In the shallow-water regime and without smallness assumption on the wave amplitude we derive, by a variational approach in the Lagrangian formalism, the Green–Naghdi equations (1.1). The second equation in (1.1) is a transport equation, the free surface is advected by the fluid flow. We show that the first equation of the system (1.1) yields the critical points of an action functional in the space of paths with fixed endpoints, within the Lagrangian formalism.

Keywords: Green–Naghdi equations; shallow-water waves; variational methods.

Mathematics Subject Classification 2000: 76B15, 70G75, 76M30

1. Introduction

The Green–Naghdi equations [14] model shallow-water waves whose amplitude is not necessarily small and represent a higher-order correction to the classical shallow-water equations. The shallow-water regime is characterized by the undisturbed water depth h_0 being much smaller than the horizontal wave length λ. The Green–Naghdi equations can be written in the following non-dimensionalized form:

$$\begin{cases} u_t + uu_x + HH_x = \dfrac{1}{3H}[H^3(uu_{xx} + u_{xt} - u_x^2)]_x \\ H_t + (Hu)_x = 0, \end{cases} \tag{1.1}$$

with $x \in \mathbf{R}$ and $t \in \mathbf{R}$, and where $u(x,t)$ represents the depth-averaged[a] horizontal velocity and $H(x,t)$ is the free upper surface. Subscripts here, and later, denote partial derivatives. Green and Naghdi considered in [14] the three-dimensional water-wave problem with a

[a]The depth-averaged value of a quantity $q(x,z,t)$ is defined by $\bar{q}(x,t) := \frac{1}{H(x,t)}\int_0^{H(x,t)} q(x,z,t)dz$.

free surface and a variable bottom, and without imposing any condition that the fluid motion should be irrotational. The model equations were not derived by a formal asymptotic expansion, but instead by imposing the condition that the horizontal velocity is independent of the vertical coordinate z, that the vertical velocity has only a linear dependence on z and by using the mass conservation equation and the energy equation in integral form plus invariance under rigid-body translation. For one horizontal x-coordinate and for a flat bottom, the obtained equations have the form (1.1).

The system (1.1) was originally derived in 1953 by Serre [26, Sec. V] by integrating over z on the interval $[0, H(x,t)]$ the Euler equations for the two-dimensional water-wave problem with a free surface (the pressure at the free surface is taken to be 0) and a flat bottom, and by making the assumption that the horizontal component of fluid velocity is independent of the vertical coordinate z being equal to its depth-averaged value. More than ten years later Su and Gardner [28] obtained the system (1.1) by depth-averaging the two-dimensional irrotational water-wave problem, by using asymptotic expansion in the small shallowness parameter $\delta = \frac{h_0}{\lambda}$ and by retaining terms as far as $O(\delta^4)$. In the literature, Eqs. (1.1) are referred to as the Serre equations, or the Su–Gardner equations but usually they are called the Green–Naghdi equations. Throughout this paper we will call them the Green–Naghdi equations.

The Green–Naghdi equations are mathematically well-posed in the sense that they admit solutions over the relevant time scale for any initial data reasonably smooth (see [2, 22]). The solution of the Green–Naghdi equations provides a good approximation of the solution of the full water-wave problem, the difference between both solutions remaining of order $O(\delta^4)$ as long as the wave does not exhibit any kind of singularity such as wave breaking (see [3, 22]).

The Green–Naghdi equations (1.1) have a Hamiltonian formulation (see [8, 16]):

$$\begin{pmatrix} m_t \\ H_t \end{pmatrix} = -\begin{pmatrix} \partial m + m\partial & H\partial \\ \partial H & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \mathcal{H}_{GN}}{\partial H} \\ \frac{\partial \mathcal{H}_{GN}}{\partial H} \end{pmatrix}, \tag{1.2}$$

where m is the momentum density defined by $\frac{\partial \mathcal{H}_{GN}}{\partial u}$, and $\mathcal{H}_{GN}$ is the total energy (kinetic plus potential) given by (see, for example, (3.11), (3.12) below):

$$\frac{1}{2}\int_{-\infty}^{\infty} \left(Hu^2 + \frac{1}{3}H^3 u_x^2 + (H-1)^2 \right) dx. \tag{1.3}$$

Looking for the traveling wave solutions of the form

$$u(x,t) = u(x-ct), \quad H(x,t) = H(x-ct), \tag{1.4}$$

with c the speed of the traveling wave, one finds that Eqs. (1.1) have a solitary wave solution [26, p. 863–864; 28, p. 539]:

$$\begin{aligned} H(x,t) &= 1 + (c^2-1)\mathrm{sech}^2\left[\frac{\sqrt{3}}{2}\frac{\sqrt{c^2-1}}{c}(x-ct) \right], \\ u(x,t) &= c\left(1 - \frac{1}{H(x,t)}\right). \end{aligned} \tag{1.5}$$

These waves exist for all c such that the following condition:

$$c^2 > 1 \tag{1.6}$$

is satisfied. In [20, 21], the eigenvalue problem obtained from linearizing the equations about solitary wave solutions is investigated and it is established that small-amplitude solitary wave solutions of the Green–Naghdi equations are linearly stable.

The Green–Naghdi equations (1.1) have also the following periodic cnoidal wave solution [26] (see also [6, 13]):

$$\begin{aligned} H(x,t) &= H_2 + (H_3 - H_2)\mathrm{cn}^2\left[\frac{\sqrt{3}}{2}\frac{\sqrt{H_3 - H_1}}{\mathcal{C}}(x - ct); k\right], \\ u(x,t) &= c - \frac{\mathcal{C}}{H(x,t)}, \end{aligned} \tag{1.7}$$

where $0 < H_1 < H_2 < H_3$ are the roots of the polynomial $-H^3 + \mathcal{A}H^2 - \mathcal{B}H + \mathcal{C}^2$, for some constants of integration $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$. In (1.7), $\mathrm{cn}(\cdot, k)$ is the cn-Jacobi elliptic function with the elliptic modulus k, $0 < k^2 < 1$,

$$k^2 := \frac{H_3 - H_2}{H_3 - H_1}. \tag{1.8}$$

The stability of these waves is further investigated in [6]: it is established that the waves with sufficiently small amplitude are stable and the waves with sufficiently large amplitude are unstable.

We remark that the cubic polynomial $-H^3 + \mathcal{A}H^2 - \mathcal{B}H + \mathcal{C}^2$ can have one real zero, denoted H_0, and two complex conjugate zeros. We have $H_0 > 0$, because the leading coefficient of this cubic polynomial is smaller than zero and its constant term is greater than zero. We denote by p and q the real coefficients such that

$$-H^3 + \mathcal{A}H^2 - \mathcal{B}H + \mathcal{C}^2 = -(H - H_0)(H^2 + pH + q). \tag{1.9}$$

Then, the Green–Naghdi equations (1.1) have also the following periodic travelling wave solution:

$$\begin{aligned} H(x,t) &= H_0 - \sqrt{H_0^2 + pH_0 + q}\quad \frac{1 - \mathrm{cn}\left[\frac{\sqrt{3}(H_0^2 + pH_0 + q)^{\frac{1}{4}}}{\mathcal{C}}(x - ct); k\right]}{1 + \mathrm{cn}\left[\frac{\sqrt{3}(H_0^2 + pH_0 + q)^{\frac{1}{4}}}{\mathcal{C}}(x - ct); k\right]}, \\ u(x,t) &= c - \frac{\mathcal{C}}{H(x,t)}, \end{aligned} \tag{1.10}$$

with the elliptic modulus k, $0 < k^2 < 1$,

$$k^2 := \frac{1}{2}\left(1 + \frac{H_0 + \frac{p}{2}}{\sqrt{H_0^2 + pH_0 + q}}\right). \tag{1.11}$$

We observe that for that $(x - ct)$'s for which the periodic Jacobi elliptic function cn satisfies the equation:

$$1 + \mathrm{cn}\left[\frac{\sqrt{3}(H_0^2 + pH_0 + q)^{\frac{1}{4}}}{\mathcal{C}}(x - ct); k\right] = 0 \tag{1.12}$$

the solution (1.10) has vertical asymptotes in the positive direction.

The purpose of the present paper is to derive the Green–Naghdi equations (1.1) by a variational approach in the Lagrangian formalism. We consider the two-dimensional irrotational water-wave problem with a free surface and a flat bottom. In Sec. 2 we present an appropriate non-dimensionalization of this water-wave problem and subsequently the classical shallow-water equations (2.13), or (2.15) in view of the notation (2.14). The two important parameters ϵ and δ that arise in water-waves theories are used to define various approximations of the governing equations and the boundary conditions. The role of δ independent of ϵ is useful in the description of arbitrary amplitude shallow-water waves, that is, $\delta \to 0$, ϵ fixed. Section 3 is dedicated to the variational derivation of the Green–Naghdi equations (1.1). In the shallow-water regime, for an arbitrary fixed ϵ, for a velocity field with a horizontal component (2.10) independent of the vertical coordinate z and a vertical component (2.11) having only a linear dependence on z, the kinetic energy of the fluid has the expression (3.11) and the potential energy calculated with respect to the undisturbed water level is given by (3.12). The second equation in (1.1) is a transport equation, the free surface is advected, or Lie transported (in the geometry literature), by the fluid flow. We will show that the first equation of the system (1.1) yields the critical points of an action functional in the space of paths with fixed endpoints, within the Lagrangian formalism. In the Eulerian formalism, the Lagrangian function integrated over time in the action functional, defined as the kinetic energy minus the potential energy, has the form (3.13). In order to get the Lagrangian (3.14), we will transport (3.13) from the Eulerian picture to the tangent bundle which represents the velocity phase space in the Lagrangian formalism, this transport being made taking into account the second equation of the system (1.1). We point out that the Lagrangian (3.13) as well as (3.14) are not metrics.

2. The Classical Shallow-Water Equations

The two-dimensional water-wave problem considered here models an irrotational, incompressible and inviscid fluid flow with a free surface and a flat bottom. Let $h_0 > 0$ be the undisturbed depth of the fluid, let $z = h_0 + \eta(x,t)$ represent the free upper surface of the fluid, with $\eta(x,t)$ the displacement of the free surface from the undisturbed state, and let $z = 0$ be the flat bottom. The velocity field $(u(x,z,t), v(x,z,t))$ — no motion takes place in the y-direction — satisfies Euler's equations of inviscid motion:

$$\begin{aligned} u_t + uu_x + vu_z &= -\frac{1}{\rho}p_x, \\ v_t + uv_x + vv_z &= -\frac{1}{\rho}p_z - g, \end{aligned} \tag{2.1}$$

$p(x,z,t)$ denotes the pressure, g is the constant gravitational acceleration in the negative z direction and ρ is the constant density. We set the constant water density $\rho = 1$. The assumption of incompressibility implies the equation of mass conservation:

$$u_x + v_z = 0. \tag{2.2}$$

The idealization of irrotational flow is physically relevant in the absence of non-uniform currents in the water. Under the assumption that the fluid is irrotational, we also have the equation:

$$u_z - v_x = 0. \tag{2.3}$$

The boundary conditions at the free surface are constant pressure:

$$p = p_0 \quad \text{on } z = h_0 + \eta(x,t), \tag{2.4}$$

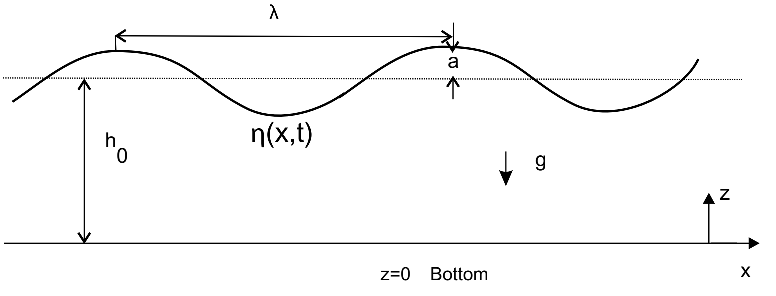
where p_0 being the constant atmospheric pressure, and the continuity of fluid velocity and surface velocity:

$$v = \eta_t + u\eta_x \quad \text{on } z = h_0 + \eta(x,t). \tag{2.5}$$

On the flat bottom $z = 0$, only one condition is required for an inviscid fluid, that is,

$$v = 0 \quad \text{on } z = 0. \tag{2.6}$$

Summing up, the exact solution for our two-dimensional water-wave problem is given by the system (2.1)–(2.6).



We define the set of non-dimensional variables (for more details see [10, 18]):

$$\begin{gathered} x \mapsto \lambda x, \quad z \mapsto h_0 z, \quad \eta \mapsto a\eta, \quad t \mapsto \frac{\lambda}{\sqrt{gh_0}} t, \\ u \mapsto \sqrt{gh_0}\, u, \quad v \mapsto h_0 \frac{\sqrt{gh_0}}{\lambda} v, \\ p \mapsto p_0 + gh_0(1 - z) + gh_0 p, \end{gathered} \tag{2.7}$$

with λ the wavelength and a the amplitude of the surface wave (see the figure above). In order to avoid new notations, we have used the same symbols for the non-dimensional variables x, z, η, t, u, v, p on the right-hand side. Therefore, in the non-dimensional variables (2.7), the water-wave problem (2.1)–(2.6) becomes:

$$\begin{aligned} u_t + uu_x + vu_z &= -p_x, \\ \delta^2(v_t + uv_x + vv_z) &= -p_z, \\ u_x + v_z &= 0, \\ u_z - \delta^2 v_x &= 0, \\ v = \epsilon(\eta_t + u\eta_x) &\quad \text{on } z = 1 + \epsilon\eta(x,t), \\ p = \epsilon\eta &\quad \text{on } z = 1 + \epsilon\eta(x,t), \\ v = 0 &\quad \text{on } z = 0, \end{aligned} \tag{2.8}$$

where $\epsilon := \frac{a}{h_0}$ is the amplitude parameter and $\delta := \frac{h_0}{\lambda}$ is the shallowness parameter.

The shallow-water approximation is obtained by requiring $\delta \to 0$, for arbitrary fixed ϵ. For $\delta = 0$, the leading-order equations become:

$$\begin{aligned} u_t + uu_x + vu_z &= -p_x, \\ p_z &= 0, \\ u_x + v_z &= 0, \\ u_z &= 0, \\ v &= \epsilon(\eta_t + u\eta_x) \quad \text{on } z = 1 + \epsilon\eta(x,t), \\ p &= \epsilon\eta(x,t) \quad\quad\;\; \text{on } z = 1 + \epsilon\eta(x,t), \\ v &= 0 \quad\quad\quad\quad\quad\; \text{on } z = 0. \end{aligned} \tag{2.9}$$

The system of Eqs. (2.9) reduces to:

$$u = u(x,t), \tag{2.10}$$

$$v = -zu_x, \tag{2.11}$$

$$p = \epsilon\eta(x,t) \tag{2.12}$$

and

$$\begin{cases} u_t + uu_x + \epsilon\eta_x = 0, \\ \epsilon\eta_t + [(1+\epsilon\eta)u]_x = 0. \end{cases} \tag{2.13}$$

In the study of the full governing equations for water waves, the relations between the free surface, the fluid velocity and the pressure are more subtle. Various properties of the pressure and the velocity beneath Stokes waves are proved in [9, 12, 15, 29].

If we denote by

$$H(x,t) := 1 + \epsilon\eta(x,t), \tag{2.14}$$

then, the system of Eqs. (2.13) becomes:

$$\begin{cases} u_t + uu_x + H_x = 0, \\ H_t + (Hu)_x = 0. \end{cases} \tag{2.15}$$

The set of hyperbolic partial differential equations (2.15) are the so-called classical shallow-water equations (see, for example, [27]). Equations (2.15) can be written in Hamiltonian form relative to a symplectic structure introduced by Manin [23]. The second Hamiltonian structure for the system (2.15) was obtained by Cavalcante and McKean [7]. In fact, the system (2.15), as a particular case of the system of polytropic gas equations in 1 + 1, i.e. in one spatial and one temporal dimension, is Hamiltonian with respect to three distinct Hamiltonian structures [24]. These Hamiltonian structures are compatible and thus, the system of Eqs. (2.15) is completely integrable [25]. The infinite sequence of integrals of motion (the conserved quantities) that the Hamiltonian structures give rise to was found much earlier by Benney [5]. Equations (2.15) provide a good approximation to the exact solution of the water-wave problem; for a rigorous justification with a precise control of the estimated error see [3].

3. Variational Derivation of the Green–Naghdi Shallow-Water Equations

In what follows we consider ϵ arbitrary fixed, there is no smallness assumption on the wave amplitude. We are looking for a higher-order correction to the classical shallow-water equations (2.13), or (2.15) in view of the notation (2.14). We observe that the second equation in (2.15) is exactly the second equation of the Green–Naghdi system (1.1). The first equation of the Green–Naghdi system (1.1) we will derive directly from a variational principle in the Lagrangian formalism.

In the Lagrangian formalism one focuses the attention on the motion of each individual particle of the mechanical system. We denote by M the ambient space whose points are supposed to represent the particles at $t = 0$. A diffeomorphism of M represents the rearrangement of the particles with respect to their initial positions. The set of all diffeomorphisms, denoted $\mathrm{Diff}(M)$, can be regarded (at least formally) as a Lie group. The motion of the mechanical system is described by a time-dependent family of orientation-preserving diffeomorphisms $\gamma(\cdot, t) \in \mathrm{Diff}(M)$. For a particle initially located at X, the velocity at time t is

$$\gamma_t(X,t) := \frac{\partial \gamma(X,t)}{\partial t}, \tag{3.1}$$

this being the material velocity used in the Lagrangian description. The spatial velocity, used in the Eulerian description, is the flow velocity

$$u(x,t) := \gamma_t(X,t), \tag{3.2}$$

at the location $x = \gamma(X,t)$, at time t, that is,

$$u(\cdot,t) = \gamma_t \circ \gamma^{-1}. \tag{3.3}$$

In the Lagrangian description, the velocity phase space is the tangent bundle $T\ \mathrm{Diff}(M)$. In the Eulerian description, the spatial velocity is in the tangent space at the identity Id of $\mathrm{Diff}(M)$, that is, it is an element of the Lie algebra of $\mathrm{Diff}(M)$. For further details see, for example, [4, 11, 19].

In our problem, $\mathrm{u}(x,t)$ can be regarded as a time-dependent vector field on $\mathbf{R}$, that is, it belongs to the Lie algebra of $\mathrm{Diff}(\mathbf{R})$. Thus, in the Lagrangian formalism of our problem we take $M = \mathbf{R}$ and add the technical assumption that the smooth functions defined on $\mathbf{R}$ with value in $\mathbf{R}$ vanish rapidly at $\pm\infty$ together with as many derivatives as necessary. The configuration space of our problem is $\mathrm{Diff}(\mathbf{R})$. For a motion $\gamma(\cdot,t) \in \mathrm{Diff}(\mathbf{R})$ we have its Lagrangian velocity given by (3.1) and its Eulerian velocity given by (3.3). γ is the flow of the time-dependent vector field u.

The other unknown of our problem is $H(x,t)$, which for a fixed t can be regarded as a real function on $\mathbf{R}$, $H(\cdot,t) \in \mathcal{F}(\mathbf{R})$. We settle that the evolution equation of $H(x,t)$ is the second equation in (2.15). This equation is a transport equation, the free surface is advected, or Lie transported (in the geometry literature), by the fluid flow. With the formula of the Lie derivative of a 1-form along a vector field in view (see, for example, [1, Sec. 2.2]), the second equation in (2.15) expresses the fact that the 1-form $\mathrm{H}(x,t) := H(x,t)dx$ is advected, or Lie transported, by the vector field $\mathrm{u}(x,t) := u(x,t)\partial_x$, that is,

$$\frac{\partial \mathrm{H}}{\partial t} + \mathrm{L}_{\mathrm{u}}\mathrm{H} = 0, \tag{3.4}$$

where L_{u} denotes the Lie derivative with respect to the vector field u.

Equation (3.4) is an equation in the Eulerian picture. With the aid of the pull back map γ^*, we can write the Lagrangian form of Eq. (3.4), that is,

$$\gamma^*\left(\frac{\partial \mathrm{H}}{\partial t} + \mathrm{L}_{\mathrm{u}}\mathrm{H}\right) = 0. \tag{3.5}$$

We use further the interpretation of the Lie derivative of a time-dependent 1-form along a time-dependent vector field in terms of the flow of the vector field (see, for example, [1, Sec. 2]), and we get the equation

$$\frac{d}{dt}[\gamma^*(\mathrm{H})] = \gamma^*(\mathrm{L}_{\mathrm{u}}\mathrm{H}) + \gamma^*\left(\frac{\partial \mathrm{H}}{\partial t}\right) \overset{(3.5)}{=} 0. \tag{3.6}$$

We denote this time invariant 1-form in the reference configuration by

$$\mathrm{H}_0 := \gamma^*(\mathrm{H}), \quad \mathrm{H}_0(X,t) = \mathrm{H}_0(X,0). \tag{3.7}$$

By the definition of the pull back map (see, for example, [1, Sec. 2]), we get between the components of the 1-forms $\mathrm{H}_0(X,t) := H_0(X,t)dX$ and $\mathrm{H}(x,t) := H(x,t)dx$ the following relation

$$H_0 = (H \circ \gamma)J_\gamma, \tag{3.8}$$

where $J_\gamma := \frac{\partial\gamma}{\partial X}$ is the Jacobian of γ, or,

$$H = (H_0 \circ \gamma^{-1})J_{\gamma^{-1}}. \tag{3.9}$$

In Lagrangian description, the equation of motion is the equation satisfied by a critical point of a certain functional $\mathfrak{a}(\gamma)$, called the action,

$$\mathfrak{a}(\gamma) := \int_0^T \mathcal{L}(\gamma, \gamma_t)dt \tag{3.10}$$

defined on all paths $\{\gamma(\cdot,t),\ t \in [0,T]\}$ in $\mathrm{Diff}(M)$, having fixed endpoints. $\mathcal{L}$ is a scalar function defined on $T\ \mathrm{Diff}(M)$, called Lagrangian.

In the shallow-water regime, for a velocity field with a horizontal component (2.10) which is independent of the vertical coordinate z and a vertical component (2.11) which has only a linear dependence on z, the kinetic energy has the expression:

$$\begin{aligned} E_c(u,\eta) &= \frac{1}{2}\int_{-\infty}^{\infty}\left(\int_0^{1+\epsilon\eta}[u^2 + (1+\epsilon\eta)^2u_x^2]dz\right)dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\left[(1+\epsilon\eta)u^2 + \frac{1}{3}(1+\epsilon\eta)^3u_x^2\right]dx \\ &\overset{(2.14)}{=} \frac{1}{2}\int_{-\infty}^{\infty}\left(Hu^2 + \frac{1}{3}H^3u_x^2\right)dx =: E_c(u,H). \end{aligned} \tag{3.11}$$

In non-dimensional variables, with ρ and g settled at 1, we define the gravitational potential energy at the free surface $z = 1 + \epsilon\eta(x,t)$, gained by the fluid parcel when it is vertically displaced from its undisturbed position with $\epsilon\eta(x,t)$, by

$$E_p(\eta) = \int_{-\infty}^{\infty}\left(\int_0^{1+\epsilon\eta}(z-1)dz\right)dx = \frac{1}{2}\int_{-\infty}^{\infty}(\epsilon\eta)^2 dx$$
$$\overset{(2.14)}{=} \frac{1}{2}\int_{-\infty}^{\infty}(H-1)^2 dx =: E_p(H). \tag{3.12}$$

We require in (3.11) and (3.12) that $u(x,t)$ and $u_x(x,t)$ decay rapidly at $\pm$ infinity and $H(x,t) \to 1$ as $x \to \pm\infty$, at any instant t.

The Lagrangian for our problem will be obtained by transporting the Lagrangian from the Eulerian picture, which by (3.11) and (3.12) has the expression:

$$\mathfrak{L}(u,H) = E_c(u,H) - E_p(H) = \frac{1}{2}\int_{-\infty}^{\infty}\left[Hu^2 + \frac{1}{3}H^3u_x^2 - (H-1)^2\right]dx, \tag{3.13}$$

to all tangent spaces $T\ \mathrm{Diff}(\mathbf{R})$, this transport being made taking into account (3.3) and (3.9). For each function $H_0 \in \mathcal{F}(\mathbf{R})$ independent of time, we define the Lagrangian $\mathcal{L}_{H_0}$: $T\ \mathrm{Diff}(\mathbf{R}) \to \mathbf{R}$ by

$$\mathcal{L}_{H_0}(\gamma,\gamma_t) := \frac{1}{2}\int_{-\infty}^{\infty}\Big\{[(H_0\circ\gamma^{-1})J_{\gamma^{-1}}](\gamma_t\circ\gamma^{-1})^2$$
$$+\frac{1}{3}[(H_0\circ\gamma^{-1})J_{\gamma^{-1}}]^3[\partial_x(\gamma_t\circ\gamma^{-1})]^2 - [(H_0\circ\gamma^{-1})J_{\gamma^{-1}} - 1]^2\Big\}dx. \tag{3.14}$$

The Lagrangian $\mathcal{L}_{H_0}$ depends smoothly on H_0 and it is right invariant under the action of the subgroup

$$\mathrm{Diff}(\mathbf{R})_{H_0} = \{\psi \in \mathrm{Diff}(\mathbf{R}) \mid (H_0\circ\psi^{-1})J_{\psi^{-1}} = H_0\}, \tag{3.15}$$

that is, if we replace the path $\gamma(t,\cdot)$ by $\gamma(t,\cdot)\circ\psi(\cdot)$, for a fixed time-independent ψ in $\mathrm{Diff}(\mathbf{R})_{H_0}$, then $\mathcal{L}_{H_0}$ is unchanged.

The action on a path $\gamma(t,\cdot)$, $t \in [0,T]$, in $\mathrm{Diff}(\mathbf{R})$ is

$$\mathfrak{a}(\gamma) := \int_0^T \mathcal{L}_{H_0}(\gamma,\gamma_t)dt. \tag{3.16}$$

The critical points of the action (3.16) in the space of paths with fixed endpoints, satisfy

$$\frac{d}{d\varepsilon}\mathfrak{a}(\gamma+\varepsilon\varphi)\bigg|_{\varepsilon=0} = 0, \tag{3.17}$$

for every path $\varphi(t,\cdot)$, $t \in [0,T]$, in $\mathrm{Diff}(\mathbf{R})$ with endpoints at zero, that is, $\varphi(0,\cdot) = 0 = \varphi(T,\cdot)$ and such that $\gamma + \varepsilon\varphi$ is a small variation of γ on $\mathrm{Diff}(\mathbf{R})$. Taking into account (3.14) and (3.16), the condition (3.17) becomes

$$\begin{aligned}
\int_0^T \int_{-\infty}^{\infty} \Big\{ & \frac{1}{2}(\gamma_t \circ \gamma^{-1})^2 J_{\gamma^{-1}} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} [H_0 \circ (\gamma + \varepsilon\varphi)^{-1}] \\
& + \frac{1}{2}(\gamma_t \circ \gamma^{-1})^2 (H_0 \circ \gamma^{-1}) \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} [J_{(\gamma+\varepsilon\varphi)^{-1}}] \\
& + [(H_0 \circ \gamma^{-1}) J_{\gamma^{-1}}](\gamma_t \circ \gamma^{-1}) \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} [(\gamma_t + \varepsilon\varphi_t) \circ (\gamma + \varepsilon\varphi)^{-1}] \\
& + \frac{1}{2}[\partial_x(\gamma_t \circ \gamma^{-1})]^2 (H_0 \circ \gamma^{-1})^2 J^3_{\gamma^{-1}} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} [H_0 \circ (\gamma + \varepsilon\varphi)^{-1}] \\
& + \frac{1}{2}[\partial_x(\gamma_t \circ \gamma^{-1})]^2 (H_0 \circ \gamma^{-1})^3 J^2_{\gamma^{-1}} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} [J_{(\gamma+\varepsilon\varphi)^{-1}}] \\
& + \frac{1}{3}[(H_0 \circ \gamma^{-1}) J_{\gamma^{-1}}]^3 \partial_x(\gamma_t \circ \gamma^{-1}) \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} [\partial_x((\gamma_t + \varepsilon\varphi_t) \circ (\gamma + \varepsilon\varphi)^{-1})] \\
& - (H_0 \circ \gamma^{-1}) J^2_{\gamma^{-1}} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} [H_0 \circ (\gamma + \varepsilon\varphi)^{-1}] \\
& - (H_0 \circ \gamma^{-1})^2 J_{\gamma^{-1}} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} [J_{(\gamma+\varepsilon\varphi)^{-1}}] + (J_{\gamma^{-1}}) \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} [H_0 \circ (\gamma + \varepsilon\varphi)^{-1}] \\
& + (H_0 \circ \gamma^{-1}) \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} [J_{(\gamma+\varepsilon\varphi)^{-1}}] \Big\} dx dt = 0. \qquad (3.18)
\end{aligned}$$

After calculation (for more details see, for example, [17]), we get

$$\begin{aligned}
\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} [H_0 \circ (\gamma + \varepsilon\varphi)^{-1}] &= (H_{0_x} \circ \gamma^{-1}) \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} (\gamma + \varepsilon\varphi)^{-1} \\
&= -(\varphi \circ \gamma^{-1}) \partial_x (H_0 \circ \gamma^{-1}), \qquad (3.19)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} [J_{(\gamma+\varepsilon\varphi)^{-1}}] &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} [\partial_x (\gamma + \varepsilon\varphi)^{-1}] \\
&= -\frac{\partial_x(\varphi \circ \gamma^{-1})}{\gamma_x \circ \gamma^{-1}} + \frac{\gamma_{xx} \circ \gamma^{-1}}{(\gamma_x \circ \gamma^{-1})^3} (\varphi \circ \gamma^{-1}) \\
&= -(J_{\gamma^{-1}}) \partial_x (\varphi \circ \gamma^{-1}) - \partial_x (J_{\gamma^{-1}})(\varphi \circ \gamma^{-1}), \qquad (3.20)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} [(\gamma_t + \varepsilon\varphi_t) \circ (\gamma + \varepsilon\varphi)^{-1}] &= \partial_t (\varphi \circ \gamma^{-1}) + (\gamma_t \circ \gamma^{-1}) \partial_x (\varphi \circ \gamma^{-1}) \\
&\quad - (\varphi \circ \gamma^{-1}) \partial_x (\gamma_t \circ \gamma^{-1}), \qquad (3.21)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} [\partial_x((\gamma_t + \varepsilon\varphi_t) \circ (\gamma + \varepsilon\varphi)^{-1})] &= \partial_{tx} (\varphi \circ \gamma^{-1}) + (\gamma_t \circ \gamma^{-1}) \partial_x^2 (\varphi \circ \gamma^{-1}) \\
&\quad - [\partial_{xx} (\gamma_t \circ \gamma^{-1})](\varphi \circ \gamma^{-1}). \qquad (3.22)
\end{aligned}$$

Thus, with (3.19)–(3.22) in view, the condition (3.18) becomes:

$$\int_0^T \int_{-\infty}^{\infty} \left\{ \left[-\frac{1}{2}u^2 H_x - uu_x H - \frac{1}{2}u_x^2 H^2 H_x - \frac{1}{3}u_x u_{xx} H^3 + HH_x - H_x \right] (\varphi \circ \gamma^{-1}) \right.$$
$$+ \left[\frac{1}{2}u^2 H - \frac{1}{2}u_x^2 H^3 + H^2 - H \right] \partial_x(\varphi \circ \gamma^{-1}) + [uH]\partial_t(\varphi \circ \gamma^{-1})$$
$$\left. + \frac{1}{3}u_x H^3 \partial_{tx}(\varphi \circ \gamma^{-1}) + \frac{1}{3}uu_x H^3 \partial_{xx}(\varphi \circ \gamma^{-1}) \right\} dxdt = 0, \tag{3.23}$$

where $u = \gamma_t \circ \gamma^{-1}$ and $H = (H_0 \circ \gamma^{-1})J_{\gamma^{-1}}$. We integrate by parts with respect to t and x in the above formula, we take into account that $u \to 0$, $u_x \to 0$, $H \to 1$ at $\pm\infty$ and φ has endpoints at zero, and we get

$$\int_0^T \int_{-\infty}^{\infty} \left\{ \left[-u^2 H_x - 2uu_x H + 2u_x^2 H^2 H_x + \frac{4}{3}u_x u_{xx} H^3 - HH_x - u_t H - uH_t \right. \right.$$
$$\left. \left. + \left(\frac{1}{3}u_x H^3 \right)_{tx} + \frac{1}{3}(uu_{xx}H^3)_x + (uu_x H^2 H_x)_x \right] (\varphi \circ \gamma^{-1}) \right\} dxdt = 0. \tag{3.24}$$

With H satisfying the second equation in (2.15), the condition (3.24) becomes:

$$-\int_0^T \int_{-\infty}^{\infty} \left\{ \left[(u_t + uu_x + H_x)H - \frac{1}{3}[H^3(uu_{xx} + u_{xt} - u_x^2)]_x \right] (\varphi \circ \gamma^{-1}) \right\} dxdt = 0. \tag{3.25}$$

Therefore, we proved the following theorem.

Theorem 1. *For an irrotational shallow-water flow, the non-dimensional horizontal velocity of the water $u(x,t)$ and the non-dimensional free upper surface $H(x,t) = 1 + \epsilon\eta(x,t)$, for ϵ arbitrary fixed, satisfy the Green–Naghdi shallow-water system* (1.1).

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