Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

SMALL-AMPLITUDE EQUATORIAL WATER WAVES WITH CONSTANT VORTICITY: DISPERSION RELATIONS AND PARTICLE TRAJECTORIES

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(Communicated by the associate editor name)

Abstract. We consider the two-dimensional equatorial water-wave problem with constant vorticity in the f-plane approximation. Within the framework of small-amplitude waves, we derive the dispersion relations and we find the analytic solutions of the nonlinear differential equation system describing the particle paths below such waves. We show that the solutions obtained are not closed curves. Some remarks on the stagnation points are also provided.

1. Introduction.

In geophysical flows the Coriolis force due to Earth's rotation has an important effect. For a physical and a mathematical point of view on the influence of Earth's rotation on geophysical flows, see, for example, [34, 15]. In fact, by describing the motion in Earth's rotating frame of reference, which is a non-inertial frame of reference, there appear the Coriolis force and the centrifugal force. It turns out that the centrifugal force plays no role in the fluid motion, the Coriolis force being responsible for the unique character of geophysical motions [34, 15].
In the study of a thin layer of water closed to Earth's surface, the Coriolis force will add additional terms in the Euler equations written in the rotating frame. The Equator is a special region dynamically because there the Coriolis force vanishes. In the theoretical analysis of flows near the Equator, in order to get the governing equations much more tractable, the Coriolis force is approximated. In the equatorial β-plane approximation, the Coriolis parameter, denoted by f which is equal to twice the rotation rate of the Earth multiplied by the sine of the geographic latitude, is set to vary linearly with the northward meridional distance from the Equator, the linear coefficient of variation being denoted by the Greek letter beta (see [34, 15]). In

2000 *Mathematics Subject Classification.* Primary: 35Q86, 37N10; Secondary: 76B15, 76F10.
Key words and phrases. geophysical water waves, equatorial f-plane approximation, small-amplitude waves, constant vorticity, particle paths, disperion relations.
The paper was supported by the ERC Advanced Grant "Nonlinear studies of water flows with vorticity" (NWFV).

the equatorial f-plane approximation, the Coriolis parameter is assumed to be zero (see [15]). The physical relevance of the f-plane approximation for equatorial waves is discussed in [5]. The existence of steady two-dimensional periodic geophysical surface waves propagating westward in the equatorial oceanic regions, was proved by bifurcation techniques in [7].
In this paper we consider the two-dimensional geophysical water-wave problem with constant vorticity in the equatorial f-plane approximation.

We remind the reader that if one neglects the effects caused by the Earth's rotation there are only a few explicit solutions to the water-wave problem. For gravity deep-water waves an explicit solution was found first by Gerstner [16] and later on by Rankine [35] (for modern detailed descriptions of this solution see [1, 19]). Recently, Gerstner's approach was extended in [6] to geophysical water waves. The exact solution presented in [6] is obtained in the equatorial β-plane approximation and describes equatorial trapped waves propagating eastward in a stratified inviscid fluid. An exact solution for equatorial geophysical water waves with an underlying current is presented in [20]. We also mention that the paper [8] describes (by means of an extension of Gerstner's idea) the eastward propagating fluctuations of the thermocline near the Equator. In the equatorial f-plane approximation, Gerster's solution can be adapted to describe geophysical deep-water waves traveling over an uniform horizontal current [32] or over still water [30]. Gerstner's solution admits extensions that describe the propagation of edge waves along a sloping beach cf. [2] and [31].
Beneath Gerstner type waves the fluid particles move on circles with depth-dependent radii [1, 19, 6, 30]. Moreover, the flow is rotational and the vorticity decays with depth. The influence of the underlying current on the flow beneath the waves is investigated in [32].

The fact that for Gerstner's waves the fluid particles move on circles is in agreement with the classical description of the particle paths within the framework of linear water-wave theory [33, 27, 28, 36]. However, even within the linear water-wave theory, the ordinary differential equation system describing the motion of the fluid particles is nevertheless nonlinear and its analysis in the non-geophysical case brought out the conclusion that the particle paths are not closed curves. For the particle paths within different types of progressive water waves, in finite depth or deep water, in the framework of linear theory or in the framework of full nonlinear theory of periodic symmetric waves, and in the presence or not of the background currents and vorticity, see the following references: [3]-[4], [9]-[14], [17]-[18], [21]-[26], [29], [38].

In this article we study in the equatorial $f-$plane approximation the particle trajectories of small-amplitude geophysical water waves with constant vorticity. The structure of the paper is the following: we start by presenting the mathematical model, then in Section 3, we transform the geophysical water-wave problem into a non-dimensionalized system. Using two types of scaling, we rescale this problem and then in Section 4 we solve the obtained linear problems. In the non-geophysical framework for water waves with constant vorticity one can consider two different scalings [25]: one made around still water, the other one made around a laminar flow. For the first scaling the constant vorticity is also scaled, for the second scaling the constant vorticity remains unscaled. The same types of scaling we will consider here: the scaling (13) with the scaled constant vorticity (14), and the scaling (17) around the laminar flow (18). The laminar flow (18) is that solution of the system

(12) characterized by a flat surface and for which every particle moves horizontally, with a speed that depends linearly on the depth. We observe that in the considered geophysical framework, the pressure of the laminar flow is different from zero. Then, we solve the linear problems (16) and (20). In the original physical variables, we get the small-amplitude geophysical water waves with constant vorticity (31) and (41), respectively. The two scalings yield different dispersion relations for the propagation speed of the linear wave: for the first scaling the explicit dispersion relation (32) and for the second scaling the implicit dispersion relation (42). If the laminar flow satisfies the condition (43) (condition satisfied, for example, if the laminar flow is westward at the surface) then, we get also for the second scaling an explicit dispersion relation (44) (the expression with minus in front of the square root was obtained by a different approach in [7]). We note that, comparing with the result obtained in the non-geophysical case studied in [25], we obtain, corresponding to each type of scaling, the same formal expressions for the velocity field, the major differences being the expressions of the pressure and the dispersion relations.
Section 5 is dedicated to the study of the particle trajectories. Information about the shape of the particle paths below the obtained small-amplitude geophysical water waves with constant vorticity is achieved by means of analytic solutions of the nonlinear differential equation system (49) describing the particle motion. Notice that in the geophysical framework the parameter A from (49), which has the expression (47) or (48), depending on which scaling we refer to, depends, through the propagation wave speed, on the rotation rate of the Earth and on the constant vorticity. The sign of A is discussed in Thm 5.1.. The important result of Section 5 is that all the solutions of the system (49) are not closed curves (Thm 5.2.).
We observe that the dispersion relations are obtained even in the presence of stagnation points. At the end, we make some remarks on the stagnation points of the small-amplitude geophysical water waves with constant vorticity.

2. The governing equations.

For geophysical water waves the forces with dominating influence are the gravity and the Coriolis force induced by the Earth's rotation. We take the Earth to be a perfect sphere of radius 6371 km and with a constant rotational speed $\omega = 73 \cdot 10^{-6}$rad/s round the polar axis towards east. A natural framework to describe our problem is a rotating one. We take this frame to have the origin at a point on the Earth's surface, the $x-$axis horizontally due east, the $y-$axis horizontally due north and the $z-$axis upward. Let $z = 0$ be the lower boundary of the water layer, and let $z = h_0 + \eta(t, x, y)$ be its upper free boundary, where with h_0 we denoted the mean depth of the water. In the region $0 \leq z \leq h_0 + \eta(t, x, y)$ the governing equations in the $f-$plane approximation near the Equator are cf. [5] and [15] the Euler equations

$$\begin{cases} u_t + uu_x + wu_y + vu_z + 2\omega v &= -p_x/\rho, \\ w_t + uw_x + ww_y + vw_z &= -p_y, \\ v_t + uv_x + wv_y + vv_z - 2\omega u &= -p_z/\rho - g, \end{cases} \tag{1}$$

and, under the assumption of constant density, the equation of mass conservation in the form

$$u_x + w_y + v_z = 0. \tag{2}$$

Here (u, w, v) is the velocity field of the fluid, p is the pressure, t represents time, ρ is the constant density of the water and $g = 9,8m/s^2$ is the constant gravitational acceleration at the Earth's surface.

In this paper we confine our study to two-dimensional periodic flows, independent upon the $y-$coordinate and with $w \equiv 0$ throughout the flow. Such flows are possible in this setting; in particular, the vorticity equation (see [34]) ensures that the vorticity $\gamma = (0, u_z - v_x, 0)$ is preserved. To simplify the notations, we will identify γ with the scalar $u_z - v_x$. Being interested in water waves with constant vorticity, let us denote it by γ_0, we have then the following relation

$$u_z - v_x = \gamma_0. \tag{3}$$

The free surface decouples the motion of the water from that of the air (see the discussion in [4]), a fact that is expressed by the dynamic boundary condition

$$p = p_0 \qquad \text{on} \quad z = h_0 + \eta(t, x), \tag{4}$$

where p_0 is the constant atmospheric pressure. Moreover, the free surface of the wave consists at each moment of the same fluid particles, so that we obtain the kinematic boundary condition

$$v = \eta_t + u\eta_x \qquad \text{on} \quad z = h_0 + \eta(t, x). \tag{5}$$

Since we assume that the fluid bed is impermeable, we impose the no-flux condition

$$v = 0 \qquad \text{on} \quad z = 0. \tag{6}$$

Summarizing, the governing equations for two-dimensional geophysical water waves with constant vorticity, in the equatorial f-plane approximation, are encompassed by the nonlinear free-boundary problem

$$\left\{ \begin{array}{rcll} u_t + uu_x + vu_z + 2\omega v & = & -p_x/\rho & \text{in} \quad 0 < z < h_0 + \eta(t,x), \\ w_t + uv_x + vv_z - 2\omega u & = & -p_z/\rho - g & \text{in} \quad 0 < z < h_0 + \eta(t,x), \\ u_x + v_z & = & 0 & \text{in} \quad 0 < z < h_0 + \eta(t,x), \\ u_z - v_x & = & \gamma_0 & \text{in} \quad 0 < z < h_0 + \eta(t,x), \\ p & = & p_0 & \text{on} \quad z = h_0 + \eta(t,x), \\ v & = & \eta_t + u\eta_x & \text{on} \quad z = h_0 + \eta(t,x), \\ v & = & 0 & \text{on} \quad z = 0. \end{array} \right. \tag{7}$$

3. **Non-dimensionalization and scaling.** In this section we introduce first a non-dimensionalization of the variables. For this purpose we use the undisturbed depth of the water h_0, as the vertical scale, a typical *wavelength* λ, as the horizontal scale, and a typical *amplitude* of the surface wave a (for more details see [33]). Then, we make the following change of variables

$$\begin{gathered} x \mapsto \lambda x, \quad z \mapsto h_0 z, \quad \eta \mapsto a\eta, \quad t \mapsto \frac{\lambda}{\sqrt{gh_0}} t, \\ u \mapsto \sqrt{gh_0} u, \quad v \mapsto h_0 \frac{\sqrt{gh_0}}{\lambda} v, \end{gathered} \tag{8}$$

where, to avoid new notations, we have used the same symbols for the non dimensional variables x, z, η, t, u, and v, on the right-hand side. With these transformations at hand we define furthermore a new pressure function

$$p \mapsto p_0 + gh_0(1 - z) + gh_0 p, \tag{9}$$

where again we use the same symbol for the non-dimensional variable p on the right-hand side. Here, $p_0 + gh_0(1-z)$ is the hydrostatic pressure distribution which describes the changing of the pressure within a stationary fluid ($u \equiv v \equiv 0$). The variable p on the right-hand side measures the pressure perturbation induced by a passing wave.

The natural scaling for the vorticity is

$$\gamma_0 \mapsto \frac{\sqrt{gh_0}}{h_0}\gamma_0, \tag{10}$$

where we have used the same symbol for the non-dimensional γ_0 on the right-hand side. It is also natural to non-dimensionalize the constant rotational speed of the Earth by

$$\omega \mapsto \frac{\sqrt{gh_0}}{h_0}\omega. \tag{11}$$

Therefore, in non-dimensional variables (8)-(11), the geophysical water-wave problem (7) transforms into

$$\begin{array}{rcll} u_t + uu_x + vu_z + 2\omega v &=& -p_x & \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), \\ \delta^2(v_t + uv_x + vv_z) - 2\omega u &=& -p_z & \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), \\ u_x + v_z &=& 0 & \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), \\ u_z - \delta^2 v_x &=& \gamma_0 & \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), \\ v &=& \varepsilon(\eta_t + u\eta_x) & \text{on} \quad z = 1 + \varepsilon\eta(t,x), \\ p &=& \varepsilon\eta & \text{on} \quad z = 1 + \varepsilon\eta(t,x), \\ v &=& 0 & \text{on} \quad z = 0, \end{array} \tag{12}$$

where we denoted by $\varepsilon = a/h_0$ the *amplitude parameter* and with $\delta = h_0/\lambda$ the *shallowness parameter*. The shallowness parameter measures the wavelength compared with the depth, therewith small δ models shallow-water waves. Moreover, the amplitude parameter measures the relative size of the waves, thus small ε models a small disturbance of the underlying flow.

3.1. **First type of scaling.** Having now non-dimensionalized the system (7) let us continue with the scaling transformation. We notice that, on $z = 1 + \varepsilon\eta$, both v and p are proportional to ε. This is consistent with the fact that as $\varepsilon \to 0$ we must have $v \to 0$ and $p \to 0$. We will consider first the following scaling of the non-dimensional variables

$$u \mapsto \varepsilon u, \quad v \mapsto \varepsilon v, \quad p \mapsto \varepsilon p, \tag{13}$$

where we avoided again the introduction of a new notation. For this scaling of u and v, we also get

$$\gamma_0 \mapsto \varepsilon\gamma_0. \tag{14}$$

This scaling of the non-dimensionalized variables yields the problem

$$\begin{array}{rcll} u_t + \varepsilon(uu_x + vu_z) + 2\omega v &=& -p_x & \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), \\ \delta^2[v_t + \varepsilon(uv_x + vv_z)] - 2\omega u &=& -p_z & \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), \\ u_x + v_z &=& 0 & \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), \\ u_z - \delta^2 v_x &=& \gamma_0 & \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), \\ v &=& \eta_t + \varepsilon u\eta_x & \text{on} \quad z = 1 + \varepsilon\eta(t,x), \\ p &=& \eta & \text{on} \quad z = 1 + \varepsilon\eta(t,x), \\ v &=& 0 & \text{on} \quad z = 0. \end{array} \tag{15}$$

Fixing δ and letting $\varepsilon \to 0$, we obtain a linear approximation of the problem (15), that is,

$$\begin{array}{rcll} u_t + 2\omega v &=& -p_x & \text{in} \quad 0 < z < 1, \\ \delta^2 v_t - 2\omega u &=& -p_z & \text{in} \quad 0 < z < 1, \\ u_x + v_z &=& 0 & \text{in} \quad 0 < z < 1, \\ u_z - \delta^2 v_x &=& \gamma_0 & \text{in} \quad 0 < z < 1, \\ v &=& \eta_t & \text{on} \quad z = 1, \\ p &=& \eta & \text{on} \quad z = 1, \\ v &=& 0 & \text{on} \quad z = 0. \end{array} \tag{16}$$

3.2. **Second type of scaling.** Instead of the scaling (13), we can also consider the following one around a laminar flow

$$u \mapsto U + \varepsilon u, \quad v \mapsto V + \varepsilon v, \quad p \mapsto P + \varepsilon p, \tag{17}$$

where (U, V, P) is the solution to the system (12), characterized by a flat surface $\eta = 0$ and for which every particle moves horizontally, with a speed that depends linearly on the depth, that is,

$$U = \gamma_0 z + \alpha, \quad V = 0, \quad P = \omega\gamma_0 z^2 + 2\omega\alpha z - \omega\gamma_0 - 2\omega\alpha, \tag{18}$$

where α is a constant. Here γ_0 remains unscaled. Thus, the geophysical water-wave problem (7) writes in the new scaling as

$$\begin{array}{rcll} u_t + \varepsilon(uu_x + vu_z) + (\gamma_0 z + \alpha)u_x + (\gamma_0 + 2\omega)v &=& -p_x & \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), \\ \delta^2[v_t + \varepsilon(uv_x + vv_z) + (\gamma_0 z + \alpha)v_x] - 2\omega u &=& -p_z & \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), \\ u_x + v_z &=& 0 & \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), \\ u_z - \delta^2 v_x &=& 0 & \text{in} \quad 0 < z < 1 + \varepsilon\eta(t,x), \\ v - [\eta_t + \varepsilon u\eta_x + \varepsilon\gamma_0\eta\eta_x + (\gamma_0 + \alpha)\eta_x] &=& 0 & \text{on} \quad z = 1 + \varepsilon\eta(t,x), \\ p - [1 - 2\omega(\gamma_0 + \alpha)]\eta + \varepsilon\omega\gamma_0\eta^2 &=& 0 & \text{on} \quad z = 1 + \varepsilon\eta(t,x), \\ v &=& 0 & \text{on} \quad z = 0. \end{array} \tag{19}$$

Fixing again δ and letting $\varepsilon \to 0$, we obtain a linear approximation of the problem (19), that is,

$$\begin{array}{rcll} u_t + (\gamma_0 z + \alpha)u_x + (\gamma_0 + 2\omega)v &=& -p_x & \text{in} \quad 0 < z < 1, \\ \delta^2[v_t + (\gamma_0 z + \alpha)v_x] - 2\omega u &=& -p_z & \text{in} \quad 0 < z < 1, \\ u_x + v_z &=& 0 & \text{in} \quad 0 < z < 1, \\ u_z - \delta^2 v_x &=& 0 & \text{in} \quad 0 < z < 1, \\ v &=& \eta_t + (\gamma_0 + \alpha)\eta_x & \text{on} \quad z = 1, \\ p &=& [1 - 2\omega(\gamma_0 + \alpha)]\eta & \text{on} \quad z = 1, \\ v &=& 0 & \text{on} \quad z = 0. \end{array} \tag{20}$$

We notice that the forth equation in the system (20), which represents the vorticity equation, becomes in these scaled variables the vorticity equation for an irrotational flow.

4. **Solutions of the linear problems.** In this section we will determine the exact solutions of the problems (16) and (20).

4.1. **The linear problem (16).** In order to find solutions of the problem (16) we take first a closer look to the third and fourth equations of system (16). Differentiating the third equation of the system with respect to the z variable and the fourth equation with respect to the x variable, and subtracting the resulting equations we obtain that

$$v_{zz} + \delta^2 v_{xx} = 0. \tag{21}$$

Using the method of *separation of variables* and taking into account the boundary conditions for v (namely the fifth and the last relation in (16)) and the periodicity of v with respect to the $x-$ variable we find the following solution of the equation (21)

$$\begin{cases} u(t,x,z) &= \dfrac{\delta}{k\sinh(k\delta)}\cosh(k\delta z)\eta_{tx} + \gamma_0 z + \mathcal{F}(t,x), \\ v(t,x,z) &= \dfrac{1}{\sinh(k\delta)}\sinh(k\delta z)\eta_t, \end{cases} \tag{22}$$

where $k \geq 0$ is a constant (that might depend on t) and $\mathcal{F}(t,x)$ is a arbitrary function which has to be found (for more details see e.g. [25], §4). Plugging relation (22) into the third equation of system (16) we get

$$\left[\frac{\delta}{k\sinh(k\delta)}\eta_{txx} + \frac{k\delta}{\sinh(k\delta)}\eta_t\right]\cosh(k\delta z) = -\frac{\partial\mathcal{F}(t,x)}{\partial x}. \tag{23}$$

Since the last relation must hold for each $z \in [0,1]$ and $x \in \mathbb{R}$ it follows that

$$\frac{\partial\mathcal{F}(t,x)}{\partial x} = 0, \tag{24}$$

meaning that $\mathcal{F}$ depends only on t. Moreover, the relation (23) implies also that

$$\eta_{txx} + k^2\eta_t = 0,$$

We are looking for periodic traveling wave solutions, thus we take $k = 2\pi$ and we pick for the last equation the solution

$$\eta(t,x) = \cos[2\pi(x-ct)], \tag{25}$$

where c is the unknown wave speed. Thus, the relation (22) becomes

$$\begin{cases} u(t,x,z) &= \dfrac{2\pi\delta c}{\sinh(2\pi\delta)}\cosh(2\pi\delta z)\cos[2\pi(x-ct)] + \gamma_0 z + \mathcal{F}(t), \\ v(t,x,z) &= \dfrac{2\pi c}{\sinh(2\pi\delta)}\sinh(2\pi\delta z)\sin[2\pi(x-ct)]. \end{cases} \tag{26}$$

Having this solution at hand we find from the first two equations of system (16) the following relation for the pressure

$$\begin{aligned} p(t,x,z) =& \frac{1}{\sinh(2\pi\delta)}\left[2\pi\delta c^2\cosh(2\pi\delta z) + 2\omega c\sinh(2\pi\delta z)\right]\cos[2\pi(x-ct)] \\ &+ \omega\gamma_0 z^2 + 2\omega\mathcal{F}(t)z - \mathcal{F}'(t)x - \mathcal{C}, \end{aligned}$$

where $\mathcal{C}$ is a constant that might depend on time and $\mathcal{F}'$ is the derivative of $\mathcal{F}$. Since $p = \eta = \cos[2\pi(x-ct)]$ on $z = 1$ (see the sixth relation in (16)) we get

$$[2\pi\delta\coth(2\pi\delta)c^2 + 2\omega c - 1]\cos[2\pi(x-ct)] = \mathcal{F}'(t)x - 2\omega\mathcal{F}(t) - \omega\gamma_0 - \mathcal{C}. \tag{27}$$

The relation (27) has to be satisfied for all $x \in \mathbb{R}$ therewith, we obtain that $\mathcal{F}(t) = c_0$, where c_0 is a constant, $\mathcal{C} = 2\omega c_0 + \omega\gamma_0$, and

$$c = \frac{-\omega \pm \sqrt{\omega^2 + 2\pi\delta\coth(2\pi\delta)}}{2\pi\delta\coth(2\pi\delta)}. \tag{28}$$

A positive (respectively negative) c means that the wave moves eastwards (respectively westwards). Summarizing, the solution of the linear problem (16) is

$$\left\{\begin{array}{rcl} u(t,x,z) & = & \dfrac{2\pi\delta c}{\sinh(2\pi\delta)}\cosh(2\pi\delta z)\cos[2\pi(x-ct)]+\gamma_0 z+c_0, \\ v(t,x,z) & = & \dfrac{2\pi c}{\sinh(2\pi\delta)}\sinh(2\pi\delta z)\sin[2\pi(x-ct)], \\ p(t,x,z) & = & \left[\dfrac{2\pi\delta c^2}{\sinh(2\pi\delta)}\cosh(2\pi\delta z)+\dfrac{2\omega c}{\sinh(2\pi\delta)}\sinh(2\pi\delta z)\right]\cos[2\pi(x-ct)] \\ & + & \omega\gamma_0 z^2+2\omega c_0 z-\omega(2c_0+\gamma_0), \\ \eta(t,x) & = & \cos[2\pi(x-ct)], \end{array}\right. \tag{29}$$

where c is given by (28).

We return now to the original physical variables. We dimensionalize the wave speed c obtained above by

$$c\mapsto\sqrt{gh_0}c, \tag{30}$$

where we have used the same symbol c for the dimensional wave speed on the left-hand side, and from (8)-(11), we obtain:

Theorem 4.1. *The small-amplitude periodic geophysical water waves with constant vorticity are characterized, for the scaling (13), by*

$$\left\{\begin{array}{rcl} u(t,x,z) & = & \varepsilon\dfrac{Kh_0 c}{\sinh(Kh_0)}\cosh(Kz)\cos[K(x-ct)]+\varepsilon\gamma_0 z+\varepsilon\sqrt{gh_0}c_0, \\ v(t,x,z) & = & \varepsilon\dfrac{Kh_0 c}{\sinh(Kh_0)}\sinh(Kz)\sin[K(x-ct)], \\ p(t,x,z) & = & p_0+g(h_0-z) \\ & & +\varepsilon\left[\dfrac{Kh_0c^2}{\sinh(Kh_0)}\cosh(Kz)+\dfrac{2h_0\omega c}{\sinh(Kh_0)}\sinh(Kz)\right]\cos[K(x-ct)] \\ & & +\varepsilon\omega\gamma_0 z^2+2\varepsilon\sqrt{gh_0}\omega c_0 z-\varepsilon\omega h_0\sqrt{gh_0}\left(2c_0+\frac{h_0}{\sqrt{gh_0}}\gamma_0\right), \\ \eta(t,x) & = & \varepsilon h_0\cos[K(x-ct)], \end{array}\right. \tag{31}$$

with c given by the following dispersion relation

$$c=\frac{1}{K}\left[-\omega\tanh(Kh_0)\pm\sqrt{gK\tanh(Kh_0)+[\omega\tanh(Kh_0)]^2}\right], \tag{32}$$

$K:=2\pi/\lambda$ being the wave number.

Remark 1. Comparing with the result obtained in the non-geophysical case studied in [25], we remark that, for this type of scaling (13), the velocity field has the same formal expression but the expression of the pressure as well as the propagation speed of the linear wave are different.

4.2. **The linearized problem (20).** The purpose of this subsection is to find the solution to the linear problem (20). Using the same method as in Section 4.1 we find that the velocity field (u,v) is given by

$$\left\{\begin{array}{rcl} u(t,x,z) & = & \dfrac{\delta}{k\sinh(k\delta)}\cosh(k\delta z)[\eta_{tx}+(\gamma_0+\alpha)\eta_{xx}]+\mathfrak{F}(t,x), \\ v(t,x,z) & = & \dfrac{1}{\sinh(k\delta)}\sinh(k\delta z)[\eta_t+(\gamma_0+\alpha)\eta_x], \end{array}\right. \tag{33}$$

where $\mathfrak{F}(t,x)$ is an arbitrary function which has to be found and $k \geq 0$ (see e.g. [25], §4). Since again the equation of mass conservation (the third equation in (20)) has to be satisfied for all $z \in [0,1]$ and $x \in \mathbb{R}$ it follows that $\partial\mathfrak{F}(t,x)/\partial x = 0$, meaning that $\mathfrak{F} = \mathfrak{F}(t)$ and

$$[\eta_t + (\gamma_0 + \alpha)\eta_x]_x + k^2[\eta_t + (\gamma_0 + \alpha)\eta_x] = 0. \tag{34}$$

Taking $k = 2\pi$ and the periodic solution of (34) in the form

$$\eta(t,x) = \cos[2\pi(x - ct)], \tag{35}$$

with c to be determined, the system (33) can be written as

$$\begin{cases} u(t,x,z) &= \dfrac{2\pi\delta(c-\gamma_0-\alpha)}{\sinh(2\pi\delta)}\cosh(2\pi\delta z)\cos[2\pi(x-ct)] + \mathfrak{F}(t), \\ v(t,x,z) &= \dfrac{2\pi(c-\gamma_0-\alpha)}{\sinh(2\pi\delta)}\sinh(2\pi\delta z)\sin[2\pi(x-ct)]. \end{cases} \tag{36}$$

Plugging (36) into the first two relations of system (20) we find the following expression for the pressure

$$\begin{aligned} p =& \frac{(c-\gamma_0-\alpha)}{\sinh(2\pi\delta)}\left[2\pi\delta(c-\gamma_0 z-\alpha)\cosh(2\pi\delta z) + (\gamma_0+2\omega)\sinh(2\pi\delta z)\right]\cos[2\pi(x-ct)] \\ &- \mathfrak{F}'(t)x + 2\omega\mathfrak{F}(t)z + \mathfrak{C}, \end{aligned}$$

where $\mathfrak{C}$ is a constant.
Since on $z = 1$ (see the sixth relation in (20))

$$p = [1 - 2\omega(\gamma_0+\alpha)]\eta = [1 - 2\omega(\gamma_0+\alpha)]\cos[2\pi(x-ct)],$$

we get

$$\begin{aligned} &\{2\pi\delta\coth(2\pi\delta)(c-\gamma_0-\alpha)^2 + (\gamma_0+2\omega)(c-\gamma_0-\alpha) - 1 + 2\omega(\gamma_0+\alpha)\}\cos[2\pi(x-ct)] \\ &= \mathfrak{F}'(t)x - 2\omega\mathfrak{F}(t) - \mathfrak{C}. \end{aligned}$$

The last relation has to be satisfied for all $x \in \mathbb{R}$ therewith, we get $\mathfrak{F}(t) = d_0$, where d_0 is a constant, $\mathfrak{C} = -2\omega d_0$ and

$$2\pi\delta\coth(2\pi\delta)(c-\gamma_0-\alpha)^2 + (\gamma_0+2\omega)(c-\gamma_0-\alpha) - 1 + 2\omega(\gamma_0+\alpha) = 0. \tag{37}$$

If

$$(\gamma_0+2\omega)^2 + 8\pi\delta[1-2\omega(\gamma_0+\alpha)]\coth(2\pi\delta) > 0, \tag{38}$$

then, from (37) we get

$$\begin{aligned} c &= (\gamma_0+\alpha) - (\gamma_0+2\omega)\frac{\tanh(2\pi\delta)}{4\pi\delta} \\ &\pm\sqrt{\left[(\gamma_0+2\omega)\frac{\tanh(2\pi\delta)}{4\pi\delta}\right]^2 + [1-2\omega(\gamma_0+\alpha)]\frac{\tanh(2\pi\delta)}{2\pi\delta}}. \end{aligned} \tag{39}$$

Summarizing, the solution of the linear problem (20) is

$$\begin{cases} u(t,x,z) &= \dfrac{2\pi\delta(c-\gamma_0-\alpha)}{\sinh(2\pi\delta)}\cosh(2\pi\delta z)\cos[2\pi(x-ct)]+d_0, \\ v(t,x,z) &= \dfrac{2\pi(c-\gamma_0-\alpha)}{\sinh(2\pi\delta)}\sinh(2\pi\delta z)\sin[2\pi(x-ct)], \\ p(t,x,z) &= \Big[2\pi\delta(c-\gamma_0-\alpha)(c-\gamma_0 z-\alpha)\dfrac{\cosh(2\pi\delta z)}{\sinh(2\pi\delta)} \\ & \quad +(\gamma_0+2\omega)(c-\gamma_0-\alpha)\dfrac{\sinh(2\pi\delta z)}{\sinh(2\pi\delta)}\Big]\cos[2\pi(x-ct)] \\ & \quad +2\omega d_0 z-2\omega d_0, \\ \eta(t,x) &= \cos[2\pi(x-ct)], \end{cases} \tag{40}$$

where c satisfies (37).

We return also in this case to the original physical variables. Taking into account (30) and (8)-(11), we get:

Theorem 4.2. *The small-amplitude periodic geophysical water waves with constant vorticity are characterized, for the scaling (17) around the laminar flow (18), by*

$$\begin{cases} u(t,x,z) &= \varepsilon\dfrac{Kh_0(c-h_0\gamma_0-\sqrt{gh_0}\alpha)}{\sinh(Kh_0)}\cosh(Kz)\cos[K(x-ct)] \\ & \quad +\gamma_0 z+\sqrt{gh_0}\alpha+\varepsilon\sqrt{gh_0}d_0, \\ v(t,x,z) &= \varepsilon\dfrac{Kh_0(c-h_0\gamma_0-\sqrt{gh_0}\alpha)}{\sinh(Kh_0)}\sinh(Kz)\sin[K(x-ct)], \\ p(t,x,z) &= p_0+g(h_0-z) \\ & \quad +\varepsilon\Big[Kh_0\left(c-h_0\gamma_0-\sqrt{gh_0}\alpha\right)\left(c-\gamma_0 z-\sqrt{gh_0}\alpha\right)\dfrac{\cosh(Kz)}{\sinh(Kh_0)} \\ & \quad +h_0(\gamma_0+2\omega)\left(c-h_0\gamma_0-\sqrt{gh_0}\alpha\right)\dfrac{\sinh(Kz)}{\sinh(Kh_0)}\Big]\cos[K(x-ct)] \\ & \quad +\omega\gamma_0 z^2+2\omega\sqrt{gh_0}\left(\alpha+\varepsilon d_0\right)z-h_0^2\omega\gamma_0-2\omega h_0\sqrt{gh_0}\left(\alpha+\varepsilon d_0\right), \\ \eta(t,x) &= \varepsilon h_0\cos[K(x-ct)], \end{cases} \tag{41}$$

with c satisfying the following dispersion relation

$$K\coth(Kh_0)(c-h_0\gamma_0-\sqrt{gh_0}\alpha)^2+(\gamma_0+2\omega)(c-h_0\gamma_0-\sqrt{gh_0}\alpha)-g+2\omega(\gamma_0h_0+\sqrt{gh_0}\alpha)=0, \tag{42}$$

$K:=2\pi/\lambda$ being the wave number.

Remark 2. We note that, comparing with the result obtained in the non-geophysical case studied in [25], for a type of scaling around a laminar flow, we obtain the same formal expression for the velocity field, the major differences being again the expression of the pressure and the dispersion relation.

Remark 3. If

$$(\gamma_0+2\omega)^2+4K[g-2\omega(\gamma_0h_0+\sqrt{gh_0}\alpha)]\coth(Kh_0)>0, \tag{43}$$

then, from (42) we get

$$\begin{aligned} c &= h_0\gamma_0+\sqrt{gh_0}\alpha+\frac{1}{2K}\Big[-(\gamma_0+2\omega)\tanh(Kh_0) \\ & \quad \pm\sqrt{(\gamma_0+2\omega)^2\tanh^2(Kh_0)+4K[g-2\omega(\gamma_0h_0+\sqrt{gh_0}\alpha)]\tanh(Kh_0)}\Big]. \end{aligned} \tag{44}$$

The expression with minus in front of the square root was obtained by a different approach in [7], see Thm 3.2.. We observe that, if the laminar flow is westward at $z = h_0$, that is,

$$U(h_0) = \gamma_0 h_0 + \sqrt{gh_0}\alpha < 0, \tag{45}$$

then, the condition (43) is satisfied.

For the rest of the paper, we assume that the laminar flow is such that the condition (43) is fulfilled, and thus, the wave propagation speed c is for this scaling given explicitly by (44).

5. **Particle trajectories.** In this section we study the particle trajectories in the linear wave (31) and (41), respectively. First we compare the results obtained when solving the problems (16) and (20) and observe that the velocity field has in both cases the form

$$\begin{array}{rcl} u(t,x,z) & = & A\cosh(Kz)\cos[K(x-ct)] + Bz + C, \\ v(t,x,z) & = & A\sinh(Kz)\sin[K(x-ct)], \end{array} \tag{46}$$

where A, B, C and the wave speed c are in the case of the first scaling (13)

$$\begin{array}{rcl} A & = & \varepsilon\dfrac{Kh_0 c}{\sinh(Kh_0)}, \qquad B = \varepsilon\gamma_0, \qquad C = \varepsilon\sqrt{gh_0}c_0, \\ c & = & \dfrac{1}{K}\left[-\omega\tanh(Kh_0) \pm \sqrt{gK\tanh(Kh_0) + [\omega\tanh(Kh_0)]^2}\right], \end{array} \tag{47}$$

and for the second scaling (17) are given by

$$\begin{array}{rcl} A & = & \varepsilon\dfrac{Kh_0(c - h_0\gamma_0 - \sqrt{gh_0}\alpha)}{\sinh(Kh_0)}, \quad B = \gamma_0, \quad C = \sqrt{gh_0}\alpha + \varepsilon\sqrt{gh_0}d_0, \\ c & = & h_0\gamma_0 + \sqrt{gh_0}\alpha + \dfrac{1}{2K}\left[-(\gamma_0 + 2\omega)\tanh(Kh_0)\right. \\ & & \left.\pm\sqrt{(\gamma_0+2\omega)^2\tanh^2(Kh_0) + 4K[g - 2\omega(\gamma_0 h_0 + \sqrt{gh_0}\alpha)]\tanh(Kh_0)}\right]. \end{array} \tag{48}$$

The trajectory $((x(t), z(t))$ of a fluid particle located iniatially (at time $t = 0$) at the point (x_0, z_0) beneath the small-amplitude wave with the velocity field (46) is obtained by solving the following nonlinear system of equations

$$\left\{\begin{array}{rcl} \dfrac{dx}{dt} & = & A\cosh(Kz)\cos[K(x-ct)] + Bz + C, \\ \dfrac{dz}{dt} & = & A\sinh(Kz)\sin[K(x-ct)], \end{array}\right. \tag{49}$$

where A, B, C and c are having either the form (47) or (48), depending on which scaling we refer to.

Let us see if A is always different from zero and on what depends its sign.

Theorem 5.1. *(1) In the first scaling case,*

$$A \neq 0 \quad \textit{for any} \quad \gamma_0.$$

The sign of A is independent of the sign and of the value of the constant vorticity γ_0. We have that $A > 0$ (respectively $A < 0$) if we choose in (47) the square root with plus (respectively minus).

(2) In the second scaling case around a laminar flow for which the constant vorticity γ_0 satisfies (43), we get that

$$A \neq 0 \quad \textit{for any} \quad \gamma_0 \neq \sqrt{\frac{g}{h_0}}\left(\frac{1}{2\omega}\sqrt{\frac{g}{h_0}} - \alpha\right).$$

The sign of A is now dependent on the value of the constant vorticity γ_0. More precisely, if we choose the square root with plus in (48), then

$$A > 0 \quad \textit{if and only if} \quad \gamma_0 + 2\omega < 0 \textit{ or } \gamma_0 < \sqrt{\frac{g}{h_0}}\left(\frac{1}{2\omega}\sqrt{\frac{g}{h_0}} - \alpha\right)$$

$$A < 0 \quad \textit{if and only if} \quad \gamma_0 + 2\omega > 0 \textit{ and } \gamma_0 > \sqrt{\frac{g}{h_0}}\left(\frac{1}{2\omega}\sqrt{\frac{g}{h_0}} - \alpha\right)$$

and if we choose in (48) the square root with minus then we have

$$A > 0 \quad \textit{if and only if} \quad \gamma_0 + 2\omega < 0 \textit{ and } \gamma_0 > \sqrt{\frac{g}{h_0}}\left(\frac{1}{2\omega}\sqrt{\frac{g}{h_0}} - \alpha\right)$$

$$A < 0 \quad \textit{if and only if} \quad \gamma_0 + 2\omega > 0 \textit{ or } \gamma_0 < \sqrt{\frac{g}{h_0}}\left(\frac{1}{2\omega}\sqrt{\frac{g}{h_0}} - \alpha\right).$$

(3) If the second scaling is around a westward laminar flow, then we get that

$$A \neq 0 \quad \textit{for any } \gamma_0 \textit{ which satisfies } (45),$$

and $A > 0$ (respectively $A < 0$) if we choose in (48) the square root with plus (respectively minus).

Proof. (1) Because K and h_0 are greater then zero, the sign of A depends only on the sign of the wave speed c from (47). A positive (respectively negative) c means that the wave moves eastwards (respectively westwards). The proof follows by direct computations.

(2) The sign of A depends now on the sign of $c - h_0\gamma_0 - \sqrt{gh_0}\alpha$ from (48). The proof follows by direct computations.

(3) In this case, γ_0 satisfies (45). This yields $\gamma_0 < \sqrt{\frac{g}{h_0}}\left(\frac{1}{2\omega}\sqrt{\frac{g}{h_0}} - \alpha\right)$ and we apply (2) from above. For this type of flows, we could regard the expression $c - h_0\gamma_0 - \sqrt{gh_0}\alpha$ as "the speed" of a wave which is westward if we take the square root with minus in (48) and is eastward if we take the square root with plus in (48). □

Remark 4. We remind that in the non-geophysical case studied in [25], the sign of A is, for both scalings, independent of sign and of the value of the constant vorticity γ_0. This sign depends only on how one chooses the square roots in the dispersion relations: if one chooses the square root with minus, that is, one considers left-going waves, one gets a negative A, if one chooses the square root with plus, that is, that is, one considers right-going waves, one gets a positive A.

Theorem 5.2. *The particle trajectories beneath small-amplitude geophysical water waves with constant vorticity are not closed curves.*

Proof. In order to obtain this result on the shape of the particle trajectories we will provide the exact solutions of the nonlinear system (49). Using the transformation

$$X = K(x - ct), \qquad Z = Kz, \tag{50}$$

the system (49) can be written as

$$\begin{cases} \dfrac{dX}{dt} &= KA\cosh(Z)\cos(X) + BZ + K(C - c), \\ \dfrac{dZ}{dt} &= KA\sinh(Z)\sin(X). \end{cases} \tag{51}$$

From the second relation in (51) we obtain, after integration, that

$$\log\left[\tanh\left(\frac{Z}{2}\right)\right] = \int KA\sin(X(t))\,dt.$$

Assuming that

$$\int KA\sin(X(t))\,dt < 0 \tag{52}$$

we get

$$Z(t) = 2\,\mathrm{arctanh}\,(a)\,, \tag{53}$$

with

$$a = a(t) := \exp\left(\int KA\sin(X(t))\,dt\right), \quad 0 < a < 1. \tag{54}$$

Inserting this expression for Z in the first relation of (51) we obtain

$$\frac{dX}{dt} = KA\frac{1+a^2}{1-a^2}\cos(X) + 2B\,\mathrm{arctanh}(a) + K(C-c). \tag{55}$$

The relation (54) yields

$$KA\sin(X(t)) = \frac{1}{a(t)}\frac{da}{dt}.$$

From this last relation it follows that

$$KA\cos(X)\frac{dX}{dt} = \frac{1}{a^2}\left[\frac{d^2a}{dt^2}a - \left(\frac{da}{dt}\right)^2\right];$$

$$K^2A^2\cos^2(X) = K^2A^2 - \frac{1}{a^2}\left(\frac{da}{dt}\right)^2.$$

Plugging these expressions in (55) we get the following equation

$$\begin{aligned}\frac{d^2a}{dt^2} + \frac{2a}{1-a^2}\left(\frac{da}{dt}\right)^2 - K^2A^2a\frac{1+a^2}{1-a^2}\\ - \sqrt{K^2A^2a^2 - \left(\frac{da}{dt}\right)^2}\,[2B\,\mathrm{arctanh}(a) + K(C-c)] = 0.\end{aligned}$$

Making the substitution $\xi^2(a) := K^2A^2a^2 - (da/dt)^2$ the last equation becomes

$$\xi\frac{d\xi}{da} + \frac{2a}{1-a^2}\xi^2 + [2B\,\mathrm{arctanh}(a) + K(C-c)]\xi = 0. \tag{56}$$

We observe that a solution of this equation is $\xi = 0$ which implies

$$\sin(X(t)) = \pm 1.$$

Therefore a solution of system (49) is given by

$$\begin{aligned} x(t) &= ct + c_1 \\ z(t) &= \frac{2}{K}\,\mathrm{arctanh}\left(\exp(-|KAt + c_2|)\right), \end{aligned} \tag{57}$$

where the constants c_1 and c_2 are determined by the initial conditions $(x(0), z(0)) := (x_0, z_0)$, the wave speed c and A being given by (47) or (48), depending on which scaling we use. The solution (57) has a horizontal asymptote at $z = 0$ and is obviously not a closed curve. This peakon-like solution appears also in the non-geophysical case below small-amplitude gravity water waves with constant vorticity (see [24], [25]). As we have already mentioned, the expression of the wave speed c (which appears in A too) has a different expression in the non-geophysical case.
The other solutions of the equation (56) satisfy

$$\frac{d\xi}{da} + \frac{2a}{1-a^2}\xi = -[2B\,\mathrm{arctanh}(a) + K(C-c)]. \tag{58}$$

Using the methods of variation of constants we found that the solution of the non-homogeneous equation (58) is given by

$$\xi(a) = (1-a^2)[\beta - K(C-c)\,\mathrm{arctanh}(a) - B\,\mathrm{arctanh}^2(a)],$$

β being an integration constant. Taking into account the substitution $\xi^2(a) := K^2A^2a^2 - (da/dt)^2$ we obtain from the previous relation, after separating the variables, that

$$\pm \frac{da}{(1-a^2)\sqrt{K^2A^2\frac{a^2}{(1-a^2)^2} - [\beta - K(C-c)\operatorname{arctanh}(a) - B\operatorname{arctanh}^2(a)]^2}} = dt.$$

With (53) in view, the latter relation can be written as

$$\pm \frac{dZ}{\sqrt{K^2A^2\sinh^2(Z) - [2\beta - K(C-c)Z - \frac{B}{2}Z^2]^2}} = dt.$$

Moreover, from the second relation in (49) and the transformation (50) we obtain the solution

$$\begin{aligned} x(t) &= ct + \frac{1}{K}\arcsin\left[\frac{1}{KA\sinh(Z(t))}\frac{dZ(t)}{dt}\right] \\ z(t) &= \frac{1}{K}Z(t), \end{aligned} \tag{59}$$

where $Z(t)$ is the solution of the equation

$$\pm \frac{dZ}{\sqrt{[KA\sinh(Z)]^2 - [2\beta - K(C-c)Z - \frac{B}{2}Z^2]^2}} = dt. \tag{60}$$

The constant β together with the constant which appears from the integration of the equation (60) are determined by the initial conditions $(x(0), z(0)) := (x_0, z_0)$. A, B, C and the wave speed speed c given by (47) or (48), depending on which scaling we use. The solution (59) appears also in the non-geophysical case below small-amplitude gravity water waves with constant vorticity (see [25]), but there the expression of the wave speed c (which appears in A too) has a different expression.
The solution (59) is not a closed curve too. Indeed, if we suppose that there exists $t_2 > t_1$ with $Z(t_1) = Z(t_2)$, then from (60) it follows that

$$\begin{aligned} \frac{dZ(t_1)}{dt_1} &= \pm\sqrt{K^2A^2\sinh^2(Z(t_1)) - [2\beta - K(C-c)Z(t_1) - \frac{B}{2}Z^2(t_1)]^2} \\ &= \pm\sqrt{K^2A^2\sinh^2(Z(t_2)) - [2\beta - K(C-c)Z(t_2) - \frac{B}{2}Z^2(t_2)]^2} = \frac{dZ(t_2)}{dt_2}. \end{aligned}$$

Thus, in the fixed frame we get

$$z(t_2) = z(t_1) \quad \text{and} \quad x(t_2) - x(t_1) = c(t_2 - t_1) \neq 0.$$

□

The exact solutions found above allow us to determine the possible stagnation points in small-amplitude geophysical water waves with constant vorticity.

Proposition 1. *For the solution (57) a stagnation point appears only for $t \to \pm\infty$. For the solution (59) the stagnation points can be find by solving the following equation in the unknown Z:*

$$|KA\sinh(Z)| = \left|2\beta - K(C-c)Z - \frac{B}{2}Z^2\right|. \tag{61}$$

Proof. Differentiating in (57) with respect to t we obtain

$$x'(t) = c$$

$$z'(t) = \begin{cases} \dfrac{2A\exp(KAt + c_2)}{1 - \exp(2(KAt + c_2))}, & \text{if } KAt + c_2 < 0 \\ -\dfrac{2A\exp(-(KAt + c_2))}{1 - \exp(-2(KAt + c_2))}, & \text{if } KAt + c_2 > 0. \end{cases}$$

Thus, the stagnation points in the fluid, i.e. the points where $x'(t) = c$ and $z'(t) = 0$, occur only if $t \to \pm\infty$. At this point the path of the particle has a horizontal tangent and the location of this stagnation point is on the bottom $z = 0$.

Differentiating in (59) with respect to t we obtain

$$\begin{aligned} x'(t) &= c + \frac{1}{K}\frac{\sinh(Z)\dfrac{d^2Z}{dt^2} - \cosh(Z)\left(\dfrac{dZ}{dt}\right)^2}{\sinh(Z)\sqrt{[KA\sinh(Z)]^2 - \left(\dfrac{dZ}{dt}\right)^2}} \\ z'(t) &= \frac{1}{K}\frac{dZ}{dt}, \end{aligned} \tag{62}$$

where dZ/dt is given by (60). Taking into account (60), if Z satisfies (61) then we get

$$\frac{dZ}{dt} = 0, \qquad \frac{d^2Z}{dt^2} = 0,$$

and therewith the system (62) becomes

$$x'(t) = c, \qquad z'(t) = 0.$$

Thus, for the solution (59) the stagnation points are obtained by solving the equation (61). This equation can be solved graphically. Depending on the signs and on the values of the parameters A, B, C, c and β, the equation (61) can have one, two, three, four or six solutions (see [25]). Which of these solutions are inside the fluid and their nature can be obtained by a further study. □

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