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VARIATIONAL DERIVATION OF THE CAMASSA-HOLM SHALLOW WATER EQUATION WITH NON-ZERO VORTICITY

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Abstract. We describe the physical hypotheses underlying the derivation of an approximate model of water waves. For unidirectional surface shallow water waves moving over an irrotational flow as well as over a non-zero vorticity flow, we derive the Camassa-Holm equation by an interplay of variational methods and small-parameter expansions.

1. **Introduction.**

The Camassa-Holm equation reads

$$U_t + 2\kappa U_x + 3UU_x - U_{txx} = 2U_xU_{xx} + UU_{xxx} \tag{1}$$

with $x \in \mathbf{R}$, $t \in \mathbf{R}$, $U(x,t) \in \mathbf{R}$. Subscripts here, and later, denote partial derivatives. The constant κ is related to the critical shallow water speed. For κ=0 the equation (1) possesses peaked soliton solutions [5]. The physical derivation of (1) as a model for the evolution of a shallow water layer under the influence of gravity, is due to Camassa and Holm [5]. See also Refs. [8], [18], [25] for alternative derivations within the shallow water regime. In [30] it was shown that the equation (1) describes a geodesic flow on the one dimensional central extension of the group of diffeomorphisms of the circle (for the case κ=0, a geodesic flow on the diffeomorphism group of the circle, see also [14]). It should be mentioned that, prior to Camassa and Holm, Fokas and Fuchssteiner [19] obtained formally, by the method of recursion operators in the context of hereditary symmetries, families of integrable equations containing (1). These equations are bi-Hamiltonian generalizations of the KdV equation and possess infinitely many conserved quantities in involution [19]. In order to see how the equation (1) relates with one of equations in the families introduced in [19], see [21], §2.2. Also, the equation (5.3) in [20] is the equation (1) but with errors in the coefficients.

The Camassa-Holm equation attracted a lot of interest, due to its complete integrability [6], [15], [22], [32] (for the periodic case), [2], [9], [13] and the citations therein (for the integrability on the line), the existence of waves that exist for all times as well as of breaking waves [11] and the presence of peakon solutions [5]. Of interest is also the fact that its solitary waves are solitons, recovering their form and speed after nonlinear interaction [3], [27]. These waves are smooth if $\kappa \neq 0$ and peaked if $\kappa = 0$, with both wave forms being stable and thus physically recognizable [16], [17]. An important question to be answered is whether solutions

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of the water wave problem can really be approximated by solutions of the Camassa-Holm equation. In [29] it is shown that suitable solutions of the water wave problem and solutions of the Camassa-Holm equation stay close together for long times, in the case $\epsilon = \delta^2$, ϵ being the amplitude parameter and δ the shallowness parameter. In [29] it is also shown that the peakon equation cannot strictly be derived from the Euler equation and hence it is at most a phenomenological model.

In the references [18], [23], [25], the derivation of the Camassa-Holm equation is made under the assumption that the fluid motion is irrotational. This assumption does not appear explicitly in the initial derivation [5]. This derivation is based on the Green-Naghdi equations and on the choice of a submanifold in the Hamiltonian representation of the Green-Naghdi equations. For the Green-Naghdi equations, one does not have to impose any condition that the fluid is irrotational, but, as shown in [25], in order to have a consistency for the second assumption one should impose an irrotational condition on the flow. Thus, another important question to be answered is the question whether a Camassa-Holm equation can be obtained when the fluid below the free surface has non-zero vorticity. By examining the underlying flow one has some important and new consequences. In [26] it is considered the Camassa-Holm equation for water waves moving over a shear flow. The calculation is completed for the case of a linear shear, that is, the underlying flow has constant vorticity. In this case it is shown that the equation for the surface wave is not a Camassa-Holm equation, but the Camassa-Holm equation can exist for a simple nonlinear function of the horizontal velocity component of the perturbed flow field, at a certain depth.

The purpose of the present paper is to derive by a variational approach in the Lagrangian formalism, the Camassa-Holm equation for unidirectional surface shallow water waves when the fluid below the free surface has non-zero vorticity. We will consider in turn the cases of an irrotational flow (for this case see also [23]), a rotational flow with constant vorticity and finally, an arbitrary flow. For the linear shear case, we show that the displacement of the free surface from the flat state, satisfies the Camassa-Holm equation (1) with $\kappa \neq 0$. For an arbitrary unidirectional shallow water flow, we get that the displacement of the free surface from the flat state, satisfies a generalized Camassa-Holm equation.

2. **Nondimensionalisation and scaling of the governing equations for water waves.** We consider water moving in a domain with a free upper surface at $z = h_0 + \eta(x,t)$, for a constant $h_0 > 0$, and a flat bottom at $z = 0$. The undisturbed water surface is $z = h_0$. Let $(u(x,z,t), v(x,z,t))$ be the velocity of the water - no motion takes place in the y-direction. The fluid is acted on only by the acceleration of gravity g, and the effects of surface tension are ignored. For gravity water waves, the appropriate equations of motion are Euler's equations (EE)(see [24]). Another realistic assumption for gravity water wave problem is the incompressibility (constant density ρ) (see [28]), which implies the equation of mass conservation (MC). The boundary conditions for the water wave problem are the kinematic boundary conditions as well as the dynamic boundary condition. The kinematic boundary conditions (KBC) express the fact that the same particles always form the free water surface and that the fluid is assumed to be bounded below by a hard horizontal bed $z = 0$. The dynamic boundary condition (DBC) expresses the fact that on the free surface the pressure is equal to the constant atmospheric pressure denoted p_0. Summing up, the exact solution for the water-wave problem is given by the system

$$\left.\begin{array}{ll} \begin{array}{c} u_t + uu_x + vu_z = -\frac{1}{\rho}p_x \\ v_t + uv_x + vv_z = -\frac{1}{\rho}p_z - g \end{array} & \text{(EE)} \\ u_x + v_z = 0 & \text{(MC)} \\ \begin{array}{c} v = \eta_t + u\eta_x \ \text{ on } \ z = h_0 + \eta(x,t) \\ v = 0 \ \text{ on } z = 0 \end{array} & \text{(KBC)} \\ p = p_0, \ \text{ on } z = h_0 + \eta(x,t) & \text{(DBC)} \end{array}\right. \tag{2}$$

where $p(x, z, t)$ denotes the pressure. While a mathematical study of exact solutions to the governing equations (2) for water waves can be pursued (see for example [31] for the periodic steady solutions in the irrotational case and [17] for the periodic steady solutions in the case of non-zero vorticity), to reach detailed information about qualitative features of water waves it is useful to derive approximate models which are more amenable to an in-depth analysis.

We non-dimensionalise the set of equations (2) using the undisturbed depth of water h_0, as the vertical scale, a typical wavelength λ, as the horizontal scale, and a typical amplitude of the surface wave a (for more details see [24], [25]). An appropriate choice for the scale of the horizontal component of the velocity is $\sqrt{gh_0}$. Then, the corresponding time scale is $\frac{\lambda}{\sqrt{gh_0}}$ and the scale for the vertical component of the velocity is $h_0\frac{\sqrt{gh_0}}{\lambda}$. Thus, we define the set of non-dimensional variables

$$\begin{array}{c} x \mapsto \lambda x, \quad z \mapsto h_0 z, \quad \eta \mapsto a\eta, \quad t \mapsto \frac{\lambda}{\sqrt{gh_0}}t, \\ u \mapsto \sqrt{gh_0}u, \quad v \mapsto h_0\frac{\sqrt{gh_0}}{\lambda}v \end{array} \tag{3}$$

where, to avoid new notations, we have used the same symbols for the non-dimensional variables x, z, η, t, u, v, in the right-hand side. The partial derivatives will be replaced by

$$\begin{array}{c} u_t \mapsto \frac{gh_0}{\lambda}u_t, \quad u_x \mapsto \frac{\sqrt{gh_0}}{\lambda}u_x, \quad u_z \mapsto \frac{\sqrt{gh_0}}{h_0}u_z, \\ v_t \mapsto \frac{gh_0^2}{\lambda^2}v_t, \quad v_x \mapsto h_0\frac{\sqrt{gh_0}}{\lambda^2}v_x, \quad v_z \mapsto \frac{\sqrt{gh_0}}{\lambda}v_z, \end{array} \tag{4}$$

Let us now define the non-dimensional pressure. If the water would be stationary, that is, $u \equiv v \equiv 0$, from the first two equations and the last condition with $\eta = 0$, of the system (2), we get for a non-dimensionalised z, the hydrostatic pressure $p_0 + \rho gh_0(1 - z)$. Thus, the non-dimensional pressure is defined by

$$p \mapsto p_0 + \rho gh_0(1 - z) + \rho gh_0 p \tag{5}$$

and

$$p_x \mapsto \rho\frac{gh_0}{\lambda}p_x, \quad p_z \mapsto -\rho g + \rho g p_z \tag{6}$$

Taking into account (3), (4), (5), and (6), the water-wave problem (2) writes in non-dimensional variables, as

$$\begin{array}{c} u_t + uu_x + vu_z = -p_x \\ \delta^2(v_t + uv_x + vv_z) = -p_z \\ u_x + v_z = 0 \\ v = \epsilon(\eta_t + u\eta_x) \ \text{ and } \ p = \epsilon\eta \ \text{ on } \ z = 1 + \epsilon\eta(x,t) \\ v = 0 \ \text{ on } z = 0 \end{array} \tag{7}$$

where we have introduced the amplitude parameter $\epsilon = \frac{a}{h_0}$ and the shallowness parameter $\delta = \frac{h_0}{\lambda}$. The small-amplitude shallow water is obtained in the limits

$\epsilon \to 0$, $\delta \to 0$. We observe that, on $z = 1 + \epsilon\eta$, both v and p are proportional to ϵ. This is consistent with the fact that as $\epsilon \to 0$ we must have $v \to 0$ and $p \to 0$, and it leads to the following scaling of the non-dimensional variables

$$p \mapsto \epsilon p, \quad (u, v) \mapsto \epsilon(u, v) \tag{8}$$

where we avoided again the introduction of a new notation. The problem (7) becomes

$$\begin{array}{c} u_t + \epsilon(uu_x + vu_z) = -p_x \\ \delta^2[v_t + \epsilon(uv_x + vv_z)] = -p_z \\ u_x + v_z = 0 \\ v = \eta_t + \epsilon u\eta_x \ \text{ and } \ p = \eta \ \text{ on } \ z = 1 + \epsilon\eta(x, t) \\ v = 0 \ \text{ on } z = 0 \end{array} \tag{9}$$

The two important parameters ϵ and δ that arise in water-waves theories, are used to define various approximations of the governing equations and the boundary conditions. The amplitude parameter ϵ is associated with the nonlinearity of the wave, so that small ϵ implies a nearly-linear wave theory. The shallowness parameter δ is associated with the dispersion of the wave, it measures the deviation of the pressure, in the water below the wave, away from the hydrostatic pressure distribution.

3. **Variational derivation of the Camassa-Holm shallow water equation.** In contrast to the KdV equation, which is a classical integrable model for shallow water waves, the Camassa-Holm equation possesses not only solutions that are global in time but models also wave breaking. The only way that singularities can arise in finite time in a smooth solution is in the form of breaking waves, that is, the solution remains bounded but its slope becomes unbounded [7]. Even if a wave breaks, there is a procedure to continue uniquely the solution after wave breaking [4].

There are different approaches which lead to the Camassa-Holm equation in the shallow water regime. The original derivation of this equation [5] consists in making approximations by relating $m := u - u_{xx}$ and η in the Green-Naghdi Hamiltonian system and preserving the momentum part of its Hamiltonian structure. The Camassa-Holm equation can be also obtained from the governing equations (9), by the use of a double asymptotic expansion, valid for $\epsilon \to 0$, $\delta \to 0$, retaining terms $O(1)$, $O(\epsilon)$, $O(\delta^2)$, $O(\epsilon\delta^2)$ (see for example [25], [26]). As a result, a single nonlinear equation for η will be obtained and thus all variables will be expressed through the solution of this equation. In what follows we will consider a derivation of the model from the governing equations for the water-wave problem by a variational approach in the Lagrangian formalism [8], [23].

One observes that the parameter δ can be removed from the system (9) (see [25]), this being equivalent to use only h_0 as the length scale of the problem. In order to do this, the non-dimensional variables x, t and v from (3) are replaced by

$$x \mapsto \frac{\sqrt{\epsilon}}{\delta} x, \quad t \mapsto \frac{\sqrt{\epsilon}}{\delta} t, \quad v \mapsto \frac{\delta}{\sqrt{\epsilon}} v \tag{10}$$

Therefore the equations in the system (9) are recovered, but with δ^2 replaced by ϵ in the second equation of the system, that is, this equation writes as

$$\epsilon[v_t + \epsilon(uv_x + vv_z)] = -p_z \tag{11}$$

The classical approximation is the linearized problem obtained by requiring the amplitude of the surface to be small letting $\epsilon \to 0$. The system (9) with the second equation given by (11) becomes linear:

$$\begin{gathered} u_t + p_x = 0 \\ p_z = 0 \\ u_x + v_z = 0 \\ v = \eta_t \ \text{ and } \ p = \eta \ \text{ on } \ z = 1 \\ v = 0 \ \text{ on } z = 0 \end{gathered} \tag{12}$$

3.1. **The case of an irrotational flow.** Firstly, we consider that the fluid flow is irrotational, i.e. the flow has zero vorticity. Then, in addition to the system (2), we also have the equation

$$u_z - v_x = 0 \tag{13}$$

Here the velocity components u and v are written in the physical (dimensional) variables. If we non-dimensionalise this equation using (3), (4), we obtain

$$u_z = \delta^2 v_x \tag{14}$$

After scaling (8) and transformation (10), the equation (14) writes as

$$u_z = \epsilon v_x \tag{15}$$

Therefore, in the limit $\epsilon \to 0$, we get in addition to the system (12), the equation

$$u_z = 0 \tag{16}$$

Thus, under the assumption that the fluid is irrotational, u is independent of z, that is,

$$u = u(x, t) \tag{17}$$

From the second equation in (12), we get that p also does not depend on z. Because $p = \eta(x, t)$ on $z = 1$, we have

$$p = \eta(x, t) \qquad \text{for any} \ \ 0 \le z \le 1 \tag{18}$$

Therefore, using the first equation in (12), and taking into account (17), we obtain

$$u = -\int \eta_x(x, t)dt + \mathcal{F}(x) \tag{19}$$

where $\mathcal{F}$ is an arbitrary function. Differentiating (19) with respect to x and using the third equation in (12), we get, after an integration against z,

$$v = -zu_x = z\left(\int \eta_{xx}(x, t)dt - \tilde{\mathcal{F}}'(x)\right) \tag{20}$$

We underline the fact that in our approximation the vertical velocity component maintains a dependence on the z-variable. In analyzing the motion of the fluid particles, this means that the particles below the surface may perform a vertical motion. This is in agreement with recent general results obtained in [10] for the periodic steady waves (Stokes waves) and in [12] for the solitary waves, both being solutions of the full Euler equations.

Making $z = 1$ in (20), and taking into account that $v = \eta_t$ on $z = 1$, we get after a differentiation with respect to t, that η has to satisfy the equation

$$\eta_{tt} - \eta_{xx} = 0 \tag{21}$$

The general solution of this equation is $\eta(x, t) = f(x - t) + g(x + t)$, where f and g are differentiable functions. It is convenient first to restrict ourselves to waves which propagate in only one direction, thus, we choose

$$\eta(x, t) = f(x - t) \tag{22}$$

Therefore, for u and v in (19), (20) we have $u(x,t) = \eta(x,t) + \mathcal{F}(x)$, $v(x,z,t) = -z\left(\eta_x + \mathcal{F}'(x)\right)$. The condition $v = \eta_t$ on $z = 1$, yields

$$\mathcal{F}(x) = \text{const} := c_0 \tag{23}$$

Thus, for the irrotational case the solution of the system (12) plus the equation (16), can be written into the form

$$\eta(x,t) = f(x-t), \quad u(x,t) = \eta(x,t) + c_0, \quad v(x,z,t) = -z\eta_x(x,t) = -zu_x \tag{24}$$

The solutions to the shallow water problem are determined by the evolution of the function $\eta(x,t)$, which represents the displacement of the free surface from the undisturbed (flat) state. The solution (24) describes the linear, non-dispersive surface wave.

By consistently neglecting the ϵ contribution, we will derive using variational methods in the Lagrangian formalism (see [8], [23]), the equation (1) governing unidirectional propagation of shallow water waves.

In the Lagrangian picture of a mechanical system, one focuses the attention on the motion of each individual particle of the mechanical system. We denote by M the ambient space whose points are supposed to represent the particles at $t = 0$. A diffeomorphism of M represents the rearrangement of the particles with respect to their initial positions. The set of all diffeomorphisms, denoted $\mathrm{Diff}(M)$, can be regarded (at least formally) as a Lie group. The motion of the mechanical system is described by a time-dependent family of orientation-preserving diffeomorphisms $\gamma(t,\cdot) \in \mathrm{Diff}(M)$. For a particle initially located at x, the velocity at time t is $\gamma_t(t,x)$, this being the material velocity used in the Lagrangian description. The spatial velocity, used in the Eulerian description, is the flow velocity $w(t,X) = \gamma_t(t,x)$ at the location $X = \gamma(t,x)$ at time t, that is, $w(t,\cdot) = \gamma_t \circ \gamma^{-1}$. In the Lagrangian description, the velocity phase space is the tangent bundle $T\mathrm{Diff}(M)$. In the Eulerian description, the spatial velocity is in the tangent space at the identity Id of $\mathrm{Diff}(M)$, that is, it is an element of the Lie algebra of $\mathrm{Diff}(M)$. The Lagrangian $\mathcal{L}$ is a scalar function defined on $T\mathrm{Diff}(M)$ and the equation of motion is the equation satisfied by a critical point of the action $\mathfrak{a}(\gamma) = \int_0^T \mathcal{L}(\gamma, \gamma_t)dt$ defined on all paths $\{\gamma(t,\cdot),\, t \in [0,T]\}$ in $\mathrm{Diff}(M)$ having fixed endpoints.

For our problem, we saw in (24) that the vertical component v of the velocity is completely determined by the horizontal component $u(x,t)$ of the velocity, which is a vector field on $\mathbf{R}$, that is, it belongs to the Lie algebra of $\mathrm{Diff}(\mathbf{R})$. Thus, in the Lagrangian formalism of our problem we take $M = \mathbf{R}$ and add the technical assumption that the smooth functions defined on $\mathbf{R}$ with value in $\mathbf{R}$ vanish rapidly at $\pm\infty$ together with as many derivatives as necessary (see [7] for a possible choice of weighted Sobolev spaces). For the one-dimensional periodic motion, one takes $M = \mathbf{S}^1$ the unit circle. In what follows we focus on the first situation.

Following Arnold's approach to Euler equations on diffeomorphism groups [1], the action for our problem will be obtained by transporting the kinetic energy to all tangent spaces of $\mathrm{Diff}(\mathbf{R})$ by means of right translations. For small surface elevation, the potential energy is negligible compared to the kinetic energy. Taking into account (24), the kinetic energy on the surface is

$$K = \frac{1}{2}\int_{-\infty}^{\infty}(u^2+v^2)dx = \frac{1}{2}\int_{-\infty}^{\infty}\left[u^2 + (1+\epsilon\eta)^2 u_x^2\right]dx \approx \frac{1}{2}\int_{-\infty}^{\infty}\left(u^2 + u_x^2\right)dx \tag{25}$$

to the order of our approximation (see [8]). We observe that if we replace the path $\gamma(t,\cdot)$ by $\gamma(t,\cdot)\circ\psi(\cdot)$, for a fixed time-independent ψ in Diff($\mathbf{R}$), then the spatial velocity is unchanged $\gamma_t\circ\gamma^{-1}$. Transforming K to a right-invariant Lagrangian, the action on a path $\gamma(t,\cdot)$, $t\in[0,T]$, in Diff($\mathbf{R}$) is

$$\mathfrak{a}(\gamma)=\frac{1}{2}\int_0^T\int_{-\infty}^{\infty}\{(\gamma_t\circ\gamma^{-1})^2+[\partial_x(\gamma_t\circ\gamma^{-1})]^2\}dxdt \tag{26}$$

The critical points of the action (26) in the space of paths with fixed endpoints, verify

$$\frac{d}{d\varepsilon}\mathfrak{a}(\gamma+\varepsilon\varphi)\Big|_{\varepsilon=0}=0, \tag{27}$$

for every path $\varphi(t,\cdot)$, $t\in[0,T]$, in Diff($\mathbf{R}$) with endpoints at zero, that is, $\varphi(0,\cdot)=0=\varphi(T,\cdot)$ and such that $\gamma+\varepsilon\varphi$ is a small variation of γ on Diff($\mathbf{R}$). Taking into account (26), the condition (27) becomes

$$\int_0^T\int_{-\infty}^{\infty}\left\{(\gamma_t\circ\gamma^{-1})\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[(\gamma_t+\varepsilon\varphi_t)\circ(\gamma+\varepsilon\varphi)^{-1}\right]\right.$$
$$\left.+\partial_x(\gamma_t\circ\gamma^{-1})\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[\partial_x\left((\gamma_t+\varepsilon\varphi_t)\circ(\gamma+\varepsilon\varphi)^{-1}\right)\right]\right\}dxdt=0 \tag{28}$$

After calculation (see [8]), we get

$$\begin{aligned}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[(\gamma_t+\varepsilon\varphi_t)\circ(\gamma+\varepsilon\varphi)^{-1}\right]&=\varphi_t\circ\gamma^{-1}-(\varphi\circ\gamma^{-1})\partial_x(\gamma_t\circ\gamma^{-1})\\&=\partial_t(\varphi\circ\gamma^{-1})+(\gamma_t\circ\gamma^{-1})\partial_x(\varphi\circ\gamma^{-1})\\&\quad-(\varphi\circ\gamma^{-1})\partial_x(\gamma_t\circ\gamma^{-1})\end{aligned} \tag{29}$$

and

$$\begin{aligned}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[\partial_x\left((\gamma_t+\varepsilon\varphi)\circ(\gamma+\varepsilon\varphi)^{-1}\right)\right]&=\partial_x(\varphi_t\circ\gamma^{-1})-\partial_x(\gamma_t\circ\gamma^{-1})\partial_x(\varphi\circ\gamma^{-1})\\&\quad-(\varphi\circ\gamma^{-1})\partial_x^2(\gamma_t\circ\gamma^{-1})\\&=\partial_{tx}(\varphi\circ\gamma^{-1})+(\gamma_t\circ\gamma^{-1})\partial_x^2(\varphi\circ\gamma^{-1})\\&\quad-(\varphi\circ\gamma^{-1})\partial_x^2(\gamma_t\circ\gamma^{-1})\end{aligned} \tag{30}$$

where are used the formulas of the type

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[(\gamma+\varepsilon\varphi)^{-1}\right]=-\frac{\varphi\circ\gamma^{-1}}{\gamma_x\circ\gamma^{-1}} \tag{31}$$

$$\partial_x(\gamma_t\circ\gamma^{-1})=\frac{\gamma_{tx}\circ\gamma^{-1}}{\gamma_x\circ\gamma^{-1}} \tag{32}$$

$$\partial_t(\varphi\circ\gamma^{-1})=\varphi_t\circ\gamma^{-1}+\left(\varphi_x\circ\gamma^{-1}\right)\partial_t(\gamma^{-1})=\varphi_t\circ\gamma^{-1}-(\gamma_t\circ\gamma)\partial_x(\varphi\circ\gamma^{-1}) \tag{33}$$

Thus, denoting $\gamma_t\circ\gamma^{-1}=u$, from (29), (30), the condition (28) writes as

$$\int_0^T\int_{-\infty}^{\infty}\left\{u\left[\partial_t(\varphi\circ\gamma^{-1})+u\partial_x(\varphi\circ\gamma^{-1})-(\varphi\circ\gamma^{-1})u_x\right]\right.$$
$$\left.+u_x\left[\partial_{tx}(\varphi\circ\gamma^{-1})+u\partial_x^2(\varphi\circ\gamma^{-1})-(\varphi\circ\gamma^{-1})u_{xx}\right]\right\}dxdt=0 \tag{34}$$

We integrate by parts with respect to t and x in the above formula, we take into account that φ has endpoints at zero, the smooth functions defined on $\mathbf{R}$ with values in **R**, together with as many derivatives as necessary, vanish rapidly at $\pm\infty$, and we obtain

$$-\int_0^T\int_{-\infty}^{\infty}(\varphi\circ\gamma^{-1})\left[u_t+3uu_x-u_{txx}-2u_xu_{xx}-uu_{xxx}\right]dxdt=0 \tag{35}$$

Therefore, we proved:

Theorem 3.1. *For an irrotational unidirectional shallow water flow, the horizontal velocity component of the water $u(x,t)$ satisfies the Camassa-Holm equation (1) for $\kappa=0$.*

Let us see now which equation fulfill the displacement $\eta(x,t)$ of the free surface from the flat state. Taking into account (24), the kinetic energy on the surface is

$$\begin{aligned} K=\frac{1}{2}\int_{-\infty}^{\infty}(u^2+v^2)dx &= \frac{1}{2}\int_{-\infty}^{\infty}\left[(\eta+c_0)^2+(1+\epsilon\eta)^2\eta_x^2\right]dx \\ &\approx \frac{1}{2}\int_{-\infty}^{\infty}\Big[(\eta+c_0)^2+\eta_x^2\Big]dx \end{aligned} \tag{36}$$

to the order of our approximation. Transforming K to a right-invariant Lagrangian, the action on a path $\Gamma(t,\cdot)$, $t\in[0,T]$, in Diff(**R**) is

$$\mathfrak{a}(\Gamma)=\frac{1}{2}\int_0^T\int_{-\infty}^{\infty}\{(\Gamma_t\circ\Gamma^{-1}+c_0)^2+[\partial_x(\Gamma_t\circ\Gamma^{-1})]^2\}dxdt \tag{37}$$

The critical points of the action (37) in the space of paths with fixed endpoints, verify

$$\frac{d}{d\varepsilon}\mathfrak{a}(\Gamma+\varepsilon\Phi)\Big|_{\varepsilon=0}=0, \tag{38}$$

for every path $\Phi(t,\cdot)$, $t\in[0,T]$, in Diff(**R**) with endpoints at zero, that is, $\Phi(0,\cdot)=0=\Phi(T,\cdot)$ and such that $\Gamma+\varepsilon\Phi$ is a small variation of Γ on Diff(**R**). Taking into account (37), the condition (38) becomes

$$\begin{aligned}\int_0^T\int_{-\infty}^{\infty} &\left\{\left(\Gamma_t\circ\Gamma^{-1}+c_0\right)\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[(\Gamma_t+\varepsilon\Phi_t)\circ(\Gamma+\varepsilon\Phi)^{-1}\right]\right. \\ &\left.+\partial_x(\Gamma_t\circ\Gamma^{-1})\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[\partial_x\left((\Gamma_t+\varepsilon\Phi_t)\circ(\Gamma+\varepsilon\Phi)^{-1}\right)\right]\right\}dxdt=0\end{aligned} \tag{39}$$

After calculation, denoting $\Gamma_t\circ\Gamma^{-1}=\eta$, (39) writes as

$$\begin{aligned}\int_0^T\int_{-\infty}^{\infty} &\left\{(\eta+c_0)\left[\partial_t(\Phi\circ\Gamma^{-1})+\eta\partial_x(\Phi\circ\Gamma^{-1})-(\Phi\circ\Gamma^{-1})\eta_x\right]\right. \\ &\left.+\eta_x\left[\partial_{tx}(\Phi\circ\Gamma^{-1})+\eta\partial_x^2(\Phi\circ\Gamma^{-1})-(\Phi\circ\Gamma^{-1})\eta_{xx}\right]\right\}dxdt=0\end{aligned} \tag{40}$$

We integrate by parts with respect to t and x in the above formula, we take into account that Φ has endpoints at zero, the smooth functions defined on **R** with values in **R**, together with as many derivatives as necessary, vanish rapidly at $\pm\infty$, and we obtain

$$-\int_0^T\int_{-\infty}^{\infty}(\Phi\circ\Gamma^{-1})\left[\eta_t+3\eta\eta_x+2c_0\eta_x-\eta_{txx}-2\eta_x\eta_{xx}-\eta\eta_{xxx}\right]dxdt=0 \tag{41}$$

We thus have:

Theorem 3.2. *For an irrotational unidirectional shallow water flow, the displacement $\eta(x,t)$ of the free surface from the flat state, satisfies the Camassa-Holm equation (1) for $\kappa=c_0$.*

3.2. **The case of a linear shear flow.** We assume now that the underling flow is rotational with a constant vorticity ω_0, that is, we are in the case of linear shear flow. Then, in addition to the system (2), we also have the equation

$$u_z - v_x = \text{const} := \omega_0 \tag{42}$$

If we non-dimensionalise this equation using (3), (4), we scale using (8) and we transform using (10), the equation (42) writes as

$$u_z = \epsilon v_x + \frac{\omega_0\sqrt{gh_0}}{g} \tag{43}$$

Therefore, for the case of a linear shear flow, in the limit $\epsilon \to 0$, we get in addition to the system (12), instead of (16), the equation

$$u_z = \frac{\omega_0\sqrt{gh_0}}{g} \tag{44}$$

The relation (18) remains the same but instead of (19) we have now

$$u = -\int \eta_x(x,t)dt + \mathcal{F}(x) + \frac{\omega_0\sqrt{gh_0}}{g}z \tag{45}$$

where $\mathcal{F}$ is an arbitrary function. Using the third equation in (12), we get again (20). Following the same procedure as in the irrotational case, presented after the relation (20), we obtain the solution of the system (12) plus the equation (44) into the form

$$\eta(x,t) = f(x-t), \quad u(x,z,t) = \eta(x,t) + \frac{\omega_0\sqrt{gh_0}}{g}z + c_0, \quad v(x,z,t) = -z\eta_x(x,t) \tag{46}$$

We will derive using the variational methods in Lagrangian formalism, the equation of the water's free surface $\eta(x,t)$. Taking into account (46), we observe that $\eta(x,t)$ determined completely the velocity components u and v. $\eta(x,t)$ can be regarded as a a vector field on $\mathbf{R}$, that is, it belongs to the Lie algebra of Diff($\mathbf{R}$). The kinetic energy on the surface is

$$\begin{aligned}
\mathcal{K} &= \frac{1}{2}\int_{-\infty}^{\infty}(u^2+v^2)dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\left\{[\eta + \frac{\omega_0\sqrt{gh_0}}{g}(1+\epsilon\eta) + c_0]^2 + (1+\epsilon\eta)^2\eta_x^2\right\}dx \\
&\approx \frac{1}{2}\int_{-\infty}^{\infty}\left[(\eta + \frac{\omega_0\sqrt{gh_0}}{g} + c_0)^2 + \eta_x^2\right]dx
\end{aligned} \tag{47}$$

to the order of our approximation. Transforming $\mathcal{K}$ to a right-invariant Lagrangian, the action on a path $\Gamma(t,\cdot)$, $t \in [0,T]$, in Diff($\mathbf{R}$) is

$$\mathfrak{a}(\Gamma) = \frac{1}{2}\int_0^T \int_{-\infty}^{\infty}\{(\Gamma_t \circ \Gamma^{-1} + \frac{\omega_0\sqrt{gh_0}}{g} + c_0)^2 + [\partial_x(\Gamma_t \circ \Gamma^{-1})]^2\}dxdt \tag{48}$$

The critical points of the action (48) in the space of paths with fixed endpoints, verify (38), for every path $\Phi(t,\cdot)$, $t \in [0,T]$, in Diff($\mathbf{R}$) with endpoints at zero.

Taking into account (48), the condition (38) becomes

$$\int_0^T \int_{-\infty}^{\infty} \left\{ \left(\Gamma_t \circ \Gamma^{-1} + \frac{\omega_0\sqrt{gh_0}}{g} + c_0 \right) \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left[(\Gamma_t + \varepsilon\Phi_t) \circ (\Gamma + \varepsilon\Phi)^{-1} \right] \right.$$
$$\left. + \partial_x(\Gamma_t \circ \Gamma^{-1}) \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \left[\partial_x \left((\Gamma_t + \varepsilon\Phi) \circ (\Gamma + \varepsilon\Phi)^{-1} \right) \right] \right\} dxdt = 0 \quad (49)$$

After calculation, denoting $\Gamma_x \circ \Gamma^{-1} = \eta$, (49) writes as

$$\int_0^T \int_{-\infty}^{\infty} \left\{ (\eta + \frac{\omega_0\sqrt{gh_0}}{g} + c_0) \left[\partial_t(\Phi \circ \Gamma^{-1}) + \eta\partial_x(\Phi \circ \Gamma^{-1}) - (\Phi \circ \Gamma^{-1})\eta_x \right] \right.$$
$$\left. + \eta_x \left[\partial_{tx}(\Phi \circ \Gamma^{-1}) + \eta\partial_x^2(\Phi \circ \Gamma^{-1}) - (\Phi \circ \Gamma^{-1})\eta_{xx} \right] \right\} dxdt = 0 \quad (50)$$

We integrate by parts with respect to t and x in the above formula, we take into account that Φ has endpoints at zero, the smooth functions defined on $\mathbf{R}$ with values in $\mathbf{R}$, together with as many derivatives as necessary, vanish rapidly at $\pm\infty$, and we obtain

$$-\int_0^T \int_{-\infty}^{\infty} (\Phi \circ \Gamma^{-1}) \left[\eta_t + 3\eta\eta_x + 2\left(\frac{\omega_0\sqrt{gh_0}}{g} + c_0 \right) \eta_x - \eta_{txx} \right.$$
$$\left. - 2\eta_x\eta_{xx} - \eta\eta_{xxx} \right] dxdt = 0 \quad (51)$$

Thus, we get:

Theorem 3.3. *For a rotational unidirectional shallow water flow with constant vorticity, the displacement $\eta(x,t)$ of the free surface from the flat state, satisfies the Camassa-Holm equation (1) for $\kappa = \frac{\omega_0\sqrt{gh_0}}{g} + c_0$.*

3.3. **The case of an arbitrary flow.** We consider now an arbitrary underlying flow. From the second equation in (12), and from $p = \eta(x,t)$ on $z = 1$, we get (18). Therefore, using the first equation in (12), we obtain

$$u = -\int \eta_x(x,t)dt + \mathcal{F}(x,z) \quad (52)$$

where $\mathcal{F}$ is an arbitrary function. Differentiating (52) with respect to x and using the third equation in (12), we get, after an integration against z,

$$v = z \int \eta_{xx}(x,t)dt - \mathcal{G}(x,z) + \mathcal{G}(x,0) \quad (53)$$

where $\mathcal{G}_z(x,z) = \mathcal{F}_x(x,z)$ and we have also taken into account the last condition in the system (12). Making $z = 1$ in (53), and taking into account that $v = \eta_t$ on $z = 1$, we get after a differentiation with respect to t, that η has to satisfy the equation (21). We restrict ourselves to waves which propagate in only one direction, thus, we choose η into the form (22). Therefore, for u and v in (52), (53) we get

$$u(x,z,t) = \eta(x,t) + \mathcal{F}(x,z), \quad v(x,z,t) = -z\eta_x(x,t) - \mathcal{G}(x,z) + \mathcal{G}(x,0) \quad (54)$$

with

$$\mathcal{G}_z(x,z) = \mathcal{F}_x(x,z), \quad \mathcal{G}(x,1) = \mathcal{G}(x,0) \quad (55)$$

arbitrary functions.

Using the variational methods in Lagrangian formalism, we will derive now the equation of the water's free surface $\eta(x,t)$, which can be regarded as a a vector field on $\mathbf{R}$, that is, it belongs to the Lie algebra of Diff($\mathbf{R}$). Taking into account (54), and the second condition in (55), the kinetic energy on the surface will be now

$$\begin{aligned}\mathcal{K} = \frac{1}{2}\int_{-\infty}^{\infty}(u^2+v^2)dx &= \frac{1}{2}\int_{-\infty}^{\infty}\left\{[\eta+\mathcal{F}(x,(1+\epsilon\eta))]^2\right.\\ &\quad \left.+\left[-(1+\epsilon\eta)\eta_x-\mathcal{G}(x,(1+\epsilon\eta))+\mathcal{G}(x,0)\right]^2\right\}dx\\ &\approx \frac{1}{2}\int_{-\infty}^{\infty}\left\{[\eta+\mathcal{F}(x,1)]^2+\eta_x^2\right\}dx \qquad (56)\end{aligned}$$

to the order of our approximation. Transforming $\mathcal{K}$ to a right-invariant Lagrangian, the action on a path $\Gamma(t,\cdot)$, $t\in[0,T]$, in Diff($\mathbf{R}$) is

$$\mathfrak{a}(\Gamma)=\frac{1}{2}\int_0^T\int_{-\infty}^{\infty}\left\{\left[\Gamma_t\circ\Gamma^{-1}+\mathcal{F}(x,1)\right]^2+[\partial_x(\Gamma_t\circ\Gamma^{-1})]^2\right\}dxdt \qquad (57)$$

The critical points of the action (57) in the space of paths with fixed endpoints, verify (38), for every path $\Phi(t,\cdot)$, $t\in[0,T]$, in Diff($\mathbf{R}$) with endpoints at zero. Taking into account (57), the condition (38) becomes

$$\begin{aligned}\int_0^T\int_{-\infty}^{\infty} &\left\{\left[\Gamma_t\circ\Gamma^{-1}+\mathcal{F}(x,1)\right]\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[(\Gamma_t+\varepsilon\Phi_t)\circ(\Gamma+\varepsilon\Phi)^{-1}\right]\right.\\ &\left.+\partial_x(\Gamma_t\circ\Gamma^{-1})\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left[\partial_x\left((\Gamma_t+\varepsilon\Phi)\circ(\Gamma+\varepsilon\Phi)^{-1}\right)\right]\right\}dxdt=0 \quad (58)\end{aligned}$$

After calculation, denoting $\Gamma_t\circ\Gamma^{-1}=\eta$, (58) writes as

$$\begin{aligned}\int_0^T\int_{-\infty}^{\infty} &\left\{[\eta+\mathcal{F}(x,1)]\left[\partial_t(\Phi\circ\Gamma^{-1})+\eta\partial_x(\Phi\circ\Gamma^{-1})-(\Phi\circ\Gamma^{-1})\eta_x\right]\right.\\ &\left.+\eta_x\left[\partial_{tx}(\Phi\circ\Gamma^{-1})+\eta\partial_x^2(\Phi\circ\Gamma^{-1})-(\Phi\circ\Gamma^{-1})\eta_{xx}\right]\right\}dxdt=0 \quad (59)\end{aligned}$$

We integrate by parts with respect to t and x in the above formula, we take into account that Φ has endpoints at zero, the smooth functions defined on $\mathbf{R}$ with values in $\mathbf{R}$, together with as many derivatives as necessary, vanish rapidly at $\pm\infty$, and we obtain

$$\begin{aligned}-\int_0^T\int_{-\infty}^{\infty}(\Phi\circ\Gamma^{-1})\,[\eta_t+3\eta\eta_x+2\mathcal{F}(x,1)\eta_x+\mathcal{F}_x(x,1)\eta-\eta_{txx}\\ -2\eta_x\eta_{xx}-\eta\eta_{xxx}]\,dxdt=0 \qquad (60)\end{aligned}$$

In conclusion, we proved:

Theorem 3.4. *For an arbitrary unidirectional shallow water flow, the displacement $\eta(x,t)$ of the free surface from the flat state, satisfies a generalized Camassa-Holm equation*

$$F'(x)U+U_t+[3U+2F(x)]\,U_x-U_{txx}=2U_xU_{xx}+UU_{xxx} \qquad (61)$$

with $x\in\mathbf{R}$, $t\in\mathbf{R}$, $U(x,t)\in\mathbf{R}$, $F(x)\in\mathbf{R}$.

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