# Moderate smoothness of most Alexandrov surfaces 

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#### Abstract

We show that, in the sense of Baire category, most Alexandrov surfaces with curvature bounded below by $\kappa$ have no conical points. We use this result to prove that at most points of such surfaces, the lower and the upper Gaussian curvatures are equal to $\kappa$ and $\infty$ respectively.


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## 1 Introduction

In this paper, an Alexandrov surface will mean a compact 2-dimensional Alexandrov space with curvature bounded below, without boundary. For the precise definition and basic properties, see [7] or [19]. It is known that these surfaces are 2-dimensional topological manifolds. It is also known that, endowed with the Gromov-Hausdorff distance, the set of all Alexandrov surfaces, together with their lower dimensional limits, is a complete metric space (see the next section for details).

By the work of A. D. Alexandrov (for the existence) and A. V. Pogorelov (for the unicity), any Alexandrov surface with curvature bounded below by 0 and homeomorphic to the sphere $S^{2}$ can be realized as a unique (up to rigid motions) convex surface (i.e., boundary of a compact convex set with interior points) in $\mathbb{R}^{3}$. Therefore, the intrinsic geometry of convex surfaces can be seen as a particular case of the geometry of Alexandrov surfaces. Nevertheless, due to Pogorelov's rigidity theorem, the proofs for intrinsic properties of convex surfaces generally involved extrinsic arguments.

We recall here a few intrinsic properties of convex surfaces. The space $\mathcal{S}$ of all convex surfaces, endowed with the usual Pompeiu-Hausdorff metric, is a Baire space. In any Baire space, a property enjoyed by all elements except those in a first category set is said to be typical. We also say that most elements enjoy a such property. For typical properties of convex surfaces, we refer to [10] or [27]. For Baire category results in some variations of the space $\mathcal{S}$, see e.g. 25] or 20].

The study of typical properties of convex surfaces started with a result of V. Klee [11], rediscovered and completed by P. Gruber [9]: most convex
surfaces are of differentiability class $C^{1} \backslash C^{2}$ and strictly convex; see also [24]. In particular, most convex surfaces have no conical points. Theorem 3.1 is a generalisation of this latter fact to Alexandrov surfaces.

The description of most convex surfaces was successively improved by several authors. For example, R. Schneider [17], T. Zamfirescu [21], [22], [26], K. Adiprasito [1], K. Adiprasito and T. Zamfirescu [2], studied lower and upper directional curvatures. Consider a convex surface $S$ of differentiability class $C^{1}$ and a point $x \in S$. The lower and upper curvature at $x$ in direction $\tau$ are defined by $\gamma_{i}^{\tau}(x)=\liminf _{z \rightarrow x} \frac{1}{r_{z}}$ and $\gamma_{s}^{\tau}(x)=\lim \sup _{z \rightarrow x} \frac{1}{r_{z}}$ respectively, where $r_{z}$ is the radius of the circle through $x$ and $z \in S$ whose center belongs to the line normal to $S$ at point $x$, and $\tau$ is the direction tangent to $S$ at point $x$ "toward" $z$. See any of the aforementioned paper for the precise definition. T. Zamfirescu proved that, on most convex surfaces $S$, at each point $x \in S$, $\gamma_{i}^{\tau}(x)=0$ or $\gamma_{s}^{\tau}(x)=\infty$, for any tangent direction $\tau$ at $x$ [22]. Moreover, both equalities hold simultaneously at most point $x \in S$ [21]. Still on most convex surfaces, $\gamma_{s}^{\tau}(x)=0$ almost everywhere, in any tangent direction $\tau$ [22]. See [26] for other results of the same flavour. One can also mention more recent works on the existence of umbilical points of infinite curvature [2], [18].

The notion of directional curvature is essentially extrinsic, and admits no counterpart in the framework of Alexandrov surfaces. Nevertheless, the above results inspired our investigations about Gaussian curvatures, though the technics involved are very different. It should be noticed that the relations between directional and Gaussian curvatures are hitherto not well understood, and it remains unclear whether Corollary 4.5 can be deduced from the above results.
T. Zamfirescu [23] discovered that on most convex surfaces, most points are interior to no geodesic, and his result was very recently extended by K. Adiprasito and himself [3] to Alexandrov surfaces, thus showing that most Alexandrov surfaces are not Riemannian. This seems to be the first found typical property for Alexandrov surfaces.

In this paper we present the space of Alexandrov surfaces (§2), and study the existence of conical points on most Alexandrov surfaces ( $\S 3$ ); this enables us to determine the lower and upper curvatures at most points on most Alexandrov surfaces ( $\S 4$ ).

In another paper [16], we study the properties of geodesics on most Alexandrov surfaces.

In any Alexandrov space $A$, a shortest path between two points is called a segment. The open (resp. closed) ball of radius $r$ centered at $x$ will be denoted by $B^{A}(x, r)$ (resp. $\left.\bar{B}^{A}(x, r)\right)$. When no confusion is possible, the superscript $A$ will be omitted. Given a subset $C$ of $A$, we denote by $\mathcal{H}^{2}(C)$ its 2-dimensional Hausdorff measure, and by $L(C)$ its length (i.e., 1-dimensional Hausdorff measure).

The length of the space of directions at a point $p \in A$ is called the total angle at $p$. The singular curvature of $p$, denoted by $\omega(p)$, is defined as $2 \pi$ minus the total angle at $p$. It is known that $\omega(p) \geq 0$. A point with non-zero singular curvature is said to be conical.

## 2 The space of Alexandrov surfaces

The results presented in this section seem to be known, but not so easy to find in the literature.

If $X$ and $Y$ are compact metric spaces, a correspondence between $X$ and $Y$ is a relation $R$ such that for any $x \in X$ there is at least one $y \in Y$ satisfying $x R y$, and conversely, for any $y \in Y$ there is at least one $x \in X$ such that $x R y$. The distortion $\operatorname{dis}(R)$ of $R$ is defined by

$$
\operatorname{dis}(R)=\sup \left\{\left|d\left(x_{1}, x_{2}\right)-d\left(y_{1}, y_{2}\right)\right| \mid x_{1}, x_{2} \in X, y_{1}, y_{2} \in Y, x_{1} R y_{1}, x_{2} R y_{2}\right\}
$$

One way to define the Gromov-Hausdorff distance $d_{G H}$ is to put

$$
d_{G H}(X, Y)=\frac{1}{2} \inf _{R} \operatorname{dis}(R),
$$

where the infimum is taken over all correspondences between $X$ and $Y$. It is known that $d_{G H}$ is a metric on the set $\mathfrak{M}$ of all compact metric spaces up to isometry, and that $\mathfrak{M}$ is complete with respect to this distance [14]. We will often use the following technical lemma.

Lemma 2.1. 15] Let $\left\{X_{n}\right\}_{n \in \mathbb{N}}$ be a sequence of elements of $\mathfrak{M}$ converging to $X$, and $\left\{\varepsilon_{n}\right\}_{n \in \mathbb{N}}$ a sequence of positive numbers. Then there exist a compact metric space $Z$, an isometric embedding $g: X \rightarrow Z$, and for each positive integer $n$, an isometric embedding $f_{n}: X_{n} \rightarrow Z$, such that $d_{H}^{Z}\left(f_{n}\left(X_{n}\right), g(X)\right)<d_{G H}\left(X_{n}, X\right)+\varepsilon_{n}$.

The set $\mathcal{A}^{d}(\kappa) \subset \mathfrak{M}$ of (isometry classes of) compact Alexandrov spaces of curvature at least $\kappa$ and dimension at most $d$ is known to be closed in $\mathfrak{M}$, and therefore complete [6, 10.8.25]. Hence the set $\mathcal{A}(\kappa)=\mathcal{A}^{2}(\kappa) \backslash \mathcal{A}^{1}(\kappa)$ of all Alexandrov surfaces is open in a Baire space, and consequently is itself a Baire space.
$\mathcal{A}(\kappa)$ obviously contains the set $\mathcal{R}(\kappa)$ of smooth compact Riemannian surfaces of Gaussian curvature at least $\kappa$ everywhere.

Polyhedra are other examples of Alexandrov surfaces. Let $M_{\kappa}$ denote the simply connected 2 -dimensional manifold with constant curvature $\kappa$. A surface $P$ obtained by gluing a finite collection $\left\{T_{i}\right\}$ of geodesic triangles of $M_{\kappa}$ will be called a $\kappa$-polyhedron. Here, gluing means identifying parts of their boundaries of equal length, in such a way that the resulting (topological) space is a manifold. There is a natural notion of length for curves on the gluing which induces an intrinsic metric. A point of a polyhedron which is not the image of some vertex of some triangle $T_{i}$ admits a neighborhood which is isometric to a ball of $M_{\kappa}$. Hence, there are only a finite number of points which do not admit such a neighborhood, called the vertices of $P$. For a more precise definition of polyhedron, see [4]. The set of $\kappa$-polyhedra will be denoted by $\mathcal{P}(\kappa)$.

It's easy to see that a $\kappa$-polyhedron is an Alexandrov surface (with curvature bounded below by any $\kappa^{\prime} \leq \kappa$ ) if and only if, at each vertex $p$, image of the points $p_{1} \in \partial T_{i_{1}}, \ldots, p_{k} \in \partial T_{i_{1}}$, the sum over $j$ of the angles of $\partial T_{i_{j}}$ at $p_{j}$ is at most $2 \pi$.

It is known that Alexandrov surfaces are topological manifolds [7]. Moreover they are two dimensional manifold with bounded integral curvature, as
defined in [4]; this follows from the existence of a curvature measure on Alexandrov surfaces [12]. In [4], it is proved that any two dimensional manifold with bounded integral curvature can be decomposed into a finite union of geodesic triangles whose interiors are disjoint. The following lemma is a particular case of a stronger theorem in [4].

Lemma 2.2. Let $M$ be a two dimensional manifold with bounded integral curvature. Let $\tau_{n}=\left\{T_{n}^{i}\right\}_{i=1}^{q_{n}}$ be a decomposition of $M$ into geodesic triangles with disjoint interiors, such that $\max _{1 \leq i \leq q_{n}} \operatorname{diam}\left(T_{n}^{i}\right) \underset{n \rightarrow \infty}{\longrightarrow} 0$. Denote by $P_{n}$ the 0-polyhedron obtained by replacing each triangle $T_{n}^{i}$ by a Euclidean one with the same side lengths. Then $P_{n}$ converges to $M$ with respect to the Gromov-Hausdorff convergence.

Lemma 2.3. $\mathcal{P}(\kappa) \cap \mathcal{A}(\kappa)$ is dense in $\mathcal{A}(\kappa)$.
Proof. Let $S \in \mathcal{A}(\kappa)$. Let $\left\{P_{n}^{0}\right\}$ be a sequence of 0-polyhedra constructed by gluing Euclidean triangles corresponding to finer and finer triangulations of $S$, and let $P_{n}^{\kappa}$ be the $\kappa$-polyhedron obtained from $P_{n}^{0}$ by replacing each Euclidean triangle by a triangle of $M_{\kappa}$. Since $S \in \mathcal{A}(\kappa)$, each angle of a face of $P_{n}^{\kappa}$ is not larger than the corresponding angle in $S$. It follows that the sum of angles glued at each vertex of $P_{n}^{\kappa}$ is less than or equal to the corresponding sum in $S$, which in turn is less than or equal to $2 \pi$. Hence $P_{n}^{\kappa} \in \mathcal{A}(\kappa)$. By Lemma 2.2, $P_{n}^{0}$ converges to $S$, so it is sufficient to prove that $d_{G H}\left(P_{n}^{0}, P_{n}^{\kappa}\right)$ tends to 0 . Let $\Theta^{\kappa}(A, B, C, s, t)$ be the distance on $M_{\kappa}$ between points $p$ and $q$, where $a, b, c, p, q \in M_{\kappa}$ are such that (see Figure (1)

$$
\begin{array}{ll}
d(a, b)=B, & d(a, p)=s B, \\
d(a, c)=C, & d(p, b)=(1-s) B, \\
d(b, c)=A . &
\end{array}
$$

It is well-known, that

$$
\lim _{\delta \rightarrow 0} \frac{\Theta^{\kappa}(\delta A, \delta B, \delta C, s, t)}{\Theta^{0}(\delta A, \delta B, \delta C, s, t)}=1
$$

and the convergence is uniform with respect to the variables $A, B, C, s, t$.


Figure 1: Definition of $\Theta^{\kappa}$.

Let $R_{n}$ be the correspondence between $P_{n}^{0}$ and $P_{n}^{\kappa}$ defined as follows: $x R_{n} y$ if and only if $x$ and $y$ belong to corresponding triangles.

Let $\delta_{n}$ be the maximum of the diameters of the faces of $P_{n}^{0}$. Choose $\varepsilon>0$. For $n$ large enough,

$$
\frac{\Theta^{0}\left(\delta_{n} A, \delta_{n} B, \delta_{n} C, s, t\right)}{\Theta^{\kappa}\left(\delta_{n} A, \delta_{n} B, \delta_{n} C, s, t\right)}>1-\varepsilon
$$

Take $x, y \in P_{n}^{0}, x^{\prime}, y^{\prime} \in P_{n}^{\kappa}$ such that $x R x^{\prime}$ and $y R y^{\prime}$. Let $\sigma_{n}$ be a segment on $P_{n}^{0}$ between $x$ and $y$; this segment crosses the edges (i.e., sides of triangles) of the decomposition at the points $x_{n}^{1}, \ldots, x_{n}^{k}$. These points have natural counterparts $x_{n}^{\prime 1}, \ldots, x_{n}^{\prime k}$ in $P_{n}^{\kappa}$ (on the same edge, at the same distances from its endpoints). Let $\sigma_{n}^{\prime} \subset P_{n}^{\kappa}$ be a simple path composed of the segments from $x_{n}^{\prime i}$ to $x_{n}^{\prime i+1}(i=1, \ldots, k-1)$. For $n$ large enough, we have

$$
\begin{aligned}
d_{P_{n}^{0}}(x, y)=L(\sigma) & \geq(1-\varepsilon) L\left(\sigma^{\prime}\right)-2 \delta_{n} \geq(1-\varepsilon) d_{P_{n}^{\kappa}}\left(x_{n}^{\prime 1}, x_{n}^{\prime k}\right)-2 \delta_{n} \\
& \geq(1-\varepsilon) d_{P_{n}^{\kappa}}\left(x^{\prime}, y^{\prime}\right)-4 \delta_{n} \geq(1-2 \varepsilon) d_{P_{n}^{\kappa}}\left(x^{\prime}, y^{\prime}\right) .
\end{aligned}
$$

An opposite inequality can be obtained in exactly the same way. Hence, dis $\left(R_{n}\right)$ tends to 0 .
Lemma 2.4. $\mathcal{R}(\kappa)$ is dense in $\mathcal{A}(\kappa)$.
Proof. By Lemma 2.3, it is sufficient to approximate a ball $B=B(o, R)$ of a $\kappa$-polyhedron, centered at a vertex $o$ of positive curvature $\omega$, by a Riemannian ball which has constant curvature $\kappa$ near its boundary. By applying a homothety, we can assume that $\kappa=0$ or $\pm 1$. We treat only the case $\kappa=1$, the reader will easily adapt the proof for cases $\kappa=0,-1$.

Note that $B \backslash\{o\}$ is isometric to $] 0, R[\times \mathbb{R} / 2 \pi \mathbb{Z}$ equipped with the metric

$$
\begin{equation*}
d s^{2}=d r^{2}+\left(1-\frac{\omega}{2 \pi}\right)^{2} \sin (r)^{2} d \theta^{2} \tag{1}
\end{equation*}
$$

For any $\lambda, \varepsilon>0$, we define $k_{\lambda, \varepsilon}: \mathbb{R} \rightarrow \mathbb{R}$ by

$$
k_{\lambda, \varepsilon}(t)= \begin{cases}1 & \text { if } t \notin[\varepsilon, 2 \varepsilon], \\ \frac{\lambda^{2}}{\varepsilon^{2}} & \text { if } t \in[\varepsilon, 2 \varepsilon] .\end{cases}
$$

There is a unique $C^{1}$ function $f_{\lambda, \varepsilon}: \mathbb{R} \rightarrow \mathbb{R}$ such that $f(0)=0, f^{\prime}(0)=1$, and, for any $t \neq \varepsilon, 2 \varepsilon$,

$$
\begin{equation*}
f_{\lambda, \varepsilon}^{\prime \prime}(t)+k_{\lambda, \varepsilon}(t) f_{\lambda, \varepsilon}=0 \tag{2}
\end{equation*}
$$

Explicitly, for $t>2 \varepsilon$,

$$
f_{\lambda, \varepsilon}(t)=A_{\lambda, \varepsilon} \sin t+B_{\lambda, \varepsilon} \cos t,
$$

where

$$
\begin{aligned}
& A_{\lambda, \varepsilon}=\frac{1}{2 \varepsilon \lambda}\binom{3\left(\varepsilon^{2}-\lambda^{2}\right) \sin \varepsilon \sin \lambda \cos ^{2} \varepsilon+2 \varepsilon \lambda \cos \lambda \cos \varepsilon}{+\sin \varepsilon\left(\varepsilon^{2}+\lambda^{2}+\left(\lambda^{2}-\varepsilon^{2}\right) \sin ^{2} \varepsilon\right) \sin \lambda} \\
& B_{\lambda, \varepsilon}=\frac{1}{\varepsilon \lambda}\left(\cos \varepsilon\left(\lambda^{2}+\left(\varepsilon^{2}-\lambda^{2}\right) \cos 2 \varepsilon\right) \sin \lambda-\varepsilon \lambda \cos \lambda \sin \varepsilon\right)
\end{aligned}
$$

Note that

$$
\begin{aligned}
& \lim _{\varepsilon \rightarrow 0} A_{\lambda, \varepsilon}=\cos \lambda-\lambda \sin \lambda, \\
& \lim _{\varepsilon \rightarrow 0} B_{\lambda, \varepsilon}=0 .
\end{aligned}
$$

If we replace in (2) $k_{\lambda, \varepsilon}$ by a smooth function $k_{\lambda, \varepsilon, \eta}$ which equals $k_{\lambda, \varepsilon}$ outside $[\varepsilon-\eta, \varepsilon+\eta] \cup[2 \varepsilon-\eta, 2 \varepsilon+\eta]$, then the corresponding solution of (22) tends to $f_{\lambda, \varepsilon}$ when $\eta$ tends to 0 . It follows that, for arbitrarily small $\tau>0$, one can find a smooth function $f$ such that $f=\sin$ on a small enough neighborhood of $0,-f^{\prime \prime} / f \geq 1$, and for $t>\tau, f(t)=\left(1-\frac{\omega}{2 \pi}\right) \sin (t+\phi)$, with $|\phi|<\tau$. Now, the metric $d r^{2}+f(r)^{2} d \theta^{2}$ is a suitable approximation of (11).

## 3 Conical points

The goal of this section is to prove the following result.
Theorem 3.1. Most Alexandrov surfaces in $\mathcal{A}(\kappa)$ have no conical points.
Before proving this theorem, we need a lemma.
Lemma 3.2. Let $Z$ be a compact metric space, and $A_{n}$ a sequence of Alexandrov surfaces, isometrically embedded in $Z$, converging to $A \subset Z$ for the Hausdorff distance of $Z$. Let $p_{n}$ be a point on $A_{n}$ and assume that $p_{n}$ converges to $p \in A$.
i) If $\sigma_{n}, \gamma_{n}$ are segments of $A_{n}$ emanating from $p_{n}$ and converging to segments $\sigma, \gamma \subset A$, then $\liminf \measuredangle\left(\sigma_{n}, \gamma_{n}\right) \geq \measuredangle(\sigma, \gamma)$.
ii) $\omega(p) \geq \lim \sup \omega\left(p_{n}\right)$.
iii) If $\omega(p)=0$ then $\lim \measuredangle\left(\sigma_{n}, \gamma_{n}\right)=\measuredangle(\sigma, \gamma)$.

Proof. Following [7, we denote by $\tilde{\measuredangle} a b c$ the angle at point $\tilde{b}$ of a geodesic triangle $\tilde{a} \tilde{b} \tilde{c} \subset M_{\kappa}$ such that $d(\tilde{a}, \tilde{b})=d(a, b), d(\tilde{b}, \tilde{c})=d(b, c)$ and $d(\tilde{a}, \tilde{c})=$ $d(a, c)$. We recall that the angle $\measuredangle(\sigma, \gamma)$ between two segments $\sigma, \gamma$ emanating for $p$ is by definition the limit of $\angle \sigma(t) p \gamma(s)$ when $s$ and $t$ both tend to 0 .

Assume that the result fails, that is, there exists $\varepsilon>0$ such that for arbitrarily large $n$, we have

$$
\measuredangle(\sigma, \gamma)>\measuredangle\left(\sigma_{n}, \gamma_{n}\right)+\varepsilon .
$$

Let $\tau>0$ be small enough to ensure that

$$
\measuredangle(\sigma, \gamma) \leq \tilde{\measuredangle} \sigma(\tau) p \gamma(\tau)+\frac{\varepsilon}{3} .
$$

Let $q_{n}, r_{n} \in A_{n}$ be such that $\max \left(d\left(q_{n}, \sigma(\tau)\right), d\left(r_{n}, \gamma(\tau)\right)\right) \leq d_{H}^{Z}\left(A, A_{n}\right)$. For $n$ large enough, we have

$$
\left|\tilde{\measuredangle} \sigma(\tau) p \gamma(\tau)-\tilde{\measuredangle} q_{n} p q_{n}\right|<\frac{\varepsilon}{3} .
$$

Now,

$$
\begin{aligned}
\measuredangle\left(\sigma_{n}, \gamma_{n}\right) & \geq \tilde{\measuredangle} q_{n} p r_{n} \geq \tilde{\measuredangle} \sigma(\tau) p \gamma(\tau)-\frac{\varepsilon}{3} \\
& \geq \measuredangle(\sigma, \gamma)-\frac{2 \varepsilon}{3} \geq \measuredangle\left(\sigma_{n}, \gamma_{n}\right)+\frac{\varepsilon}{3},
\end{aligned}
$$

and a contradiction is obtained, thus proving the first statement.
Consider a segment $\zeta$ emanating from $p$, and such that $2 \pi-\omega(p)=$ $\measuredangle(\sigma, \gamma)+\measuredangle(\gamma, \zeta)+\measuredangle(\zeta, \sigma)$. Choose $s, g, z$, some interior points of $\sigma, \gamma$, and $\zeta$ respectively. Take $s_{n}, g_{n}, z_{n} \in A_{n}$ converging to $s, g, z$ respectively. Let $\sigma_{n}$ (resp. $\gamma_{n}, \zeta_{n}$ ) be a segment between $p_{n}$ and $s_{n}$ (resp. $g_{n}, z_{n}$ ). Then $\sigma_{n}$ (resp. $\gamma_{n}, \zeta_{n}$ ) converges to the part of $\sigma$ (resp. $\gamma, \zeta$ ) between $p$ and $s$ (resp. $g, z$ ), for there is only one segment between these points. By (i),

$$
\begin{aligned}
\limsup \omega\left(p_{n}\right) & =2 \pi-\liminf \left(\measuredangle\left(\sigma_{n}, \gamma_{n}\right)+\measuredangle\left(\gamma_{n}, \zeta_{n}\right)+\measuredangle\left(\zeta_{n}, \gamma_{n}\right)\right) \\
& \leq 2 \pi-(\measuredangle(\sigma, \gamma)+\measuredangle(\gamma, \zeta)+\measuredangle(\zeta, \sigma))=\omega(p) .
\end{aligned}
$$

This proves (ii).
Now, if $\omega(p)=0$, the above inequality must be an equality, implying (iii).

Proof of Theorem [3.1. Let $\mathcal{M}(a), a>0$, be the set of Alexandrov surfaces with curvature bounded below by $\kappa$ which admits a point of singular curvature greater than or equal to $a$.

We claim that $\mathcal{M}(a)$ is closed. Take a sequence $\left\{A_{n}\right\}$ of surfaces of $\mathcal{M}(a)$ converging to $A$ in $\mathcal{A}(\kappa)$. By Lemma [2.1, we can assume that each $A_{n}$ and $A$ are all isometrically embedded in some compact metric space $Z$, and that $A_{n}$ converges to $A$ with respect to the Hausdorff distance of $Z$. Let $p_{n} \in A_{n}$ be a point of curvature at least $a$. Let $p \in A$ be a limit point of the sequence $\left\{p_{n}\right\}$; by Lemma 3.2, $\omega(p) \geq a$ and $A \in \mathcal{M}(a)$.

Moreover, due to the density of $\mathcal{R}(\kappa)$ in $\mathcal{A}(\kappa), \mathcal{M}(a)$ has empty interior. Hence

$$
\bigcup_{n \in \mathbb{N}} \mathcal{M}\left(\frac{1}{n+1}\right)=\{A \in \mathcal{A}(\kappa) \mid \exists p \in A \omega(p)>0\}
$$

is meager, and the conclusion follows.

## 4 Lower and upper curvatures

For any Alexandrov space $A$ with curvature bounded below, and any geodesic triangle $\Delta$ in $A$, denote by $\sigma_{0}(\Delta)$ the area of the Euclidean triangle with sides of the same length as $\Delta$, and by $e_{0}(\Delta)$ the excess of $\Delta$ (the sum of its angles minus $\pi$ ). Let us denote by $E(x, \delta, a)$ the set of all triangles of diameter less than $\delta$, such that $x$ is interior to $\Delta$, and such that each angle of $\Delta$ is at least $a$.

The lower and the upper curvature at $x, \underline{K}(x)$ and $\bar{K}(x)$, were defined by A. D. Alexandrov (see the survey [5]) by

$$
\underline{K}(x)=\lim _{\delta \rightarrow 0} \inf _{\Delta \in E(x, \delta, 0)} \frac{e_{0}(\Delta)}{\sigma_{0}(\Delta)}, \quad \bar{K}(x)=\lim _{\delta \rightarrow 0} \sup _{\Delta \in E(x, \delta, 0)} \frac{e_{0}(\Delta)}{\sigma_{0}(\Delta)}
$$

Y. Machigashira defined the lower and the upper curvature at $x, \underline{G}(x)$ and $\bar{G}(x)$, with slightly more sophisticated formulas [12]:

$$
\underline{G}(x)=\lim _{a \rightarrow 0} \underline{G}_{a}(x), \quad \bar{G}(x)=\lim _{a \rightarrow 0} \bar{G}_{a}(x),
$$

where

$$
\underline{G}_{a}(x)=\lim _{\delta \rightarrow 0} \inf _{\Delta \in E(x, \delta, a)} \frac{e_{0}(\Delta)}{\mathcal{H}^{2}(\Delta)}, \quad \bar{G}_{a}(x)=\lim _{\delta \rightarrow 0} \sup _{\Delta \in E(x, \delta, a)} \frac{e_{0}(\Delta)}{\mathcal{H}^{2}(\Delta)} .
$$

This new definition allowed him to prove that, for any Alexandrov surface, $\underline{G}(x)=\bar{G}(x)$ almost everywhere; therefore a variant of the Gauss-Bonnet theorem holds.

Let $\mathcal{A}^{+}(0)$ be the space of Alexandrov surfaces with curvature bounded below by 0 which have a positive Euler characteristic. The aim of this section is to prove the following result.

Theorem 4.1. For most surfaces $A \in \mathcal{A}(\kappa)$, at most points $x \in A, \underline{K}(x)=$ $\underline{G}(x)=\kappa$.

If $\kappa \neq 0$, then for most surfaces $A \in \mathcal{A}(\kappa)$, at most points $x \in A, \bar{K}(x)=$ $\bar{G}(x)=\infty$. Otherwise, for most surfaces $A \in \mathcal{A}^{+}(0)$, at most points $x \in A$, $\bar{K}(x)=\bar{G}(x)=\infty$.

Remark 1. The restriction of the second statement is necessary if $\kappa=0$. Indeed, one can easily prove that a topological torus (or a Klein bottle) in $\mathcal{A}(0)$ is necessarily flat. Since by Pelerman's stability theorem (see for instance [6, 10.10.5]) the tori form an open set in $\mathcal{A}(0)$, it is not possible to expect any kind of roughness for most surfaces in $\mathcal{A}(0)$. This fact was already pointed out by K. Adiprasito and T. Zamfirescu [3].

We begin the proof of Theorem 4.1 with the following lemma.
Lemma 4.2. Let $A \in \mathcal{A}(\kappa), x \in A$ be a non-conical point, and a be a (small) fixed positive number. Let $\left\{\Delta_{n}\right\}$ be a sequence of geodesic triangles of $E(x, 1, a)$ converging to $x$. Then $\frac{\sigma_{0}\left(\Delta_{n}\right)}{\mathcal{H}^{2}\left(\Delta_{n}\right)}$ converges to 1 .

Proof. This follows from the fact that $\Delta_{n}$ is $\varepsilon_{n}$-isometric (see [7) to a Euclidean triangle for a sequence $\left\{\varepsilon_{n}\right\}$ tending to 0 . For details, see [12, Lemma 2.4].

As a consequence of Lemma 4.2, one can substitute $\sigma_{0}$ to $\mathcal{H}^{2}$ in the definition of $\underline{G}_{a}(x)$ and $\bar{G}_{a}$ (both definitions give $\infty$ if $x$ is conical). It follows that for all $x \in A$ and any fixed (small) number $a>0$, we have

$$
\underline{K}(x) \leq \underline{G}(x) \leq \underline{G}_{a}(x) \leq \bar{G}_{a}(x) \leq \bar{G}(x) \leq \bar{K}(x) .
$$

A small geodesic triangle $\Delta$ has a well defined interior, and a well defined exterior. Among all triangles sharing the same vertices, one has the smallest (with respect to inclusion) interior; we denote it by $\underline{\Delta}$. Similarly, the triangle with the same vertices as $\Delta$ and the largest interior will be denoted by $\bar{\Delta}$. Let $F(x, \delta, a)$ be the set of those triangles $\Delta$ such that $x \in \operatorname{int} \underline{\Delta}, \operatorname{diam} \Delta<\delta$, and each angle of $\underline{\Delta}$ is greater than $a$. We set

$$
\begin{array}{ll}
\underline{H}_{a}(x)=\lim _{\delta \rightarrow 0} \underline{H}_{a, \delta}(x), & \underline{H}_{a, \delta}(x)=\inf _{\Delta \in F(x, \delta, a)} \frac{e_{0}(\bar{\Delta})}{\sigma_{0}(\Delta)}, \\
\bar{H}_{a}(x)=\lim _{\delta \rightarrow 0} \bar{H}_{a, \delta}(x), & \bar{H}_{a, \delta}(x)=\sup _{\Delta \in F(x, \delta, a)} \frac{e_{0}(\underline{\Delta})}{\sigma_{0}(\Delta)} .
\end{array}
$$

Since $e_{0}(\bar{\Delta}) \geq e_{0}(\Delta) \geq e_{0}(\underline{\Delta})$ and $F(x, \delta, a) \subset E(x, \delta, a)$, it is clear that $\underline{G}_{a}(x) \leq \underline{H}_{a}(x)$ and $\bar{H}_{a}(x) \leq \bar{G}_{a}(x)$ for all points $x$ in $A$. Moreover, on surfaces without conical points, $\underline{H}_{a, \delta}$ and $\bar{H}_{a, \delta}$ are respectively upper and lower semi-continuous in the following strong sense.

Lemma 4.3. Let $Z$ be a compact metric space. Let $\left\{A_{n}\right\}$ be a sequence of Alexandrov surfaces with curvature bounded below by $\kappa$, embedded in $Z$ and converging to $A \in \mathcal{A}(\kappa)$ with respect to the Hausdorff distance in $Z$. Assume that $A$ has no conical points. Let $x_{n} \in A_{n}$ converges to $x \in A$. Then

$$
\limsup _{n \rightarrow \infty} \underline{H}_{a, \delta}\left(x_{n}\right) \leq \underline{H}_{a, \delta}(x), \quad \liminf _{n \rightarrow \infty} \bar{H}_{a, \delta}\left(x_{n}\right) \geq \bar{H}_{a, \delta}(x) .
$$

Proof. By Lemma 3.2, in the case of surfaces without conical points, the angle between two segments of $A_{n}$ tends to the angle between the limit segments.

Fix $\varepsilon>0$; there exists a triangle $\Delta$ of $F(x, \delta, a)$ such that

$$
\begin{equation*}
\frac{e_{0}(\bar{\Delta})}{\sigma_{0}(\Delta)} \leq \underline{H}_{a, \delta}(x)+\frac{\varepsilon}{2} . \tag{3}
\end{equation*}
$$

Let $u, v, w$ be the vertices of $\Delta$. Obviously, one can choose $\Delta$ such that $\Delta=\underline{\Delta}$. Let $u_{n}, v_{n}, w_{n}$ be points of $A_{n}$ converging to $u, v$, and $w$ respectively. Let $\overline{\Delta_{n}}=\overline{\Delta_{n}}$ be the fattest geodesic triangle with vertices $u_{n}, v_{n}, w_{n}$. Let $\Delta^{\prime}$ (resp. $\Delta^{\prime \prime}$ ) be a limit triangle of the sequence $\left\{\underline{\Delta_{n}}\right\}$ (resp. $\Delta_{n}$ ).

Since the diameter is continuous with respect to the Hausdorff distance, $\operatorname{diam} \Delta_{n}<\delta$ for $n$ large enough. Due to the choice of $\Delta$, the angles of $\Delta^{\prime}$ are greater than the corresponding angles of $\Delta$, hence, by continuity of angles, for $n$ large enough, the angles of $\underline{\Delta}_{n}$ are also greater than $a$. The point $x$ belongs to int $\Delta \subset \operatorname{int} \Delta^{\prime}$, hence, for $n$ large enough, $x_{n} \in \underline{\Delta_{n}}$. It follows that $\Delta_{n} \in F\left(x_{n}, \delta, a\right)$, and therefore

$$
\begin{equation*}
\underline{H}_{a, \delta}\left(x_{n}\right) \leq \frac{e_{0}\left(\Delta_{n}\right)}{\sigma_{0}\left(\Delta_{n}\right)} . \tag{4}
\end{equation*}
$$

For $n$ large enough, the continuity of angles implies

$$
\begin{equation*}
\frac{e_{0}\left(\Delta_{n}\right)}{\sigma_{0}\left(\Delta_{n}\right)} \leq \frac{e_{0}\left(\Delta^{\prime \prime}\right)}{\sigma_{0}(\Delta)}+\frac{\varepsilon}{2} \leq \frac{e_{0}(\bar{\Delta})}{\sigma_{0}(\Delta)}+\frac{\varepsilon}{2} . \tag{5}
\end{equation*}
$$

Gathering (4), (5) and (3), we get for $n$ large enough

$$
\underline{H}_{a, \delta}\left(x_{n}\right) \leq \underline{H}_{a, \delta}(x)+\varepsilon .
$$

This holds for any $\varepsilon>0$, whence the conclusion concerning $\underline{H}_{a, \delta}$.
The case of $\bar{H}_{a, \delta}$ is similar.
Lemma 4.4. Put $\mathcal{B}=\mathcal{A}(\kappa)$ if $\kappa \neq 0$, and $\mathcal{B}=\mathcal{A}^{+}(0)$ otherwise. For any $A \in \mathcal{B}$ and any $\varepsilon>0$, there exists a $\kappa$-polyhedron $P \in \mathcal{B}$ such that $d_{G H}(A, P) \leq \varepsilon$ and the conical points of $P$ form an $\varepsilon$-net in $P$.

Proof. Assume first that $\kappa>0$. Applying to $A \in \mathcal{A}(\kappa)$ a global homothety (of scaling factor slightly less than 1) yields a surface $A^{\prime} \in A\left(\kappa^{\prime}\right)$ with $\kappa^{\prime}>\kappa$, arbitrarily close to $A$. By angle comparison, the vertices of any $\kappa$-polyhedral approximation $P$ of $A^{\prime}$ (see Lemma 2.3) are conical points.

The case $\kappa^{\prime}<0$ is similar (with a scaling factor more than 1 ).
Now consider $A \in \mathcal{A}^{+}(0)$. Suppose first that $A$ is homeomorphic to the sphere. A polyhedral approximation $P$ of $A$ is also homeomorphic to the sphere; by Alexandrov's existence Theorem, it can be realized as a convex polyhedron in $\mathbb{R}^{3}$. If the vertices of $P$ are too far from each other, one can add new ones in the following way. Choose a point $p$ near $P$, outsides of $P$, close to the point you want to "make" conical, and take the convex hull of $P \cup\{p\}$. If $A$ is homeomorphic to the projective plane, one can do the same construction with the universal covering of $P$, by adding pair of points $p, p^{\prime}$ symmetric to each other.

Now we are in a position to prove Theorem 4.1.
Proof of Theorem 4.1. Denote by $\mathcal{A}^{0}(\kappa)$ the set of Alexandrov surfaces with curvature bounded below by $\kappa$ without conical points. Observe that $\mathcal{A}^{0}(\kappa)$ is residual in $\mathcal{A}(\kappa)$ by Theorem 3.1, and so is itself a Baire space. For $A \in \mathcal{A}^{0}(\kappa)$, we define

$$
C_{a, \delta, \varepsilon}^{A}=\left\{x \in A \mid \underline{H}_{a}, \delta(x) \geq \kappa+\varepsilon\right\}
$$

and

$$
\begin{aligned}
\mathcal{M} & =\left\{A \in \mathcal{A}^{0}(\kappa) \mid\left\{x \in A \mid \underline{H}_{a}(x)>\kappa\right\} \text { is not meager }\right\} \\
& =\bigcup_{p \in \mathbb{N}^{*}}\left\{A \in \mathcal{A}^{0}(\kappa) \left\lvert\,\left\{x \in A \left\lvert\, \underline{H}_{a, \frac{1}{p}}(x)>\kappa\right.\right\}\right. \text { is not meager }\right\} \\
& =\bigcup_{p \in \mathbb{N}^{*}} \bigcup_{q \in \mathbb{N}^{*}}\left\{A \in \mathcal{A}^{0}(\kappa) \left\lvert\, C_{a, \frac{1}{p}, \frac{1}{q}}^{A}\right. \text { is not meager }\right\} .
\end{aligned}
$$

Applying Lemma 4.3 (with $A_{n}=A$ ), we get that the sets $C_{a, \delta, \varepsilon}^{A}$ are closed, and consequently, they are meager if and only if they have empty interior. It follows that

$$
\mathcal{M}=\bigcup_{p \in \mathbb{N}^{*}} \bigcup_{q \in \mathbb{N}^{*}} \bigcup_{r \in \mathbb{N}^{*}} \mathcal{M}_{p, q, r}
$$

where

$$
\mathcal{M}_{p, q, r} \stackrel{\text { def }}{=}\left\{A \in \mathcal{A}^{0}(\kappa) \mid \exists x \in A \text { s.t. } \bar{B}^{A}\left(x, \frac{1}{r}\right) \subset C_{a, \frac{1}{p}, \frac{1}{q}}^{A}\right\} .
$$

We claim that $\mathcal{M}_{p, q, r}$ is closed in $A^{0}(\kappa)$. Let $\left\{A_{n}\right\}$ be a sequence of surfaces of $\mathcal{M}_{p, q, r}$, converging to $A \in \mathcal{A}^{0}(\kappa)$. For each $n$, there exists $x_{n} \in A_{n}$ such that $B_{n} \stackrel{\text { def }}{=} \bar{B}^{A_{n}}\left(x, \frac{1}{r}\right) \subset C_{a, 1 / p, 1 / q}^{A_{n}}$. By selecting a subsequence, we may assume that $B_{n}$ converges to a ball $B=\bar{B}^{A}(x, 1 / r)$. Now, a point $x \in B$ is limit of points of $B_{n}$ and, by Lemma 4.3, $\underline{H}_{a, 1 / p}(x) \geq \kappa+1 / q$, whence $B \subset C_{a, 1 / p, 1 / q}^{A}$ and $A \in \mathcal{M}_{p, q, r}$. This proves the claim.

Arbitrarily close to any surface of $\mathcal{M}_{p, q, r}$ is a $\kappa$-polyhedron, and arbitrarily close to this polyhedron, is a Riemannian surface whose Gaussian curvature is $\kappa$ outside a finite number of arbitrarily small closed balls (see Lemma 2.4). Consequently $\mathcal{M}_{p, q, r}$ has no interior points. Hence $\mathcal{M}$ is meager in $\mathcal{A}^{0}(\kappa)$, and so in $\mathcal{A}(\kappa)$.

Put $\mathcal{B}^{0}=\mathcal{B} \cap \mathcal{A}^{0}$, where $\mathcal{B}$ is the set defined in Lemma 4.4. For $A \in \mathcal{B}^{0}$, we define

$$
D_{a, \delta, q}^{A}=\left\{x \in A \mid \bar{H}_{a}, \delta(x) \leq q\right\}
$$

and

$$
\begin{aligned}
\mathcal{N} & =\left\{A \in \mathcal{B}^{0} \mid\left\{x \in A \mid \bar{H}_{a}(x)<\infty\right\} \text { is not meager }\right\} \\
& =\bigcup_{p \in \mathbb{N}^{*}}\left\{A \in \mathcal{B}^{0} \left\lvert\,\left\{x \in A \left\lvert\, \bar{H}_{a, \frac{1}{p}}(x)<\infty\right.\right\}\right. \text { is not meager }\right\} \\
& =\bigcup_{p \in \mathbb{N}^{*}} \bigcup_{q \in \mathbb{N}^{*}}\left\{A \in \mathcal{B}^{0} \left\lvert\, D_{a, \frac{1}{p}, q}^{A}\right. \text { is not meager }\right\} \\
& =\bigcup_{p \in \mathbb{N}^{*}} \bigcup_{q \in \mathbb{N}^{*}} \bigcup_{r \in \mathbb{N}^{*}} \mathcal{N}_{p, q, r},
\end{aligned}
$$

where

$$
\mathcal{N}_{p, q, r} \stackrel{\text { def }}{=}\left\{A \in \mathcal{B}^{0} \left\lvert\, \exists x \in A \bar{B}^{A}\left(x, \frac{1}{r}\right) \subset D_{a, \frac{1}{p}, q}^{A}\right.\right\}
$$

One proves that the sets $\mathcal{N}_{p, q, r}$ are closed in the same way as for $\mathcal{M}_{p, q, r}$. By Lemma 4.4, for any $\varepsilon>0$, arbitrarily closed to any surface of $\mathcal{M}_{p, q, r}$ is a $\kappa$-polyhedron $P$, whose vertices form an $\varepsilon$-net. Arbitrarily close to this
polyhedron is a smooth surface with a region of Gaussian curvature more than $q$ close to each vertex of $P$ (see the proof of Lemma [2.4). Hence $\mathcal{N}_{p, q, r}$ has empty interior in $\mathcal{B}^{0}, \mathcal{N}$ is meager in $\mathcal{B}^{0}$, and consequently in $\mathcal{B}$.

We close the paper with a new result in convex geometry, whose proof is a simplified version of the proof of Theorem 4.1. Notice that the GromovHausdorff topology differs from the Pompeiu-Hausdorff one (usually considered in the framework of convexity), so Corollary 4.5 cannot be directly obtained from Theorem 4.1.

Corollary 4.5. On most convex surfaces $S \in \mathcal{S}$, at most point $x \in S, \underline{K}(x)=$ $\underline{G}(x)=0$ and $\bar{K}(x)=\bar{G}(x)=\infty$.

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