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Criteria for farthest points on convex surfaces

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We provide a sharp, sufficient condition to decide if a point y on a convex surface S is a farthest point (i.e., is at maximal intrinsic distance from some point) on S, involving a lower bound π on the total curvature ω_y at y, $\omega_y \geq \pi$. Further consequences are obtained when $\omega_y > \pi$, and sufficient conditions are derived to guarantee that a convex cap contains at least one farthest point. A connection between simple closed quasigeodesics O of S, points $y \in S \setminus O$ with $\omega_y \geq \pi$, and the set $\mathbb{F}$ of all farthest points on S, is also investigated.

1 Introduction

A *convex surface* S is the boundary of a *convex body* (compact convex set with interior points) in Euclidean space $\mathbb{R}^3$, or a doubly covered planar convex body.

The metric ρ of the convex surface S is defined, for any points x, y in S, as the length $\rho(x, y)$ of a *segment* (i.e., shortest path on S) from x to y. For any point x in S, let ρ_x denote the *distance function from* x given by $\rho_x(y) = \rho(x, y)$, F_x the set of all *farthest points from* x (i.e., global maxima of ρ_x), and F the induced multivalued mapping. For simplicity, we shall denote by $\mathbb{F}$ the set of all farthest points on S, $\mathbb{F} = \bigcup_{x \in S} F_x$, and often make no distinction between a set $F_x = \{y\}$ and the point y.

Chapter A35 of the book [4] of H. Croft, K. Falconer and R. Guy details several questions of H. Steinhaus, who asked for characterizations of the sets of farthest points.

Despite the simplicity of the notion, few examples of completely determined sets of farthest points seem to be known (see [8], [9], [15], [17], [20]), a possible reason being the difficulty to determine by direct computation these sets on general surfaces. This “quasi-poverty” constitutes a first motivation for presenting criteria to recognize farthest points, or caps to which they belong, on convex surfaces.

Denote by $\mathcal{S}$ the space of all convex surfaces, endowed with the usual Pompeiu–Hausdorff metric. Another motivation for our work comes from the conjecture of T. Zamfirescu [23], that the set $\mathcal{S}_2$, of all surfaces $S \in \mathcal{S}$ on which there exists a point x with disconnected set of relative maxima, is dense in $\mathcal{S}$. In view of Theorems 6 and 7 in [21], for proving this conjecture it would be sufficient to show that densely many surfaces have the mapping F properly multivalued. Our Corollaries 2.2 and 3.2 give two sufficient conditions to decide this.

The survey [18] presents general properties of farthest points on convex surfaces.

The space T_y of all unit tangent directions at $y \in S$ is a topological circle in the unit sphere S^2. The *complete angle at* y, denoted by θ_y, is the length of T_y, while the *total curvature* ω_y *at* y is defined by $\omega_y = 2\pi - \theta_y$.

An interesting criterion was obtained by J. Rouyer [16]: if y is a point in a polyhedral convex surface such that $F_y^{-1} = \{x\}$ and $\omega_y > \frac{N-2}{N-1}\pi$, where $N > 1$ is the number of segments from x to y, then F is properly multivalued. Here, $F_y^{-1} = \{x \in S : y \in F_x\}$.

We provide a sufficient condition, $\omega_y \geq \pi$, to conclude that y is a farthest point, and derive more consequences if $\omega_y > \pi$ (Theorem 2.1). In the case of strict inequality, $\mathrm{int}F_y^{-1} \neq \emptyset$ and y is an isolated point of $\mathbb{F}$ in S. Both

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conclusions of Theorem 2.1 are optimal. Theorem 2.1 may be compared to the mentioned result of J. Rouyer [16]; we use a stronger lower bound for ω_y, but drop off all other hypotheses and obtain improved results. It may also be compared to Theorems 3 and 10 in [9], where it is obtained, under much more restrictive conditions (doubly covered polygons two of whose vertices have large curvatures), that $\mathbb{F}$ has very few points.

Another criterion, settled in [6], locates—roughly speaking—$\mathbb{F}$ around points at maximal distance to a simple closed quasigeodesic O (see § 4 for the definition).

Our Theorem 3.1 complements this result, by showing that $\omega_y \geq \pi$ is also sufficient for $y \in S \setminus O$ to be at maximal distance to O, and thus a "central point" of $\mathbb{F}$. Its Corollary 3.2 states that, if O is a simple closed quasigeodesic of length $\lambda(O)$ on a convex surface S, $y \in S \setminus O$ a point with $\omega_y \geq \pi$, and $\rho(y, O) > \lambda O$, then the mapping F is properly multivalued. Theorem 3.1 is also optimal.

While $\omega_y > \pi$ easily implies y is a strict local maximum for the distance function from any point in $S \setminus \{y\}$, the passage from local to global involves nontrivial arguments. The proofs of Theorems 2.1 and 3.1 are rather lengthy, and therefore are left for the last part of the corresponding sections. We shall employ methods of A. D. Alexandrov [2], [3], and often refer precisely to the results we use. Nevertheless, knowledge of large parts of [2] or [3] would be of considerable help for the reader.

In the final part of the paper, by the use of Theorem 2.1, we derive in Theorem 4.2 a criterion to conclude that a small cap C of a convex surface (see §4 for the definition) contains at least one farthest point. For example (see Corollary 4.4), this happens if the boundary length of C is at most π times the height h of C, and $7h$ is not larger than the distance between supporting planes to S parallel to $\mathrm{bd}C$.

Among all farthest points, those realizing the diameter of the surface received a special interest. A complementary paper [10] will complete these results by presenting criteria to recognize diameter points.

Finally we set some notation. $\lambda(C)$ will always denote the length of the curve C, while V will stand for the vertex set of a polyhedral surface. $B(x, r)$ will denote the closed intrinsic ball of radius r around x, and $S(x, r)$ its boundary.

2 Total curvature and farthest points

The goal of this section is to provide a criterion to recognize a farthest point, and to present remarks and examples concerning this criterion.

Theorem 2.1 *For any point y in any convex surface S with $\omega_y \geq \pi$, the set F_y^{-1} is nonempty and arcwise connected. If moreover $\omega_y > \pi$, then y is an isolated point of $\mathbb{F}$ in S and F_y^{-1} has interior points.*

Being quite long, the proof of Theorem 2.1 is left for the last part of this section, and it is given as a sequence of several lemmas.

The next result follows easily from Theorem 2.1, the upper semicontinuity of F, and Brouwer's fixed point theorem. Another application of Theorem 2.1 is presented in Section 4.

Corollary 2.2 *If there exists a point y in a convex surface S with $\omega_y > \pi$ then the mapping F is properly multivalued.*

Remark 2.3 The first part of Theorem 2.1 is tight, as follows from the example of D_ε's, doubly-covered isosceles triangles $xy_\varepsilon z$ with $||x - y_\varepsilon|| = ||z - y_\varepsilon||$ and $\angle xy_\varepsilon z = \pi/2 + \varepsilon/2 > \pi/2$. Indeed, $y_\varepsilon \notin \mathbb{F}$ but $D_\varepsilon \to D_0$ as $\varepsilon \to 0$ (x, z fixed), $y_\varepsilon \to y_0$ with $\pi - \varepsilon = \omega_{y_\varepsilon} \to \omega_{y_0} = \pi$, and $y_0 \in \mathbb{F}$ in D_0.

Remark 2.4 The second part of Theorem 2.1 is also tight best possible. J. Rouyer [15] determined explicitly the set $\mathbb{F}$ for (the boundary of) a regular tetrahedron T; in particular, each vertex v of T has curvature π, belongs to $\mathbb{F}$, F_v^{-1} is a tree with three leaves, and $\mathbb{F}$ is arcwise connected.

Example 2.5 Theorem 2.1 directly applies to tetrahedra. Since the total curvature of any convex surface is 4π, at least one vertex of any tetrahedron and all vertices of isosceles tetrahedra are farthest points.

The remainder of this section is devoted to the proof of Theorem 2.1.

A *triangle* in a convex surface is a collection of three segments $\gamma_1, \gamma_2, \gamma_3$ such that γ_i, γ_{i+1} have the common endpoint a_{i+2} ($i = 1, 2, 3$ modulo 3). We shall denote the triangle by $\gamma_1\gamma_2\gamma_3$ or, if the segments are clear from he context, by $a_1a_2a_3$.

The following comparison result can be found in [2, p. 215].

Lemma 2.6 *Let $\gamma_1\gamma_2\gamma_3$ be a triangle in a convex surface S and $\overline{\gamma}_1\overline{\gamma}_2\overline{\gamma}_3$ a planar triangle with $\lambda\gamma_i = \lambda\overline{\gamma}_i$. Then $\angle\overline{\gamma}_i\overline{\gamma}_{i+1} \leq \angle\gamma_i\gamma_{i+1}$, $i = 1, 2, 3 \pmod 3$, and equality holds if and only if $\gamma_1\gamma_2\gamma_3$ is isometric to $\overline{\gamma}_1\overline{\gamma}_2\overline{\gamma}_3$.*

Two segments joining the points x and y are called *consecutive* if their union bounds a domain no point of which is interior to another segment from x to y.

Lemma 2.7 *Consider points x, y in a convex surface S such that $\omega_y > \pi$. Then all points z in $S \setminus \{y\}$ (if any) with $\rho(x, z) \geq \rho(x, y)$ belong to the interior of only one of the digons bounded by consecutive segments from x to y.*

Proof. Assume, for simplicity, that there are only two segments joining x to y, say γ^1_{xy} and γ^2_{xy}. Suppose the conclusion is false and take points v, w in S outside $\mathrm{int}B(x, \rho(x, y))$, separated by the closed curve $\Lambda = \gamma^1_{xy} \cup \gamma^2_{xy}$. Choose segments γ_{vx}, γ_{wx} and γ_{yv}, γ_{yw}. The interiors of any two of the following four geodesic triangles on S, $T_1 = \gamma^1_{xy}\gamma_{yv}\gamma_{vx}$, $T_2 = \gamma^1_{xy}\gamma_{yw}\gamma_{wx}$, $T_3 = \gamma^2_{xy}\gamma_{yv}\gamma_{vx}$ and $T_4 = \gamma^2_{xy}\gamma_{yw}\gamma_{wx}$, are disjoint (see Fig. 1 left).

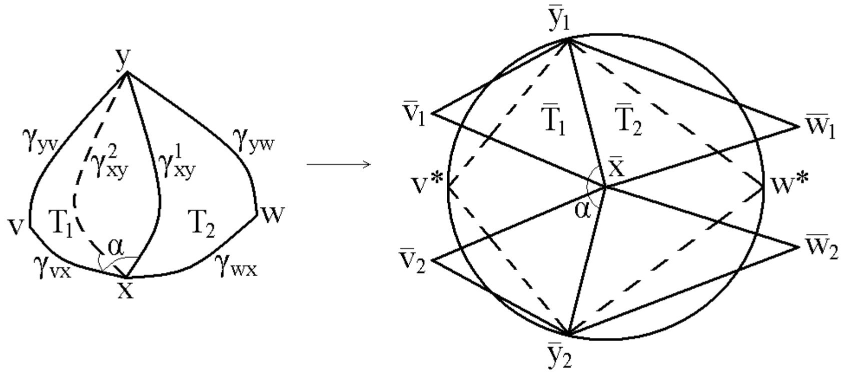

Fig. 1 If $\rho(x, v) \geq \rho(x, y)$ and $\rho(x, w) \geq \rho(x, y)$ then $\omega_y \leq \pi$

Construct four nonoverlapping planar triangles $\overline{T}_i$, respectively isometric to T_i $(i = 1, \dots, 4)$, $\overline{T}_1 = \bar{x}\bar{y}_1\bar{v}_1$, $\overline{T}_2 = \bar{x}\bar{y}_1\overline{w}_1$, $\overline{T}_3 = \bar{x}\bar{y}_2\bar{v}_2$ and $\overline{T}_4 = \bar{x}\bar{y}_2\overline{w}_2$, such that $\angle\bar{y}_1\bar{x}\bar{y}_2 = \angle\gamma^1_{xy}\gamma^2_{xy}$. Moreover, the circular order of $\overline{T}_i$ around $\bar{x}$ is the circular order of T_i around x, $i = 1, \dots, 4$. By Lemma 2.6, the angles of each $\overline{T}_i$ are at most equal to the corresponding angles of T_i. Then, by our initial assumption, $\bar{v}_1$, $\bar{v}_2$, $\overline{w}_1$ and $\overline{w}_2$ are at Euclidean distance from x larger than $\rho(x, y) = ||\bar{x} - \bar{y}_1|| = ||\bar{x} - \bar{y}_2||$. Choose points v^*, w^* on the circle $S(\bar{x}, \rho(x, y))$ on the same side of $\bar{y}_1\bar{y}_2$ as $\bar{v}_1$, $\bar{v}_2$ and respectively $\overline{w}_1$, $\overline{w}_2$, such that $\angle v^*\bar{y}_j\bar{x} \leq \angle\bar{v}_j\bar{y}_j\bar{x}$, $\angle w^*\bar{y}_j\bar{x} \leq \angle\overline{w}_j\bar{y}_j\bar{x}$ (for $j = 1, 2$) and the quadrilateral $v^*\bar{y}_1w^*\bar{y}_2$ is convex (see Fig. 2 right). Then $\angle v^*\bar{y}_jw^* \leq \angle\bar{v}_j\bar{y}_j\bar{x} + \angle\bar{x}\bar{y}_j\overline{w}_j$ $(j = 1, 2)$, and we obtain by addition

$$\pi \leq \angle v^*\bar{y}_1w^* + \angle v^*\bar{y}_2w^* \leq \angle\bar{v}_1\bar{y}_1\bar{w}_1 + \angle\bar{v}_2\bar{y}_2\bar{w}_2 = 2\pi - \omega_y < \pi,$$

a contradiction which establishes the lemma. □

For any point $x \in S$, the *cut locus* $C(x)$ *of* x is the set of all endpoints (different from x) of maximal (by inclusion) segments starting at x; it is known to be a tree.

Recall that a *tree* is a set T any two points of which can be joined by a unique arc included in T. A point $y \in T$ is called a *leaf* of T if $T \setminus \{y\}$ is connected, and a *junction point* of T if $T \setminus \{y\}$ has at least 3 components. A tree is *finite* if it has finitely many leaves.

Lemma 2.8 *If S is a polyhedral convex surface and y a point in S such that $\omega_y > \pi$ then there exists a point x in S having y as farthest point.*

Proof. The cut locus $C(y)$ of y is a finite tree whose edges are segments, because S is polyhedral (see [1]). Moreover, the points of S joined to y by more than two segments are precisely the junction points of $C(y)$, and the leaves of $C(y)$ are vertices of S and are joined to y by precisely one segment.

Choose a leaf of $C(y)$, say x_0, and assume $y \notin F_{x_0}$.

Move a point $x = x_t$ at constant speed along $C(y)$, starting from x_0 ($t \geq 0$). Assume that, at all times t, $y \notin F_{x_t}$ (otherwise there is nothing to prove). At each time $t_1 > 0$ we define the direction to follow by x (in the future) starting from the (present) position x_{t_1}, and a digon $S_{\leq t_1}$.

Define $S_{\leq 0}$ to be the unique (by the choice of x_0) segment $\gamma_{x_0 y}$.

If $x_1 = x_{t_1}$ is joined to y by precisely two segments, say $\gamma^1_{x_1 y}$ and $\gamma^2_{x_1 y}$, then $S_{\leq t_1}$ is the digon of S bounded by $\gamma^1_{x_1 y} \cup \gamma^2_{x_1 y}$ which contains x_0. The variable point x_t will move from the position x_1 along the edge of $C(y)$ containing x_1, to locally increase the distance from $S_{\leq t_1}$.

Assume now that x_1 is joined to y by $N > 2$ segments, say $\gamma^1_{x_1 y}, \ldots, \gamma^N_{x_1 y}$, and let S_i denote the digon of S bounded by the (consecutive) segments $\gamma^i_{x_1 y}$ and $\gamma^{i+1}_{x_1 y}$ (for $i = 1, \ldots, N \ (\mathrm{mod} N + 1)$), with the indices such that $S_1 \supset \gamma_{x_0 y}$. Then, by Lemma 2.7, all points z in S at distance to x_1 $\rho(x_1, z) \geq \rho(x_1, y)$ belong to the interior of only one such digon, say S_{i_1}. Put $S_{\leq t_1} = \mathrm{cl}\,(S \setminus S_{i_1})$. Here again, the direction of motion from x_1 should locally increase the distance from $S_{\leq t_1}$.

Finally, when x arrives at a leaf x_T of $C(y)$ (and this certainly happens at some time T), it stops there. In this case, clearly $\lim_{t \uparrow T} x_t = x_T$ and, for t close to T and x_t on the same edge of $C(y)$ as x_T, the unique segment $\gamma_{x_T y}$ is the limit of precisely two segments $\gamma^1_{x_t y}$, $\gamma^2_{x_t y}$, bounding $S_{\leq t}$. Define in the obvious way $S_{\leq T} = \lim_{t \uparrow T} S_{\leq t}$.

We claim that, at any time t,

$$F_{x_t} \subset S_{>t} = S \setminus S_{\leq t} \,. \tag{2.1}$$

For $t = 0$, (2.1) is clear from the assumption $y \notin F_{x_0}$.

Assume that, at some time t_1, the point $x_1 = x_{t_1}$ is not a junction point of $C(y)$ and $F_{x_1} \subset S_{>t_1}$. Suppose there exists a time $t_2 > t_1$ such that $x_2 = x_{t_2}$ belongs to the same edge of $C(y)$ as x_1, and $F_{x_2} \subset S_{\leq t_2}$. Move continuously from x_1 to x_2 and observe that $S_{\leq t_1}$ increases continuously to $S_{\leq t_2}$. Then, by the upper semicontinuity of the mapping F and by Lemma 2.7, there exists some time t_*, $t_1 < t_* < t_2$, such that $F_{x_*} \cap \mathrm{bd} S_{\leq t_2} \neq \emptyset$ ($x_* = x_{t_*}$), hence $y \in F_{x_*}$, a contradiction with $y \notin F_{x_t}$ for all t. Therefore, $F_{x_2} \subset S_{>t_2}$ for all points $x_2 = x_{t_2}$ on the same edge of $C(y)$ as x_1, provided $F_{x_1} \subset S_{>t_1}$.

Assume finally that the point $x_1 = x_{t_1}$ is a junction point of $C(y)$. Then $\lim_{t \uparrow t_1} S_{\leq t} \subset S_{\leq t_1}$ and, by the definition of $S_{\leq t_1}$, if $F_{x_t} \subset S \setminus S_{\leq t}$ for all $t < t_1$ then also $F_{x_1} \subset S \setminus S_{\leq t_1}$, so the claim is established.

Since the boundary of $S_{\leq t}$ consists of segments starting at y, $C(y)$ intersects it only at x_t. Therefore, from $\lim_{t \uparrow t_1} S_{\leq t} \subset S_{\leq t_1}$, at the time $T > 0$ when x_T reaches a leaf of $C(y)$ the set $S_{\leq T}$ will completely contain $C(y)$, so $S_{>T} = S \setminus S_{\leq T} = \emptyset$, because x_T is joined to y by precisely one segment.

But, by the claim that $F_{x_t} \subset S_{>t}$ at all times t, this is also true for $t = T$, $\emptyset \neq F_{x_T} \subset S_{>T} = \emptyset$ and a contradiction is reached. $\square$

We are now in the position to prove the first part of Theorem 2.1.

Lemma 2.9 *For any point y in any convex surface S with $\omega_y \geq \pi$, the set F^{-1}_y is nonempty and arcwise connected.*

P r o o f. The general arcwise connectedness of the set F^{-1}_y is proven in [19].

To show that $F^{-1}_y \neq \emptyset$, consider a sequence of polyhedral convex surfaces S_n converging to S such that $y = y_n \in S_n$ and $\omega_{y_n} > \pi$ ($n \geq 1$). This is possible, for example, by choosing sets $V_n = \{v^1_n, \ldots, v^{m_n}_n\} \subset V_{n+1} \subset S$, with m_n integers such that $m_1 \geq 4$ and $m_{n+1} = 2m_n$. Then $S_n = \mathrm{bd}(\mathrm{conv} V_n) \to S$ provided V_n becomes dense in S, and the curvature condition is fulfilled if $S_n \neq S$ around y.

By Lemma 2.8, there exist points $x_n \in S_n$ with $y_n \in F_{x_n}$. Possibly passing to a subsequence, we can assume $x_n \to x \in S$. By the upper semicontinuity of the mapping F, the limit set of F_{x_n} is included in F_x, so $y \in F_x$ and we are finished. $\square$

Lemma 2.10 *Let x, y be two points in S. The set $\{\alpha_i\}_{i \in I}$, consisting of all angles formed at y by pairs of consecutive segments from y to x, is at most countable, and if it is not void then $\sup_{i \in I} \alpha_i$ is attained.*

P r o o f. Fix a homeomorphism $f : T_y \to S^1$. The interiors of any two angles, determined by distinct pairs of consecutive segments, are disjoint. Therefore, the set $\{\alpha_i\}_{i \in I}$ is at most countable. Suppose that $\sup_{i \in I} \alpha_i$ is not attained. Then there exists a sequence of angles $\{\alpha_{i_n}\}_{n \geq 1}$ converging to $\sup_{i \in I} \alpha_i > 0$, whence $\sum_{n=1}^{\infty} \alpha_{i_n} = \infty$. But this contradicts the fact that $\Sigma_{i \in I} \alpha_i \leq \theta_y \leq 2\pi$. $\square$

Aside from its concrete estimation, the conclusion of the next lemma follows directly from the first variation formula ([13, Theorem 3.5]).

Lemma 2.11 *Let x, y be points in a convex surface S, and α_i the angles at y between consecutive segments to x, $i \in I \neq \emptyset$. If $\max_{i\in I} \alpha_i < \pi$ then y is a strict maximum for the restriction of the distance function ρ_x to $B\left(y, 2\rho(x, y) \cos\left(\max_{i\in I} \alpha_i/2\right)\right)$.*

P r o o f. To prove y is a strict local maximum for ρ_x, it suffices to show that, for some neighbourhood V_y of y, $\rho_x(y) > \rho_x(z)$ holds for all $z \in V_y \cap C(x)$. Take $V_y = B\left(y, 2l \cos\left(\max_{i\in I} \alpha_i/2\right)\right)$, where $l = \rho(x, y)$.

For any point $z \in C(x) \setminus \{y\}$, there exists a digon D bounded by two consecutive segments from x to y, such that $z \in D$. Let α_1 be the angle of D at y. Since $\alpha_1 \leq \max_{i\in I} \alpha_i < \pi$, we get

$$0 < \rho(y, z) < 2l \cos\left(\max_{i\in I} \alpha_i/2\right) \leq 2l \cos(\alpha_1/2).$$

One of the two angles formed at y by a segment from z to y, with the two segments bounding D, is at most $\alpha_1/2$; denote it by β. Consider a planar triangle $\bar{x}\bar{y}\bar{z}$ with $||\bar{x} - \bar{y}|| = l$, $||\bar{y} - \bar{z}|| = \rho(y, z)$ and the angle at $\bar{y}$ equal to β. Since $\beta \leq \alpha_1/2 < \pi/2$, we have

$$||\bar{y} - \bar{z}|| = \rho(y, z) < 2l \cos(\alpha_1/2) \leq 2l \cos\beta,$$

hence the angle at $\bar{z}$ is larger than β and $||\bar{x} - \bar{z}|| < ||\bar{x} - \bar{y}||$.

By the convexity of the metric of S (see [2] or [3]), we also have $\rho(x, z) \leq ||\bar{x} - \bar{z}||$, so we obtain $\rho(x, z) < \rho(x, y)$, i.e., y is a strict local maximum for ρ_x. □

Lemma 2.12 ([22]) *Suppose that the set F_x contains more than one point for some point x in S. Then F_x is contained in a minimal* (*by inclusion*) *arc $J_x \subset C(x)$, and any geodesic triangle on S with vertices in F_x is obtuse or right.*

Lemma 2.13 ([23], [21]) *Suppose that $x \in S$ and $\mathrm{card} F_x \geq 2$. Let $y \in F_x$ be an endpoint of the arc J_x defined by Lemma 2.12, and $\varepsilon > 0$. Then there exist an arc A starting at x and a number $k > 0$, such that for any $v \in A$ and any $u \in B(v, k\rho(v, x))$, we have $F_u \subset B(y, \varepsilon)$.*

We now have all we need to prove the last part of Theorem 2.1.

Lemma 2.14 *Suppose the point y in the convex surface S satisfies $\omega_y > \pi$. Then y is an isolated point of $\mathbb{F}$ in S and F_y^{-1} has interior points.*

P r o o f. By Lemma 2.9, we can choose a point $x \in S$ with $y \in F_x$. Assume $F_x \setminus \{y\} \neq \emptyset$ (otherwise the proof is slightly easier), and observe that y is an endpoint of the arc J_x defined by Lemma 2.12, as follows either from Lemma 2.12 itself or from Lemma 2.7. Consider positive numbers

$$\varepsilon_1 < \cos(\theta_y/2), \quad \varepsilon_2 = \frac{\cos(\theta_y/2) - \varepsilon_1}{\cos(\theta_y/2) + \varepsilon_1}.$$

Apply Lemma 2.13 for $\varepsilon \leq \varepsilon_1\rho(x, y)$ and v such that (for k as in its statement) $U = B(v, k\rho(v, x)) \subset B(x, \varepsilon_2\rho(x, y))$, so we have $F_U \subset B(y, \varepsilon_1\rho(x, y))$. From the choice of ε_2 we obtain, for any point u in $U \subset B(x, \varepsilon_2\rho(x, y))$,

$$(\cos(\theta_y/2) + \varepsilon_1)\rho(x, u) < (\cos(\theta_y/2) - \varepsilon_1)\rho(x, y),$$

whence

$$\begin{aligned}(\cos(\theta_y/2) + \varepsilon_1)\rho(x, u) + \varepsilon_1\rho(x, y) &< \cos(\theta_y/2)\rho(x, y)\\ &\leq \cos(\theta_y/2)(\rho(x, u) + \rho(u, y)).\end{aligned}$$

By subtracting $\cos(\theta_y/2)\rho(x, u)$, we further have

$$\varepsilon_1\rho(x, y) \leq \varepsilon_1\rho(x, u) + \varepsilon_1\rho(x, y)) < \rho(u, y) \cos(\theta_y/2), \tag{2.2}$$

and therefore

$$F_u \subset B(y, \varepsilon_1 \rho(x, y)) \subset B(y, \rho(u, y) \cos(\theta_y/2)).$$

Since the maximal angle between two consecutive segments from u to y is smaller than $\theta_y < \pi$, and $\cos|_{[0,\pi]}$ is a strictly decreasing function, Lemma 2.11 applied to u yields $\rho(u, y) > \rho(v, w)$, for each $w \in B(y, \rho(u, y) \cos(\theta_y/2))$. Thus $F_u = y$, and consequently $U \subset F_y^{-1}$.

To see that y is an isolated point of $\mathbb{F}$, assume there exists a sequence of points $y_n \in F_S \setminus \{y\}$ converging to y. By the upper semicontinuity of F^{-1} [18], the limit set of the sequence of sets $\left\{F_{y_n}^{-1}\right\}_n$ is a subset of F_y^{-1}. Possibly passing to a subsequence, we may consider a sequence of points $x_n \in F_{y_n}^{-1}$, convergent to the point $x' \in F_y^{-1}$. For n large enough, we have

$$x_n \in B(x', \varepsilon_2 \rho(x', y)), \quad y_n \in B(y, \varepsilon_1 \rho(x', y)).$$

By replacing x with x' and u with x_n in the inequalities (2.2), we get $\varepsilon_1 \rho(x', y) < \rho(x_n, y) \cos(\theta_y/2)$, i.e.

$$y_n \in B(y, \rho(x_n, y) \cos(\theta_y/2)).$$

Now Lemma 2.11 applied to x_n yields $\rho(x_n, y_n) < \rho(x_n, y)$, contradicting $y_n \in F_{x_n}$, and the proof is complete. □

3 Farthest points from simple closed quasigeodesics

The sufficient condition $\omega_y \geq \pi$ to decide y is a farthest point also proves useful in the situation considered in this section. For the reader's convenience, we recall first the definition of a quasigeodesic.

Consider a piecewise geodesic Γ which is a Jordan arc, say $\Gamma = \bigcup_{i=0}^{n} \Gamma_{a_i a_{i+1}}$, where $\Gamma_{a_i a_{i+1}}$ is a geodesic arc joining the points $a_i, a_{i+1} \in S$ $(i = 0, \dots, n)$. Then a *right* and a *left side* can be consistently locally defined along $\Gamma \setminus \{a_0, a_{n+1}\}$. Denote by α_i and β_i the angle between $\Gamma_{a_i a_{i-1}}$ and $\Gamma_{a_i a_{i+1}}$ to the right and to the left of Γ, respectively. The *right* and *left swerve* of Γ are the numbers $s_r(\Gamma) = \Sigma_{i=1}^{n}(\pi - \alpha_i)$, $s_l(\Gamma) = \Sigma_{i=1}^{n}(\pi - \beta_i)$.

Consider now a Jordan arc A which has definite directions at its endpoints p, q, and Γ a piecewise geodesic from p to q which is a Jordan arc and lies to the right of, or on, A. Denote by δ_p and δ_q the angles between Γ and A at p and q. Then $\lim(\delta_p + \delta_q + s_r(\Gamma))$ exists when Γ approaches A from the right (see [2, p. 353]) and it is called the *right swerve* of A ([3, p. 109]). The *left swerve* is defined similarly.

A *quasigeodesic arc* is a Jordan arc which has definite directions at each point and whose every subarc has nonnegative right and left swerves ([3, p. 114]).

A segment connecting two points with complete angles $\leq \pi$ forms, traversed back and forth, a *degenerate* closed quasigeodesic.

Any geodesic is clearly a quasigeodesic, and if S has bounded specific curvature (in particular, if it is smooth) then the converse is also true (see [3, pp. 114 and 27]). Notice that a convex surface may have no closed geodesic [2], [5], but it always has at least three simple closed quasigeodesics [14].

The distance from the point x in S to a closed subset K of S is given by $\rho(x, K) = \min_{y \in K} \rho(x, y)$.

Theorem 3.1 *If O is a simple closed quasigeodesic of a convex surface S and y a point in $S \setminus O$ such that $\omega_y \geq \pi$, then y is at maximal distance to O in the domain of S bounded by O to which it belongs.*

The following statement is a direct consequence of the previous result, Theorem 2.1 and [6, Theorem 3].

Corollary 3.2 *Let O be a simple closed quasigeodesic on a convex surface S, $y \in S \setminus O$ a point with $\omega_y \geq \pi$, and S' the subset of S bounded by O that does not contain y. Then $\mathbb{F}$ is included in the union of two intrinsic balls of radius $\lambda(O)$, centered at y and at a farthest point from O in S'. If moreover $\rho(y, O) > \lambda(O)$ then $\mathbb{F}$ is disconnected and the mapping F is properly multivalued.*

Remark 3.3 Theorem 3.1 is optimal, as follows from this easy example. Consider a convex quadrilateral $Q_\varepsilon = abcy_\varepsilon$ such that $\angle y_\varepsilon ab = \angle abc = \pi/2$ and $\angle a y_\varepsilon c = \pi/2 + \varepsilon/2$, with ε an arbitrarily small positive number. Take points $p \in [ab]$ and $q \in [ay]$ such that $pq \parallel ab$. Then the line-segment $[pq]$ provides a simple closed geodesic O on the double D_ε of Q_ε and, on the half-surface H_ε of D_ε bounded by O which contains y_ε, c is at larger distance to O than y_ε. Since D_ε converges to a doubly-covered rectangle D_0 as $\varepsilon \to 0$ $(a, b, c$ fixed), $y_\varepsilon \to y_0$ with $\pi - \varepsilon = \omega_{y_\varepsilon} \to \omega_{y_0} = \pi$, and $y_0 \in D_0$ is at largest distance to $O \subset D_0$ in $H_0 = \lim_{\varepsilon \to 0} H_\varepsilon$.

The rest of this section consists of several lemmas, each of which to be used for, or being a part of, the proof of Theorem 3.1. The first one is an elementary result.

Lemma 3.4 *Consider a convex planar n-gon $xyw_1 \dots w_{n-2}$ such that*

$$\angle xyw_1 \leq \pi/2, \quad \angle yw_1w_2 = \angle xw_{n-2}w_{n-3} = \pi/2$$

and all other angles are at least $\pi/2$. Then $||x - w_{n-2}|| \leq ||y - w_1||$.

Proof. Move continuously a point u along the line-segment $[xy]$, from x to y. Denote by u_0 the foot of u on the curve $W = w_1 \dots w_{n-2}$; if several, take the closest to w_1 (see Fig. 2). Observe that $\angle xuu_0 \leq \pi/2$,

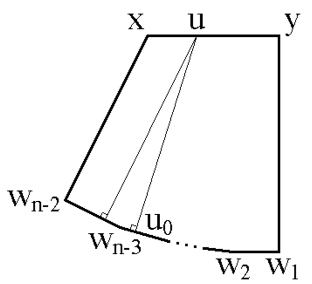

Fig. 2 $||x - w_{n-2}|| \leq ||y - w_1||$

because the function $f(u) = \angle xuu_0$ is increasing and its value at y is $\leq \pi/2$. Therefore, the distance from u to W increases as $||u - x||$ increases, and the conclusion follows when considering the extreme positions of u. □

An *arc* A is the image set of $[0, 1]$ through a homeomorphism (denoted by the same letter) $A : [0, 1] \to S$, and its *relative interior* $\mathrm{relint} A$ is the image set of $]0, 1[$.

Notice that a point x may have several *projections on* (i.e., points realizing the distance to) a quasigeodesic.

Lemma 3.5 ([6]) *Let G be a quasigeodesic on a convex surface S, x a point in $S \setminus G$ and $x_0 \in G$ a projection of x on G. If $x_0 \in \mathrm{relint}\, G$ then there exists at most one segment joining x_0 to x on each side of G, and it is orthogonal to G.*

Lemma 3.6 *Let S be a polyhedral convex surface, O a simple closed quasigeodesic on a S, $y \in S \setminus O$ a point with $\omega_y \geq \pi$ and S_y the subset of S bounded by O containing y. Then y is at maximal distance to O in S_y.*

Proof. Assume there exists a point x in S_y at larger distance to O than y. Consider a shortest path joining x to y in the relatively convex domain S_y, say $\gamma_{xy} \subset S_y$, and observe that $\gamma_{xy} \cap O = \emptyset$.

Choose projections x_0, y_0 of x, y on O and segments $\gamma_{x_0x}, \gamma_{yy_0}$; then, by Lemma 3.5, $\gamma_{x_0x} \perp O$, $\gamma_{yy_0} \perp O$.

The complete angle at y is $2\pi - \omega_y \leq \pi$, so at least one of the quasigeodesic domains of S_y determined by O and $\gamma_{x_0x} \cup \gamma_{xy} \cup \gamma_{yy_0}$, say D, has an angle at most $\pi/2$ at y. The Gauss–Bonnet theorem (see [3, p. 105]) applied to D gives, for the angle $\angle^D \gamma_{xx_0}\gamma_{xy}$ in D between γ_{xx_0} and γ_{xy}, $\angle^D \gamma_{xx_0}\gamma_{xy} \geq \pi/2$, and equality occurs if and only if D is a rectangle (and thus $\rho(x, O) = \rho(y, O)$).

Observe that no vertex in S_y, and therefore in $D \subset S_y$, has curvature strictly larger than π, because the curvature of S_y is at most 2π and we already have $\omega_y \geq \pi$.

Assume the number of vertices in $V_D = V \cap \mathrm{int}\, D$ is $N + 1$, and apply the following procedure. Another application of the idea of this procedure is given in [7].

To start with, set $D^0 = D$ and $O^0 = O \cap D^0$.

3.1 Procedure flattening

for $i = 0$ to N do
  choose $v^i \in V_D$ such that $\rho(v^i, O^i) = \min_{v \in V_D} \rho(v, O^i)$
  choose $v_0^i \in O^i$ such that $\rho(v^i, O^i) = \rho(v^i, v_0^i)$
  cut D^i along the (uniquely determined, by Lemma 3.5) segment $\gamma_{v_0^i v^i}$
  glue to D^i a flat isosceles triangle T^i whose equal sides

have length $\rho(v^i, v_0^i)$ and the angle between them is equal to ω_{v^i},
such that after gluing the point v^i has zero curvature
define D^{i+1} as the result of gluing T^i to D^i
define O^{i+1} as the polygonal chain of segments in the boundary of D^i
obtained by inserting the base of T^i to O^i.

Fig. 3 illustrates the Procedure flattening for the case $N+1=2$. The projections of the vertices $v^0, v^1 \in D$ onto $O \cap D$ are denoted by v_0^0 and v_0^1, and the segments joining them to their foots are dashed. The left part of Fig. 3.1 illustrates the initial configuration, and the right part the final result. The unfolding is realized in the plane yv^0v^1. The edges are thick, but the edges that allowed rotations (thus dissapearing) are thin in the right part of Fig. 3.1. The segments joining points to their foots are medium sized. The inserted triangles T^0 and T^1 are filled with dotted, respectively dashed, lines.

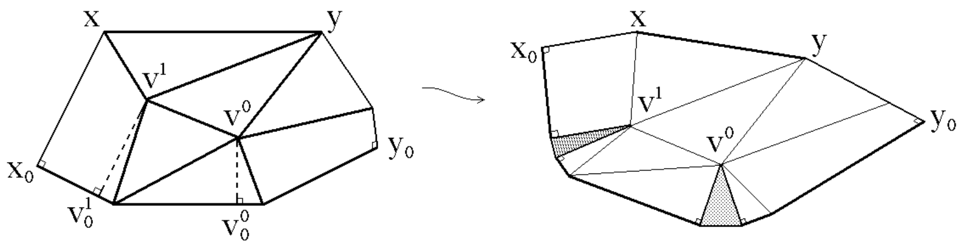

Fig. 3 Procedure flattening for $N+1=2$ vertices, v^0 and v^1

Since S is polyhedral, O is a union of line-segments (by the definition of a quasigeodesic), as well as O^i $(i = 1, \ldots, N)$.

Notice that, by the first variation formula, no segment $\gamma_{v_0^i v^i}$ cuts $\gamma_{x_0 x} \cup \gamma_{y_0 y}$. Therefore, the angles of the polygonal chain of segments O^N are all at least $\pi/2$.

Also notice D^N is (isometric to) a planar convex polygon, since its interior contains no vertex, so the isometric image of Γ_{xy} in the boundary of D^N is a line-segment. Finally apply Lemma 3.4 (with $w_1 = y_0$, $w_{n-2} = x_0$) to obtain $\rho(y, O) \geq \rho(x, O)$. □

We shall need the following version of Alexandrov's gluing theorem (see [2, p. 362]).

Lemma 3.7 *If a 2-manifold M results from gluing together several quasigeodesic polygonal domains such that the sum of the angles at vertices glued together is not larger than 2π, then M is of positive curvature.*

We shall also use Pogorelov's rigidity theorem ([14, p. 167]), which says that

Lemma 3.8 *Any two isometric convex surfaces are congruent.*

Lemma 3.9 *Assume O is a simple closed quasigeodesic of a convex surface S and y a point in $S \setminus O$ such that $\omega_y \geq \pi$. Then we can approximate S with polyhedral convex surfaces P_n $(n \geq 1)$ with the following properties: there exist points $y_n \in P_n$ with $\omega_{y_n} \geq \pi$ and simple closed (quasi)geodesics $O_n \subset P_n$ such that $y_n \notin O_n$ and $y_n \to y$, $O_n \to O$ as $n \to \infty$.*

Proof. We show how to obtain the necessary approximation in three steps. Denote by ${}^1S$ the domain of S bounded by O to which y belongs,

First, cut S along O and glue between the two pieces a right cylinder C_ε of height ε and base isometric to O. By Lemmas 3.7 and 3.8, the result S_ε is (isometric to) a convex surface. Let ${}^1S_\varepsilon$ be the domain of S_ε isometric to ${}^1S$, and y_ε the image of y in ${}^1S_\varepsilon$. Clearly $S_\varepsilon \to S$, $C_\varepsilon \to O$ and $y_\varepsilon \to y$ as $\varepsilon \to 0$. Moreover, there exist simple closed (quasi)geodesics $O_\varepsilon \to O$ on $C_\varepsilon \subset S_\varepsilon$.

Second, approximate C_ε by a right cylinder over a polygonal curve. For, consider a simple closed polygonal curve O_ε^n approximating O_ε, such that $O_\varepsilon^n \subset \operatorname{conv} C_\varepsilon$. Denote by C_ε^n the cylinder over O_ε^n parallel to C_ε. Let S_ε^n be the convex surface that contains C_ε^n and is isometric to S_ε outside C_ε^n. It is well defined, by Lemmas 3.7 and 3.8. Clearly, S_ε^n contains simple closed (quasi)geodesics inside C_ε^n, which we may and shall assume to be O_ε^n. Since $y \notin O$, and consequently $y_\varepsilon \notin O_\varepsilon$, we may also assume $y_\varepsilon \in S_\varepsilon^n \setminus O_\varepsilon$. Put $B_\varepsilon^n = \mathrm{bd} C_\varepsilon^n \subset S_\varepsilon^n$ and denote by ${}^1D_\varepsilon^n$, ${}^2D_\varepsilon^n$ the two domains of S_ε^n determined by B_ε^n, with $y_\varepsilon \in {}^1D_\varepsilon^n$.

At the third step, approximate ${}^iD^n_\varepsilon$ by (locally convex) polyhedral surfaces ${}^iP^n_\varepsilon$ such that $\bigcup_i \mathrm{bd}\ {}^iP^n_\varepsilon = B^n_\varepsilon$, $y_\varepsilon \in {}^1P^n_\varepsilon$ and the total curvature at y_ε in ${}^1P^n_\varepsilon$ is at least π. Denote by P^n_ε the convex surface obtained by gluing in the obvious way ${}^iP^n_\varepsilon$ to C^n_ε. By the construction, we have $P^n_\varepsilon \to S_\varepsilon$ and $O^n_\varepsilon \to O_\varepsilon$ as $n \to \infty$. Moreover, $y_\varepsilon \in P^n_\varepsilon \cap S_\varepsilon$.

Finally define $P^n = P^n_{1/n}$ and observe that $P^n \to S$, $O^n_{1/n} \to O$ and $y_{1/n} \to y$ as $n \to \infty$, because $S_{1/n} \to S$. Moreover, the total curvature at $y_{1/n} \in P^n$ is always at least π. □

With all these preparations, the statement of Theorem 3.1 follows easily.

Proof. Assume O is a simple closed quasigeodesic of a convex surface S and y a point in $S \setminus O$ such that $\omega_y \geq \pi$. In order to show that y is at maximal distance to O in the domain ${}^1S$ of S bounded by O to which it belongs, we approximate S with polyhedral convex surfaces P_n $(n \geq 1)$ as in Lemma 3.9. Let ${}^1P_n$ be the domain of P_n bounded by O_n to which y_n belongs $(n \geq 1)$. By Lemma 3.6, $\rho_n(y_n, O_n) \geq \rho_n(z, O_n)$ for all $z \in {}^1P_n$, and we reach the desired conclusion by passing to the limit. □

4 Caps and farthest points

In this section we derive criteria to conclude that small caps contain farthest points.

A *cap* of the convex surface S is a closed subset C of S with interior points, whose boundary $\mathrm{bd}C$ is included in a plane P, such that any normal to P at a point x on $\mathrm{bd}C$ intersects C in exactly that one point x. The *height* of the cap C is the Euclidean distance between P and a plane parallel to P and supporting C.

The metric projection of a point $x \in \mathbb{R}^3$ onto a convex body K is a point $x_0 \in K$ realizing the Euclidean distance from x to K, $\mathrm{dist}(x, K) = ||x - x_0||$. The next result is well-known (see, for example, [3, p. 80]).

Lemma 4.1 *The length of a curve in $\mathbb{R}^3$ is at least as long as its metric projection onto a convex body.*

The *radius* of the closed subset M of S is $\mathrm{rad}(M) = \sup_{x\in M} \rho(x, \mathrm{bd}M)$. The *diameter* of $M \subset S$ is $\mathrm{diam}(M) = \sup_{x,y\in M} \rho(x, y)$.

Theorem 4.2 *If the cap C of the convex surface S has diameter not larger than the radius of $\mathrm{cl}(S \setminus C)$, and it contains a point y such that $\mathrm{dist}(y, \mathrm{bd}C) \geq \pi^{-1}\lambda(\mathrm{bd}C)$, then C contains a farthest point of S.*

Proof. Put $l = \min_{z\in \mathrm{bd}C} ||y - z||$. The cone V at y over $\mathrm{bd}C$ is interior to S and convex, as well as the cone $V_l = \{v \in V; ||y - v|| \leq l\}$, because $\mathrm{bd}C$ is a planar convex curve. Since the metric projection of $\mathrm{bd}C$ onto $\mathrm{conv}V_l$ is the set $Q = \{v \in V; ||y - v|| = l\}$, Lemma 4.1 yields $\lambda(Q) \leq \lambda(\mathrm{bd}C)$.

The complete angle α_y of V (or V_l) at y is less than π, because

$$\alpha_y l = \lambda(Q) \leq \lambda(\mathrm{bd}C) \leq \pi l,$$

and the total curvature of y in S_0 is $\geq \pi$.

The surface $S_0 = (S \setminus C) \cup V$ is clearly convex, hence we can apply Theorem 2.1 to find a point $x \in S_0$, such that the set of farthest point from x on S_0 contains y. Observe that $x \in S_0 \cap S$, because otherwise, for the centre z of some intrinsic ball of radius $r \geq \mathrm{diam}(C)$, one easily obtains $||x - y|| \leq \rho(z, \mathrm{bd}C) < \rho^{S_0}(z, x)$.

We have, for $u \in S \setminus C$,

$$\rho(x, y) \geq \rho^{S_0}(x, y) \geq \rho^{S_0}(x, u) = \rho(x, u),$$

and the conclusion follows. □

Assume that the cap C of height h of the convex surface S has boundary length at most πh. If h is much smaller than $\mathrm{diam}\, S$, or $\mathrm{diam}\, C$ is much smaller than $\mathrm{diam}(S)$, then C contains at least one farthest point on S. The meaning of “being much smaller than” can be made precise, as for example in the following corollary of Theorem 4.2.

Let $\mathrm{d_{ex}}(M)$ denote the *extrinsic diameter* of the closed subset M of S, $\mathrm{d_{ex}}(M) = \sup_{x,y\in M} ||x - y||$.

N. P. Makuha [11] showed the following nice inequality.

Lemma 4.3 *For any convex surface S the inequality $\mathrm{diam}(S) \leq \frac{\pi}{2}\mathrm{d_{ex}}(S)$ holds, with equality if and only if S is a surface of revolution having constant width.*

Corollary 4.4 *Assume a cap C of a convex surface S has height h and boundary length at most πh. If the distance between the two supporting planes of S parallel to* $\mathrm{bd}C$*, is not smaller than* $h\left(1+\pi\sqrt{1+\frac{\pi^2}{4}}\right)$*, then C contains at least one farthest point on S.*

Proof. Assume that the points $y, z \in C$ are realizing the extrinsic diameter of C. Then at least one of them, say z, is on the boundary of C. To see this, suppose that $y, z \notin \mathrm{bd}C$. Let P denote the plane containing $\mathrm{bd}C$, $P_{yz}^{\perp}$ the plane orthogonal to P through yz, and put $\{x_1, x_2\} = P_{yz}^{\perp} \cap \mathrm{bd}C$. Assume that $\mathrm{dist}(y, P) \geq \mathrm{dist}(z, P)$. Then it follows from elementary considerations that $||y - z|| < \max\{||y - x_1||, ||y - x_2||\}$, contradicting the choice of y and z.

Denote by y_0 the orthogonal projection of y onto P. Because the length l of $\mathrm{bd}C$ satisfies $l \leq \pi h$, we have

$$\mathrm{d_{ex}}(C) = ||y - z|| = \sqrt{||y - y_0||^2 + ||z - y_0||^2} \leq \sqrt{h^2 + \frac{l^2}{4}} \leq h\sqrt{1 + \frac{\pi^2}{4}}. \tag{4.1}$$

Denote by S_C the double of C, i.e., the convex surface obtained by gluing together two isometric copies of C. Then

$$\mathrm{diam}(C) \leq \mathrm{diam}(S_C), \quad \mathrm{d_{ex}}(C) \leq \mathrm{d_{ex}}(S_C) \leq 2\mathrm{d_{ex}}(C), \tag{4.2}$$

the last inequality following from the triangle inequality.

On the other hand, by Lemma 4.3,

$$\mathrm{d_{ex}}(S_C) \leq \mathrm{diam}(S_C) \leq \frac{\pi}{2}\mathrm{d_{ex}}(S_C), \tag{4.3}$$

and putting together (4.3), (4.2) and (4.1) we obtain

$$\mathrm{diam}(C) \leq \mathrm{diam}(S_C) \leq \frac{\pi}{2}\mathrm{d_{ex}}(S_C) \leq \pi\mathrm{d_{ex}}(C) \leq \pi h\sqrt{1 + \frac{\pi^2}{4}}. \tag{4.4}$$

Let P', P'' denote the supporting planes of S parallel to P, and v', v'' their contact points with S. Assume that $v' \in C$, and put $\{z\} = v'v'' \cap P$. Then, since $z \in \mathrm{conv}(\mathrm{bd}(C))$,

$$\mathrm{rad}(S \setminus C) \geq \max_{x \in \mathrm{bd}C} ||v'' - x|| > ||v'' - z|| \geq h_C - h. \tag{4.5}$$

From $h_C \geq h\left(1+\pi\sqrt{1+\frac{\pi^2}{4}}\right)$, (4.4) and (4.5) we finally get

$$\mathrm{rad}(S \setminus C) \geq h_C - h > \pi h\sqrt{1 + \frac{\pi^2}{4}} \geq \mathrm{diam}(C)$$

and Theorem 4.2 ends the proof. □

Remark 4.5 Good candidates to verify the hypotheses of Theorem 4.2 or Corollary 4.4 are the caps close to cones of complete angle $> \pi$. For example, any cap C_r of height r containing a parallel cap C_ϵ such that ϵ is "much smaller" than r and the total curvatures of C_r and C_ϵ are "almost equal" and larger than π.

We even believe that each convex cap $C \subset S$ of height h "much smaller" than $\mathrm{diam}(S)$ and total curvature $\omega(C) > \pi$ contains a farthest point on S, but this is not proven yet.

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