Continuous Flattening of Convex Polyhedra

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Abstract. A *flat folding* of a polyhedron is a folding by creases into a multilayered planar shape. It is an open problem of E. Demaine et al., that every flat folded state of a polyhedron can be reached by a continuous folding process. Here we prove that every convex polyhedron possesses infinitely many continuous flat folding processes. Moreover, we give a sufficient condition under which every flat folded state of a convex polyhedron can be reached by a continuous folding process.

1 Introduction

We use the terminology *polyhedron* for a closed polyhedral surface which is permitted to touch itself but not self-intersect (and so a doubly covered polygon is a polyhedron). A *flat folding* of a polyhedron is a folding by creases into a multilayered planar shape.

The results presented here are related to the following problem proposed by Erik Demaine et al. (see Open Problem 18.1 in [5]): *Can every flat folded state of a polyhedron be reached by a continuous folding process?*

Notice that, if a polyhedron P is flattened by a continuous folding process (see Definition 1) with polyhedra $\{P_t : 0 \le t \le 1\}$, then the crease pattern in P for $\{P_t : 0 \le t \le 1\}$ is an infinite set of line segments. This follows from Cauchy's rigidity theorem and Sabitov's result on the volume invariance under flexing [9]-[10].

The existence of flat folded states for polyhedra homeomorphic to the 2-sphere was proved by the method of disk packing (see §18.3 in [5]), and for some special classes of convex polyhedra was also proved by the method of straight skeletons (see §18.4 in [5], and [4]).

Section 2 of this work is devoted to preliminaries. We also briefly present there (Theorem 1) the method to continuously flatten the Platonic polyhedra onto their original faces, proposed in [6].

⋆ Supported by Grand-in-Aid for Scientific Research (No.23540098), JSPS.

⋆⋆ Supported by Grand-in-Aid for Scientific Research (No.23540160), JSPS.

⋆⋆⋆ Partially supported by the grant PN-II-ID-PCE-2011-3-0533 of CNCS-UEFISCDI.

In Sect. 3 we propose a method to flatten general convex polyhedra by continuous folding processes (Theorem 2). We employ Alexandrov's gluing theorem and the structure of cut loci (see Definition 2) for the proofs.

In Sect. 4 we give a sufficient condition, under which every flat folded state of a convex polyhedron can be reached by a continuous folding process (Theorem 3).

We end the paper with a few remarks and open questions (Sect. 5).

2 Preliminaries

We start with the definition of a continuous folding process for a polyhedron.

Definition 1. *Let P be a polyhedron in the Euclidean space $\mathbb{R}^3$. We say that a family of polyhedra $\{P_t : 0 \leq t \leq 1\}$ is a* continuous folding process *from $P = P_0$ to P_1 if it satisfies the following conditions:*

(1) *for each $0 \leq t \leq 1$, there exists a polyhedron P'_t obtained from P by subdividing some faces of P (i.e., some faces of P'_t may be included in the same face of P, but P'_t is congruent to P) such that P_t is combinatorially equivalent to P'_t and the corresponding faces of P'_t and P_t are congruent,*

(2) *the mapping $[0, 1] \ni \tau \longmapsto P_\tau \in \{P_t : 0 \leq t \leq 1\}$ is continuous.*

Moreover, if P_1 is a flat folded polyhedron, we say that P is flattened by a continuous folding process *and we call P_1 a* flat folded polyhedron (or state) of *P.*

In the case of Platonic polyhedra, two of us proved the next result [6], which will serve as a contrast in Section 4.

Theorem 1. *For the five Platonic polyhedra there are continuous flat folding processes onto their original faces.*

Figure 1 shows how to continuously flatten the cube and the regular octahedron on their faces (see [6] for details or for the other Platonic polyhedra). Theorem 1 was proved by using a key lemma: any rhombus can be folded into a shape as showed in Fig.2 (2), with distances $|f(b)f(d)| = l$ and $|f(a)f(c| = m$ for any given $0 \leq l \leq |bd|$ and $0 \leq m \leq |ac|$, where we denote by $|xy|$ the Euclidean metric distance between $x, y \in \mathbb{R}^3$.

Our main tools here are the Alexandrov's gluing theorem (stated below) and the cut loci, to which the remaining of this section is devoted.

Alexandrov's gluing theorem. *Consider a topological sphere S obtained by gluing planar polygons (i.e., naturally identifying pairs of sides of the same length) such that at most 2π angle is glued at each point. Then S, endowed with the intrinsic metric induced by the distance in $\mathbb{R}^2$, is isometric to a polyhedral convex surface $P \subset \mathbb{R}^2$, possibly degenerated. Moreover, P is unique up to rigid motion and reflection in $\mathbb{R}^3$.* (See [2], p.100.)

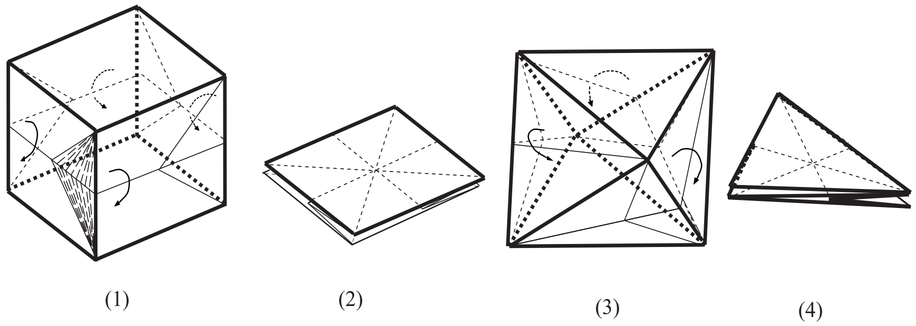

Fig. 1. (1) The cube; (2) the flatted cube on its face; (3) the regular octahedron; (4) the flatted octahedron on its face.

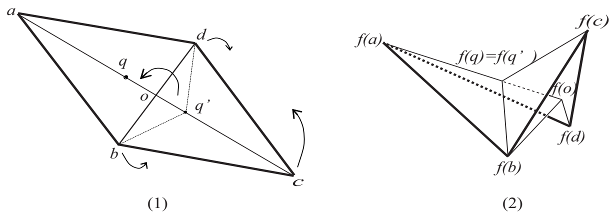

Fig. 2. An example of a folded rhombus.

Definition 2. *Let P be a convex polyhedron. The* cut locus $C(x) = C(x, P)$ *of the point x on P is defined as the set of endpoints (different from x) of all nonextendable shortest paths (on the surface P) starting at x.*

Figure 3 provides two examples of cut loci for points on the cube.

We summarize in our first lemma various known properties of cut loci.

Lemma 1. *(i) $C(x)$ is a tree whose leaves (endpoints) are vertices of P, and all vertices of P, excepting x (if the case), are included in $C(x)$. Notice that we allow vertices of degree two in $C(x)$.*

(ii) The junction points in $C(x)$ are joined to x by as many shortest paths as their degree in the tree.

(iii) The edges of $C(x)$ are shortest paths on P.

(iv) Assume the shortest paths γ and γ' from x to $y \in C(x)$ are bounding a domain D of P, which intersects no other shortest path from x to y. Then the arc of $C(x)$ at y towards D bisects the angle of D at y.

(v) If P has n vertices then $C(x)$ is a tree with $O(n)$ vertices, and it can be constructed in time $O(n^2)$.

Proof. (i)-(ii) and (iv) These are well known.

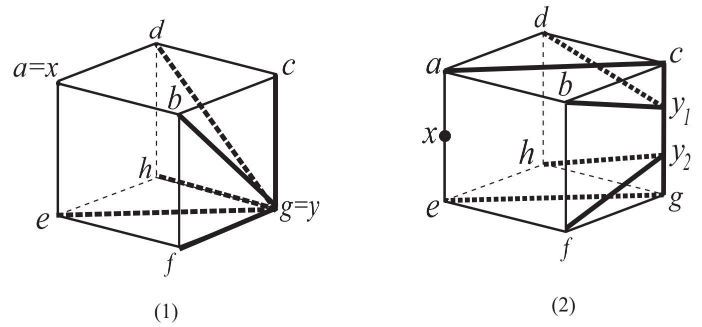

Fig. 3. (1) The cut locus of the vertex $x = a$ on a cube; (2) the cut locus on a cube with respect to the midpoint x of the edge ae.

(iii) This is Lemma 2.4 in [1].

(v) The first part is clear. For the second part, we use the algorithm of J. Chen and Y. Han [3] (see [7] for a public implementation). □

We will use the cube to illustrate our method. This method depends upon shortest paths from a point x on the cube to particular points on $C(x)$, y in Fig. 4(1), and y_1 and y_2 in Fig. 4(2).

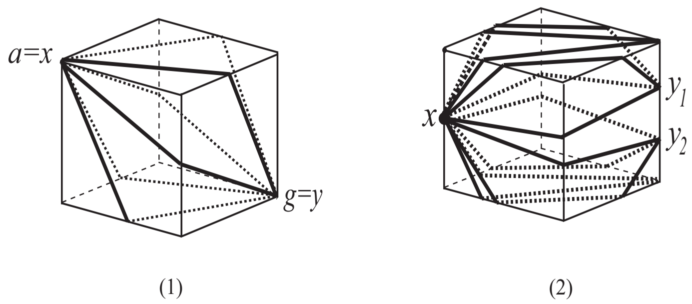

Fig. 4. (1) Shortest paths joining $x = a$ to $y = g$; (2) shortest paths joining x to y_1, y_2, and to the cube vertices that are interior points of $C(x)$.

We call an edge of $C(x)$ an *leaf edge* if it is incident to a leaf of $C(x)$.

3 Continuous Flattening Processes for Convex Polyhedra

In this section we provide a method to continuously flatten any convex polyhedron P, based on cut loci and Alexandrov's gluing theorem. Toward this goal, we further describe the structure of cut loci.

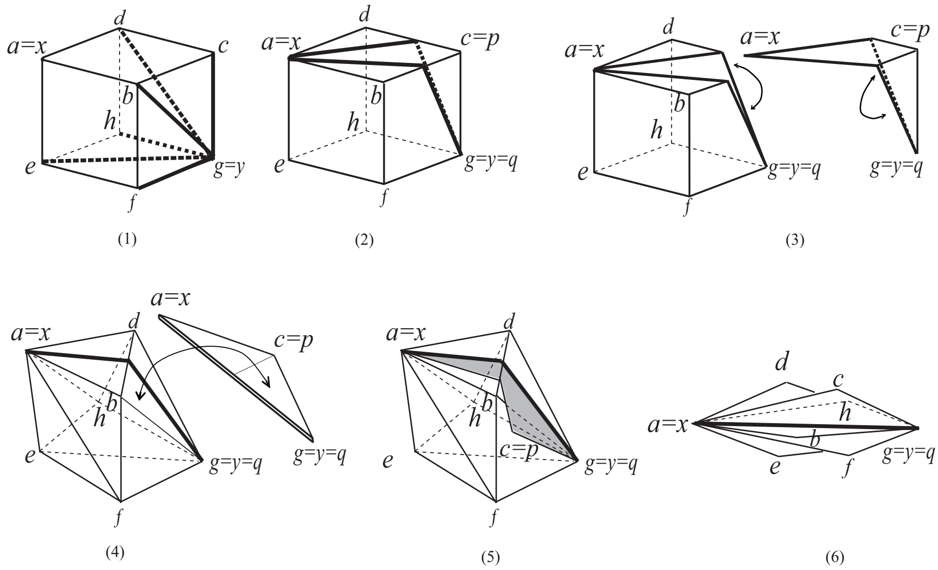

Fig. 5. (1) The cut locus of $x = a$; (2) two shortest paths γ_1 and γ_2 joining x to g, enclosing precisely one leaf edge $E = uv = gc$; (3) two parts of the cube, separated by $\gamma_1 \cup \gamma_2$; (4) two resulting surfaces obtained by gluing (the images of) γ_1 and γ_2; (5) the resulting polyhedron from the cube after gluing γ_1 to γ_2; (6) the flat folded state of the cube finally obtained.

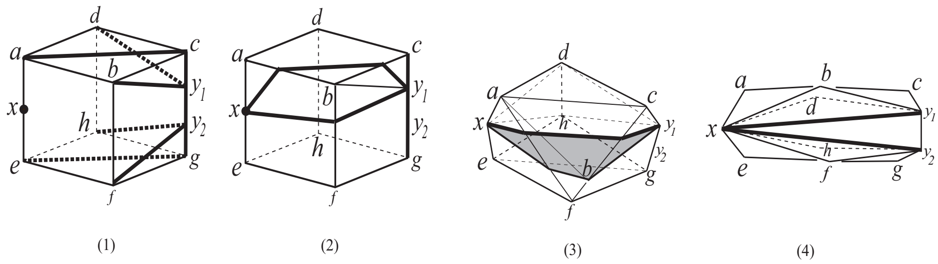

Fig. 6. (1) The cut locus of the midpoint x of ae; (2) two shortest paths γ_1 and γ_2 joining x to y_1, and enclosing precisely one leaf edge $E = uv = y_1b$; (3) the resulting polyhedron after gluing γ_1 to γ_2; (4) the flat folded state of the cube finally obtained.

We first give a high-level view of the method, presenting two different ways to flatten the cube, illustrated in Figs. 5 and 6. We start with an arbitrary point x on P, and determine its cut locus and all segments from x to the junction points of $C(x)$, see Figs 5 (1) and 6 (1). Every leaf edge E of $C(x)$ is included in some region T of P bounded by two consecutive segments from x to a junction point of $C(x)$ ($E = cg$ in Figure 5 (2/3) and $E = by_1$ in Figure 6 (2)). T can be flattened to a doubly covered triangle T_E, and $P \setminus T$ can be "zipped" to some convex polyhedron Q_E (by Alexandrov's gluing theorem). Therefore, P is isometric to $P_E = T_E \cup Q_E$, consisting of a bent doubly covered triangle T_E (shaded in the figures) attached to some convex polyhedron Q_E (Lemma 2), see Figs. 5 (4/5) and 6 (3). The cut locus $C(x, Q_E)$ of x on Q_E is precisely the truncation of the cut locus $C(x, P)$ (Lemma 3), hence we can iterate the process until $C(x)$ is merely a path, in which case the resulting polyhedron is already flattened (Lemma 4), see Figs. 5 (6) and 6 (4). Lemma 5 shows the continuity of this procedure.

Let γ_1 and γ_2 be shortest paths on P from $x \in P$ to $y \in C(x)$; cut along $\gamma_1 \cup \gamma_2$ and keep one half-surface P'. By *gluing* γ_1 *to* γ_2 we mean to identify the points on γ_1 and respectively γ_2 at equal distance to x.

Lemma 2. *Let x be a point on a convex polyhedron P. Each leaf edge $E = uv$ of the cut locus $C(x)$, starting at the leaf v of $C(x)$, is bounded by two shortest paths γ_1 and γ_2 from x to u, whose union encloses precisely one leaf, v, of $C(x)$. The region T of P, enclosed by $\gamma_1 \cup \gamma_2$ and containing v, can be flattened to a doubly covered triangle T_E, and the remaining part of P corresponds to a convex polyhedron Q_E by gluing γ_1 to γ_2. The original polyhedron P is isometric to the polyhedron $P_E = Q_E \cup T_E$, where we attach T_E to Q_E, such that γ_1 and γ_2 are touching each other but are included in distinct layers.*

Proof. Let $E = uv$ be any leaf edge with a leaf v of the cut locus $C(x)$. Then there are d shortest paths joining x to u on P, where d is the degree of the point u in the tree $C(x)$, and exactly two of them, say γ_1 and γ_2, enclose precisely one leaf v. Figs. 5(2) and 6(2) show the regions of the cube corresponding to the leaf edges cg and by_1 in the respective cut loci. The region T of P, bounded by $\gamma_1 \cup \gamma_2$ and containing v, has no other vertex of P inside, hence it consists of two flat congruent triangles with edges xv, uv and γ_i ($i = 1, 2$). Therefore, T can be flattened to some doubly covered triangle T_E, by gluing γ_1 to γ_2.

The remaining part of P corresponds to a convex polyhedron Q_E by Alexandrov's gluing theorem. Hence P is isometric to $P_E = Q_E \cup T_E$, where we attach T_E to Q_E such that γ_1 and γ_2 are touching each other but are included in distinct layers in P_E. Notice that, although flattened, T_E may not lie in a plane. $\square$

For an leaf edge E of $C(x, P)$ we will use the notation Q_E and P_E for the polyhedra introduced in Lemma 2.

Lemma 3. *Let x be a point in a convex polyhedron P, and let $E = uv$ be an leaf edge of the cut locus $C(x)$, incident to the the leaf v of $C(x)$. The cut locus*

$C(x, Q_E)$ of x on Q_E is (isometric to) the truncation of the cut locus $C(x, P)$ with respect to the cuts along γ_1 and γ_2 defined in Lemma 2, and the gluing along them.

Proof. This follows from the definition of cut locus and the property (iv) in Lemma 1. $\square$

Lemma 4. *If the cut locus $C(x)$ of a point x in a convex polyhedron P is a path, then P is a doubly covered polygon.*

Proof. Since all vertices of P except possibly x are included in $C(x)$, P is a doubly covered polygon, by Lemmas 2 and 3. $\square$

Lemma 5. *Let x be a point in a convex polyhedron P, whose cut locus $C(x)$ is not a path, and let $E = uv$ be an leaf edge of $C(x)$, incident to the the leaf v of $C(x)$. There is a continuous folding process from P to P_E.*

Proof. Let p_t be a point moving continuously from v to u along the edge $E = uv$, as t increases from 0 to 1, and denote $E_t = p_t v$. There are two shortest paths $\gamma_{t,1}$ and $\gamma_{t,2}$ joining x to p_t, enclosing precisely one leaf v of $C(x)$. By cutting along $\gamma_{t,1} \cup \gamma_{t,2}$ and gluing γ_{t_1} to γ_{t_2}, we obtain a doubly covered triangle $T_t = T_{E_t}$ and a convex polyhedron $Q_t = Q_{E_t}$, by Alexandrov's gluing theorem. Let $P_t = P_{E_t} = Q_t \cup T_t$ be defined similarly to P_E in Lemma 2. Here T_t is flipped clockwise about the point x and it touches Q_t, in order to avoid any conflict later.

We establish below a property of Q_t.

The structure of Q_t is given to us via Alexandrov's gluing theorem. We know the vertices of Q_t (they are x, p_t, and vertices of P), but not its edges. Nevertheless,
(i) the edges of Q_t between vertices (corresponding to vertices) of P are a subset of the collection of all shortest paths between pairs of such vertices;
(ii) the edges of Q_t from its vertex x to vertices (corresponding to vertices) of P are a subset of the collection of all shortest paths from x to such vertices;
(iii) the edges of Q_t from its vertex p_t to vertices (corresponding to vertices) of P are a subset of the collection of all shortest paths from p_t to such vertices;
(iv) the edge of Q_t between x and p_t corresponds to $\gamma_{t,1} \cup \gamma_{t,2}$.

Denote by G the subset of P consisting of all shortest paths joining pairs of vertices of P, or joining x to vertices of P.

Consider now a neighborhood N of u, and points $y_0, z_0 \in N$ such that the triangle $\Delta_0 = \triangle u y_0 z_0$ intersects $G \cup C(x)$ only at u. This is possible, because both G and $C(x)$ are composed of finitely many shortest paths on P, hence $N \cap (G \cup C(x))$ consists of finitely many shortest paths on N.

Let now $G'_t \subset P$ consist of all shortest paths from p_t to the vertices of P, and define $G_t = G \cup G'_t \cup \gamma_{t,1} \cup \gamma_{t,2}$.

It follows that $G_t \cap N$ is the union of finitely many line-segments, hence there exist points $y_t, z_t \in N \setminus G_t$ such that the triangle $\Delta_t = \triangle u y_t z_t$ intersects $G_t \cup C(x)$ only at u (see Fig. 7).

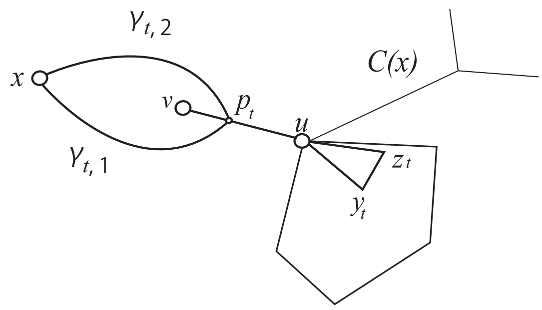

Fig. 7. A triangle $\triangle u y_t z_t$ such that $\triangle u y_t z_t \cap G_t = \{u\}$.

We claim that there are at most finitely many values $\tau_1, \tau_2, ..., \tau_m \in]0, 1[$ such that, for any number t in $]\tau_i, \tau_{i+1}[$, and any vertex v of P, the number of shortest paths from p_t to v does not depend on t ($i = 1, ..., m-1$). To prove the claim, recall that each edge of $C(v)$ is a shortest path on P, and so is uv (see Lemma 1). Therefore, for any edge E of $C(v)$, $E \cap uv$ is either $\emptyset$, or a point, or an arc A. Moreover, all points in $uv \setminus E$ are joined to v by precisely one shortest path, and if $E \cap uv = A$ then all interior points to A are joined to v by precisely two shortest paths.

By the above argument, the number of shortest paths from v to an exceptional point p_{τ_i} is at least equal to the number of shortest paths from v to p_t, for t sufficiently close to τ_i.

The above claim and the upper semi-continuity of shortest paths show that, for any $t \in [\tau_i, \tau_{i+1}[$, y_t and z_t can be choosen such that Δ_t depends continuously on t. Moreover, if t reaches an exceptional value τ_i, $0 < t < \tau_i$, then we may choose y_t and z_t such that $\Delta_{\tau_i} \subset \Delta_t$.

In conclusion, Δ_t exists and depends lower semi-continuously on $t \in [0, 1]$.

We show next how to realize P_t in $\mathbb{R}^3$, for $0 \leq t \leq 1$. Of course, it suffices to show how to realize Q_t.

For $t = 0$, we realize Q_0 (and hence $P_0 = P$) in $\mathbb{R}^3$, satisfying the conditions: *(i)* $u = (0, 0, 0)$, $y_0 = (y_{0,1}, 0, 0)$, and $z_0 = (z_{0,1}, z_{0,2}, 0)$, for some real numbers $y_{0,1} \geq 0$, $z_{0,1}$, $z_{0,2}$; and *(ii)* Q_0 is included in the half-space $z \geq 0$.

For $0 < t < 1$, we realize Q_t in the half-space $z \geq 0$ such that Δ_t is realized as the corresponding subset of N.

Assume now that $t = 1$. Let $\{t_n\}_{n \geq 1}$ be a sequence converging to 1, with $0 < t_n < 1$. Since the family $\{Q_{t_n}\}_{n \geq 1}$ is bounded with respect to the Hausdorff metric on the space of all compact sets in $\mathbb{R}^3$, there exists a subsequence which converges to a compact set R_1, by Blashke's convergence theorem. The unicity in Alexandrov's gluing theorem shows now that R_1 does not depend on the choice of the converging sequence $t_n \to 1$, hence we may realize Q_1 by R_1.

Concluding, if Q_t is close to Q_s then their corresponding faces are close to each other, in particular those including Δ_t and Δ_s, and hence their realizations in $\mathbb{R}^3$ are close to each other.

Finally, we notice that the mapping from $0 \leq t \leq 1$ to the 1-parameter family of compact sets $\{Q_t : 0 \leq t \leq 1\}$ is continuous with respect to the Hausdorff metric. Let s be a real number with $0 \leq s \leq 1$, and $\{s_n\}_{n \geq 1}$ a sequence converging to s, with $0 < s_n < 1$. The family $\{Q_{s_n}\}_{n \geq 1}$ is bounded with respect to the Hausdorff metric on the space of all compact sets in $\mathbb{R}^3$, hence there exists a subsequence which converges to a compact set R_s, by Blashke's convergence theorem, and the unicity in Alexandrov's gluing theorem ends the proof. □

Theorem 2. *For every convex polyhedron there exist infinitely many continuous flat folding processes.*

Proof. Let P be a convex polyhedron and let x be a point in P.

Step 1. Determine the cut locus $C(x)$, which is a tree (see Lemma 1).

Step 2. Flatten the region T of P corresponding to an leaf edge E of $C(x)$ (see Lemma 2). The remaining part of P, after flattening T as above, is realized as a convex polyhedron Q_E, by Alexandrov's gluing theorem. Therefore, the result P_E after this flattening is isometric to P, and consists of the polyhedron Q_E attached to the doubly covered triangle T_E. T_E should be laid clockwise about the point x in order to avoid conflict.

Step 3. Iterate Step 2 for Q_E instead of P, until $C(x, Q_E)$ is reduced to a path; i.e., until Q_E is a doubly covered polygon. Lemma 3 guarantees the iterations are possible, while Lemma 4 establishes the final form of Q.

Figs. 5(6) and 6(4) show the flat folded states of the cube after flattening all such regions corresponding to leaf edges of $C(x, P)$.

Since there are $O(n)$ vertices in $C(x)$, where n is the number of vertices of P (see Lemma 1), we have to flatten $O(n)$ regions of P corresponding to leaf edges of $C(x, Q)$ one by one, and therefore the flattening process ends after $O(n)$ iterations.

All folding processes corresponding to leaf edges are continuous by Lemma 5, so P is continuously folded to a flat folded state. □

4 Continuous Flattening Processes for Simple Flat Folded States

In this section we give a sufficient condition for a flat folded state of a convex polyhedron, to be reached by a continuous folding process.

Definition 3. *A 2-covered convex polygon consists of two copies of a convex polygon glued along some of their corresponding edges (the other edges are "cut").*

Figure 8(3) provides examples of 2-covered convex polygons. We will always regard such surfaces with boundary as having two congruent layers touching at their corresponding points, but glued along only some edges.

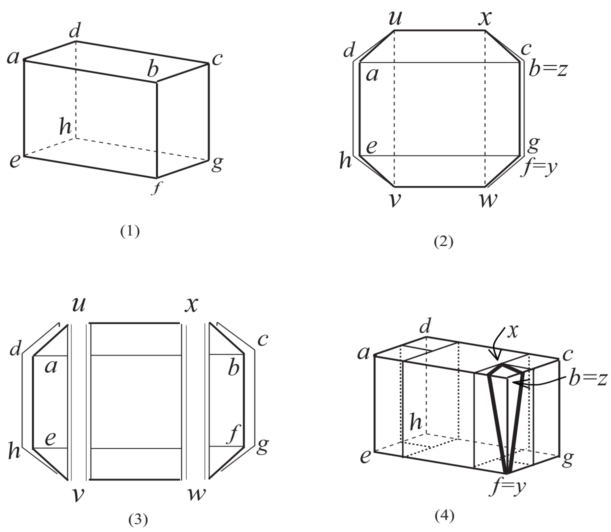

Fig. 8. (1) A rectangular box B; (2) a simple flat folded state of B obtained by pushing in two side faces of B; (3) a decomposition of the flat folded state into four congruent 2-covered trapezoids and one 2-covered rectangle; (4) the region enclosed by two shortest paths from x to y, corresponding to the doubly covered triangle xfb.

Definition 4. *A flat folded state P_f of a convex polyhedron is called simple if it enjoys the following properties:*
(i) P_f can be decomposed into a finite number of 2-covered convex polygons $\{R_i : 1 \leq i \leq k\}$,
(ii) for any $1 \leq i \leq k$, if $R_1, R_2, \cdots, R_{i-1}$ were cut off from P_f, then R_i can also be cut off from the remaining part of P_f by precisely one cut along an edge of R_i.

Notice that all flat folded states obtained in Theorem 2 are simple, while Theorem 1 provides examples of non-simple flat folded states; for example, the flat folded states shown in Figs. 1(2) and 1(4) are not simple because they do not satisfy the condition (ii). The folded state P_f showed in Fig. 1(2) or Fig. 1(4) can be decomposed into twelve 2-covered triangles R_i, but no R_i can be cut off from P_f by precisely one cut along an edge of R_i.

Figure 8(2) shows a flat folded state of a rectangular box B, where two side faces of B are pushed in. It is simple, because it can be decomposed into five 2-covered convex polygons as shown in Fig. 8(3), namely four congruent cut doubly trapezoids and one 2-covered rectangle. However, it cannot be obtained by the cut locus method described in the previous section.

Theorem 3. *Every simple flat folded state of a convex polyhedron can be reached by a continuous folding process.*

Proof. Let P_f be a simple flat folded state of a convex polyhedron P, decomposed into a finite collection of 2-covered convex polygons $\{R_i : 1 \leq i \leq k\}$ by cutting P_f along some edge of R_i, one by one. By subdividing R_i $(1 \leq i \leq k)$ if necessary, we can assume, without loss of generality, that all R_i are 2-covered triangles.

We prove the result by induction over k.

If $k = 1$ then $P = P_f$ and the conclusion holds.

Suppose now that the statement is true n, and assume $k = n+1$ for P_f. Let R_1 be a 2-covered triangle $\triangle xyz$ with the edge xy cut and edges xz and yz glued. Consider the cut locus $C(x) = C(x, P)$. Then $E = yz$ is a leaf edge of $C(x)$. By Lemma 5, there exists a continuous folding process from P to the polyhedron $P_E = Q_E \cup R_1$. Notice that Q_E has the flat folded state $Q_{E,f}$ consisting of 2-covered convex triangles $\{R_i : 2 \leq i \leq n+1\}$, and $Q_{E,f}$ satisfies the condition (ii) in Definition 4, with the edge xy of R_2 glued. By the induction's assumption, $Q_{E,f}$ can reached by a continuous folding process from Q_E. Therefore, P_f can reached by a continuous folding process from P. □

5 Remarks and Open Questions

Our first result in this paper proposes an algorithmic method to continuously flatten convex polyhedra. Here, the assumption of convexity is essential at two points. First, for the existence of the flat folded state we used the fact that each edge of the cut locus is a shortest path (Lemma 1 (iii)), property which is not true on non-convex surfaces. Second, for the continuity of the process, we used the uniqueness in Alexandrov's gluing theorem, which fails for non-convex surfaces. Since essentially the same argument is employed to prove our second result, a sufficient condition for the existence of continuous flattening processes, our proof there also fails for non-convex surfaces.

Our approach raises several questions concerning the structure of cut loci.

Our flattening procedure starts with the cut locus $C(x)$ of the point x in P, and at each step it treats (more precisely, it eliminates) all leaf edges of $C(x)$.

Question 1. For which polyhedra can one find points x whose cut locus has precisely one ramification point (i.e., $C(x)$ is homeomorphic to a star graph)? Generally, what is the minimal number of steps (i.e., of ramification points in $C(x)$) to end the procedure on a convex polyhedron P with n vertices, if the point x varies on P?

Consider now the continuous flattening process. It is based on a point continuously moving along $C(x)$. Suppose the movement is at constant speed. Then the flattening time is proportional to the length $\lambda C(x)$ of $C(x)$, so it seems of particular interest to find lower and upper bounds on $\lambda C(x)$.

Question 2. Can one locate on each P a point x with minimal length cut locus?

The starting point of our investigation is the question of Erik Demaine et al. (see Open Problem 18.1 in [5]), on the existence of continuous folding processes for all flat folded states of (not necessarily convex) polyhedra. This problem remains open, and can be rephrased –and widened– in a different framework, as follows. Consider an *abstract convex polyhedron* P (i.e., one obtained according to Alexandrov's gluing theorem). It has a unique *isometric embedding* in $\mathbb{R}^3$ as a convex surface, but many other non-convex *realizations* in $\mathbb{R}^3$ (see [8] for the precise definitions).

Question 3. Let $\mathcal{R}$ denote the space of all realizations of P in $\mathbb{R}^3$, with the topology induced by the Hausdorff metric on the space of all compact sets in $\mathbb{R}^3$. Is $\mathcal{R}$ arcwise connected?

Acknowledgement. The authors are indebted to Joseph O'Rourke for his careful reading of a preliminary version of this paper and his valuable suggestions, particularly improving the proof of Lemma 5.

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