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On Two Conjectures of Steinhaus

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Abstract. We disprove two conjectures of H. Steinhaus by showing that: (1) there is a convex surface S such that for any point x on S and any point y in the set F_x of farthest points from x, there are at most two segments from x to y; (2) the properties $|F_x| = 1$ and $F_{F_x} = x$ do not characterize the sphere.

Mathematics Subject Classification (1991): 52A15.

Key words: convex surface, centrally symmetric, intrinsic distance, (geodesic) segment, farthest points.

1. The First Conjecture

In a very interesting book by Croft, Falconer and Guy ([2], p. 44 (iii)) we find the following conjecture of Steinhaus: on each closed convex surface (whose class of differentiability remains unprecise), there exist points x and y in the set F_x of farthest points from x, with at least three segments (i.e. shortest paths) joining them. We give here an example of a convex surface S disproving Steinhaus's conjecture. First we construct it and then prove that S provides a suitable example.

In the following, we consider closed convex surfaces in the Euclidean space $\mathbb{R}^3$. The coordinates in $\mathbb{R}^3$ will be x_1, x_2, x_3. For two points x, y on a surface, the geodesic (intrinsic) distance between them will be denoted by $\delta(x, y)$. $|A|$ denotes the cardinality of the set A. Also, we shall not distinguish between a point and the set containing exactly that point.

Let C_1 be the arc of the circle of centre $o' = (0, -\alpha, 0)$ and radius 1 lying in the half-plane $x_2 \geqslant 0$ ($\alpha \in (0, 1)$). Let C_2 be the arc symmetric to C_1 with respect to the x_1-axis (see Figure 1).

Denote $\{a, b\} = C_1 \cap C_2$ and let d be the middle point of C_1. Take the points v_1 and v_2 on the x_3-axis, symmetric with respect to the origin o and consider the boundary S of the convex hull of the set $\{v_1, v_2\} \cup C_1 \cup C_2$. We choose v_1 and v_2 far enough such that the length $l(C_1)$ of C_1 be less than the distance from d to v_i and such that $\beta < \pi$, where β denotes the total angle of the tangent cone at v_i ($i = 1, 2$). For any point u on C_1 we have $l(C_1) < \delta\ (u, v_1)$, since $\|d\| < \|u\|$.

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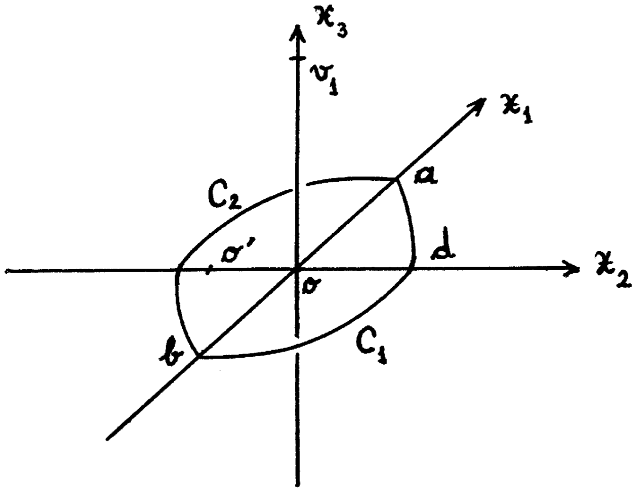

Figure 1.

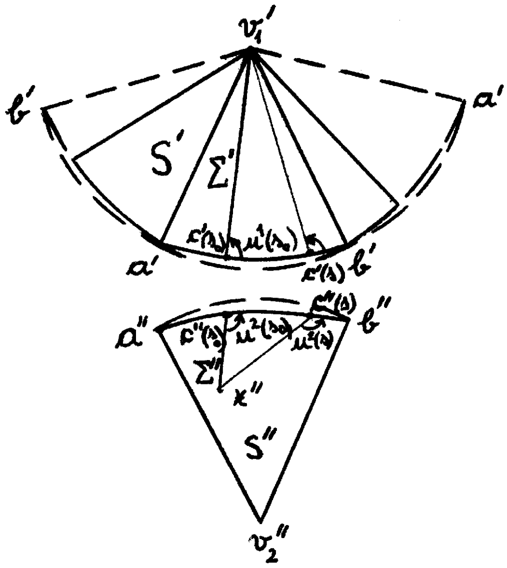

Figure 2.

PROPOSITION 1. *On the surface S constructed above, from an arbitrary point $x \in S$ to any point $y \in F_x$ there are at most two segments.*

Proof. The surfaces $S_+ = \{(x_1, x_2, x_3) \in S | x_3 \geqslant 0\}$ and $S_- = \{(x_1, x_2, x_3) \in S | x_3 \leqslant 0\}$ are unfoldable. Let us consider the unfoldings S' and S'' as in Figure 2, where we marked by a 'prime' and a 'double prime' the points or arcs of S' and S'' corresponding to those of S. On each unfolding the segments of S became line segments. In the following we shall implicitly refer to these unfoldings.

Let x be a point on S. We can assume that $x \in S_- \cap \{(x_1, x_2, x_3) | x_2 \geqslant 0\}$. If $x = a$ (or $x = b$) then $F_x = \{v_1, v_2\}$ and from x to each v_i ($i = 1, 2$) there is precisely one segment. If x lies on the segment av_2 and is different from a then there are two segments from x to v_1, symmetric with respect to the plane x_1ox_3. Suppose now $x \notin av_2$; let $s \in [0, l(C_1)]$ be the canonical parameter on the curve C_1 and $c(s)$ the corresponding point on C_1 ($c(0) = a$). Let $u^1(s)$ be the angle made in $c(s)$ by C_1 with the segment $\Gamma_{c(s)v_1}$ (from $c(s)$ to v_1) and let $u^2(s)$ be the angle made in $c(s)$ by the segment $\Gamma_{c(s)x}$ with C_1. Since the angles u^1 and u^2 are increasing functions of s, their sum is also. A path Σ from x to v_1 is a segment if and only if the opposite angles made with C_1 in the point $c(s_0)$ given by $\{c(s_0)\} = \Sigma \cap C_1$ are equal. Equivalently, Σ is a segment from x to v_1 if and only if $u^1(s_0) + u^2(s_0) = \pi$. Since $u^1(0) + u^2(0) < \pi$ and $u^1(l(C_1)) + u^2(l(C_1)) > \pi$ there is an unique value $s_0 \in [0, l(C_1)]$ with $u^1(s_0) + u^2(s_0) = \pi$. Therefore the proof is finished if we show that $F_x \subseteq \{v_1, v_2\}$, with equality if and only if $x \in C_1 \cup C_2$ (because of the symmetry with respect to the plane x_1ox_2). Equivalently, we have to check that all points of the surface S_+ are at distance at most $\delta(x, v_1)$ from x. If z is an arbitrary point on S_+ then we have

$$\begin{aligned}
\delta(x, z) &\leqslant \delta(x, c(s_0)) + \delta(c(s_0), z) \\
&\leqslant \delta(x, c(s_0)) + \max\{\delta(c(s_0), v_1), l(C_1)\} \\
&\leqslant \delta(x, c(s_0)) + \delta(c(s_0), v_1) = \delta(x, v_1),
\end{aligned}$$

with equality precisely for $z = v_1$. □

This conjecture of Steinhaus remains open for the smooth case. It remains also open if restricted to those convex surfaces S with $|F_x| = 1$ for all $x \in S$.

T. Zamfirescu [5] conjectured that on any convex surface S there are two points x, y such that from x to y there are at least three segments. This conjecture is also open.

2. The Second Conjecture

In the same book of Croft, Falconer and Guy ([2], p. 44 (ii)) we find another conjecture of Steinhaus saying that a convex surface S is a sphere if it verifies the properties (SC): $|F_x| = 1$ and $F_{F_x} = x$ for any point x on S. We shall disprove this conjecture by showing that the regular tetrahedron satisfies the above two conditions. Thus two new problems arise naturally:

PROBLEM A. *Characterize the family $\mathcal{SC}$ of all convex surfaces which verify Steinhaus's conditions (SC).*

Problem B. *Sharpen (SC) to obtain a characterization of the sphere.*

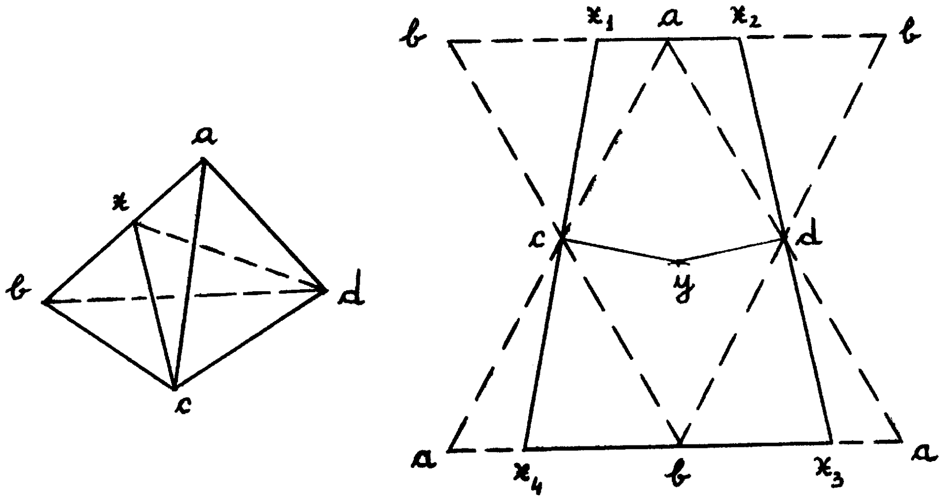

Figure 3.

In the last part of this section we give a partial answer to Problem A by defining a set $\mathcal{R}$ of convex surfaces and proving that $\mathcal{R}$ is a subfamily of $\mathcal{SC}$.

By a regular tetrahedron we mean here the boundary of the regular simplex in $\mathbb{R}^3$.

THEOREM 2. *All points x on a regular tetrahedron satisfy $F_{F_x} = x$ and $|F_x| = 1$.*

Proof. Let x be a point on a regular tetrahedron $T = abcd$. We shall prove separately the above two properties. We shall treat the cases:

(1) x is a vertex;
(2) x is interior to an edge;
(3) x is interior to a face.

If the natural correspondence between elements on the tetrahedron T and their correspondents on unfoldings is one-to-one then we shall make no distinction between them.

Property $|F_x| = 1$

(1) Assume that $x = a$ and let us consider for T an equilateral triangle $a_1a_2a_3$ as unfolding, obtained by cutting T along the edges ab, ac and ad and straightening. Let y be the centre of the circumscribed circle of this equilateral triangle. One can easily see that any other point $z \in T$ is closer to a than y, hence we have $F_a = y$.

(2) Assume that our point x is interior to an edge, say ab, and is closer to a than to b or is the middle point of the segment ab. Let us cut T along the segments xc, xd and ab and consider an unfolding as in Figure 3. We get the isosceles trapezoid $x_1x_2x_3x_4$. The centre y of the circumscribed circle of this trapezoid is uniquely determined by x and $F_x = y$.

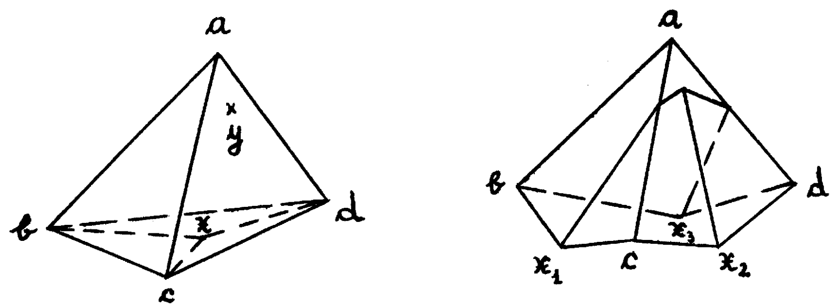

Figure 4.

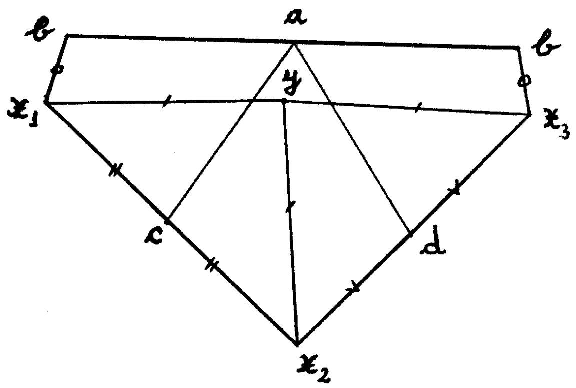

Figure 5.

(3) Assume that x is interior to the triangle bcd (see Figure 4). We cut T along the segments xb, xc and xd and we straighten the full triangles xbc, xcd and xdb to obtain the respectively coplanar points a, b, c, x_1; a, c, d, x_2 and a, d, b, x_3. Let T' be the obtained (nonclosed) surface. Let y be a point in F_x. If y is the vertex a then x must be the centre of the triangle bcd and y is uniquely determined by x. If y belongs to an edge (say ac) or to a face (say acd) then there are three segments from x to y, namely one from each x_i to y, $i = 1, 2, 3$. This follows from a known, more general fact: from any point x on a convex surface S to any point $y \in F_x$ where the tangent cone has full angle 2π there are at least three segments. Assume that y is interior to the face acd (the proof for the case $y \in ac$ is similar). If we cut T' (see Figure 4) along the edge ab and we straighten such that the faces abx_1c, acx_2d and adx_3b come into the same plane then we get a planar picture T'' as in Figure 5.

Then c is the midpoint of x_1x_2, d is the midpoint of x_2x_3 and y is the circumcentre of $x_1x_2x_3$.

Property $F_{F_x} = x$

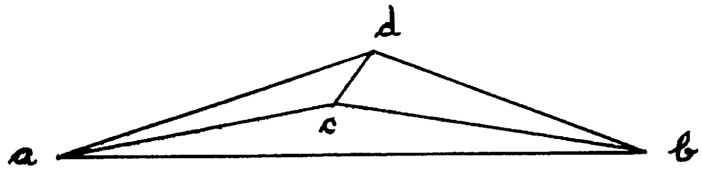

Figure 6.

This follows at once from the fact that y is the circumcentre of $a_1a_2a_3$ in case 1, of $x_1x_2x_3x_4$ in Case 2, and of $x_1x_2x_3$ in Case 3.

This ends the proof of Theorem 2. □

Remark 3. One can prove that $|F_x| \leqslant 2$ for any arbitrary tetrahedron $\mathbf{T}$, for all points $x \in \mathbf{T}$.

Remark 4. The property $F_{F_x} = x$ is not true for an arbitrary tetrahedron. One can see it on an 'almost degenerate' tetrahedron $abcd$ with c and d close to the middle of ab (see Figure 6).

STEINHAUS' CONDITIONS IN THE PRESENCE OF SYMMETRY

The regular tetrahedrons are far from spheres in the space of all nondegenerate convex surfaces endowed with the Pompeiu–Haussdorf metric $\rho(S_1, S_2) = \max\{\sup_{x\in S_1} \inf_{y\in S_2} \|x-y\|, \sup_{x\in S_2} \inf_{y\in S_1} \|x-y\|\}$. Moreover, the proof for the regular tetrahedron is quite elementary, but this example does not cover the smooth case. Next we shall study Steinhaus' conditions (SC) in the presence of a symmetry centre and discover a whole class $\mathcal{R}$ of convex surfaces which verify (SC), including the ellipsoids of revolution with axes $a = b > c$. Thus $\mathcal{R}$ intersects all neighborhoods of a sphere.

From now on we shall denote by a 'prime' all symmetric objects.

THEOREM 5. *Let S be a centrally symmetric convex surface $(S = S')$ and $x \in S$. The following statements are equivalent:*

(i) $F_y = x$ *for all* $y \in F_x$;
(ii) $F_x = x'$.

Moreover, both of them imply $|F_x| = 1$.

Proof. That (ii) implies (i) is immediate.

We now show that (ii) implies (i). Suppose there is a point $y \in F_x$ such that $y \neq x'$. By hypothesis and by symmetry we have $F_y = x \neq y'$, $F_{x'} = y'$, $F_{y'} = x'$ and $\delta(x, y) = \delta(x', y')$. Denote by Σ a segment between x and x'. By symmetry, Σ' is a segment between x' and x and the curve $\Sigma \cup \Sigma'$ divides the surface S in two symmetric (topologically open) half-surfaces S_1 and S_1'. Since $y' \neq x$ and $\delta(x, y) \geqslant \delta(x, x')$, $y \notin \Sigma \cup \Sigma'$. Suppose that $y \in S_1$, hence $y' \in S_1'$ and let Λ be a

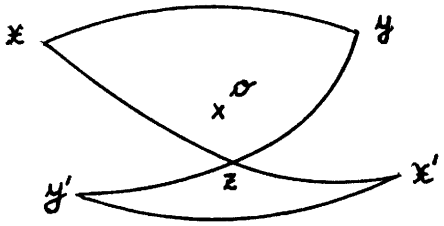

Figure 7.

segment from y to y'. We have $(\Lambda \cap \Sigma) \cup (\Lambda \cap \Sigma') \neq \phi$; suppose for example that $\Lambda \cap \Sigma = \{z\}$ (see Figure 7).

By applying twice the triangle inequality, we get $\delta(x, y) \leqslant \delta(x, z) + \delta(z, y)$ and $\delta(x', y') \leqslant \delta(x', z) + \delta(z, y')$. Therefore $\delta(x, y) + \delta(x', y') \leqslant \delta(x, z) + \delta(z, y) + \delta(x', z) + \delta(z, y') = \delta(x, x') + \delta(y, y')$. But $\delta(x, y) \geqslant \delta(x, x')$ and $\delta(x', y') = \delta(y, x) \geqslant \delta(y, y')$. Hence $\delta(x, y) = \delta(x, z) + \delta(z, y)$. Since geodesics do not admit bifurcations, this implies that a segment from x to y strictly includes Σ. But now Σ' provides such a bifurcation, and a contradiction is found. □

We propose here the following open problem: *on any centrally symmetric convex surface S if $|F_x| = 1$ for all $x \in S$ then $F_x = x'$.*

PROPOSITION 6. *Let S be a convex surface with a symmetry centre o. If the points $x, y \in S$ determine the intrinsic diameter of S then they are symmetric to each other: $y = x'$.*

Proof. Similar to that of Theorem 5. Suppose there is a point $y \in F_x$ such that $y \neq x'$. By hypothesis and by symmetry we have $x \in F_y$, $x \neq y'$, $F_{x'} = y'$, $x' \in F_{y'}$ and $\delta(x, y) = \delta(x', y')$. Denote by Σ a segment between x and x'. By symmetry, Σ' is a segment between x' and x and the curve $\Sigma \cup \Sigma'$ divides the surface S in two symmetric (topologically open) half-surfaces S_1 and S_1'. Since $y' \neq x$ and $\delta(x, y) \geqslant \delta(x, x')$, $y \notin \Sigma \cup \Sigma'$. Suppose that $y \in S_1$, hence $y' \in S_1'$, and let Λ be a segment from y to y'. We have $(\Lambda \cap \Sigma) \cup (\Lambda \cap \Sigma') \neq \phi$; suppose for example that $\Lambda \cap \Sigma = \{z\}$ (see Figure 7). By applying twice the triangle inequality, we get $\delta(x, y) \leqslant \delta(x, z) + \delta(z, y)$ and $\delta(x', y') \leqslant \delta(x', z) + \delta(z, y')$. Therefore $\delta(x, y) + \delta(x', y') \leqslant \delta(x, z) + \delta(z, y) + \delta(x', z) + \delta(z, y') = \delta(x, x') + \delta(y, y')$. But $\delta(x, y) \geqslant \delta(x, x')$ and $\delta(x', y') = \delta(y, x) \geqslant \delta(y, y')$. Hence $\delta(x, y) = \delta(x, z) + \delta(z, y)$. Since geodesics do not admit bifurcations, this implies that a segment from x to y strictly includes Σ. But now Σ' provides such a bifurcation, and a contradiction is found. □

COROLLARY 7. *If S is a convex surface of revolution with symmetry centre then the diameter of S is equal to the half-length of a meridian.*

Proof. Let $x, y \in S$ such that $\operatorname{diam} S = \delta(x, y)$. By Proposition 6 we have $y = F_x = x'$. Since all meridians are geodesics and between any two symmetrical points there is a meridian $\mathcal{M}$ we get $\delta(x, x') \leqslant \frac{1}{2}l(\mathcal{M})$. But the only segments between the poles of the surface are the meridians, therefore $\delta(x, y) = \frac{1}{2}l(\mathcal{M})$. $\square$

Let $\varphi: [0, a] \to \mathbb{R}$ be a convex function differentiable at 0 with $-a < \varphi(0) < 0 = \varphi(a)$, $\varphi'(0) = 0$ and such that $\psi: [0, a] \to \mathbb{R}$ given by $\psi(u) = \sqrt{u^2 + \varphi(u)^2}$ is an increasing function. Let $\mathcal{R}$ be the set of all surfaces obtained by rotating the curves C given by $\pm\varphi$, i. e. with

$$x_1 = u \cos \beta, \qquad x_2 = u \sin \beta, \qquad x_3 = \pm\varphi(u) \quad (u \in [0, a], \beta \in [0, 2\pi]).$$

Each surface in $\mathcal{R}$ is convex (because φ is a convex function), is differentiable at its points with $x_1 = x_2 = 0$ (because $\varphi'(0) = 0$) and is symmetric with respect to the plane $x_1 o x_2$.

Also notice that $\mathcal{R}$ contains the ellipsoids of revolution with axis $a = b > c$.

LEMMA 8. *The length of a curve* Γ *in* $\mathbb{R}^3$ *is at least as long as its metric projection onto a convex body.*

Proof. See [1] p. 80. $\square$

THEOREM 9. $\mathcal{R} \subset \mathcal{SC}$.

Proof. Let $S \in \mathcal{R}$. First, let $u \in S$ be an equatorial point and $\mathcal{M}$ a meridian of S through u and $-u$. Let $S_{\mathcal{M}}$ be the surface obtained by the revolution of $\mathcal{M}$ around the line through u and the origin. Because ψ is an increasing function, the surface $S_{\mathcal{M}}$ lies in the convex hull of S and $S \cap S_{\mathcal{M}} = \mathcal{M}$. By Lemma 8, the only segments of S from u to $-u$ are the half-meridians. By Corollary 7 and Proposition 6, $F_u = -u$.

Let $x \in S$ be now an arbitrary (but not equatorial) point. Let Γ be segment from x to $-x$ and let u be the intersection point between Γ and the equator. By the symmetry of S we get $\delta(-x, u) = \delta(x, -u)$. Therefore the length of Γ is larger than or equal to the length of a half-meridian: $l(\Gamma) = \delta(u, x) + \delta\ (x, -u) \geqslant \frac{1}{2}(\mathcal{M})$. Hence $\delta(x, -x) = \frac{1}{2}(\mathcal{M})$ and we obtain $F_x = -x$ for all $x \in S$. Applying Theorem 5 we get $S \in \mathcal{SC}$. $\square$

Remark 10. As shown in the above proof, for all surfaces $S \in \mathcal{R}$ the only geodesic segments from an arbitrary point $x \in S$ to x' are half-meridians.

Acknowledgement

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References

1. Busemann, H.: *Convex Surfaces*, Interscience Publishers, New York, 1958.
2. Croft, H. T., Falconer, K. J. and Guy, R. K.: *Unsolved Problems in Geometry*, Springer-Verlag, New York, 1991.

3. Steinhaus, H.: On shortest path on closed surfaces, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astr. Phys.* **6** (1958), 303–308.
4. Steinhaus, H.: Problem 291, *Colloq. Math.* **7** (1959), 110.
5. Zamfirescu, T.: Points joined by three shortest paths on convex surfaces, *Proc. Amer. Math. Soc.* **123** (1995), 3513–3518.