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Quasigeodesics and farthest points on convex surfaces

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Abstract. We prove a quadrilateral comparison result for convex surfaces, involving projections onto quasigeodesics. As an application, we locate the set of all farthest points (i.e., points at maximal intrinsic distance from some point) in a convex surface, with respect to a simple closed quasigeodesic.

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1 Introduction

A *convex surface* S is the boundary of a *convex body* (compact convex set with interior points) in the Euclidean space $\mathbb{R}^3$, or a doubly covered planar convex body. The metric ρ of S is defined, for any points x, y in S, as the length $\rho(x, y)$ of a *segment* (i.e., shortest path on S) from x to y. The distance from the point x in S to a closed subset K of S is given by $\rho(x, K) = \min_{y \in K} \rho(x, y)$.

Several comparison results are well-known for convex surfaces, see for example [1], [14]; basically, they compare triangles or hinges in a given surface to those in the plane. We propose, with our Theorem 1 and Corollary 2, a quadrilateral type comparison.

Theorem 1 states, for any convex surface, that the length of a segment γ_{uv} between points u, v on the same side of, and at equal distance to, a simple quasigeodesic δ is at most equal to the length of any subarc $\delta_{u_0 v_0}$ of δ between projections u_0, v_0 of u, v onto the relative interior of δ, provided that $\gamma_{uv} \cap \delta_{u_0 v_0} = \emptyset$. Corollary 2 was inspired by the property of nonpositively curved manifolds, that the projection onto closed convex subsets is distance decreasing (see [2] p. 9); it stipulates, for any convex domain D on any convex surface, that the projection onto $\operatorname{bd} D$ of any two points in D at equal distance to $\operatorname{bd} D$ is distance increasing.

As an application, we repeatedly use Theorem 1 for proving a criterion to locate farthest points on convex surfaces, with respect to simple closed quasigeodesics. We will give later the precise definitions; here we just notice that any geodesic of a convex surface S is a quasigeodesic, and if S has bounded specific curvature (in particular, if S is

differentiable) then the converse is also true (see [3] pp. 114 and 27).

For any point x in S, denote by ρ_x the *distance function from* x, $\rho_x(y) = \rho(x, y)$ for any $y \in S$, by F_x the set of all *farthest points from* x (i.e., points of global maximum for ρ_x), and by F the corresponding multivalued mapping. Put $\mathbb{F} = \bigcup_{x \in S} F_x$, the set of all farthest points. Chapter A35 in the book [4] of H. Croft, K. Falconer and R. Guy indicates several questions of H. Steinhaus, asking for characterizations of the sets of farthest points.

Our Theorem 2 contributes to this topic, by stating that all farthest points are located around points at maximal distance from a simple closed quasigeodesic O, in intrinsic balls of radii $\lambda(O)$, where $\lambda(C)$ stands for the length of the curve C. As consequences of Theorem 2, Corollaries 3 to 6 give sufficient conditions for the (closed) set $\mathbb{F}$ to be disconnected, and thus for the existence of points with multiple farthest points. Refer to [21] for an overview on this topic.

For the reader's convenience, in the remaining part of this introduction we recall some more definitions and give additional notation. A *geodesic* is a curve $\gamma : [0, l] \to S$ which is locally segment: for each $s \in [0, l]$ there exists $\varepsilon > 0$ such that the restriction of γ to $[s-\varepsilon, s+\varepsilon] \cap [0, l]$ is a segment (i.e., a shortest path) between its extremities. (We restrict our attention in this paper to compact geodesics.)

Consider a broken geodesic γ which is a Jordan arc, say $\gamma = \bigcup_{i=0}^{n} \gamma_{a_i a_{i+1}}$, where $\gamma_{a_i a_{i+1}}$ is a geodesic joining the points $a_i, a_{i+1} \in S$ $(i = 0, \ldots, n)$. Then a *right* and a *left side* can be consistently defined locally along $\gamma \setminus \{a_0, a_{n+1}\}$. Denote by α_i and β_i the angles between $\gamma_{a_i a_{i-1}}$ and $\gamma_{a_i a_{i+1}}$ to the right and to the left of γ, respectively. The *right* and *left swerve* of γ are the numbers $s_r(\gamma) = \sum_{i=1}^{n}(\pi - \alpha_i)$, $s_l(\gamma) = \sum_{i=1}^{n}(\pi - \beta_i)$.

Consider now a Jordan arc A which has definite tangent directions at its endpoints p, q, and γ a broken geodesic from p to q which is a Jordan arc and lies to the right of, or on, A. Denote by θ_p and θ_q the angles between γ and A at p and q. Then $\lim(\theta_p + \theta_q + s_r(\gamma))$ exists when γ approaches A from the right (see [1] p. 353) and it is called the *right swerve* of A ([3], p. 109). The *left swerve* is defined similarly.

A *quasigeodesic arc* (respectively a *closed quasigeodesic curve*) is a Jordan arc (respectively a closed Jordan curve) which has definite tangent directions at each point and every subarc of which has nonnegative right and left swerves ([3], p. 114). A *quasigeodesic* is a curve each subarc of which is quasigeodesic. A segment connecting two points with complete angles $\leq \pi$ forms, traversed back and forth, a *degenerate* closed quasigeodesic.

A *convex subset* of a convex surface is defined to contain with any two points a segment joining them.

A *domain* of a convex surface is a subset homeomorphic to a closed disk; it is called (*quasi*)*geodesic* if its boundary consists of finitely many (quasi)geodesic arcs. A (quasi)-geodesic *quadrilateral* is a domain bounded by four (quasi)geodesic arcs. A *triangle* on a convex surface is a domain whose boundary consists of three segments. We shall denote a quasigeodesic domain by the sequence of its boundary arcs or, if those are clear from the context, by the sequence of their intersection points.

The angle towards the domain D made by the boundary arcs γ and γ' at their common point $a \in D$ will be denoted by $\angle^D \gamma\gamma'$.

For the points u, v on the Jordan arc A, A_{uv} will denote the subarc of A joining them.

As usual, $\operatorname{diam}(S)$ is the intrinsic diameter of S, $\operatorname{diam}(S) = \max_{x,y\in S} \rho(x,y)$, $\operatorname{relint} A$ is the *relative interior* of A, and $||\cdot||$ is the standard norm in $\mathbb{R}^3$.

2 Basic results

We shall need the following version of Alexandrov's gluing theorem (see [1] p. 362 or [3] p. 154).

Lemma 1. *The* 2*-manifold resulting from gluing together along the boundary, by identifying the points via the natural isometry, two isometric copies of a quasigeodesic* (*respectively convex*) *domain of a convex surface is isometric to a convex surface.*

We shall also use Pogorelov's rigidity theorem ([15] p. 167).

Lemma 2. *Any two isometric convex surfaces are congruent.*

For the following result, see [1] p. 215.

Lemma 3. *For any triangle $\gamma_1\gamma_2\gamma_3$ in a convex surface denote by $\overline{\gamma}_1\overline{\gamma}_2\overline{\gamma}_3$ a planar triangle with $\lambda(\gamma_i) = \lambda(\overline{\gamma}_i)$. Then $\angle\overline{\gamma}_i\overline{\gamma}_{i+1} \leq \angle\gamma_i\gamma_{i+1}$, $i = 1,2,3 \pmod 3$, and equality holds if and only if the triangles $\gamma_1\gamma_2\gamma_3$ and $\overline{\gamma}_1\overline{\gamma}_2\overline{\gamma}_3$ are isometric.*

When they exist, the left and right tangent directions of the arc A at $p \in A$ will be denoted by τ_p^- and τ_p^+, respectively.

In the following lemma, the notation $\angle\gamma\tau$ stands for the angle at the point w, between the tangent direction τ to S at w and the tangent direction to the segment γ starting at w.

Lemma 4. *Let S be a convex surface, C a curve on S having tangent directions τ_y^-, τ_y^+ at every $y \in \operatorname{relint} C$, and let x be a point in $S \setminus C$.*

a) *Suppose there exist points $z, v \in \operatorname{relint} C$ such that $\max_{\gamma_{zx}} \angle\gamma_{zx}\tau_z^+ < \pi/2$ and $\min_{\gamma_{vx}} \angle\gamma_{vx}\tau_v^- > \pi/2$, where γ_{zx} and γ_{vx} are segments from z and v to x. Then there are points $z_m \in \operatorname{relint} C$ close to z in the direction τ_z^+, and $v_M \in \operatorname{relint} C$ close to v in the direction τ_v^-, such that $\rho(x, z_m) < \rho(x, z)$ and $\rho(x, v_M) > \rho(x, v)$.*
b) *If $x_0 \in \operatorname{relint} C$ such that $\rho(x, x_0) = \rho(x, C)$, and γ_{x_0x} is a segment joining x_0 to x, then $\min\{\angle\gamma_{x_0x}\tau_{x_0}^+, \angle\gamma_{x_0x}\tau_{x_0}^-\} \geq \pi/2$.*

Proof. The first part is a direct consequence of the first variation formula (Theorem 3.5 in [13]). The second part follows from the first one. □

Notice that a point x may have several *projections onto* (i.e., points realizing the distance to) a quasigeodesic.

Also notice that for any point x on any quasigeodesic δ, the angles between the tangent directions of δ at x on both sides of δ are at most π. This is clear for polyhedral convex surfaces from the definition of quasigeodesics, and remains true by passing to the limit.

Corollary 1. *Let δ be a quasigeodesic on a convex surface S, x a point in $S \setminus \delta$ and $x_0 \in \operatorname{relint} \delta$ a projection of x onto δ. Then there exists, on each side of δ, at most one segment joining x_0 to x, each of which is orthogonal to δ.*

Proof. Fix a segment γ_{x_0x} from x_0 to x, say on the right side of δ. The angle α_γ made by the tangent directions $\tau^-_{x_0}$, $\tau^+_{x_0}$ of δ at x_0, on the right side of δ, is at most π. Therefore, $\min\{\angle\gamma_{x_0x}\tau^-_{x_0}, \angle\gamma_{x_0x}\tau^+_{x_0}\}$ is at most $\alpha_\gamma/2 \le \pi/2$.

By the use of *b)* in Lemma 4, we also obtain $\min\{\angle\gamma_{x_0x}\tau^-_{x_0}, \angle\gamma_{x_0x}\tau^+_{x_0}\} \ge \pi/2$, so $\angle\gamma_{x_0x}\tau^-_{x_0} = \angle\gamma_{x_0x}\tau^+_{x_0} = \pi/2$. Since the complete angle at x_0 is at most 2π, the rest of the proof follows. □

3 Projections onto quasigeodesics

The main goal of this section is to establish Theorem 1, a quadrilateral comparison result which will be the key ingredient for the proofs of Corollary 2 and Theorem 2.

We start with two lemmas; the first one, an elementary property, is the final step in the proof of Theorem 1.

Lemma 5. *If p, q are two points inside a planar rectangle $xyzv$ such that $||p - v|| = ||q - z|| = ||x - v||$ then $||p - q|| \le ||x - y||$, with equality if and only if either $p = x$ and $q = y$, or $p \in yz$ and $q \in xv$.*

Proof. Put $R = xyzv$, $a = ||x - v||$, $D_v = R \cap \{y : ||y - v|| \le a\}$ and $D_z = R \cap \{y : ||y - z|| \le a\}$. Assume that $D_z \cap D_v \ne \emptyset$ (the other case is simpler).

The point t of intersection between pz and $\operatorname{bd} D_z$ divides $\operatorname{bd} D_z$ into two subarcs, say A_y containing y, and A_v meeting $xv \cup vz$.

Suppose first that $p \in R \setminus D_z$ (or, similarly, $q \in R \setminus D_v$). If $q \in A_y$ then

$$||p - q|| \le \max\{||p_q - q||, ||p - q_p||\} \le ||x - y||,$$

where $p_q \in \operatorname{bd} D_v$ and $q_p \in \operatorname{bd} D_z$ have the same distance to xy as q and respectively p. If $q \in A_v$, put $\{q_0\} = A_v \cap \operatorname{bd} R$. We obtain again

$$||p - q|| \le ||p - q_0|| \le ||x - q_0|| \le ||x - y||,$$

the last inequality following from $\angle xq_0z \ge \angle xyz = \pi/2$, $\angle xq_0y \ge \angle xyq_0$.

Suppose now that $p \in D_z$ and $q \in D_v$. If $q \in A_v$, put $\{q_0\} = A_v \cap \operatorname{bd} R$ and take the point p_0 in $\operatorname{bd} R \setminus \{x\}$ such that $||x - v|| = ||p_0 - v||$. Then again

$$||p - q|| \le \max\{||p_q - q||, ||p - q_p||\} \le ||p_0 - q_0|| \le ||x - y||,$$

where $p_q \in \operatorname{bd} D_v$ and $q_p \in \operatorname{bd} D_z$ are defined as above. Finally, $q \in A_y$ implies

$$||p - q|| \le ||p - y|| \le ||x - y||,$$

where the last inequality follows from $\angle ypv \ge \angle yxv = \pi/2$, $\angle xpy \ge \angle yxp$. □

Let δ be a quasigeodesic on a convex surface S and u, v points in $S \setminus \delta$. We say that u and v are *on the same side of* δ if there exist projections u_0, v_0 of u, v onto δ, and segments γ_{u_0u}, γ_{v_0v} on the same side of δ. Of course, two points might be on the left, and also on the right, side of δ.

The next result is a particular case of Theorem 2.

Lemma 6. *Let u and v be two points in a convex surface S, on the same side of, and at equal distances to, a geodesic δ of S. Let $u_0, v_0 \in \operatorname{relint} \delta$ be projections of u and respectively v onto δ, and let $\gamma_{uu_0}, \gamma_{vv_0}$ be segments on the same side of δ (see Corollary 1). If there exists a segment $\gamma_{uv} \subset S \setminus \delta_{u_0v_0}$ such that one of the two quadrilaterals bounded by $\gamma_{uv} \cup \gamma_{uu_0} \cup \gamma_{vv_0} \cup \delta_{u_0v_0}$ has all its angles at most π, then $\lambda(\gamma_{uv}) \leq \lambda(\delta_{u_0v_0})$. Equality holds if and only if the above quadrilateral is isometric to a Euclidean rectangle.*

Proof. The idea is to show that a quadrilateral $D = uu_0v_0v$ contains a geodesic diagonal, in order to apply Lemmas 3 and 5. However, without the assumption that one of the two quadrilaterals bounded by $\gamma_{uv} \cup \gamma_{uu_0} \cup \gamma_{vv_0} \cup \delta_{u_0v_0}$ — denote it by uu_0v_0v — has all its angles at most π, such a diagonal may not exist (e.g., all geodesics joining on S opposite vertices in uu_0v_0v are outside uu_0v_0v).

Glue two isometric copies of uu_0v_0v, say D and D^*, along their boundary by identifying the points via the natural isometry. The angle condition for uu_0v_0v assures that the resulting surface S^D, to which we will refer in the following, is (isometric to) a convex surface, by Lemmas 1 and 2.

Assume first that $\delta_{u_0v_0}$ is a segment in S^D.

Denote by γ_{vu_0} a segment on S^D joining v to u_0. Then, because of the symmetry of S^D, γ_{vu_0} is contained in one of the faces of S^D, say in D. In particular, the tangent direction of γ_{u_0v} at u_0 lies in the tangent cone T_{u_0} at u_0 between the tangent directions of $\delta_{u_0v_0}$ and γ_{u_0u}, hence

$$\angle\gamma_{u_0u}\gamma_{u_0v} + \angle\gamma_{u_0v}\delta_{u_0v_0} = \pi/2.$$

Consider a planar triangle $\bar{v}\bar{u}_0\bar{v}_0$ having the same side lengths as $vu_0v_0 \subset D$. Lemma 3 yields

$$\angle\bar{v}\bar{v}_0\bar{u}_0 \leq \angle\gamma_{v_0v}\delta_{v_0u_0} = \pi/2, quad \angle\bar{v}\bar{u}_0\bar{v}_0 \leq \angle\gamma_{u_0v}\delta_{u_0v_0}.$$

Further construct a planar triangle $\bar{v}\bar{u}_0\bar{u}_2$ exterior to $\bar{v}\bar{u}_0\bar{v}_0$, and of the same side lengths as $vu_0u \subset D$. For it we have, by Lemma 3, $\angle\bar{v}\bar{u}_0\bar{u} \leq \angle\gamma_{u_0v}\gamma_{u_0u}$. We obtain

$$\angle\bar{u}\bar{u}_0\bar{v} + \angle\bar{v}\bar{u}_0\bar{v}_0 \leq \angle\gamma_{u_0u}\gamma_{u_0v} + \angle\gamma_{u_0v}\delta_{u_0v_0} = \pi/2.$$

This together with $\angle\bar{v}\bar{v}_0\bar{u}_0 \leq \pi/2$ enable us to apply Lemma 5, for a (planar) rectangle $u^*\bar{u}_0\bar{v}_0v^*$ containing the quadrilateral $\bar{u}\bar{u}_0\bar{v}_0\bar{v}$, and to obtain

$$\lambda(\gamma_{uv}) = ||\bar{u} - \bar{v}|| \leq ||\bar{u}_0 - \bar{v}_0|| = \lambda(\delta_{u_0v_0}).$$

The equality case follows directly from Lemmas 3 and 5.

Assume now that $\delta_{u_0v_0} \subset S^D$ is a not a segment between u_0 to v_0, and denote by $\mu_{u_0v_0} \subset D$ such a segment. Notice that $\mu_{u_0v_0} \cap \gamma_{uv} = \emptyset$ (because $\gamma_{uv} \cup \{u_0, v_0\}$ is included in the boundary of the convex domain D of S^D), and $\angle\gamma_{u_0u}\mu_{u_0v_0} < \angle\gamma_{u_0u}\delta_{u_0v_0} = \pi/2$.

Take the domain Δ in D bounded by $\gamma_{uv} \cup \gamma_{uu_0} \cup \gamma_{vv_0} \cup \mu_{u_0v_0}$, and construct its double S^Δ. Applying the above argument for S^Δ instead of S^D gives

$$\lambda(\gamma_{uv}) = ||\bar{u} - \bar{v}|| \leq ||\bar{u}_0 - \bar{v}_0|| = \lambda(\mu_{u_0v_0}) < \lambda(\delta_{u_0v_0}).$$

In this case we cannot obtain equality. □

We notice here that the condition $\gamma_{uv} \subset S \setminus \delta_{u_0v_0}$ was necessary to construct the double S^D of the domain D.

We are now in the position to prove the main result of this section.

Theorem 1. *Let u and v be two points in a convex surface S, on the same side of, and at equal distance to, a quasigeodesic δ of S. If $u_0, v_0 \in \operatorname{relint} \delta$ are projections of u and respectively v onto δ, and there exists a segment γ_{uv} disjoint from $\delta_{u_0v_0}$, then $\rho(u, v) \leq \lambda(\delta_{u_0v_0})$. Equality holds if and only if a quadrilateral determined by u, v and $\delta_{v_0u_0}$ is isometric to a Euclidean rectangle.*

Proof. There exist, by Corollary 1, segments γ_{uu_0} and γ_{vv_0} on — say — the right side of δ, and they are orthogonal to δ, because $u_0, v_0 \in \operatorname{relint} \delta$. Denote by $\coprod^*$ the domain of S bounded by $\gamma_{uv} \cup \gamma_{vv_0} \cup \delta_{v_0u_0} \cup \gamma_{u_0u}$, on the right of δ. Our main idea of proof is to find a geodesic domain $D \subset \coprod^*$ to which to apply Lemma 6. Unfortunately, there is no guarantee that $\coprod^*$ is appropriate.

Assume that $\lambda(\gamma_{uv}) > \lambda(\delta_{u_0v_0})$, since otherwise there is nothing to prove. Consider a shortest path $\gamma_{u_0v_0}$ connecting u_0 to v_0 in $\coprod^*$ and observe that $\gamma_{u_0v_0}$ cannot contain both u and v.

The case $u \in \gamma_{u_0v_0}$ (or similarly $v \in \gamma_{u_0v_0}$) is easily completed, with $\gamma_{uv_0} \subset \gamma_{u_0v_0}$, by

$$\lambda(\delta_{u_0v_0}) \geq \lambda(\gamma_{u_0v_0}) = \rho(u_0, u) + \lambda(\gamma_{uv_0}) = \rho(v_0, v) + \lambda(\gamma_{uv_0}) \geq \lambda(\gamma_{uv}).$$

In this case, it is easily seen that we cannot have simultaneously equality at both inequalities above.

Therefore, we can assume that $\gamma_{u_0v_0} \cap \{u, v\} = \emptyset$ and, because geodesics do not branch, $\gamma_{u_0v_0} \cap \operatorname{bd} \coprod^* \subset \delta_{u_0v_0}$. Then either $\gamma_{u_0v_0} \cap \operatorname{relint} \delta_{u_0v_0} \neq \emptyset$, or $\gamma_{u_0v_0} \cap \operatorname{bd} \coprod^* = \{u_0, v_0\}$ and, in the last case, $\gamma_{u_0v_0}$ is a geodesic of S because of its local length-minimality. In both cases, the angles in $\coprod^*$ between $\gamma_{u_0v_0}$ and γ_{u_0u} at u_0, and between $\gamma_{v_0u_0}$ and γ_{v_0v} at v_0, clearly exist and they are at most $\pi/2$.

Let $\coprod$ denote the geodesic domain of $\coprod^*$ bounded by $\gamma_{uv} \cup \gamma_{vv_0} \cup \gamma_{v_0u_0} \cup \gamma_{u_0u}$, and let ρ^{U} be its metric naturally induced by ρ.

Take points $u_1 \in \gamma_{uu_0}$ and $v_1 \in \gamma_{vv_0}$ such that $\rho^{\mathrm{U}}(\gamma_{uu_0}, \gamma_{vv_0}) = \rho^U(u_1, v_1)$.

Observe first that we cannot have $u_1 = u$ and $v_1 = v$, because $\rho^U(u, v) > \lambda(\delta_{u_0v_0}) \geq \lambda(\gamma_{u_0v_0})$. Neither can we have $u_1 \neq u$ and $v_1 = v_0$, because of the first part of (a) in Lemma 4.

Case i) Assume that $u_1 = u_0$ and $v_1 = v_0$ and, moreover, $\rho^{\mathrm{U}}(u_1, v_1) < \rho^U(u', v')$ for any points $u' \in \gamma_{uu_0}$ and $v' \in \gamma_{vv_0}$.

Since $\rho^{\mathrm{U}}(u_0, v_0) = \lambda(\gamma_{u_0v_0}) < \rho^U(u, v)$, we may take points $u_2 \in \operatorname{relint} \gamma_{uu_0}$ close to u_0, and $v_2 \in \operatorname{relint} \gamma_{vv_0}$ close to v_0, such that $\rho(u_2, u_0) = \rho(v_2, v_0)$ and

$$\rho^U(u_2, u_0) + \rho^U(u_0, v_0) + \rho^U(v_2, v_0) < \rho(u, v).$$

Consequently,

$$\rho^U(u_2, v_2) \leq \rho^U(u_2, u_0) + \rho^U(u_0, v_0) + \rho^U(v_2, v_0) < \rho(u, v).$$

Consider a shortest path $\gamma_{u_2v_2}$ in $\coprod$. By the choice of u_2 and v_2, and by Lemma 4, $\gamma_{u_2v_2}$ does not meet $\operatorname{bd} \coprod$ except at its extremities, so it is a geodesic of S. Therefore, $u_2u_0v_0v_2$ defines a geodesic quadrilateral, all interior angles of which are less than π. Finally apply Lemma 6 to obtain $\lambda(\gamma_{u_2v_2}) \leq \lambda(\gamma_{u_0v_0})$, contradicting the case assumptions.

Case ii) Assume that $u_1 \in \operatorname{relint} \gamma_{uu_0}$ and $v_1 \in \operatorname{relint} \gamma_{vv_0}$. Consider a shortest path $\gamma_{u_1v_1} \subset \coprod$, from u_1 to v_1; by Lemma 4, $\gamma_{u_1v_1} \perp \gamma_{uu_0}$ and $\gamma_{v_1u_1} \perp \gamma_{vv_0}$. Notice that $\gamma_{u_1v_1}$ cannot pass through u or v, because $\rho^{\mathrm{U}}(\gamma_{uu_0}, \gamma_{vv_0}) = \rho^U(u_1, v_1)$.

We claim that $\gamma_{u_1v_1} \cap \gamma_{u_0v_0} = \emptyset$. Suppose this is not true, choose the first point x along $\gamma_{u_1v_1}$ in $\gamma_{u_1v_1} \cap \gamma_{u_0v_0}$, and notice that the arc γ_{u_1x} of $\gamma_{u_1v_1}$ is a geodesic. At x we have $0 < \angle^{\coprod} \gamma_{v_1u_1}\gamma_{v_0u_0} < \pi$, because quasigeodesics have definite tangent directions at each point. Consider a small convex cap around x (see the precise definition in [3] p. 84), cut it along $\gamma_{u_0v_0}$ and keep the part U included in $\coprod$. By further restricting, if necessary, we can assume it be a quasigeodesic domain. Its boundary has nonnegative swerve towards U ([3] p. 111), so we can glue two copies of U along their boundary (by identifying the points via the natural isometry) and the result is (isometric to) a convex surface S^U, by Lemmas 1 and 2. There exists a segment γ_{yz} joining, on the face U of S^U, points y, z of $\gamma_{u_1v_1}$ separated by x. Moreover, $x \notin \gamma_{yz}$, because at x we have $\angle^U \tau^-_{\gamma_{u_1v_1}} \tau^+_{\gamma_{u_1v_1}} \leq \angle^U \gamma_{u_1v_1}\gamma_{u_0v_0} < \pi$. But this contradicts the length-minimality of $\gamma_{u_1v_1}$ inside $\coprod$, and proves the claim.

Consequently, the quadrilateral $u_1u_0v_0v_1$ has all its angles $\leq \pi/2$. Since its sides are geodesic arcs, the Gauss–Bonnet theorem (see [3] p. 105) implies its excess is zero, so $u_1u_0v_0v_1$ is flat and therefore isometric to a planar rectangle.

Take the maximal (with respect to inclusion) flat rectangle $R \subset \coprod$ with one side $\gamma_{u_0v_0}$ and two sides included in $\gamma_{uu_0} \cup \gamma_{vv_0}$. We may assume, by possibly renaming two of its vertices, that $R = u_1u_0v_0v_1$. The previous argument shows now that, for any points $u' \in \gamma_{uu_0}$ and $v' \in \gamma_{vv_0}$ outside R, we have $\rho^{\mathrm{U}}(u_1, v_1) < \rho^U(u', v')$. Thus we can reduce this case to the preceding one.

Case iii) Assume finally that $u_1 = u$ and $v_1 \neq v$ (or, similarly, $u_1 \neq u$ and $v_1 = v$). Define $w \in \gamma_{uu_0}$ by $\rho(w, u_0) = \rho(v_1, v_0)$ and choose shortest paths $\gamma_{u_1v_1}$, γ_{wv_1} in $\coprod$. Then $\gamma_{u_1v_1} \perp \gamma_{uu_0}$ by Corollary 1, $\gamma_{u_1v_1} \cap \gamma_{wv_1} = \{v_1\}$, and moreover $\gamma_{wv_1} \cap \gamma_{u_0v_0} = \emptyset$ (see the proof for this at Case ii)).

Since S is nonnegatively curved, the Gauss–Bonnet theorem applied to the quadrilateral $Q = wu_0v_0v_1$ yields $\angle^Q u_0wv_1 + \angle^Q wv_1v_0 \geq \pi$, so

$$\angle^Q uwv_1 + \angle^Q wv_1u = 3\pi/2 - (\angle^Q u_0wv_1 + \angle^Q wv_1v_0) \leq \pi/2.$$

Construct nonoverlapping planar triangles $\bar{w}\bar{u}\bar{v}_1$ and $\bar{u}\bar{v}_1\bar{v}$ with the same side lengths as wuv_1 and uv_1v. Then, by Lemma 3,

$$\angle\bar{u}\bar{w}\bar{v}_1 + \angle\bar{w}\bar{v}_1\bar{u} \leq \angle^Q uwv_1 + \angle^Q wv_1u \leq \pi/2,$$

hence $\angle\bar{w}\bar{u}\bar{v}_1 \geq \pi/2$. Moreover, $\angle\bar{u}\bar{v}_1\bar{v} \leq \angle^Q uv_1v = \pi/2$. The cosine rule applied to $\bar{w}\bar{u}\bar{u}_1$ and $\bar{u}\bar{u}_1\bar{v}$ yields

$$\rho^U(w, v_1) = ||\bar{w} - \bar{v}_1|| \geq ||\bar{u} - \bar{v}|| = \lambda(\gamma_{uv}).$$

Finally observe that all angles of the quadrilateral $wu_0v_0v_1$ are at most π, and apply Lemma 6 to get $\rho^U(w, v_1) = \lambda(\gamma_{wv_1}) \leq \lambda(\delta_{u_0v_0})$.

The equality case follows from Lemma 6 and the proof. □

Remark. Theorem 1 gives the better estimate $\rho(u, v) \leq \rho(u_0, v_0)$, provided the domain of S determined by u, v and $\delta_{v_0u_0}$ contains a segment $\gamma_{u_0v_0}$.

Theorem 1 proved useful (see Corollary 2.3 in [9]) to show, for any convex surface S and any simple closed quasigeodesic of length on l on S, that $l^2 - 2l\,\mathrm{diam}(S) + 4\,\mathrm{area}(S) > 0$.

It also implies the following result, inspired by a property of nonpositively curved manifolds, that the projection onto closed convex subsets is distance decreasing (see [2] p. 9).

Corollary 2. *For any convex domain D of any convex surface, the projection onto $\mathrm{bd}\,D$ of any two points in D at equal distance to $\mathrm{bd}\,D$, is distance increasing.*

Proof. Glue two isometric copies of D along their boundary (by identifying the points via the natural isometry) to obtain a convex surface S with planar symmetry (see Lemmas 1 and 2), where (the image of) $\mathrm{bd}\,D$ is a simple closed quasigeodesic (see [3] p. 154). Theorem 1 and the above remark directly apply to D to obtain the conclusion. □

4 Farthest points and simple closed quasigeodesics

In this section we apply Theorem 1 for proving a criterion to locate the set $\mathbb{F}$ of all farthest points on a convex surface (Theorem 2), and we derive sufficient conditions for this set being disconnected, and thus for the existence of points with multiple farthest points.

A motivation for this section comes from H. Steinhaus' questions to characterize the sets of farthest points (see §A35 in [4]). Next we briefly mention some of the results recently obtained in this direction, and refer the reader to [21] for an overview.

On any convex surface S, for *nearly all* (in the sense of porosity) points x in S, F_x consists of a single point [24].

Let $\mathcal{S}$ denote the space of all convex surfaces, endowed with the usual Pompeiu–Hausdorff metric. A more particular motivation comes from a conjecture of T. Zamfirescu [24], that the set $\mathcal{S}_2$, of all surfaces $S \in \mathcal{S}$ on which there exists a point x with disconnected set of relative maxima, is dense in $\mathcal{S}$. By Theorems 6 and 7 in [23], for proving this

conjecture it suffices to show that densely many surfaces have the mapping F properly multivalued.

Four types of results should be considered, when trying to decide which points are (not) farthest points. First, there are classes of surfaces with explicit determination of all (or some) sets of farthest points [16], [18], or with bijective (or even involutive) mapping F, see [7], [11], [20], [22]. Second, there are a few criteria to recognize farthest points by the use of the total curvature at a point, see [8] and [10]. Third, there are results showing that some points on a surface are not farthest points, by considering the total curvature [17], the relationship between farthest points and other critical points of distance functions [23], or a bound on the Gauß curvature along a loop of the surface [23]. And fourth, there are criteria to locate farthest points on convex surfaces. We mention here our Theorem 2 with its corollaries, as well as the last results in [10], stating — roughly speaking — that any small cap of boundary length small enough with respect to its radius does contain at least one farthest point.

Let $B(x, l)$ denote the open intrinsic ball of radius l centered at x.

Theorem 2. *Let O be a simple closed quasigeodesic on a convex surface S, S_1 and S_2 the subsets of S bounded by O, and $x_1 \in S_1$, $x_2 \in S_2$ points at maximal distance from O in S_1, respectively S_2. Then for any point y in $\mathbb{F}$ one has $\min_{i=1,2} \rho(y, x_i) < \lambda(O)$.*

If, moreover, the intrinsic balls $B(x_1, \lambda(O))$ and $B(x_2, \lambda(O))$ are disjoint then each contains farthest points; consequently, the set $\mathbb{F}$ is disconnected and the mapping F is properly multivalued.

Proof. Put $l = \lambda(O)$ and suppose there exists a point $v \in \mathbb{F} \setminus \bigcup_{i=1,2} B(x_i, l)$, say $v \in S_1$. Choose $z \in S$ such that $v \in F_z$.

Denote by x_0, z_0 and v_0 the projections of x_1, z and respectively v onto O, and choose segments $\gamma_{x_1 x_0}$, $\gamma_{z z_0}$ and $\gamma_{v v_0}$; by Lemma 4, all these segments are orthogonal to O.

Take $x_v \in \gamma_{x_1 x_0}$ such that $\rho(x_v, x_0) = \rho(v, v_0)$. By Theorem 1, $\rho(x_v, v)$ is at most equal to the length of the smaller arc of O joining x_0 to v_0, so $\rho(x_v, v) \leq l/2$ and thus, by the triangle inequality,

$$\rho(x_1, x_v) \geq \rho(x_1, v) - \rho(x_v, v) \geq l/2.$$

Assume first that $z \in S_2$. Choose a segment $\gamma_{x_1 z}$ and put $\{y\} = O \cap \gamma_{x_1 z}$. Clearly $O \cap \gamma_{x_1 z} \neq \emptyset$, because either $z \in O$, or O separates z and x_1. We have

$$\begin{aligned}
\rho(z, x_1) &= \rho(z, y) + \rho(y, x_1) \\
&\geq \rho(z, z_0) + \rho(x_0, x_1) \\
&\geq \rho(z, z_0) + l/2 + \rho(x_v, x_0) \\
&\geq \rho(z, z_0) + \rho(z_0, v_0) + \rho(v_0, v) \geq \rho(z, v).
\end{aligned}$$

To end the proof of this case, observe that $\rho(z, x_1) > \rho(z, v)$. Indeed, equality would imply $\gamma_{x_1 z} \perp O$, by $z_0 = x_0$, and $\gamma_{x_1 z} \subset O$, by $\rho(z_0, v_0) = l/2$ and $\rho(z, z_0) + \rho(z_0, v_0) + \rho(v_0, v) = \rho(z, v)$, which is impossible.

Assume now that $z \in S_1$ and $\rho(z, O) \leq \rho(v, O)$, and denote by v_z the point of γ_{vv_0} given by $\rho(v_z, v_0) = \rho(z, z_0)$. We have $l/2 \geq \rho(z, v_z)$, by Theorem 1. Since

$$\rho(x_1, x_0) = \rho(x_1, O) \leq \rho(x_1, z) + \rho(z, z_0),$$

we obtain

$$\begin{aligned}\rho(z, x_1) &\geq \rho(x_1, x_0) - \rho(z, z_0)\\ &\geq l/2 + \rho(x_v, x_0) - \rho(z, z_0)\\ &\geq \rho(z, v_z) + \rho(v_0, v) - \rho(v_z, v_0)\\ &= \rho(z, v_z) + \rho(v_z, v) \geq \rho(z, v).\end{aligned}$$

Observe that $\rho(z, x_1) > \rho(z, v)$ in this case too. Indeed, equality would imply the existence of some segment $\gamma_{x_1x_0}$ such that $z \in \gamma_{x_1x_0}$, and also $v \in \gamma_{x_1x_0}$, by $\rho(x_1, x_v) \geq l/2$, hence $v = x_1$ and a contradiction is obtained.

Assume finally that $z \in S_1$ and $\rho(z, O) > \rho(v, O)$. Define the level set of v by

$$C = \big\{y \in S_1 : \rho(y, O) = \rho(v, O)\big\}.$$

Then, since C is closed in S, we may consider points $x_v, z_v \in C$ such that $\rho(x_2, x_v) = \rho(x_2, C)$ and respectively $\rho(z, z_v) = \rho(z, C)$. Choose segments $\gamma_{x_2x_v}$ and γ_{zz_v}.

Notice that C separates z and x_2, because $\rho(z, O) > \rho(v, O)$, so $\rho(z, x_2) \geq \rho(z, z_v) + \rho(x_2, x_v)$. Theorem 1 applied to O and $\rho(x_v, v)$, respectively to O and $\rho(z_v, v)$, gives

$$\max\big\{\rho(x_v, v), \rho(z_v, v)\big\} \leq l/2.$$

We obtain again $\rho(x_2, x_v) \geq l/2$, and therefore

$$\begin{aligned}\rho(z, x_2) &\geq \rho(z, z_v) + \rho(x_2, x_v)\\ &\geq \rho(z, z_v) + l/2\\ &\geq \rho(z, z_v) + \rho(z_v, v) \geq \rho(z, v).\end{aligned}$$

To end the first part of the proof, suppose $\rho(z, x_2) = \rho(z, v)$. Then $x_v \in \gamma_{x_2v}$ and consequently $x_v = v$, by $\rho(x_1, x_v) = \rho(x_1, v) - \rho(x_v, v) = l/2$, but also $z_v = x_v$, by $\rho(z, x_2) = \rho(z, z_v) + \rho(x_2, x_v)$, and $z_v = x_v$, because $\rho(z, z_v) + \rho(z_v, v) = \rho(z, v)$. Altogether imply $v \in \gamma_{x_2v}$ for some segment γ_{x_2v}, and a contradiction is obtained.

For the last part of the theorem, if $B(x_1, \lambda(O)) \cap B(x_2, \lambda(O)) = \emptyset$ then necessarily $F_{x_1} \subset B(x_2, \lambda(O))$ and $F_{x_2} \subset B(x_1, \lambda(O))$. Moreover, the upper semicontinuity of F shows that all points z close to x_i have $F_z \subset B(x_{i+1}, \lambda(O))$, $i = 1, 2 \pmod 2$.

Finally, the upper semicontinuity of F and Brouwer's fixed point theorem imply (see [23]) that the mapping F is properly multivalued as soon as $\mathbb{F} \neq S$. □

Remarks. i) If the convex surface S is smooth enough then there exist at least three simple closed geodesics on S, by the classical result of L. A. Lusternik and L. G. Schnirelman.

Contrasting to this is the theorem of P. Gruber [6], that *most* (in the sense of Baire category) convex surfaces have no closed geodesic. A. V. Pogorelov [15] proved that any $S \in \mathcal{S}$ has at least three simple closed quasigeodesics.

ii) The estimate of Theorem 2 is not optimal, since $\mathbb{F}$ is closed. It can however be improved by successive applications of Theorem 2 to different simple closed quasigeodesics.

As a direct consequence of Lemma 1, each domain D of S whose boundary is a simple closed quasigeodesic is *relatively convex*; i.e., it is isometric to some convex domain (see [3] p. 155). Then any two (possibly *degenerate*) simple closed quasigeodesics on a convex surface either intersect each other or the region they bound is isometric to a zone between parallel circles on a circular cylinder. For, if they do not cut each other, the region between them is relatively convex, as intersection of two relatively convex domains, and the arguments of [3] p. 112 entirely apply.

iii) Let $\{x_i^{\alpha_i}\}_{\alpha_i \in A_i}$ denote the set of all points in S_i at maximal distance to O, $i = 1, 2$. Theorem 2 can be rephrased as

$$\mathbb{F} \subset \bigcup_{i=1,2} \bigcap_{\alpha_i \in A_i} B(x_i^{\alpha_i}, \lambda(O)).$$

Notice that $\rho(x_i^{\alpha_1}, x_i^{\alpha_2}) \leq l/2$, by Theorem 1, because $\rho(x_i^{\alpha_1}, O) = \rho(x_i^{\alpha_2}, O)$. Equality holds if and only if the region of S between O and the (necessarily unique) segment from $x_i^{\alpha_1}$ to $x_i^{\alpha_2}$ is isometric to a cylinder, by the Gauss–Bonnet theorem.

iv) A necessary condition for the conclusion of Theorem 2 to be non-trivial (i.e., $\mathbb{F} \subseteq \bigcup_{i=1,2} \bigcap_{\alpha_i} B(x_i^{\alpha_i}, \lambda(O)) \neq S$) is that $\lambda(O_s) < \operatorname{diam}(S)$, where O_s denotes the shortest non-trivial closed quasigeodesic of S.

Various estimates on $\lambda(O_s)$ were obtained for smooth convex (hyper)surfaces (see for example [5], [12], [19] and the references therein); some of them also hold, by passing to the limit, for arbitrary convex surfaces. Estimates on $\lambda(O_s)$ are good to verify whether $\operatorname{diam}(S) > \lambda(O_s)$, but they may also be useful in conjunction with criteria to find some small neighborhood of O_s in S, or to decide that a point in S is at maximal distance to O_s in the domain bounded by O_s that it belongs to, *without* having O_s explicitly determined. This last problem is treated in [10].

Theorem 2 is particularly useful in the presence of additional hypotheses.

Corollary 3. *Let S be a convex surface of revolution, E the equator of S, and M a meridian of S. If $\lambda(M) \geq 2\lambda(E)$ then the set $\mathbb{F}$ is disconnected and the mapping F is properly multivalued.*

Proof. The equator E (the largest parallel circle) of S is easily seen to be a closed quasigeodesic. Thus, $B(x_1, \lambda(E)) \cap B(x_2, \lambda(E)) = \emptyset$ if and only if $\lambda(M) \geq 2\lambda(E)$, and Theorem 2 directly applies to get the conclusion. □

Corollary 4. *Suppose the convex surface S is symmetric with respect to the plane Π, and the Euclidean distance w_Π between planes supporting S and parallel to Π satisfies $w_\Pi \geq 2\lambda(\Pi \cap S)$. Then the set $\mathbb{F}$ is disconnected and the mapping F is properly multivalued.*

Proof. The planar symmetry of S with respect to Π directly implies that $O = \Pi \cap S$ is a simple closed quasigeodesic of S.

Let x_1, x_2 be points in S symmetric with respect to Π and at maximal intrinsic distance from O. Denote by z_1, z_2 the contact points of S with its supporting planes parallel to Π, hence they are symmetric to each other and $w_\Pi = 2\,\mathrm{dist}(z_i, \Pi)$, $i = 1, 2$. We have:

$$\lambda(O) \leq \mathrm{dist}(z_i, \Pi) \leq \rho(z_i, O) \leq \rho(x_i, O),$$

hence $B(x_i, \lambda(O)) \cap O = \emptyset$ and Theorem 2 directly applies to get the conclusion. □

Corollary 5. *If there exist a simple closed quasigeodesic O of the convex surface S and a farthest point x from O such that $\rho(x, O) \geq 2\lambda(O)$, then the set $\mathbb{F}$ is disconnected and the mapping F is properly multivalued.*

Proof. Denote by S_1 the subset of S bounded by O which contains x, put $S_2 = \mathrm{cl}(S \setminus S_1)$, and take a point x_2 at maximal distance in S_2 from O. Let v be the intersection point of O with a segment from x to x_2. Then

$$\rho(x, x_2) \geq \rho(x, v) \geq \rho(x, O) \geq 2\lambda(O),$$

hence the open balls $B(x, \lambda(O))$ and $B(x_2, \lambda(O))$ are disjoint and, by Theorem 2, the proof is complete. □

Corollary 6. *If a simple closed quasigeodesic O of the convex surface S satisfies $4\lambda(O) \leq \mathrm{diam}(S)$ then the set $\mathbb{F}$ is disconnected and the mapping F is properly multivalued.*

Proof. Choose points $y, z \in S$ realizing the diameter of S. Since $\rho(y, z) = \mathrm{diam}(S) > 2\lambda(O)$, y and z belong to different intrinsic balls, say (with the notations in Theorem 2) $y \in B(x_1, \lambda(O))$ and $z \in B(x_2, \lambda(O))$.

Suppose that $B(x_1, \lambda(O)) \cap B(x_2, \lambda(O)) \neq \emptyset$, hence $\rho(x_1, x_2) < 2\lambda(O)$. Then

$$4\lambda(O) \leq \mathrm{diam}(S) = \rho(y, z) \leq \rho(y, x_1) + \rho(x_1, x_2) + \rho(x_2, z) < 4\lambda(O),$$

and a contradiction is obtained.

Since $\mathbb{F}$ is included in two disjoint balls, Theorem 2 ends the proof. □

Open questions. The last conclusion of Theorem 2 may also be regarded from an opposite viewpoint, suggesting the following related open questions.

Suppose the set $\mathbb{F}$ of all farthest points on the convex surface S has precisely two components; are these components separated by a simple closed quasigeodesic of S?

Suppose that $\mathbb{F}$ is included in two disjoint balls of equal radius; are these balls separated by a simple closed quasigeodesic of S?

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