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Multiple farthest points on Alexandrov surfaces

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Abstract. The farthest point mapping on compact surfaces, associating to each point x of the surface the set of absolute maxima of the intrinsic distance from x, is for some surfaces single-valued and a homeomorphism, while for other surfaces it is not single-valued, and not surjective. These two big classes are not very well understood. For instance it is still unknown whether, say in the convex case, the second class is dense. For a C^2 metric on both surfaces and the space of surfaces, the first class has, however, nonempty interior. We describe various properties of the sets of critical points, and of relative and absolute maxima of distance functions, and find several connections between them. We see for example that, on smooth surfaces homeomorphic to S^2, a point cannot be critical with respect to more than one other point. Sufficient conditions for a surface to belong to the second class will be formulated and a particular Tannery surface belonging to the boundary of both classes will be presented.

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1 Introduction

We investigate the set of critical points with respect to distance functions on Alexandrov surfaces. Special attention will receive Alexandrov surfaces homeomorphic to the 2-dimensional sphere S^2, while we sometimes restrict ourselves to convex surfaces.

In this paper, by *surface* we always mean a compact 2-dimensional Alexandrov space with curvature bounded below (without boundary), as defined by Burago, Gromov and Perelman in [4]. Let $\mathcal{A}$ be the space of all surfaces.

It is well-known that our surfaces are topological manifolds. Other basic facts on surfaces, such as convergence theorems on shortest paths or on angles, the generalized Toponogov theorem, and an ample description of the topological and metric structure of the cut loci, can be found in [4], [18], [11]. Some of these basic facts will be tacitly used, while in other cases we shall recall the needed facts as lemmas.

Denote by $\mathcal{S}$ the space of all convex surfaces embedded in $\mathbb{R}^3$, and by $\mathcal{A}_0$, $\mathcal{R}_0$ the set of all Alexandrov, respectively Riemannian, surfaces homeomorphic to S^2.

For any two points x, y on the surface S, $\rho(x, y)$ means the geodesic (intrinsic) distance between them (induced by the Euclidean distance for $S \in \mathcal{S}$), and ρ_x the distance function from x: $\rho_x(y) = \rho(x, y)$. For $x \in S$ denote by F_x the set of all farthest points from x (i.e., absolute maxima of ρ_x) and by F the *farthest point mapping*, the multivalued mapping associating to any point $x \in S$ the set F_x. Similarly, M_x is the set of all relative maxima of ρ_x, Q_x the set of all critical points with respect to ρ_x, and M, respectively Q, are the corresponding mappings. Here, a point $y \in S$ is called *critical* with respect to ρ_x, or simply to x, if for any vector v tangent to S at y there exists a segment from y to x whose direction at y makes an angle not larger than $\pi/2$ with v.

If $f : S \to \mathcal{P}(S)$ is a multivalued mapping (with the set f_x as image of x), we call it *injective* if $f_x \cap f_y = \emptyset$ whenever $x \neq y$, and *surjective* if for every point $y \in S$ there exists $x \in S$ with $y \in f_x$. We say that f is *connected* if f_x is connected for each $x \in S$. When we say that f is bijective or a homeomorphism, we implicitly state that f is single-valued. Also, $f_y^{-1} = \{x \in S : y \in f_x\}$.

Several questions about farthest points proposed by H. Steinhaus (see Chapter A35 of the book [7] of H. T. Croft, K. J. Falconer and R. K. Guy) have been answered in the convex case by the second author ([24], [25], [28], [29], [30]). J. Rouyer [14], [13] showed that some of these results are also true in the framework of Riemannian geometry. The present paper is also contributing to the study of the farthest point mapping on surfaces, which Steinhaus had asked for.

The next section contains a description of the components of M_x and F_x, and points out that, generically in $\mathcal{S}$, Q_x — and therefore M_x and F_x too — are totally disconnected. In the third section we see that, on any smooth surface in $\mathcal{A}_0$, no point can be critical with respect to more than one point. Section 4 establishes the especially strong relationship between the components of M_x and those of Q_x. In the following section it is proved that the family $\mathcal{S}_2$ of those surfaces on which M is disconnected is open in $\mathcal{A}_0$. Two major results of the paper, both in Section 6, are Theorem 6 stating several implications between various connectedness and surjectivity properties of Q, M, or F, and Theorem 7 providing the generic equivalence of all properties considered in Theorem 6. The paper continues with three very different sufficient conditions for a surface to belong to $\mathcal{S}_2$. The third brings to light an exceptional type of surface belonging to the boundary of $\mathcal{S}_2$. This takes a concrete form in the next section, where a special Tannery surface is investigated. The paper ends with a few open questions.

We need more definitions and additional notation. $\operatorname{card} A$ denotes the cardinality of the set A, $\operatorname{bd} A$ denotes its boundary, $\overline{A}$ its closure, and λA its 1-dimensional Hausdorff measure (length).

For any surface S and $A \subset S$, put $\rho(x, A) = \inf_{y \in A} \rho_x(y)$ and $r_x = \sup_{y \in S} \rho_x(y)$. Also, $F_A = \cup_{x \in A} F_x$. The *radius* of the surface S is defined by $\operatorname{rad} S = \inf_{x \in S} r_x$ and its *diameter* by $\operatorname{diam} S = \sup_{x \in S} r_x$. A *domain* of S is an open connected subset of S.

Let $S \in \mathcal{A}$, $x \in S$, and $\varepsilon > 0$. Any set $B(x, \varepsilon) = \{y \in S : \rho_x(y) < \varepsilon\}$ homeomorphic to a usual open disc in the plane is called *open disc* around x. If ε is small enough, $B(x, \varepsilon)$ is indeed an open disc (see [4]). An *arc* Γ is a homeomorphic image of $[0, 1]$; its *interior* $\operatorname{int} \Gamma$ is the image of $]0, 1[$.

The union of two *segments* (i.e., shortest paths) from $x \in S$ to some point $y \in S$, which make an angle equal to π at y, will be called a *loop at* x; of course, the segments have equal lengths.

A *geodesic* is a curve which is locally a segment. More precisely, there is an interval I of $\mathbb{R}$ and a parametrization $i : I \to S$ such that any point $t \in I$ has a neighbourhood V for which $i(V)$ is a segment. A point of a surface S is called *endpoint* of S if it is not interior to any geodesic.

If, for any $x \in S$, each loop (if there is at least one) at x has length $2r_x$, we say that S is *loopy*.

If σ, σ' are two segments with precisely one common endpoint a, then $\angle\sigma\sigma'$ denotes the angle between the tangent directions of σ and σ' at a. For $a \neq b$, ab means the segment from a to b when that segment is unique or clearly identifiable from the context. $\angle xyz$ means $\angle xyyz$.

A *geodesic triangle* in a Riemannian manifold or a convex surface is a collection of three segments $\gamma_1, \gamma_2, \gamma_3$ such that γ_i, γ_{i+1} have the common endpoint a_{i+2}. The indices should be taken modulo 3. We shall denote the triangle by $(\gamma_1, \gamma_2, \gamma_3)$.

For a point $x \in S$, let C_x be the set of all points joined to x by at least two segments, and $C(x)$ the *cut locus of* x, i.e. the set of all endpoints (different from x) of maximal (with respect to inclusion) segments starting at x. Clearly, $C_x \subset C(x)$, and $C(x)$ is known to be a local tree (that is, each of its points y has a neighbourhood V in S where the component $K_y(V)$ of y is a tree), even a tree if the surface is homeomorphic to S^2 (see [10], [16], [18] for basic properties of the cut locus). Recall that a *tree* is a set T any two points of which can be joined by a unique arc included in T. The *degree* of a point y of a local tree is the number of components of $K_y(V) \setminus \{y\}$ if V is chosen such that $K_y(V)$ is a tree. A point $y \in T$ is called an *extremity* of T if it has degree 1, and a *ramification point* of T if it has degree at least 3. A tree is called *nondegenerate* if it has at least one ramification point. A Y*-tree* is a tree with precisely three extremities, and a tree is *finite* if it has finitely many extremities.

We consider isometrical surfaces as not different, and equip the space $\mathcal{A}$ with the Hausdorff–Gromov metric. In $\mathcal{S}$, where distinct surfaces may be isometric, the usual Pompeiu–Hausdorff distance is never less than the Hausdorff–Gromov distance between any two surfaces.

The space $\mathcal{S}$ is Baire, and in any Baire space *most* means "all except those in a set of first category".

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2 Arcs in Q_x, M_x, F_x

For any surface S, the space T_y of all unit tangent directions at $y \in S$ is a closed Jordan curve of length at most 2π [4]. The point y is said to be *conical* if $\lambda T_y < 2\pi$, and we call S *smooth* if it has no conical points. Notice that a surface may be smooth without being differentiable. This is, for example, the case for all convex surfaces admitting 1-singular but no 0-singular points (in the terminology of R. Schneider [17]). In 1959 V. Klee proved that most convex surfaces are smooth, even of class C^1 [9].

Lemma 1. *On $S \in \mathcal{A}$, let $y \in C(x)$ and D be a component of the complement of $C(x)$ in an open disc O around y with $y \in \overline{D}$. Then there exists a segment from x to y meeting $O \setminus \{y\}$ inside D. Consequently, if the degree of y is d, then there are at least d segments from x to y.*

Proof. Let the sequence of points $y_n \in D$ converge to y. Choose any segment σ_n from x to y_n. Since $\mathcal{A}$ is compact, $\{\sigma_n\}_{n=1}^{\infty}$ has a convergent subsequence, the limit σ being also a segment. Clearly, σ joins x to y and $\sigma \cap O \subset \overline{D}$. By the definition of the cut locus, $\operatorname{int} \sigma \cap O \subset D$. □

Lemma 2. *Let $S \in \mathcal{A}$, $x \in S$ and $y, z \in C(x)$. Let J be an arc joining y to z in $C(x)$. If $u \in \operatorname{int} J$ is a relative minimum of $\rho_x|J$ then u is the midpoint of a loop Λ at x and, except for the two subarcs of Λ, no segments connect x to u.*

This follows from the main result in [27].

For $S \in \mathcal{S}$ and $x \in S$, we know that F_x is homeomorphic to a compact subset of $\mathbb{R}$ [28], so each component of F_x is either a point or an arc. We extend this here.

Theorem 1. *For any surface $S \in \mathcal{A}$ and any point x in S, each component of M_x is homeomorphic to a connected subset of the circle S^1, hence each component of F_x is a point or an arc or a closed Jordan curve. For $S \in \mathcal{A}_0$, each component of M_x is homeomorphic to a connected subset of $\mathbb{R}$, hence each component of F_x is a point or an arc.*

Proof. Let $x \in S$ and take a component M^1 of M_x containing more than one point. Since M^1 is included in $C(x)$ which is a local tree, M^1 is itself a local tree.

Since the restriction of ρ_x to M^1 is continuous, it is a constant function, and has therefore a relative minimum at each point of M^1 of degree at least 2, in particular at any ramification point of M^1, if it has one. But such a point is also a ramification point of $C(x)$ and, by Lemma 1, is joined to x by at least three segments, in contradiction to Lemma 2.

Without any ramification points, M^1 must be an arc with possibly one or both endpoints removed, or a closed Jordan curve. The latter case is excluded for $S \in \mathcal{A}_0$, because then $C(x)$ is a tree. □

Lemma 3. *Let $S \in \mathcal{A}_0$, $x \in S$ and $y, z \in C(x)$ be distinct. Suppose Γ_y, Γ'_y are (possibly coinciding) segments from x to y and Γ_z, Γ'_z are (possibly coinciding) segments from x to z. Then there is a domain Δ with boundary $\Gamma_y \cup \Gamma'_y \cup \Gamma_z \cup \Gamma'_z$ and a Jordan arc J_{yz} in $C_x \cup \{y, z\}$ joining y to z. Moreover, every point in $\operatorname{int} J_{yz}$ belongs to Δ and can be joined to x by two segments the union of which separates y from z.*

Lemma 3 appears in [24], formulated for $\mathcal{S}$ instead of $\mathcal{A}_0$. The extension to $\mathcal{A}_0$ is straightforward.

Lemma 4. *Let $S \in \mathcal{A}$, $x \in S$, and $J \subset C(x)$ be an arc each point of which is joined to x by precisely two segments. Let y_1, y_2 be the endpoints of J, and suppose there* *exists a domain* Δ *bounded by the segments from* x *to* y_1, y_2 *and containing* $\operatorname{int} J$. *Then* $\Delta \cap C(x) \subset J$.

In case $S \in \mathcal{A}_0$, *the existence of* Δ *is guaranteed by Lemma* 3.

Proof. Suppose there is a point z in $\Delta \cap C(x) \setminus J$. Since $C(x)$ is arcwise connected, there is a minimal subarc A of $C(x)$ containing z and meeting J. Let $\{u\} = A \cap J$.

Assume first that $u \in \Delta$. Then $u \in J \setminus \{y_1, y_2\}$, so it is a ramification point of $C(x)$, whence it is joined to x by at least three segments by Lemma 1, and we obtained a contradiction.

Assume now that $u \in C(x) \setminus \Delta$. Since the interiors of the segments from x to y_1, y_2 are disjoint from $C(x)$, u must be one of the endpoints of J, say y_1, and $A \setminus \{y_1\} \subset \Delta$. By Lemma 1, there is another segment from x to y_1 arriving at y_1 between J and A, and a contradiction is obtained again. □

Lemma 5. *On most convex surfaces, most points are endpoints.*

This result was proven in [21].

Lemma 6. *On* $S \in \mathcal{A}$, *let* σ, σ' *be* (*possibly coinciding*) *segments from* x *to* y *and* D *a component of the complement of* $\sigma \cup \sigma'$ *in an open disc around* y. *Further, suppose no point of* D *belongs to any segment from* x *to* y. *Then* $C(x) \cap \overline{D}$ *admits a half-tangent at* y *which bisects the angle of* σ, σ' *at* y *toward* D.

This is Lemma 2.1 in [18].

The following statement is rather obvious for $S \in \mathcal{R}_0$.

Lemma 7. *For* $S \in \mathcal{A}$ *and* $x \in S$, *any arc* $J \subset C(x)$ *joining two points* $y, z \in S$ *with* $\rho_x(y) < \rho_x(z)$ *contains a point which is not critical with respect to* ρ_x.

Proof. Let $j : [0, \lambda J] \to J$ be a parametrization by arc-length of J, and $f(s) = \rho_x(j(s))$.

By Lemma 6, J has at any point $v \in \operatorname{int} J$ right and left tangents with directions τ^+, τ^-. There are points $\tau_1^+, \tau_2^+, \tau_1^-, \tau_2^-$ lying together with τ^+, τ^- in the order $\tau_1^+, \tau^+, \tau_2^+, \tau_1^-, \tau^-, \tau_2^-$ on T_v (where possibly $\tau_1^+ = \tau_2^-$ or $\tau_2^+ = \tau_1^-$), and there are segments from v to x with directions $\tau_1^+, \tau_2^+, \tau_1^-, \tau_2^-$ at v such that (i) the direction at v of any further segment from x to v is separated from τ^+ by τ_1^+ and τ_2^+, and from τ^- by τ_1^- and τ_2^-; (ii) $\angle \tau^+ \tau_1^+ = \angle \tau^+ \tau_2^+$ and $\angle \tau^- \tau_1^- = \angle \tau^- \tau_2^-$ (by Lemma 6).

This yields the existence of right and left derivatives f^+ and f^- of f everywhere in $]0, \lambda J[$. (More precisely, $f^+(s) = -\cos \angle \tau^+ \tau_1^+$ and $f^-(s) = \cos \angle \tau^- \tau_1^-$.) Since $\lambda T_v \leq 2\pi$, we have $\angle \tau^+ \tau_1^+ + \angle \tau^- \tau_1^- \leq \pi$, which yields $f^- \geq f^+$.

Let $q = \min\{x \in [0, \lambda J] : f(x) = f(\lambda J)\}$. There exists a point $t \in]0, q[$ with the whole graph of $f|_{[0,q]}$ in one of the two closed half-planes with the line through $(t, f(t))$ of slope $(f(q) - f(0))/q$ as boundary.

If there is such a point t with the graph above the line, then

$$f^-(t) = f^+(t) = \frac{f(q) - f(0)}{q} > 0.$$

In the contrary case, the graph is below the line, and $f^-(t) \geq f^+(t) > 0$ or $(t + \varepsilon, f(t + \varepsilon))$ lies below the line through $(t, f(t))$ and $(q, f(q))$ for $\varepsilon > 0$ small enough. In the latter situation, repeat the argument for $f|_{[t,q]}$ instead of $f|_{[0,q]}$, and we are in the previous case.

So, in any case we have found a point t satisfying $f^-(t) \geq f^+(t) > 0$, which means that $\angle\tau^-\tau_1^- < \pi/2$ and $\tau^+\tau_1^+ > \pi/2$. The second inequality shows that v is not a critical point. $\square$

Theorem 2. *On most convex surfaces S, for any $x \in S$ and $a > 0$, the set $\rho_x^{-1}(a) \cap C(x)$ is totally disconnected; consequently, F_x, and even Q_x, are totally disconnected too.*

Proof. Let $S \in \mathcal{S}$, and suppose there exist $x \in S$ and $a > 0$ such that $\rho_x^{-1}(a) \cap C(x)$ contains an arc J'.

Let y_1, y_2 be two interior points of J', and denote by J the subarc of J' joining them. Since each point $z \in J$ is a relative minimum for $\rho_x|_{J'}$, we conclude, by Lemma 2, that z is the midpoint of a loop Λ_z at x, and no segments connect x to z excepting those in Λ_z. Now, denote by Δ the domain bounded by $\Lambda_{y_1} \cup \Lambda_{y_2}$. By Lemma 4, we have $\Delta \cap C(x) \subset J$, whence no endpoint of S belongs to $\Delta \setminus J$, a fact which contradicts Lemma 5.

By Lemma 7, ρ_x would be constant on any arc in Q_x, and this we just showed to be impossible. $\square$

Further, sometimes strange, properties of most convex surfaces are surveyed in [8] and [22]. (See also [32].) It is, for example, true that on most convex surfaces there are arbitrarily long geodesics without self-intersections [23] and with huge distance between consecutive conjugate points [26].

3 The injectivity of *Q*

The goal of this section is to show that a smooth point cannot be simultaneously critical with respect to two distinct points.

Lemma 8. *On $S \in \mathcal{A}$, let σ, σ' be (possibly coinciding) segments from x to y and D a component of the complement of $\sigma \cup \sigma'$ in an open disc around y. Further, suppose no point of D belongs to any segment from x to y. If the angle between σ, σ' at y toward D is larger than π, then y is a strict relative minimum of $\rho_x|_{C(x)\cap\overline{D}}$.*

This is part of Proposition 2.4 in [18].

Lemma 9. *For any $S \in \mathcal{A}$ and $x \in S$ we have $M_x \subset Q_x$.*

For $S \in \mathcal{R}_0$, this is essentially Berger's Lemma (see [2]). For $S \in \mathcal{S}$, this is Theorem 2 in [24]. In the general case, it follows from Lemma 8.

Lemma 10. *If $S \in \mathcal{A}$, $x \in S$, $y \in Q_x$ and $\lambda T_y > \pi$, then there are at least two segments from x to y.*

This useful lemma follows directly from the definition of a critical point.

Theorem 3. *If $S \in \mathcal{A}_0$, $x, y \in S$ are distinct, and $z \in Q_x \cap Q_y$, then z is a conical point. Hence, if $S \in \mathcal{A}_0$ is smooth then Q is injective; consequently, M and F are injective too.*

Proof. Let $\mathcal{X}$, $\mathcal{Y}$ be the family of all segments from x, respectively y, to z.

If y lies on a segment in $\mathcal{X}$, then $\operatorname{card} \mathcal{Y} = 1$, whence z is conical by Lemma 10. If not, since any segment in $\mathcal{X}$ meets any segment in $\mathcal{Y}$ only at z, all segments of $\mathcal{Y}$ lie in a single component D of $S \setminus \bigcup_{\sigma \in \mathcal{X}} \sigma$.

Thus, the directions of the segments in $\mathcal{X}$ at z are separated in T_z from those of the segments in $\mathcal{Y}$ at z by the points α, β, say. The longer (if equally long, any) arc Γ from α to β in T_z does not contain the direction τ_σ at z of any segment σ of precisely one of the two families, say of $\mathcal{X}$. Then, if z were not conical, the distance in T_z from the midpoint of Γ to τ_σ would be larger than $\pi/2$ for any $\sigma \in \mathcal{X}$, contradicting $z \in Q_x$.

If S has no conical points, then, by Lemma 9,

$$F_x \cap F_y \subset M_x \cap M_y \subset Q_x \cap Q_y = \emptyset$$

for $x \neq y$. $\square$

The preceding result is not extendable to orientable surfaces of higher genus or to nonorientable surfaces.

On any flat torus not obtainable by identification of the sides of a rectangle, every point is at diametral distance from other two points. The standard projective plane, too, has F noninjective in a strong way: for any two points x, y on it, $F_x \cap F_y \neq \emptyset$.

The sets M_y^{-1} containing more than a single point are investigated in [19].

4 About the components of Q_x and M_x

We start, once again, with two preparatory lemmas. The first of them complements Lemma 8.

Lemma 11. *On $S \in \mathcal{A}$, let σ, σ' be (possibly coinciding) segments from x to y and D a component of the complement of $\sigma \cup \sigma'$ in an open disc around y. If the angle of σ, σ' at y toward D is smaller than π, then y is a strict relative maximum of $\rho_x|_{\overline{D}}$.*

This is easily proven using again Proposition 2.4 in [18].

Lemma 12. *On $S \in \mathcal{A}$, let σ, σ' be segments from x to y and D, D' the components of the complement of $\sigma \cup \sigma'$ in an open disc around y. If $y_n \in D$, $y'_n \in D'$, $y_n \to y$, $y'_n \to y$, $\rho_x(y_n) \geq \rho_x(y)$, and $\rho_x(y'_n) \geq \rho_x(y)$, then $\sigma \cup \sigma'$ is a loop at x, and no segment joins x to y except for σ, σ'.*

This follows from Lemma 11.

Theorem 4. *Consider the surface $S \in \mathcal{A}$. For any point $x \in S$, every component of Q_x is a single point or an arc or a closed Jordan curve* (*the latter possibility being excluded if $S \in \mathcal{A}_0$*). *If the component Q^1 of Q_x is an arc then Q^1, possibly with one or both endpoints removed, is a component of M_x. Hence $Q_x \setminus M_x$ is totally disconnected.*

The closure of any component of M_x is a component of Q_x. This defines a natural injection from the family of components of M_x to the family of components of Q_x. Consequently, if Q is connected then M is connected too.

Proof. Let the component Q^1 of Q_x contain more than one point. Since $C(x)$ is a local tree, Q^1 includes an arc A. By Lemma 7, ρ_x is constant on A. So, $\operatorname{int} A \subset M_x$. Since this is true for any arc in Q^1, by Theorem 1, Q^1 cannot include a nondegenerate tree, so it must be an arc or a closed Jordan curve, the latter case being excluded if $S \in \mathcal{A}_0$.

Let now M^1 be a component of M_x. By Lemma 9, $M^1 \subset Q_x$. If M^1 is an arc with an endpoint e removed, we show that still $e \in Q_x$.

Let σ, σ' be the segments obtained as limits of the pairs of segments (see Lemma 2) from x to the points $y_n \in M^1$ when $y_n \to e$.

Let O be a small open disc around e, with $M^1 \setminus O \neq \emptyset$. For n large enough, $y_n \in O$ and the domain Δ from Lemma 4 exists (J joining y_n to e). Hence every point $u \in \Delta \cap O$ lies on some segment from x to a point in M^1; therefore $\rho(x,u) \leq \rho(x,e)$.

The segments σ, σ' cannot coincide, because arbitrarily close to e there are points farther than e from x. These points must be in the component of $O \setminus (\sigma \cup \sigma')$ not meeting Δ. By Lemma 12, $\sigma \cup \sigma'$ is a loop at x, and $e \in Q_x$.

Hence $\overline{M^1} \subset Q_x$. To show that $\overline{M^1}$ is a component of Q_x it suffices to exclude the existence of a second component M^2 of M_x with $\overline{M^1} \cap \overline{M^2} \neq \emptyset$.

Suppose such a component M^2 exists. Then it is an arc with the endpoint $\{e\} = \overline{M^1} \cap \overline{M^2}$ removed, while M^1 is also an arc with the endpoint e removed. As we already know, there are precisely two segments from x to e. Since this is also true for any point $z \in M^1 \cup M^2$, by Lemma 4,

$$C(x) \cap V \subset M^1 \cup M^2 \cup \{e\}$$

for some neighbourhood V of e. Then each point of V lies on some segment from x to $M^1 \cup M^2 \cup \{e\}$. Thus, $e \in M_x$, whence M^1 is not a component of M_x, and a contradiction is obtained. □

5 Openness of the family $\mathcal{S}_2$ of surfaces with disconnected M

The set $\mathcal{S}_2$ of all surfaces $S \in \mathcal{A}_0$ on which there exists a point x with disconnected M_x has been introduced for $S \in \mathcal{S}$ by the second author in [29], where it was stated that $\mathcal{S}_2$ is open and shown that on most $S \in \mathcal{S}_2 \cap \mathcal{S}$ there exists a point x for which M_x is infinite. We prove here that $\mathcal{S}_2$ is open in $\mathcal{S}$, as well as in $\mathcal{A}_0$.

The set $\mathcal{S}_1$ of all surfaces $S \in \mathcal{A}_0$ on which F is a homeomorphism is not empty, as $\mathrm{S}^2 \in \mathcal{S}_1$. In [20] we showed that, using $\mathcal{C}^2$ metrics on both S and $\mathcal{A}_0$, $\mathcal{S}_1$ has nonempty interior. We do not know whether, equipped with the Hausdorff–Gromov distance, $\mathcal{S}_1/$isometries is nowhere dense or not. (See Question 1 in Section 9.) Theorem 6 will show that $\mathcal{S}_1$ and $\mathcal{S}_2$ are disjoint. But are they complementary sets? (See Question 4 in Section 9.)

Lemma 13. *Let $S \in \mathcal{A}_0$, $x \in S$, and $P \subset C(x)$ be an arc from y to z. If $\rho(x,P) < \rho(x,y)$ and $\rho(x,P) < \rho(x,z)$, then M_x has at least two closed components M^1, M^2 separated by a loop Λ at x, such that $\rho(x,v) < \rho(x,w)$ for any pair of points $v \in \Lambda$, $w \in M^1 \cup M^2$.*

Proof. Let $u \in P$ satisfy $\rho(x,P) = \rho(x,u)$. By Lemma 2, there is a loop Λ at x with midpoint u. Let the arc $P^* \subset C(x)$ be a maximal extension of P with respect to inclusion. The point u divides P^* into two subarcs P_1, P_2. Let M^i be the set of all absolute maxima of $\rho|_{P_i}$ $(i = 1,2)$. Each component of M^i is closed. Moreover, since $u \notin M^i$, the sets M^1, M^2 are separated by Λ. The proof ends with the remark that the components of each M^i are also components of M_x, and we find at least two such components, one in M^1 and the other in M^2. □

Lemma 14. *Suppose $S \in \mathcal{A}_0$ has a loop at $x \in S$ of length less than $2r_x$. Then, for some point $x' \in S$, $M_{x'}$ has at least two closed components M^1, M^2 separated by a loop Λ at x', such that $\rho(x',v) < \rho(x',w)$ for any pair of points $v \in \Lambda$, $w \in M^1 \cup M^2$.*

Proof. Let Λ be the loop of length less than $2r_x$, with midpoint z. Let D be a component of $S \setminus \Lambda$ containing some point $v \in F_x$. There are two consecutive segments σ, σ' from x to z in $\overline{D}$ (i.e., there is no segment from x to z separating $\operatorname{int}\sigma$ from $\operatorname{int}\sigma'$ in $\overline{D}$), with v between them. Possibly $\sigma \cup \sigma' = \Lambda$. Then $C(z)$ contains a small arc $A \subset \overline{D}$ starting at x, bisecting the angle of σ, σ' at z, and also bisecting the angle of suitably chosen segments σ_y, σ'_y from any $y \in A$ to z, by Lemma 6 (see also Proposition 2.4 in [18]). Then $\sigma_y \to \sigma$ and $\sigma'_y \to \sigma'$ as $y \to x$ (supposing $\sigma, \sigma', \sigma'_y, \sigma_y$ in this order around z). Let D_y be the component of $S \setminus (\sigma_y \cup \sigma'_y)$ included in D.

For y close enough to x, $v \in D_y$ and $\rho(y,v) > \rho(y,z)$. Clearly, the angle of σ_y, σ'_y at z toward D_y is less than the angle of σ, σ' at z toward D, which in turn was not larger than π. Thus, by Lemma 11, z is a strict relative maximum of $\rho_y|_{\overline{D_v}}$.

Let $P \subset C(y)$ be the arc from z to v. Clearly, $P \subset D_v \cup \{z\}$. Hence $\rho(y,P)$ is smaller than both $\rho(y,z), \rho(y,v)$. Now, the conclusion follows from Lemma 13. □

Theorem 5. *The set $\mathcal{S}_2$ is open in $\mathcal{A}_0$. The set $\mathcal{S}_2 \cap \mathcal{S}$ is open in $\mathcal{S}$.*

Proof. We show that $\mathcal{S}_2 \cap \mathcal{S}$ is open in $\mathcal{S}$. Let $S \in \mathcal{S}_2 \cap \mathcal{S}$. By Theorem 6 in the next section, S is not loopy, hence, by Lemma 14, for some $x \in S$, M_x has at least two closed components, say M^1, M^2, separated by a loop Λ at x. Let

$$6\varepsilon \leq \min\Big\{\rho_x(M^1) - \frac{\lambda\Lambda}{2}, \rho_x(M^2) - \frac{\lambda\Lambda}{2}\Big\}$$

and denote by $\mathcal{H}$ the Pompeiu–Hausdorff distance in $\mathcal{S}$.

Let

$$\mathcal{O} = \{S' \in \mathcal{S} : \mathcal{H}(S,S') < \varepsilon\}.$$

Take the points

$$x = a_1, a_2, \ldots, a_k, a_{k+1} = x \in \Lambda$$

with $\rho(a_i, a_{i+1}) \leq \varepsilon$ $(i = 1, \ldots, k)$.

Consider an arbitrary surface $S' \in \mathcal{O}$. We find points

$$y = b_1, b_2, \ldots, b_k, b_{k+1} = y \in S'$$

with $\|a_i - b_i\| < \varepsilon$. This yields $\rho(b_i, b_{i+1}) < 3\varepsilon$. Let σ_i be a segment from b_i to b_{i+1} and $\Sigma = \cup_{i+1}^{k} \sigma_i$. For any point $s \in \Sigma$, there is some point $a_i \in \Lambda$ such that $\|a_i - s\| < 3\varepsilon$. Therefore, on S',

$$\rho(y, s) \leq \|y - x\| + \frac{\lambda\Lambda}{2} + \|a_i - s\| < \frac{\lambda\Lambda}{2} + 4\varepsilon.$$

For $u_i \in M^1$ there exists a point $v_i \in S'$ satisfying $\|u_i - v_i\| < \varepsilon$ $(i = 1, 2)$. For ε small enough, v_1 and v_2 are separated by Σ on S'. We have

$$\rho(y, v_i) \geq \rho(x, u_i) - \|x - y\| - \|u_i, v_i\| > \rho(M^i) - 2\varepsilon.$$

It follows that $\rho(y, v_i) > \rho(y, s)$ for any $s \in \Sigma$ and $i \in \{1, 2\}$. This proves that $S' \in \mathcal{S}_2$.

The same argument with the Hausdorff–Gromov metric instead of $\mathcal{H}$ shows that $\mathcal{S}_2$ is open in $\mathcal{A}_0$. □

6 About the connectedness of Q, M, F, and the surjectivity of F

In [29] the second author introduced, for $S \in \mathcal{S}$, the *antipodal arc* J_x, which is the smallest arc in $C(x)$ including F_x. Let us extend this notion for $S \in \mathcal{A}_0$, and call the smallest tree J_x in $C(x)$ including F_x simply the *antipode* of x.

Lemma 15. *Suppose that $S \in \mathcal{A}_0$, $x \in S$ and $\operatorname{card} F_x \geq 2$. Let z be an extremity of the antipode J_x, and $\varepsilon > 0$. Then there exist an arc A starting at x and a number $k > 0$, such that for any $v \in A$ and any $u \in B(v, k\rho(v, x))$, we have $F_u \subset B(z, \varepsilon)$.*

The proof of Lemma 15 appears inside the proof of Theorem 5 in [29]. It is given for the convex case only. However, the same argument entirely applies for any $S \in \mathcal{A}_0$; in particular the antipode of x, which is now a tree, should be used at the place of the antipodal arc.

Theorem 6. *For a surface $S \in \mathcal{A}_0$, the following implications hold.*

F is single-valued $\Longleftrightarrow$ F is continuous $\Longrightarrow$ F is surjective $\Longrightarrow$ M is surjective $\Longrightarrow$ S is loopy $\Longleftrightarrow$ Q is connected $\Longleftrightarrow$ M is connected $\Longrightarrow$ F is connected.

Proof. *F is single-valued $\Longleftrightarrow$ F is continuous.*

The upper semicontinuity of arbitrary F becomes, for F single-valued, continuity. Conversely, suppose F is continuous and there exists $x \in S$ such that $\operatorname{card} F_x > 1$. By Lemma 15, there is a sequence of points $x_n \in S \setminus \{x\}$ convergent to x such that the sequence of sets F_{x_n} converges to a single-point set, therefore different from F_x, in contradiction the the continuity of F in x.

F is continuous $\Longrightarrow$ F is surjective.

Since F is continuous, F_S is closed. If F is not surjective, there is a small open disc D in $S \setminus F_S$. Clearly, $F_{S\setminus D}$ is included in $S \setminus D$. By Brouwer's fixed point theorem, $F|_{S\setminus D}$ has a fixed point, which is impossible.

F is surjective $\Longrightarrow$ M is surjective.

This follows from $F_x \subset M_x$ for any $x \in S$.

M is surjective $\Longrightarrow$ S is loopy.

Suppose S is not loopy, i.e., there is a loop Λ at $x \in S$ with midpoint y at distance less than r_x from x. By Lemma 14, for some point $x' \in S$, $M_{x'}$ has at least two closed components. Let $A' \subset C(x')$ be the (unique) arc meeting each of the two preceding components in precisely one point. Let A be the set of all absolute minima of $\rho|_{A'}$. The components of A are points or arcs. Let u be a single-point component of A or, if such components are missing, an endpoint of a component of A. Then $u \in Q_{x'} \setminus M_{x'}$ and, by Theorem 3, $u \notin M_S$.

S is loopy $\Longrightarrow$ Q is connected.

Suppose S is loopy, but Q_x has besides a component Q^0 in F_x a further component Q^1. If $y \in Q^1$ is a strict relative maximum of ρ_x or belongs to F_x, then take $z \in Q^0$ and consider the arc $J \subset C(x)$ joining y to z. Then $\rho_x|_J$ has an absolute minimum at a point u different from y and z, and $\rho_x(u) < r_x$. By Lemma 2, there is a loop at x through u of length $2\rho_x(u)$, in contradiction to the assumption that S is loopy.

Suppose now that $y \in Q^1$ is not a strict relative maximum of ρ_x and does not belong to F_x. Moreover, assume there is no loop at x with y as midpoint. Then the maximal angle at y between consecutive segments (if there are any consecutive segments) from x to y is less than π. By Lemma 11, y is a strict relative maximum of ρ_x, and we reached a contradiction. Hence there is a loop at x with midpoint y; moreover its length is less than r_x. Hence S is not loopy, and a contradiction is obtained again.

Q is connected $\Longrightarrow$ M is connected.

This is part of Theorem 4.

M is connected $\Longrightarrow$ S is loopy.

This follows from Lemma 14.

M is connected $\Longrightarrow$ F is connected.

Indeed, for any $x \in S$, ρ_x being constant on each component of M_x, it is constant on M_x; since $F_x \subset M_x$, the constant is r_x, whence $M_x = F_x$. $\square$

Theorem 7. *Let S belong to $\mathcal{A}_0$. If F_x is totally disconnected for each $x \in S$, then all implications in Theorem* 6 *are valid in both directions.*

This is in particular true for polyhedral convex surfaces and for most convex surfaces.

Proof. Indeed, by Theorem 1, for any surface $S \in \mathcal{A}_0$ and any point $x \in S$, if F_x is connected, then it is either a point or an arc. Since the second possibility is excluded, F_x contains a single point. On most convex surfaces, there is no point x with an arc in F_x, by Theorem 2.

For a polyhedral convex surface P, one can easily see that, for any point x in P, the only points in F_x which are joined to x by precisely two segments are among the vertices of P (see also [31]). If F_x contained an arc then, by Lemma 2, all points interior to that arc would be joined to x by precisely two segments, too many points to be among the vertices of P. $\square$

Remark. The set of all surfaces $S \in \mathcal{S}$ with F surjective is closed in $\mathcal{S}$.

Since a geodesic arriving at an endpoint cannot go beyond it, the endpoint is a kind of "farthest point" on that geodesic. This may suggest that endpoints of $S \in \mathcal{S}$ might always lie in F_x for some point $x \in S$. However, this is deeply false, as every surface $S \in \mathcal{S}_2 \cap \mathcal{S}$ has, by Theorem 6, an open set O disjoint from F_S, while, for most surfaces in $\mathcal{S}$, O contains lots of endpoints, by Lemma 5.

7 Sufficient conditions for a surface to belong to $\mathcal{S}_2$

We shall make use of a hinge variant of Toponogov's well-known comparison theorem (see, for example, [6]), which will now be recalled. Let M_H be the simply connected 2-dimensional space of constant curvature H.

Lemma 16. *Let M be a complete manifold with sectional curvature $K \leq H$, and let $(\gamma_1, \gamma_2, \gamma_3)$ be a geodesic triangle in M. If $H > 0$, suppose $\lambda\gamma_i \leq \pi/\sqrt{H}$ $(i = 1, 2)$.*

Then there exists in M_H a geodesic triangle $(\overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3)$ such that $\lambda\gamma_i = \lambda\overline{\gamma}_i$ $(i = 1, 2)$, $\angle\gamma_1\gamma_2 = \angle\overline{\gamma}_1\overline{\gamma}_2$, and $\lambda\gamma_3 \geq \lambda\overline{\gamma}_3$.

Theorem 8. *Suppose the surface $S \in \mathcal{A}_0$ is of class $\mathcal{C}^2$ in a neighbourhood of a loop Λ of length l. If S has Gauß curvature $K < \pi^2/l^2$ along Λ, then $S \in \mathcal{S}_2$. In particular, this is true if K is nonpositive along Λ.*

Proof. Choose $x, y \in S$ so that Λ is a loop at x with midpoint y. Using the continuity of K, we find a neighbourhood $V \subset S$ of Λ and a number $k < \pi^2/l^2$ such that $K < k$ in V.

Let $yz \in V$ be a segment orthogonal to Λ at y. Let $\bar{x}, \bar{y}, \bar{z}$ be points on the sphere M_k of curvature k, such that $\rho_0(\bar{x}, \bar{y}) = \pi/(2\sqrt{k})$, $\rho_0(\bar{y}, \bar{z}) = \rho(y, z)$ and $\angle\bar{x}\bar{y}\bar{z} = \pi/2$; here, ρ_0 is the standard metric of M_k. It follows that $\bar{y}$ belongs to the equator farthest from $\bar{x}$, whence $\rho_0(\bar{x}, \bar{z}) = \rho_0(\bar{x}, \bar{y})$. By Lemma 16, we have

$$\rho(x, z) \geq \rho_0(\bar{x}, \bar{z}) = \frac{\pi}{2\sqrt{k}} > \frac{l}{2} = \rho(x, y).$$

Hence $y \notin F_x$ and S is not loopy. By Theorem 6, $S \in \mathcal{S}_2$. $\square$

The above estimate is sharp, as it is shown by the example of an ellipsoid E with half-axes a, b, c satisfying $a = b$ and $c = 2a$, whose curvature along the equator of length l equals $1/c^2 = \pi^2/l^2$. Indeed, we proved in [20] that the mapping F is a homeomorphism on the ellipsoids for which $a = b < c < 2a$. Since the set of convex surfaces with surjective F is closed, F must be surjective on E too. By Theorem 6, $E \notin \mathcal{S}_2$. But $E \in \operatorname{bd}\mathcal{S}_2$, as follows from the next few lines.

Examples. Theorem 8 provides compact surfaces of revolution in $\mathcal{S}_2$, for example those with Gauß curvature $K < 1/(4r^2)$ along an equator Λ of radius r. Among them we find the ellipsoids with semi-axes a, b, c verifying $a = b$ and $c > 2a$.

We can also apply Theorem 8 to boundaries of rectangular boxes and to doubly covered polygons, all of which admit a loop Λ with a neighbourhood isometric to an open subset of a cylinder or a cone.

Remark. One can, of course, formulate Theorem 8 in terms of an upper bound to the curvature, in the sense of Alexandrov, in addition to the lower bound that our surfaces have by definition. But the goal being to give an easy-to-apply criterion, we chose to use Gauß curvature.

Theorem 9. *If on $S \in \mathcal{A}$ there exists a point $y \in S$ such that $\lambda T_y < \pi$, then $S \in \mathcal{S}_2$.*

Proof. By Lemma 11, y is a strict relative maximum of ρ_x for any point $x \in S \setminus \{y\}$. Choosing x close enough to y guarantees that $y \notin F_x$, whence M_x does not consist of the isolated point y alone, and is therefore disconnected. □

Theorem 10. *If $S \in \mathcal{A}_0$ and $\operatorname{rad} S = \operatorname{diam} S/2$ then $S \in \mathcal{S}_2$, except for the case of a surface S with precisely two conical points y, z where $\lambda T_y = \lambda T_z = \pi$, and with a closed geodesic Λ so that $\lambda\Lambda = \operatorname{diam} S$, $F_\Lambda = S$, and, for each point $x \in \Lambda$, $r_x = \operatorname{rad} S$ and F_x is an arc from y to z.*

Proof. Suppose $S \notin \mathcal{S}_2$, i.e., for each point $x \in S$, M_x is connected and, since $F_x \subset M_x$ and ρ_x is constant on M_x, $F_x = M_x$.

Let $x, y, z \in S$ satisfy $r_x = \operatorname{rad} S$ and $\rho(y, z) = \operatorname{diam} S$. Then

$$\operatorname{diam} S = 2r_x \geq \rho(x, y) + \rho(x, z) \geq \rho(y, z) = \operatorname{diam} S,$$

whence $y, z \in F_x$ and x belongs to a segment Γ_{yz} from y to z. Thus, $\{y, z\} \subset C(x) \setminus C_x$, whence λT_y and λT_z are, by Lemma 9, not larger than π.

By Theorem 1, F_x is an arc. The points y and z belong to F_x and are joined to x by unique segments. Since each point v interior to F_x is a relative minimum for $\rho_x|_{F_x}$ and hence is the mid-point of a loop Λ_v at x by Lemma 2, y and z must be the endpoints of the arc F_x.

Now put $\Delta = S \setminus \Gamma_{yz}$ and apply Lemma 4. We obtain $\Delta = \bigcup_{v \in \operatorname{int} F_x} \Lambda_v$, so S has no conical points, except for y, z.

Consider now the set $E(y, z)$ of all points in S at equal distance from y and z. Each arc from y to z meets $E(y, z)$.

For an arbitrary point $w \in E(y, z) \setminus \{x\}$, denote by Λ_w the loop at x through w; so Λ_w separates y from z. Since y, z are points of the tree $C(w)$, there exists a unique arc $J_w \subset C(w)$ joining them. Consider a point $w' \in J_w \cap \Lambda_w$. Then

$$\rho(w, w') \leq \lambda\Lambda_w/2 = \operatorname{rad} S.$$

We also have

$$2 \operatorname{rad} S = \rho(y, z) \leq \rho(w, y) + \rho(w, z) = 2\rho(w, y),$$

so we obtain $\rho(w,w') \le \rho(w,y)$. If $\rho(w,w') < \rho(w,y) = \rho(w,z)$ then, by Lemma 13, $S \in \mathcal{S}_2$, which, as we assumed, is not the case. Thus,

$$\rho(w,w') = \lambda\Lambda_w/2 = \operatorname{rad} S = \rho(w,y) = \rho(w,z).$$

From

$$\rho(y,z) = \operatorname{diam} S = \rho(w,y) + \rho(w,z),$$

we obtain that w is the mid-point of a segment from y to z, and the segments joining w to y and z are unique. Since M_w is connected, we have $M_w = F_w$, and since $y, z \in F_w$, we get, as before, $F_w = J_w$.

Assume now $w \notin F_x$. The equality $\rho(w,w') = \lambda\Lambda_w/2$ implies that Λ_w is a closed geodesic, consequently its directions at x make an angle of π.

Suppose there exists a point $w^* \in E(y,z) \setminus (F_x \cup \Lambda_w)$. The above arguments show that Λ_{w^*} is a closed geodesic. Then $\Lambda_w \cap \Lambda_{w^*} = \{x\}$ implies that the directions of Λ_{w^*} at x do not separate in T_x those of Λ_w and are different from them, so their angle is less than π, and a contradiction is obtained. Hence

$$E(y,z) \subset F_x \cup \Lambda_w.$$

To show that $\Lambda_w \subset E(y,z)$, let $x' \in \Lambda_w$ be chosen arbitrarily. Choose an arc A joining y to z, disjoint from $(F_x \cup \Lambda_w) \setminus \{x', y, z\}$. This arc obviously meets $E(y,z)$ in a point different from y and z. Due to the preceding inclusion, the point must be x'. We saw that $u \in E(y,z)$ yields

$$r_u = \operatorname{rad} S = \rho(u,y) = \rho(u,z).$$

So, if $u \notin \Lambda_w$, it is separated from y or z by Λ_w, say from y. Then any segment from u to y meets Λ_w at a point at distance $\operatorname{rad} S$ from y, whence $\rho(u,y) > \operatorname{rad} S$, and a contradiction is obtained.

Put $\Lambda = E(y,z)$. Finally, we have to prove that $F_\Lambda = S$. To see this, notice first that, by Lemma 4, $C(w) = F_w$ for all points $w \in \Lambda$. Let now $v \in S \setminus \{y,z\}$. Since y, z are conical, they belong to $C(v)$. The arc $J \subset C(v)$ joining y to z clearly meets Λ, say at v'. Then $v \in C(v') = F_{v'}$, by the preceding remark. And, obviously, $y, z \in F_\Lambda$, too. □

The actual existence of an exceptional surface as described in Theorem 10 is illustrated by the example considered in the next section.

8 Farthest points on a Tannery surface

We investigate here the special case of a Tannery surface, particularly interesting for our purposes.

A Riemannian surface (S, g) is called a *P_l-surface* if all of its geodesics are periodic with least common period l ([3], p. 182). A P_l-surface of revolution (i.e., having S^1 as an effective isometry group) is called a *Tannery surface* (see [3], pp. 95 and 102).

On the unit sphere S^2, let n and s be the North and South poles, and consider a point $x \in \mathrm{S}^2 \setminus \{n, s\}$. Denote by G_x the great circle through n and x, and let $2r$ be the distance from n to x (realized on G_x) and θ the angle made by the plane of G_x with a fixed plane through n and s.

Consider, for the set $U = \mathrm{S}^2 \setminus \{n, s\}$, the parametrization (r, θ) described before, with $r \in]0, \pi/2[$, $\theta \in [0, 2\pi[$, and endow U with the metric $g = 4dr^2 + \sin^2 r d\theta$. By Proposition 4.6 in [3] p. 96, the metric g extends (only) to a $\mathcal{C}^0$ metric on S^2.

We obtain, from considerations in [3] and an application of Theorem 10, the following about (S^2, g).

Theorem 11. a) (S^2, g) *is a Tannery surface* (*with parameters* $p = 2$, $q = 1$).

b) (S^2, g) *may be isometrically embedded in the Euclidean space* $\mathbb{R}^3$ *as a convex surface of revolution* (S, ρ)*, whose half-meridian* (*joining the images of the poles through the isometry, denoted again by* n *and* s) *is described by* $c(R) = \pm \int_R^1 \sqrt{\frac{4}{1-u^2} - 1}\, du$*, where* $R = \sin r$.

c) *Except for the subsegments of half-meridians, every geodesic extends to a closed geodesic of* S*. Apart from the equator* E*, which has length* 2π*, every closed geodesic* Γ *consists of* $2q = 2$ *arcs between two consecutive points of tangent contact with the parallels;* Γ *has length* $4q\pi = 4\pi$ *and turns* $p = 2$ *times. The length of a half-meridian is* 2π.

d) $\operatorname{diam} S = 2 \operatorname{rad} S$*. For each point* $x \in E$*,* F_x *is the half-meridian opposed to* x.

e) $\lambda T_n = \lambda T_s = \pi$*,* $F_E = S$*, and* $S \notin \mathcal{S}_2$.

Proof. a) and c) follow from Theorems 4.11 (p. 100) and 4.13 (p. 102) in [3], applied to the particular metric g.

Concerning b), the isometric embedding of (S^2, g) in $\mathbb{R}^3$ is a consequence of Propositions 4.18 (p. 105) and 4.20 (p. 107) (in our case $h \equiv 0$), while the convexity of the surface follows from Proposition 4.23 and Remark 4.21 (pp. 108–109), all from [3].

We prove now d). The symmetry of S and $p = 2$ imply that each closed geodesic Γ has precisely one self-intersection point x_Γ, which must lie on E, and two points z_Γ, z'_Γ on the half-meridian opposite to x_Γ, at which Γ and the half-meridian are orthogonal.

Let $x \in E$, $y \in F_x$, and σ be a segment from x to y. Then the closed geodesic Γ including σ has $x_\Gamma = x$ and y on one of the four arcs joining x to z_Γ, z'_Γ, each of length π. Hence y equals z_Γ or z'_Γ and $\rho(x, y) = \pi$. By choosing $x \in E$ and y in the half-meridian H_x opposite to x, the same argument shows that $y \in F_x$, so $F_x = H_x$.

Since $\rho(n, s) = 2\pi$ by c), and any segment not included in a meridian extends to a closed geodesic and has therefore length at most 2π, we have indeed $\operatorname{diam} S = 2 \operatorname{rad} S$.

To prove e), first note that $F_E = \bigcup_{x \in E} H_x = S$. If $x \in E$, then $M_x = H_x$. Clearly, $M_n = \{s\}$ and $M_s = \{n\}$. If $x \notin E \cup \{n, s\}$, then $C(x) = H_x$. For any point $y \in H_x$, there are precisely two closed geodesics passing through x and y, and they have of course different directions at y. Therefore, by Lemmas 8 and 11, ρ_x is strictly monotone on $H_x \setminus \{n, s\}$. It follows that M_x equals $\{n\}$ or $\{s\}$. Hence $S \notin \mathcal{S}_2$.

Finally, $\lambda T_n = \lambda T_s = \pi$ follows from Theorem 10, because every point but n and s is interior to a geodesic and therefore not conical. □

Our Tannery surface has F surjective and properly multi-valued; thus, the converse of the second implication in Theorem 6 is not true.

9 Nine open questions

We restrict our open questions to the perhaps easier but certainly important convex case, but most of them make sense in $\mathcal{A}_0$ or even $\mathcal{A}$. Let us start with the problem mentioned already in the Abstract.

Question 1. Is $\mathcal{S}_2$ dense in $\mathcal{S}$?

The case of the regular tetrahedron shows that F_S can be connected even if $S \in \mathcal{S}_2$, as J. Rouyer proved in [12]. In most of our examples, however, the set F_S is disconnected as soon as F is not surjective.

Question 2. For which convex polyhedral surfaces S is F_S connected?

Question 3. Consider a surface $S \in \mathcal{S}$ such that $F_S \neq S$. Do there always exist points $y \in \operatorname{bd} F_S$ and $x \in F_y^{-1}$ such that $\operatorname{card} F_x \geq 2$?

It is natural to ask whether $\{\mathcal{S}_1, \mathcal{S}_2\}$ forms a partition of $\mathcal{A}_0$. This reduces to the following problem.

Question 4. Do convex surfaces with F single-valued and noninjective exist?

Several possible implications between statements in Theorem 6 are neither proved nor disproved, so far.

Question 5. Does connectedness of F imply connectedness of M?

Question 6. Does surjectivity of M imply surjectivity of F?

Of course, surjectivity of M implies surjectivity of Q. But what is the relationship between the surjectivity of Q and the loopiness of S?

Question 7. Does surjectivity of Q imply loopiness of S? Or vice-versa?

J. Rouyer proved that all tetrahedra in $\mathbb{R}^3$ belong to $\mathcal{S}_2$ (see [12], [15]).

Question 8. Which convex polyhedral surfaces do not belong to $\mathcal{S}_2$?

We believe that there are no surfaces different from the Tannery surface described in Section 8 playing the exceptional role in Theorem 10. However this is not proven yet.

Question 9. Is the Tannery surface from the preceding section the unique exception in Theorem 10?

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