Rend. Lincei Mat. Appl. 22 (2011), 89–111
DOI 10.4171/RLM/589

Calculus of Variations — *Variational methods for nonlinear perturbations of singular ϕ-Laplacians*, by Cristian Bereanu, Petru Jebelean and Jean Mawhin, communicated on 14 January 2011.

Remembering Giovanni Prodi's pioneering work in nonlinear analysis.

Abstract. — Motivated by the existence of radial solutions to the Neumann problem involving the mean extrinsic curvature operator in Minkowski space

$$\operatorname{div}\Big(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\Big) = g(|x|, v) \quad \text{in } \mathscr{A}, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\mathscr{A},$$

where $0 \le R_1 < R_2$, $\mathscr{A} = \{x \in \mathbb{R}^N : R_1 \le |x| \le R_2\}$ and $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ is continuous, we study the more general problem

$$[r^{N-1}\phi(u')]' = r^{N-1} g(r, u), \quad u'(R_1) = 0 = u'(R_2),$$

where $\phi := \Phi' : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$ and the continuous function $\Phi : [-a, a] \to \mathbb{R}$ is of class C^1 on $(-a, a)$. The associated functional in the space of continuous functions over $[R_1, R_2]$ is the sum of a convex lower semicontinuous functional and of a functional of class C^1. Using the critical point theory of Szulkin, we obtain various existence and multiplicity results for several classes of nonlinearities. We also discuss the case of the periodic problem.

Key words: Neumann problem, radial solutions, mean extrinsic curvature, critical point, Palais–Smale condition, saddle point, Mountain Pass Theorem, periodic problem.

MSC 2010 Classification: 35J20, 35J60, 35J93, 35J87, 58E05.

1. Introduction

This study is essentially motivated by the existence of radial solutions to the Neumann problem involving the *mean extrinsic curvature operator in Minkowski space* (see e.g. [3]):

$$\operatorname{div}\Big(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\Big) = g(|x|, v) \quad \text{in } \mathscr{A}, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\mathscr{A}, \tag{1}$$

where $0 \le R_1 < R_2$, $\mathscr{A} = \{x \in \mathbb{R}^N : R_1 \le |x| \le R_2\}$ and $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. As usual, we have denoted by $\dfrac{\partial v}{\partial \nu}$ the outward normal derivative of v and $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^N$. Setting $r = |x|$ and $v(x) = u(r)$, the above problem (1) becomes

$$\left[r^{N-1}\left(\frac{u'}{\sqrt{1-u'^2}}\right)\right]' = r^{N-1}g(r,u), \quad u'(R_1) = 0 = u'(R_2), \tag{2}$$

and the solutions of (2) are classical radial solutions of (1).

In this paper we obtain existence results for the more general problem

$$[r^{N-1}\phi(u')]' = r^{N-1}g(r,u), \quad u'(R_1) = 0 = u'(R_2), \tag{3}$$

where $\phi := \Phi' : (-a,a) \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$ and the continuous function $\Phi : [-a,a] \to \mathbb{R}$ is of class C^1 on $(-a,a)$ and, without loss of generality, we can assume that $\Phi(0) = 0$. This kind of ϕ is called *singular ϕ-Laplacian*. Note that for $\phi(s) = \dfrac{s}{\sqrt{1-s^2}}$ one takes $\Phi(s) = 1 - \sqrt{1-s^2}$.

Our approach is a variational one and relies on Szulkin's critical point theory [13]. Using a strategy inspired from [4], we show in Proposition 1 that u is a solution of (3) provided that u is a critical point of the energy functional $I : C[R_1,R_2] \to (-\infty,+\infty]$ defined by

$$I(u) = \begin{cases} \displaystyle\int_{R_1}^{R_2} r^{N-1}\Phi(u')\,dr + \int_{R_1}^{R_2} r^{N-1}G(r,u)\,dr, & \text{if } u \in K, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $G : [R_1,R_2] \times \mathbb{R} \to \mathbb{R}$ is the primitive of g with respect to the second variable and $K = \{u \in W^{1,\infty}[R_1,R_2] : |u'| \le a \text{ a.e. on } [R_1,R_2]\}$. The functional I has the structure required by Szulkin's critical point theory, i.e., it is the sum of a proper convex, lower semicontinuous functional and of a C^1 functional. In this context, a critical point of I means a function $u \in K$ such that

$$\int_{R_1}^{R_2} r^{N-1}[\Phi(v') - \Phi(u')]\,dr + \int_{R_1}^{R_2} r^{N-1}g(r,u)(v-u)\,dr \ge 0 \quad \text{for all } v \in K.$$

In Section 2 we introduce some notations and definitions and we prove the above mentioned Proposition 1. Notice that, in contrast to [4], we replace some auxiliary result based upon Leray-Schauder theory by an elementary argument (Lemma 1) and obtain in this way a purely variational treatment of our problem. A similar methodology can be applied to obtain pure variational proofs of the results on periodic solutions in [5, 6, 12].

Section 3 deals with minimization problems for I based upon the fact that if there exists $\rho > 0$ such that

$$\inf\left\{I(u) : u \in K, \left|\int_{R_1}^{R_2} r^{N-1}u\,dr\right| \le \rho\right\} = \inf_K I,$$

then I is bounded from below and attains its infimum at some u, which solves problem (3) (Lemma 2). Theorem 1 from [4] is then an immediate consequence of this result (Corrollary 1). We also prove (Theorem 1) that if g is such that

$$\liminf_{|x|\to\infty} G(r,x) > 0, \quad \text{uniformly in } r \in [R_1, R_2],$$

then (3) has at least one solution u which minimizes I on C.

The same is also true if g is bounded and

$$\lim_{|x|\to\infty} \int_{R_1}^{R_2} r^{N-1} G(r,x)\, dr = +\infty$$

(Theorem 2). On the other hand, if $G(r,\cdot)$ is convex for any $r \in [R_1, R_2]$, then (3) has at least one solution if and only if the function

$$x \mapsto \int_{R_1}^{R_2} r^{N-1} g(r,x)\, dr$$

has at least one zero, or, equivalently, the real convex function

$$x \mapsto \int_{R_1}^{R_2} r^{N-1} G(r,x)\, dr$$

has a minimum (Theorem 3).

In Section 4 we derive some properties of the (PS)–sequences (Lemma 3) and we show that if g is bounded and

$$\lim_{|x|\to\infty} \int_{R_1}^{R_2} r^{N-1} G(r,x)\, dr = -\infty,$$

then (3) has at least one solution u which is a saddle point of I (Theorem 4). As in Section 3, if g is not necessarily bounded but the above condition upon G is replaced with the following more restrictive assumption

$$\lim_{|x|\to\infty} G(r,x) = -\infty, \quad \text{uniformly in } r \in [R_1, R_2],$$

then the same result holds true (Theorem 5).

In Section 5 we consider the problem

$$(4) \qquad [r^{N-1}\phi(u')]' = r^{N-1}[\lambda|u|^{m-2}u - f(r,u)], \quad u'(R_1) = 0 = u'(R_2),$$

where $\lambda > 0$ and $m \geq 2$ are fixed real numbers and $f : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the classical Ambrosetti–Rabinowitz condition: there exists $\theta > m$ and $x_0 > 0$ such that

$$0 < \theta F(r,x) \leq x f(r,x) \quad \text{for all } r \in [R_1, R_2] \text{ and } |x| \geq x_0.$$

We also assume that

$$\limsup_{|x|\to 0} \frac{mF(r,x)}{|x|^m} < \lambda \quad \text{uniformly in } r \in [R_1, R_2],$$

and prove that under these assumptions, problem (4) has at least one solution u which is a mountain pass critical point of the corresponding I (Theorem 6).

Section 6 is devoted to the periodic problem

$$(5) \qquad [\phi(u')]' = g(r,u), \quad u(R_1) - u(R_2) = 0 = u'(R_1) - u'(R_2).$$

Here we discuss the manner in which the above results for problems (3) and (4) can be transposed for problem (5).

2. The functional framework

In what follows, we assume that $\Phi : [-a,a] \to \mathbb{R}$ satisfies the following hypothesis:

(H_Φ) $\Phi(0) = 0$, Φ *is continuous, of class* C^1 *on* $(-a,a)$, *with* $\phi := \Phi' : (-a,a) \to \mathbb{R}$ *an increasing homeomorphism such that* $\phi(0) = 0$.

Clearly, Φ is strictly convex and $\Phi(x) \geq 0$ for all $x \in [-a,a]$.

Given $0 \leq R_1 < R_2$ and $g : [R_1,R_2] \times \mathbb{R} \to \mathbb{R}$ a *continuous* function, we denote by $G : [R_1,R_2] \times \mathbb{R} \to \mathbb{R}$ the indefinite integral of g, i.e.,

$$G(r,x) := \int_0^x g(r,\xi)\, d\xi, \quad (r,x) \in [R_1,R_2] \times \mathbb{R}.$$

We set $C := C[R_1,R_2]$, $L^1 := L^1(R_1,R_2)$, $L^\infty := L^\infty(R_1,R_2)$ and $W^{1,\infty} := W^{1,\infty}(R_1,R_2)$. The usual norm $\|\cdot\|_\infty$ is considered on C and L^∞. The space $W^{1,\infty}$ is endowed with the norm

$$\|v\| = \|v\|_\infty + \|v'\|_\infty, \quad v \in W^{1,\infty}.$$

Denoting

$$L^1_{N-1} := \left\{ v : (R_1,R_2) \to \mathbb{R} \text{ measurable} : \int_{R_1}^{R_2} r^{N-1} |v(r)|\, dr < +\infty \right\},$$

each $v \in L^1_{N-1}$ can be written $v(r) = \bar{v} + \tilde{v}(r)$, with

$$\bar{v} := \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} v(r)\, r^{N-1}\, dr, \quad \int_{R_1}^{R_2} \tilde{v}(r)\, r^{N-1}\, dr = 0.$$

If $v \in W^{1,\infty}$ then $\tilde{v}$ vanishes at some $r_0 \in (R_1,R_2)$ and

$$|\tilde{v}(r)| = |\tilde{v}(r) - \tilde{v}(r_0)| \leq \int_{R_1}^{R_2} |v'(t)|\, dt \leq (R_2 - R_1)\|v'\|_\infty,$$

so, one has that

$$\|\tilde{v}\|_\infty \leq (R_2 - R_1)\|v'\|_\infty. \tag{6}$$

Putting

$$K := \{v \in W^{1,\infty} : \|v'\|_\infty \leq a\},$$

it is clear that K is a convex subset of $W^{1,\infty}$.

Let $\Psi : C \to (-\infty, +\infty]$ be defined by

$$\Psi(v) = \begin{cases} \varphi(v), & \text{if } v \in K, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\varphi : K \to \mathbb{R}$ is given by

$$\varphi(v) = \int_{R_1}^{R_2} r^{N-1}\Phi(v')\,dr, \quad v \in K.$$

Obviously, Ψ is proper and convex. On the other hand, as shown in [4], we have that if $\{u_n\} \subset K$ and $u \in C$ are such that $u_n(r) \to u(r)$ for all $r \in [R_1, R_2]$, then $u \in K$ and

$$\varphi(u) \leq \liminf_{n\to\infty} \varphi(u_n). \tag{7}$$

This implies that Ψ is lower semicontinuous on C. Also, note that K is closed in C.

Next, let $\mathscr{G} : C \to \mathbb{R}$ be defined by

$$\mathscr{G}(u) = \int_{R_1}^{R_2} r^{N-1} G(r, u)\,dr, \quad u \in C.$$

A standard reasoning (also see [9, Remark 2.7]) shows that $\mathscr{G}$ is of class C^1 on C and its derivative is given by

$$\langle \mathscr{G}'(u), v\rangle = \int_{R_1}^{R_2} r^{N-1} g(r, u) v\,dr, \quad u, v \in C.$$

The functional $I : C \to (-\infty, +\infty]$ defined by

$$I = \Psi + \mathscr{G} \tag{8}$$

has the structure required by Szulkin's critical point theory [13]. Accordingly, a function $u \in C$ is *a critical point* of I if $u \in K$ and it satisfies the inequality

$$\Psi(v) - \Psi(u) + \langle \mathscr{G}'(u), v - u\rangle \geq 0 \quad \text{for all } v \in C,$$

or, equivalently

$$\int_{R_1}^{R_2} r^{N-1}[\Phi(v') - \Phi(u')]\,dr + \int_{R_1}^{R_2} r^{N-1} g(r,u)(v-u)\,dr \geq 0 \quad \text{for all } v \in K.$$

Now, we consider the Neumann boundary value problem (3) under the basic hypothesis (H_Φ). Recall that by a *solution* of (3) we mean a function $u \in C^1[R_1, R_2]$, such that $\|u'\|_\infty < a$, $\phi(u')$ is differentiable and (3) is satisfied.

Lemma 1. *For every $f \in C$, problem*

$$[r^{N-1}\phi(u')]' = r^{N-1}[\bar{u} + f], \quad u'(R_1) = 0 = u'(R_2) \tag{9}$$

has a unique solution u_f, which is also the unique solution of the variational inequality

$$\int_{R_1}^{R_2} r^{N-1}[\Phi(v') - \Phi(u') + \bar{u}(\bar{v} - \bar{u}) + f(v-u)]\,dr \geq 0 \quad \textit{for all } v \in K, \tag{10}$$

and the unique minimum over K of the strictly convex functional J defined by

$$J(u) = \int_{R_1}^{R_2} r^{N-1}\left[\Phi(u') + \frac{\bar{u}^2}{2} + fu\right] dr. \tag{11}$$

Proof. Problem (9) is equivalent to finding $u = \bar{u} + \tilde{u}$ with $\bar{u}$ and $\tilde{u}$ solutions of

$$\begin{cases} [r^{N-1}\phi(\tilde{u}')]' = r^{N-1}\tilde{f}, \quad \tilde{u}'(R_1) = 0 = \tilde{u}'(R_2), \\ \bar{u} = -\bar{f}, \quad \int_{R_1}^{R_2} r^{N-1}\tilde{u}(r)\,dr = 0. \end{cases} \tag{12}$$

Now the first equation gives, using the first boundary condition,

$$\tilde{u}'(r) = \phi^{-1}\left[r^{1-N}\int_{R_1}^{r} s^{N-1}\tilde{f}(s)\,ds\right]. \tag{13}$$

From (13) we get

$$\|\tilde{u}'\|_\infty < a, \quad \tilde{u}'(R_2) = \phi^{-1}\left[R_2^{1-N}\int_{R_1}^{R_2} s^{N-1}\tilde{f}(s)\,ds\right] = \phi^{-1}(0) = 0.$$

Then the unique solution of (13) is given by

$$\tilde{u}(r) = c + \int_{R_1}^{r} \phi^{-1}\left[t^{1-N}\int_{R_1}^{t} s^{N-1}\tilde{f}(s)\,ds\right] dt, \tag{14}$$

where

$$c = -\frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} r^{N-1} \int_{R_1}^{r} \phi^{-1}\left[t^{1-N} \int_{R_1}^{t} s^{N-1} \tilde{f}(s)\, ds\right] dt\, dr. \tag{15}$$

The unique solution $u_f = \bar{u} + \tilde{u}$ of (9) follows from (12), (14) and (15).

Now, if u is a solution of (9), then, taking $v \in K$, multiplying each member of the differential equation by $v - u$, integrating over $[R_1, R_2]$, and using integration by parts and the boundary conditions, we get

$$\int_{R_1}^{R_2} r^{N-1}[\phi(u')(v' - u') + \bar{u}(\bar{v} - \bar{u}) + f(v - u)]\, dr = 0,$$

which gives (10) if we use the convexity inequality for Φ

$$\Phi(v') - \Phi(u') \geq \phi(u')(v' - u').$$

The inequality $\frac{\bar{v}^2}{2} - \frac{\bar{u}^2}{2} \geq \bar{u}(\bar{v} - \bar{u})$ introduced in (10) implies that

$$\int_{R_1}^{R_2} r^{N-1}\left[\Phi(v') - \Phi(u') + \frac{\bar{v}^2}{2} + fv - \frac{\bar{u}^2}{2} - fu\right] dr \geq 0 \quad \text{for all } v \in K,$$

which shows that J has a minimum on K at u. Conversely if it is the case, then, for all $\lambda \in (0, 1]$ and all $v \in K$, we get

$$\int_{R_1}^{R_2} r^{N-1}\left\{\Phi[(1-\lambda)u' + \lambda v'] + \frac{[(1-\lambda)\bar{u} + \lambda\bar{v}]^2}{2} + f[(1-\lambda)u + \lambda v]\right\} dr$$
$$\geq \int_{R_1}^{R_2} r^{N-1}\left[\Phi(u') + \frac{\bar{u}^2}{2} + fu\right] dr,$$

which, using the convexity of Φ, simplifying, dividing both members by λ and letting $\lambda \to 0_+$, gives the variational inequality (10). Thus solving (10) is equivalent to minimizing (11) over K. Now, it is straightforward to check that J is strictly convex over K and therefore has a unique minimum there, which gives the required uniqueness conclusions of Lemma 1. □

PROPOSITION 1. *If u is a critical point of I, then u is a solution of problem* (3).

PROOF. We set

$$f_u := g(\cdot, u) - \bar{u} \in C$$

and consider the problem

$$[r^{N-1}\phi(w')]' = r^{N-1}[\overline{w} + f_u(r)], \quad w'(R_1) = 0 = w'(R_2). \tag{16}$$

By virtue of Lemma 1, problem (16) has an unique solution $\hat{u}$ and it is also the unique solution of

$$(17)\quad \int_{R_1}^{R_2} r^{N-1}[\Phi(v') - \Phi(\hat{u}') + \bar{\hat{u}}(\bar{v} - \bar{\hat{u}}) + f_u(r)(v - \hat{u})]\,dr \geq 0 \quad \text{for all } v \in K.$$

Since u is a critical point of I, we infer that

$$(18)\quad \int_{R_1}^{R_2} r^{N-1}[\Phi(v') - \Phi(u') + \bar{u}(\bar{v} - \bar{u}) + f_u(r)(v - u)]\,dr \geq 0 \quad \text{for all } v \in K.$$

It follows by uniqueness that $u = \hat{u}$. Hence, u solves problem (3). □

3. Ground state solutions

We begin by a lemma which is the main tool for the minimization problems in this section. With this aim, for any $\rho > 0$, set

$$\hat{K}_\rho := \{u \in K : |\bar{u}| \leq \rho\}.$$

Lemma 2. *Assume that there is some $\rho > 0$ such that*

$$(19)\quad \inf_{\hat{K}_\rho} I = \inf_K I.$$

Then I is bounded from below on C and attains its infimum at some $u \in \hat{K}_\rho$, which solves problem (3).

Proof. By virtue of (19) and $\inf_C I = \inf_K I$, it suffices to prove that there is some $u \in \hat{K}_\rho$ such that

$$(20)\quad I(u) = \inf_{\hat{K}_\rho} I.$$

Then, we get that u is a minimum point of I on C and, on account of [13, Proposition 1.1], is a critical point of I. The proof will be accomplished by virtue of Proposition 1.

If $v \in \hat{K}_\rho$ then, using (6) we obtain

$$|v(r)| \leq |\bar{v}| + |\tilde{v}(r)| \leq \rho + (R_2 - R_1)a.$$

This, together with $\|v'\|_\infty \leq a$ show that $\hat{K}_\rho$ is bounded in $W^{1,\infty}$ and, by the compactness of the embedding $W^{1,\infty} \subset C$, the set $\hat{K}_\rho$ is relatively compact in C. Let $\{u_n\} \subset \hat{K}_\rho$ be a minimizing sequence for I. Passing to a subsequence if necessary and using [4, Lemma 1], we may assume that $\{u_n\}$ converges uniformly to some

$u \in K$. It is easily seen that actually $u \in \hat{K}_\rho$. From (7) and the continuity of $\mathcal{G}$ on C, we obtain

$$I(u) \leq \liminf_{n\to\infty} I(u_n) = \lim_{n\to\infty} I(u_n) = \inf_{\hat{K}_\rho} I,$$

showing that (20) holds true. □

The following result is proved in [4, Theorem 1].

COROLLARY 1. *Let $f : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ be continuous and $F : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ be defined by*

$$F(r,x) := \int_0^x f(r,\xi)\,d\xi, \quad (r,x) \in [R_1, R_2] \times \mathbb{R}.$$

If there is some $\omega > 0$ such that $F(r,x) = F(r,x+\omega)$ for all $(r,x) \in [R_1, R_2] \times \mathbb{R}$, then, for any $h \in C$ with $\bar{h} = 0$, the problem

$$[r^{N-1}\phi(u')]' = r^{N-1}[f(r,u) + h(r)], \quad u'(R_1) = 0 = u'(R_2),$$

has at least one solution $u \in \hat{K}_\omega$ which is a minimizer of the corresponding energy functional I on C.

PROOF. We have

$$G(r,x) = F(r,x) + h(r)x, \quad (r,x) \in [R_1, R_2] \times \mathbb{R}.$$

Due to the ω-periodicity of $F(r,\cdot)$ and because of $\bar{h} = 0$, it holds

$$I(v + j\omega) = I(v) \quad \text{for all } v \in K \text{ and } j \in \mathbb{Z}.$$

Then, the conclusion follows from the equality

$$\{I(v) : v \in K\} = \{I(v) : v \in \hat{K}_\omega\}$$

and Lemma 2. □

THEOREM 1. *If $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that*

(21) $$\liminf_{|x|\to\infty} G(r,x) > 0, \quad \textit{uniformly in } r \in [R_1, R_2],$$

then (3) *has at least one solution which minimizes I on C.*

PROOF. Using (6) and (21) it follows that there exists $\rho > 0$ such that

$$G(r,u) > 0$$

for any $u \in K$ such that $|\bar{u}| > \rho$. It follows that $I(u) > 0$ provided that $u \in K$ and $|\bar{u}| > \rho$. The proof follows from Lemma 2, as $I(0) = 0$. □

Remark 1. An easy adaptation of the techniques in Section 2.3 of [7] shows that *the Neumann problem for the p-Laplacian* $(p > 1)$ *on a bounded domain* $\Omega \subset \mathbb{R}^N$

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) = g(x, v) \quad in\ \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad on\ \partial\Omega,$$

with $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *continuous has at least one strong solution if*

$$\liminf_{|u|\to\infty} \frac{G(x, u)}{|u|^p} > 0, \quad uniformly\ in\ x \in \overline{\Omega},$$

a condition of the type already introduced by Hammerstein [8] for the Laplacian with Dirichlet conditions. For the radial solutions of (1), Theorem 1 shows that it is sufficient that such a condition holds with $p = 0$.

Example 1. The Neumann problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}}\right) = \frac{v + h(|x|)}{1 + [v + h(|x|)]^2} + \cos v \quad \text{in } \mathscr{A}, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\mathscr{A},$$

has at least one radial solution for all $h \in C$.

Theorem 2. *Let* $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function and* $l \in L^1_{N-1}$ *be such that*

$$|g(r, x)| \le l(r) \tag{22}$$

for a.e. $r \in (R_1, R_2)$ *and all* $x \in \mathbb{R}$. *If*

$$\lim_{|x|\to\infty} \int_{R_1}^{R_2} r^{N-1} G(r, x)\, dr = +\infty, \tag{23}$$

then (3) *has at least one solution which minimizes* I *on* C.

Proof. We shall apply Lemma 2. For arbitrary $u \in K$, using (6) and (22), we estimate I as follows.

$$\begin{aligned} I(u) &= \int_{R_1}^{R_2} r^{N-1}\Phi(u')\, dr + \int_{R_1}^{R_2} r^{N-1} G(r, u)\, dr \\ &\ge \int_{R_1}^{R_2} r^{N-1} G(r, \bar{u})\, dr + \int_{R_1}^{R_2} r^{N-1}[G(r, u) - G(r, \bar{u})]\, dr \\ &= \int_{R_1}^{R_2} r^{N-1} G(r, \bar{u})\, dr + \int_{R_1}^{R_2} r^{N-1} \int_0^1 g(r, \bar{u} + s\tilde{u})\tilde{u}\, ds\, dr \\ &\ge \int_{R_1}^{R_2} r^{N-1} G(r, \bar{u})\, dr - a(R_2 - R_1) \int_{R_1}^{R_2} r^{N-1} l(r)\, dr. \end{aligned}$$

From (23) we can find $\rho > 0$ such that $I(u) > 0$ provided that $|\bar{u}| > \rho$. As by (H_Φ) we know that $\Phi(0) = 0$, one has $I(0) = 0$. Therefore, (19) is fulfilled and the proof is complete. □

REMARK 2. Condition (23) is of the type introduced by Ahmad-Lazer-Paul [1] for the Laplacian with Dirichlet conditions. The reader will observe that the conclusion of Theorem 2 still remains true if (23) is replaced by the weaker but more technical condition

$$\liminf_{|x|\to\infty} \int_{R_1}^{R_2} r^{N-1} G(r,x)\, dr > a(R_2 - R_1) \int_{R_1}^{R_2} r^{N-1} l(r)\, dr.$$

EXAMPLE 2. For every $h \in C$ such that $-\frac{\pi}{2} < \bar{h} < \frac{\pi}{2}$, the Neumann problem

$$\operatorname{div}\Big(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\Big) - \arctan v - \cos v = h(|x|) \quad \text{in } \mathscr{A}, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\mathscr{A},$$

has at least one radial solution.

THEOREM 3. *Let $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $G(r,\cdot)$ is convex for all $r \in [R_1, R_2]$. Then, problem* (3) *has at least one solution if and only if there is some $c \in \mathbb{R}$ such that*

$$\int_{R_1}^{R_2} r^{N-1} g(r,c)\, dr = 0. \tag{24}$$

PROOF. Define

$$\Gamma : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \int_{R_1}^{R_2} r^{N-1} G(r,x)\, dr$$

and note that

$$\Gamma'(x) = \int_{R_1}^{R_2} r^{N-1} g(r,x)\, dr \quad \text{for all } x \in \mathbb{R}.$$

Let us assume that (3) has a solution u. Clearly, we have

$$\int_{R_1}^{R_2} r^{N-1} g(r,u)\, dr = 0. \tag{25}$$

On account of the convexity of $G(r,\cdot)$, the function $g(r,\cdot) : \mathbb{R} \to \mathbb{R}$ is nondecreasing for any $r \in [R_1, R_2]$. Hence,

$$g(r, -\|u\|_\infty) \le g(r, u(r)) \le g(r, \|u\|_\infty) \quad \text{for all } r \in [R_1, R_2]. \tag{26}$$

From (25) and (26) we infer

$$\Gamma'(-\|u\|_{\infty}) \leq 0 \leq \Gamma'(\|u\|_{\infty}).$$

Then, by the continuity of Γ' there exists $c \in \mathbb{R}$ such that (24) holds true.

Reciprocally, assume that there exists $c \in \mathbb{R}$ such that $\Gamma'(c) = 0$. Using the fact that Γ' is nondecreasing, we have to consider the following three cases.

(i) It holds

$$\Gamma'(x) = \Gamma'(c) = 0 \quad \text{for all } x \geq c.$$

This implies that

$$g(r,x) = g(r,c) \quad \text{for all } r \in [R_1, R_2] \text{ and } x \geq c.$$

Let v be a solution of the problem

$$[r^{N-1}\phi(w')]' = r^{N-1}g(r,c), \quad w'(R_1) = 0 = w'(R_2);$$

we know that this exists by Theorem 2.3 in [3]. Setting $u = c + \|v\|_{\infty} + v$, we get that u solves problem (3).

(ii) One has that

$$\Gamma'(x) = \Gamma'(c) = 0 \quad \text{for all } x \leq c.$$

In this case the reasoning is similar to that in the case (i).

(iii) There are $x_1, x_2 \in \mathbb{R}$ with $x_1 < c < x_2$ and $\Gamma'(x_1) < 0 < \Gamma'(x_2)$. If $x \geq x_2$, then

$$\begin{aligned}\Gamma(x) &= \Gamma(x_2) + \int_{R_1}^{R_2} r^{N-1}\Big(\int_{x_2}^{x} g(r,t)\,dt\Big)\,dr \\ &\geq \Gamma(x_2) + (x - x_2)\Gamma'(x_2).\end{aligned}$$

It follows that $\Gamma(x) \to +\infty$ when $x \to +\infty$. Analogously $\Gamma(x) \to +\infty$ when $x \to -\infty$. Hence,

$$\lim_{|x|\to\infty} \Gamma(x) = +\infty. \tag{27}$$

On the other hand, by the convexity of $G(r,\cdot)$, we have

$$G(r,u) \geq 2G\Big(r, \frac{\bar{u}}{2}\Big) - G(r, -\tilde{u}) \quad \text{for all } r \in [R_1, R_2],$$

which gives

$$I(u) \geq \int_{R_1}^{R_2} r^{N-1}\Phi(u')\,dr + 2\Gamma\Big(\frac{\bar{u}}{2}\Big) - \int_{R_1}^{R_2} r^{N-1}G(r,\tilde{u})\,dr \quad \text{for all } u \in K. \tag{28}$$

The estimate (28) together with (6) and (27) show that we can find $\rho > 0$ such that $I(u) > 0$ provided that $u \in K$ and $|\bar{u}| > \rho$. Then, the proof follows from Lemma 2 as in the proof of Theorem 2. □

Remark 3. Theorem 3 can be stated equivalently as: *Let $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $G(r, \cdot)$ is convex for all $r \in [R_1, R_2]$. Then, problem (3) has at least one solution if and only if the real convex function $x \mapsto \int_{R_1}^{R_2} r^{N-1} G(r, x)\, dr$ has a minimum.* Corresponding results for the Laplacian with Neumann or Dirichlet boundary conditions have been given in [10] and [11].

Example 3. The Neumann problem with $h \in C$

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = \arctan v - h(|x|) \quad \text{in } \mathscr{A}, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\mathscr{A},$$

has at least one radial solution if and only if $-\frac{\pi}{2} < \bar{h} < \frac{\pi}{2}$.

Example 4. The Neumann problem with $h \in C$

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = \arctan v^+ - h(|x|) \quad \text{in } \mathscr{A}, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\mathscr{A},$$

has at least one radial solution if and only if $0 \le \bar{h} < \frac{\pi}{2}$.

Example 5. The Neumann problem with $h \in C$

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = e^v - h(|x|) \quad \text{in } \mathscr{A}, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\mathscr{A},$$

has at least one radial solution if and only if $\bar{h} > 0$.

4. (PS)–sequences and Saddle Point solutions

Towards the application of the minimax results obtained in Szulkin [13] to the functional I defined by (8) we have to know when I satisfies the compactness *Palais-Smale* (in short, (PS)) *condition*.

Viewing our functional framework from Section 2, we say that a sequence $\{u_n\} \subset K$ is a *(PS)–sequence* if $I(u_n) \to c \in \mathbb{R}$ and

$$\text{(29)} \qquad \int_{R_1}^{R_2} r^{N-1}[\Phi(v') - \Phi(u_n') + g(r, u_n)(v - u_n)]\, dr \ge -\varepsilon_n \|v - u_n\|_\infty \quad \text{for all } v \in K,$$

where $\varepsilon_n \to 0+$. According to [13], the functional I is said to satisfy the (PS) *condition* if any (PS)–sequence has a convergent subsequence in C.

The lemma below provides useful properties of the (PS)–sequences.

LEMMA 3. *Let $\{u_n\}$ be a (PS)–sequence. Then the following hold true:*

(i) *the sequence* $\left\{\int_{R_1}^{R_2} r^{N-1} G(r, u_n)\, dr\right\}$ *is bounded;*
(ii) *if $\{\bar{u}_n\}$ is bounded, then $\{u_n\}$ has a convergent subsequence in C;*
(iii) *one has that*

$$-\varepsilon_n \leq \int_{R_1}^{R_2} r^{N-1} g(r, u_n)\, dr \leq \varepsilon_n \quad \textit{for all } n \in \mathbb{N}. \tag{30}$$

PROOF. (i) This is immediate from the fact that $\{I(u_n)\}$ and Φ are bounded.

(ii) From (6) and $u_n \in K$, the sequence $\{\tilde{u}_n\}$ is bounded in $W^{1,\infty}$. By the compactness of the embedding $W^{1,\infty} \subset C$, we deduce that $\{\tilde{u}_n\}$ has a convergent subsequence in C. Using then the boundedness of $\{\bar{u}_n\} \subset \mathbb{R}$ it follows that $\{u_n\}$ has a convergent subsequence in C.

(iii) Taking $v = u_n \pm 1$ in (29) one obtains (30). □

THEOREM 4. *Let $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and $l \in L^1_{N-1}$ be such that* (22) *is satisfied for a.e. $r \in (R_1, R_2)$ and all $x \in \mathbb{R}$. If*

$$\lim_{|x| \to \infty} \int_{R_1}^{R_2} r^{N-1} G(r, x)\, dr = -\infty, \tag{31}$$

then (3) *has at least one solution.*

PROOF. We shall apply the Saddle Point Theorem [13, Theorem 3.5].

From (31) the functional I is not bounded from below. Indeed, if $v = c \in \mathbb{R}$ is a constant function then

$$I(c) = \int_{R_1}^{R_2} r^{N-1} G(r, c)\, dr \to -\infty \quad \text{as } |c| \to \infty. \tag{32}$$

We split $C = \mathbb{R} \oplus X$, where $X = \{v \in C : \bar{v} = 0\}$. Note that

$$I(v) \geq \int_{R_1}^{R_2} r^{N-1} G(r, \tilde{v})\, dr \quad \text{for all } v \in K \cap X,$$

which together with (6) imply that there is a constant $\alpha \in \mathbb{R}$ such that

$$I(v) \geq \alpha \quad \text{for all } v \in X. \tag{33}$$

Using (32) and (33) we can find some $R > 0$ so that

$$\sup_{S_R} I < \inf_X I,$$

where $S_R = \{c \in \mathbb{R} : |c| = R\}$.

It remains to show that I satisfies the (PS) condition. Let $\{u_n\} \subset K$ be a (PS)–sequence. Since $\{I(u_n)\}$, $\{\varphi(u_n)\}$ are bounded and, by (22) we have

$$\left|\int_{R_1}^{R_2} r^{N-1}[G(r,u_n) - G(r,\bar{u}_n)]\,dr\right| \leq \int_{R_1}^{R_2} r^{N-1} \int_0^1 |g(r,\bar{u}_n + s\tilde{u}_n)\tilde{u}_n|\,ds\,dr$$
$$\leq a(R_2 - R_1) \int_{R_1}^{R_2} r^{N-1} l(r)\,dr,$$

from

$$I(u_n) = \varphi(u_n) + \int_{R_1}^{R_2} r^{N-1} G(r,\bar{u}_n)\,dr + \int_{R_1}^{R_2} r^{N-1}[G(r,u_n) - G(r,\bar{u}_n)]\,dr$$

it follows that there exists a constant $\beta \in \mathbb{R}$ such that

$$\int_{R_1}^{R_2} r^{N-1} G(r,\bar{u}_n)\,dr \geq \beta.$$

Then by (31) the sequence $\{\bar{u}_n\}$ is bounded and Lemma 3 (ii) ensures that $\{u_n\}$ has a convergent subsequence in C. Consequently, I satisfies the (PS) condition and the conclusion follows from [13, Theorem 3.5] and Proposition 1. □

REMARK 4. Condition (31), also of the Ahmad-Lazer-Paul type [1] is, in some sense, 'dual' to condition (23).

EXAMPLE 6. For every $h \in C$ such that $-\frac{\pi}{2} < \bar{h} < \frac{\pi}{2}$, the Neumann problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}}\right) + \arctan v + \cos v = h(|x|) \quad \text{in } \mathscr{A}, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\mathscr{A},$$

has at least one radial solution.

THEOREM 5. *If $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that*

(34) $$\lim_{|x|\to\infty} G(r,x) = -\infty, \quad \textit{uniformly in } r \in [R_1, R_2],$$

then (3) *has at least one solution.*

PROOF. We keep the notations introduced in the proof of Theorem 4. Clearly, (34) implies (31) and from the proof of Theorem 4 it follows that I has the geometry required by the Saddle Point Theorem. To show that I satisfies the (PS) condition, let $\{u_n\} \subset K$ be a (PS)–sequence. If $\{|\bar{u}_n|\}$ is not bounded, we may assume going if necessary to a subsequence, that $|\bar{u}_n| \to \infty$. Using (6) and (34) we deduce that

$$G(r, u_n(r)) \to -\infty, \quad \text{uniformly in } r \in [R_1, R_2].$$

This implies

$$\int_{R_1}^{R_2} r^{N-1} G(r, u_n)\, dr \to -\infty,$$

contradicting Lemma 3 (i). Hence, $\{\bar{u}_n\}$ is bounded and by Lemma 3 (ii), the sequence $\{u_n\}$ has a convergent subsequence in C. Therefore, I satisfies the (PS) condition. The proof is complete. □

Remark 5. No result corresponding to Theorem 5 holds for the Laplacian with Neumann (or Dirichlet) boundary conditions. Indeed, if λ_k is a positive eigenvalue of $-\Delta$ on some bounded domain $\Omega \subset \mathbb{R}^N$ with Neumann boundary conditions, and φ_k a corresponding eigenfunction, the problem

$$\Delta v = -\lambda_k v + \varphi_k(x) \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

has no solution, but $-\lambda_k \dfrac{u^2}{2} + \varphi_k(x) u \to -\infty$ uniformly in $\overline{\Omega}$ when $|u| \to \infty$.

Example 7. The Neumann problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}}\right) + \frac{v + h(|x|)}{1 + [v + h(|x|)]^2} = \cos v \quad \text{in } \mathscr{A}, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\mathscr{A},$$

has at least one radial solution for all $h \in C$.

5. Mountain Pass solutions

In this section we consider problem (4) with $\lambda > 0$ and $m \geq 2$ fixed real numbers, and $f : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ a continuous function satisfying the Ambrosetti–Rabinowitz condition [2]:

(AR) *There exists* $\theta > m$ *and* $x_0 > 0$ *such that*
$0 < \theta F(r, x) \leq x f(r, x)$ *for all* $r \in [R_1, R_2]$ *and* $|x| \geq x_0$.

Note that for problem (4) the function g from the general functional framework in Section 2 is now defined in terms of f by

$$g(r, x) = \lambda |x|^{m-2} x - f(r, x) \quad \text{for all } (r, x) \in [R_1, R_2] \times \mathbb{R}$$

and accordingly, G entering in the definition of the energy functional I becomes

$$G(r,x) = \lambda\frac{|x|^m}{m} - F(r,x) \quad \text{for all } (r,x) \in [R_1,R_2] \times \mathbb{R}.$$

LEMMA 4. *Let $p \geq 1$ be a real number. Then*

$$|u(r)|^p \geq |\bar{u}|^p - pa(R_2 - R_1)|\bar{u}|^{p-1}, \quad \forall u \in K, \forall r \in [R_1,R_2] \tag{35}$$

and there are constants $\alpha_1, \alpha_2 \geq 0$ such that

$$|u(r)|^p \leq |\bar{u}|^p + \alpha_1|\bar{u}|^{p-1} + \alpha_2, \quad \forall u \in K \text{ with } |\bar{u}| \geq 1, \forall r \in [R_1,R_2]. \tag{36}$$

PROOF. The result is trivial for $p = 1$. If $p > 1$, $u \in K$ and $r \in [R_1,R_2]$, then, using the convexity of the differentiable function $s \mapsto |s|^p$, we get

$$\begin{aligned} |u(r)|^p = |\bar{u} + \tilde{u}(r)|^p &\geq |\bar{u}|^p + p|\bar{u}|^{p-2}\bar{u}\tilde{u}(r) \\ &\geq |\bar{u}|^p - p|\bar{u}|^{p-1}(R_2 - R_1)a. \end{aligned}$$

On the other hand, denoting by $\tilde{p}$ the smallest integer larger or equal to p and letting $M := a(R_2 - R_1)$, we have, for all $r \in [R_1,R_2]$,

$$\begin{aligned} |u(r)|^p = |\bar{u} + \tilde{u}(r)|^p &\leq (|\bar{u}| + M)^p = |\bar{u}|^p\left(1 + \frac{M}{|\bar{u}|}\right)^p \\ &\leq |\bar{u}|^p\left(1 + \frac{M}{|\bar{u}|}\right)^{\tilde{p}} = |\bar{u}|^p\left(1 + \sum_{k=1}^{\tilde{p}} \frac{\tilde{p}!}{k!(\tilde{p}-k)!}\frac{M^k}{|\bar{u}|^k}\right) \\ &= |\bar{u}|^p + \sum_{k=1}^{\tilde{p}} \frac{\tilde{p}!}{k!(\tilde{p}-k)!}M^k|\bar{u}|^{p-k}, \end{aligned}$$

and (36) follows easily. □

LEMMA 5. *If (AR) holds, then I satisfies the (PS) condition.*

PROOF. Let $\{u_n\} \subset K$ be a (PS)–sequence. From Lemma 3 (i) and (35) there are constants $c_1, d \in \mathbb{R}$ such that

$$\lambda\frac{R_2^N - R_1^N}{N}\frac{|\bar{u}_n|^m}{m} - c_1|\bar{u}_n|^{m-1} - \int_{R_1}^{R_2} r^{N-1}F(r,u_n)\,dr \leq d \quad \text{for all } n \in \mathbb{N}. \tag{37}$$

Using Lemma 3 (iii) and $\varepsilon_n \to 0$, we may assume that

$$-1 \leq \lambda\int_{R_1}^{R_2} r^{N-1}|u_n|^{m-2}u_n\,dr - \int_{R_1}^{R_2} r^{N-1}f(r,u_n)\,dr \leq 1 \quad \text{for all } n \in \mathbb{N}. \tag{38}$$

Suppose, by contradiction, that $\{|\bar{u}_n|\}$ is not bounded. Then, there is a subsequence of $\{|\bar{u}_n|\}$, still denoted by $\{|\bar{u}_n|\}$, with $|\bar{u}_n| \to \infty$. Let $n_0 \in \mathbb{N}$ be such that $|\bar{u}_n| \geq \max\{1, x_0 + a(R_2 - R_1)\}$ for all $n \geq n_0$. By virtue of (6) we have

$$|u_n(r)| \geq x_0 \quad \text{for all } r \in [R_1, R_2] \text{ and } n \geq n_0.$$

The (AR) condition ensures that

$$\text{(39)} \quad \operatorname{sign} \bar{u}_n = \operatorname{sign} u_n(r) = \operatorname{sign} f(r, u_n(r)) \quad \text{for all } r \in [R_1, R_2] \text{ and } n \geq n_0$$

and

$$\text{(40)} \quad -\int_{R_1}^{R_2} r^{N-1} F(r, u_n)\, dr \geq -\frac{\bar{u}_n}{\theta} \int_{R_1}^{R_2} r^{N-1} f(r, u_n)\, dr - \frac{1}{\theta} \int_{R_1}^{R_2} r^{N-1} f(r, u_n)\tilde{u}_n\, dr \quad \text{for all } n \geq n_0.$$

From (38) and (36) there are constants $c_2, c_3 \geq 0$ such that

$$\text{(41)} \quad -\frac{\bar{u}_n}{\theta} \int_{R_1}^{R_2} r^{N-1} f(r, u_n)\, dr \geq -\lambda \frac{R_2^N - R_1^N}{\theta N} |\bar{u}_n|^m - c_2 |\bar{u}_n|^{m-1} - c_3 \quad \text{for all } n \geq n_0.$$

Also, using (6), (36), (38) and (39) we can find constants $c_4, c_5, c_6 \geq 0$ so that

$$\text{(42)} \quad -\frac{1}{\theta} \int_{R_1}^{R_2} r^{N-1} f(r, u_n)\tilde{u}_n\, dr \geq -c_4 |\bar{u}_n|^{m-1} - c_5 |\bar{u}_n|^{m-2} - c_6, \quad \text{for all } n \geq n_0.$$

From (40), (41) and (42) we obtain

$$\text{(43)} \quad -\int_{R_1}^{R_2} r^{N-1} F(r, u_n)\, dr \geq -\lambda \frac{R_2^N - R_1^N}{N} \frac{|\bar{u}_n|^m}{\theta} - (c_2 + c_4)|\bar{u}_n|^{m-1} - c_5 |\bar{u}_n|^{m-2} - c_3 - c_6 \quad \text{for all } n \geq n_0.$$

Then, (43) together with $\theta > m$ imply

$$\lambda \frac{R_2^N - R_1^N}{N} \frac{|\bar{u}_n|^m}{m} - c_1 |\bar{u}_n|^{m-1} - \int_{R_1}^{R_2} r^{N-1} F(r, u_n)\, dr \to +\infty \quad \text{as } n \to \infty,$$

contradicting (37). Consequently, $\{\bar{u}_n\}$ is bounded and the proof follows from Lemma 3 (ii). □

LEMMA 6. *If (AR) holds and $c \in \mathbb{R}$, then $I(c) \to -\infty$ as $|c| \to \infty$.*

PROOF. The (AR) condition implies (see [7]) that there exists $\gamma \in C$, $\gamma > 0$, such that

$$F(r,x) \geq \gamma(r)|x|^{\theta} \quad \text{for all } r \in [R_1, R_2] \text{ and } |x| \geq x_0. \tag{44}$$

From (44) we infer

$$\begin{aligned} I(c) &= \lambda \frac{R_2^N - R_1^N}{mN}|c|^m - \int_{R_1}^{R_2} r^{N-1} F(r,c)\,dr \\ &\leq \lambda \frac{R_2^N - R_1^N}{mN}|c|^m - |c|^{\theta} \int_{R_1}^{R_2} r^{N-1}\gamma(r)\,dr, \end{aligned}$$

for all $c \in \mathbb{R}$ with $|c| \geq x_0$. Then, the conclusion follows from $\theta > m$ and $\gamma > 0$. □

LEMMA 7. *Assume that F satisfies*

$$\limsup_{x \to 0} \frac{mF(r,x)}{|x|^m} < \lambda \quad \textit{uniformly in } r \in [R_1, R_2]. \tag{45}$$

Then there exist $\alpha, \rho > 0$ such that

$$\int_{R_1}^{R_2} r^{N-1}\left[\lambda \frac{|u|^m}{m} - F(r,u)\right] dr \geq \alpha \quad \textit{for all } u \in K \cap \partial B_{\rho}, \tag{46}$$

where $\partial B_{\rho} := \{u \in C : \|u\|_{\infty} = \rho\}$.

PROOF. Assumption (45) ensures that there are constants $b < \lambda$ and $\rho > 0$ such that

$$F(r,x) \leq \frac{b}{m}|x|^m \quad \text{for all } r \in [R_1, R_2] \text{ and } |x| \leq \rho. \tag{47}$$

We claim that:

$$\inf_{u \in K \cap \partial B_{\rho}} \int_{R_1}^{R_2} r^{N-1}|u|^m\,dr > 0. \tag{48}$$

Then, by virtue of (47) we have

$$\begin{aligned} &\int_{R_1}^{R_2} r^{N-1}\left[\lambda \frac{|u|^m}{m} - F(r,u)\right] dr \\ &\qquad \geq \frac{\lambda - b}{m}\int_{R_1}^{R_2} r^{N-1}|u|^m\,dr =: \alpha \quad \text{for all } u \in K \cap \partial B_{\rho}, \end{aligned}$$

and (48) implies (46). In order to prove (48), suppose by contradiction that there exists a sequence $\{u_n\} \subset K \cap \partial B_\rho$ such that

$$\int_{R_1}^{R_2} r^{N-1}|u_n|^m \, dr \to 0 \quad \text{as } n \to \infty.$$

It is clear that $\{u_n\}$ is bounded in $W^{1,\infty}$. Passing to a subsequence if necessary, we may assume a that $\{u_n\}$ is convergent in C to some u. This implies that $\|u\|_\infty = \rho$ and

$$\int_{R_1}^{R_2} r^{N-1}|u_n|^m \, dr \to \int_{R_1}^{R_2} r^{N-1}|u|^m \, dr \quad \text{as } n \to \infty.$$

It follows that $u = 0$, contradiction with $\|u\|_\infty = \rho > 0$. Therefore, (48) holds true and the proof is complete. □

THEOREM 6. *Assume that the* (AR) *condition holds true. If* F *satisfies* (45), *then problem* (4) *has at least one nontrivial solution.*

PROOF. The proof follows immediately from Lemmas 5, 6 and 7 and the Mountain Pass Theorem [13, Theorem 3.2]) applied to the functional I. □

REMARK 6. Theorem 6 is of the type introduced by Ambrosetti and Rabinowitz [2] for nonlinear perturbations of the Laplacian with Dirichlet boundary conditions.

EXAMPLE 8. If $\theta > m \geq 2$, $\lambda > 0$ are given real numbers and $\mu \in C$ is a positive function, then the Neumann problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = \lambda|v|^{m-2}v - \mu(|x|)|v|^{\theta-2}v \quad \text{in } \mathscr{A}, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\mathscr{A},$$

has at least one nontrivial radial solution.

6. THE PERIODIC CASE

Let $\Phi : [-a, a] \to \mathbb{R}$ and $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ be as above, i.e., Φ satisfies (H_Φ) and g is continuous. The periodic problem (5) can be treated quite similarly to problem (3) with the following modifications. Taking $N = 1$, one works with

$$K_P := \{v \in W^{1,\infty} : \|v'\|_\infty \leq a, v(R_1) = v(R_2)\}$$

instead of K, and $\Psi_P : C \to (-\infty, +\infty]$ given by

$$\Psi_P(v) = \begin{cases} \int_{R_1}^{R_2} \Phi(v'), & \text{if} v \in K_P, \\ +\infty, & \text{otherwise,} \end{cases}$$

instead of Ψ. With $\mathcal{G}_P : C \to \mathbb{R}$ defined by

$$\mathcal{G}_P(u) = \int_{R_1}^{R_2} G(r,u)\,dr, \quad u \in C,$$

the energy functional $I_P : C \to (-\infty, +\infty]$ will be now $I_P = \Psi_P + \mathcal{G}_P$.

The references from [4] are replaced by the similar ones from [5].

We only state the following existence results which are obtained as the corresponding ones for problems (3) and (4) by no longer than "mutatis mutandis" arguments.

PROPOSITION 2. *If $u \in K_P$ is a critical point of I_P, then u is a solution of problem* (5).

Denoting

$$\hat{K}_{P,\rho} := \{u \in K_P : |\bar{u}| \le \rho\},$$

we have the following

LEMMA 8. *Assume that there is some $\rho > 0$ such that*

$$\inf_{\hat{K}_{P,\rho}} I_P = \inf_{K_P} I_P.$$

Then I_P is bounded from below and attains its infimum at some $u \in \hat{K}_{P,\rho}$, which solves problem (5).

By means of Lemma 8 we can easily reformulate Corollary 1, Theorem 1 and Theorem 5 for the periodic problem (5). Also we note the following versions of the other theorems.

THEOREM 7. *Assume that there exists $l \in L^1$ such that*

$$|g(r,x)| \le l(r)$$

for a.e. $r \in (R_1, R_2)$ and all $x \in \mathbb{R}$. If either

$$\liminf_{|x|\to\infty} \int_{R_1}^{R_2} G(r,x)\,dr > (R_2 - R_1)\Big(a \int_{R_1}^{R_2} l(r)\,dr\Big) \tag{49}$$

or

$$\lim_{|x|\to\infty} \int_{R_1}^{R_2} G(r,x)\,dr = -\infty,$$

then problem (5) *has at least one solution u. Moreover, if* (49) *holds true then u minimizes I_P on C.*

THEOREM 8. *Let $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $G(r, \cdot)$ is convex for all $r \in [R_1, R_2]$. Then, problem* (5) *has at least one solution if and only if there is some $c \in \mathbb{R}$ such that*

$$\int_{R_1}^{R_2} g(r, c)\, dr = 0.$$

THEOREM 9. *Let $f : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that the (AR) condition is fulfilled. If F satisfies* (45), *then the problem*

$$[\phi(u')]' = \lambda |u|^{m-2} u - f(r, u), \quad u(R_1) - u(R_2) = 0 = u'(R_1) - u'(R_2),$$

has at least one nontrivial solution for any $\lambda > 0$ and $m \geq 2$.

ACKNOWLEDGMENTS. Support of C. Bereanu from the Romanian Ministry of Education, Research, and Innovation (PN II Program, CNCSIS code RP 3/2008) is gratefully acknowledged.

REFERENCES

[1] S. AHMAD - A. C. LAZER - J. L. PAUL, *Elementary critical point theory and perturbations of elliptic boundary value problems at resonance*, Indiana Math. J. 25 (1976), 933–944.

[2] A. AMBROSETTI - P. RABINOWITZ, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. 14 (1973), 349–381.

[3] C. BEREANU - P. JEBELEAN - J. MAWHIN, *Radial solutions for Neumann problems involving mean curvature operators in Euclidean and Minkowsi spaces*, Math. Nachr. 283 (2010), 379–391.

[4] C. BEREANU - P. JEBELEAN - J. MAWHIN, *Radial solutions for Neumann problems involving mean extrinsic curvature and periodic nonlinearities*, submitted.

[5] H. BREZIS - J. MAWHIN, *Periodic solutions of the forced relativistic pendulum*, Differential Integral Equations 23 (2010), 801–810.

[6] H. BREZIS - J. MAWHIN, *Periodic solutions of Lagrangian systems of relativistic oscillators*, Comm. Appl. Anal., to appear.

[7] G. DINCĂ - P. JEBELEAN - J. MAWHIN, *Variational and topological methods for Dirichlet problems with p-Laplacian*, Portug. Math. (N.S.) 58 (2001), 339–378.

[8] A. HAMMERSTEIN, *Nichtlineare Integralgleichungen nebst Anwendungen*, Acta Math. 54 (1930), 117–176.

[9] P. JEBELEAN, *Variational methods for ordinary p-Laplacian systems with potential boundary conditions*, Adv. Differential Equations 14 (2008), 273–322.

[10] J. MAWHIN, *Problèmes de Dirichlet variationnels non linéaires*, Sémin. Math. Sup. No. 104, Presses Univ. Montréal, Montréal, 1987.

[11] J. MAWHIN, *Semi-coercive monotone variational problems*, Bull. Cl. Sci. Acad. Roy. Belgique (5) 73 (1987), 118–130.

[12] J. MAWHIN, *Periodic solutions of second order nonlinear difference systems with ϕ-Laplacian: a variational approach*, to appear.

[13] A. Szulkin, *Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire 3 (1986), 77–109.

Received 15 December 2010,
and in revised form 20 December 2010.

Cristian Bereanu
Institute of Mathematics "Simion Stoilow"
Romanian Academy
21, Calea Griviţei, RO-010702-Bucharest
Sector 1, România
cristian.bereanu@imar.ro

Petru Jebelean
Department of Mathematics
West University of Timişoara
4, Blvd. V. Pârvan RO-300223-Timişoara
România
jebelean@math.uvt.ro

Jean Mawhin
Institut de Recherche en Mathématique et Physique
Université Catholique de Louvain
2, Chemin du Cyclotron B-1348 Louvain-la-Neuve
Belgique
jean.mawhin@uclouvain.be