MATHEMATISCHE NACHRICHTEN

www.mn-journal.org

Founded in 1948 by Erhard Schmidt

Edited by R. Mennicken (Regensburg)
in co-operation with
F. Finster (Regensburg), F. Gesztesy (Columbia-Missouri),
K. Hulek (Hannover), F. Klopp (Villetaneuse),
R. Schilling (Dresden)

WILEY

REPRINT

Math. Nachr. **283,** No. 3, 379 – 391 (2010) / **DOI** 10.1002/mana.200910083

Radial solutions for Neumann problems involving mean curvature operators in Euclidean and Minkowski spaces

Cristian Bereanu[∗1], **Petru Jebelean**[∗∗ 2], and **Jean Mawhin**[∗∗∗ 3]

[1] Institute of Mathematics "Simion Stoilow" of the Romanian Academy, 21, Calea Griviţei Street, RO-010702-Bucharest, Sector 1, România

[2] Department of Mathematics, West University of Timişoara, 4, Blvd. V. Pârvan RO-300223-Timişoara, România

[3] Département de Mathématique, Université Catholique de Louvain, 2, Chemin du Cyclotron, B-1348 Louvain-la-Neuve, Belgique

Received 1 April 2009, accepted 1 May 2009
Published online 26 February 2010

Key words Mean curvature operators, Neumann problem, radial solutions, fixed point operators, Leray-Schauder degree

MSC (2000) Primary: 35J65; Secondary: 34B15

Dedicated to the memory of Erhard Schmidt

In this paper we study the existence of radial solutions for Neumann problems in a ball and in an annular domain, associated to mean curvature operators in Euclidean and Minkowski spaces. Our approach relies on the Leray-Schauder degree together with some fixed point reformulations of our nonlinear Neumann boundary value problems.

1 Introduction

The study of hypersurfaces in the Lorentz-Minkowski space with coordinates $(x_1, \ldots, x_N, t)$ and the metric $\sum_{j=1}^{N}(dx_j)^2 - (dt)^2$ leads to partial differential equations (PDE) of the type

$$\operatorname{div}\left(\frac{\nabla v(x)}{\sqrt{1-|\nabla v(x)|^2}}\right) = H(x, v(x)) \quad \text{in} \quad \Omega,$$

where Ω is a domain in $\mathbb{R}^N$ and $H : \Omega \times \mathbb{R} \to \mathbb{R}$ is a nonlinearity prescribing the mean curvature of the hypersurface. A first essential result concerning the above PDE was proved by E. Calabi [7] in the case $\Omega = \mathbb{R}^N$ and $N \le 4$. This was later extended to arbitrary dimension by S. Y. Cheng and S. T. Yau in [8]. On the other hand, if $H \equiv c > 0$ and $\Omega = \mathbb{R}^N$, then A. Treibergs [21] obtained an existence result about entire solutions for the above PDE in the presence of a pair of well ordered upper and lower-solutions, and the Dirichlet problem with H bounded has been considered by Bartnik and Simon [2].

In the paper [3] we studied the existence of radial solutions for nonlinear Dirichlet problems in the unit ball and in an annular domain from $\mathbb{R}^N$, associated with the *mean curvature operator in Euclidean space*

$$\mathcal{E}v = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right)$$

∗ e-mail: cristian.bereanu@imar.ro, Phone: +40 21 3196506, Fax: +40 21 3196505
∗∗ e-mail: jebelean@math.uvt.ro, Phone: +40 256592155, Fax: +40 256592316
∗∗∗ Corresponding author: e-mail: jean.mawhin@uclouvain.be, Phone : +32 10 47 31 71, Fax : +32 10 47 25 30

and with the *mean extrinsic curvature operator in Minkowski space*

$$\mathcal{M}v = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right).$$

Here $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^N$. The results from [3] were extended to the case of systems in [4]. The approach in [3] and [4] relies on Schauder fixed point theorem.

The aim of this paper is to present existence results of radial solutions for Neumann problems in a ball and in an annular domain, associated with the operators $\mathcal{E}$ and $\mathcal{M}$. To formulate these problems, let $R_1, R_2 \in \mathbb{R}$, $0 \le R_1 < R_2$ and let us denote by $\mathcal{A}$ the annular domain $\{x \in \mathbb{R}^N : R_1 \le |x| \le R_2\}$. For $f : [R_1, R_2] \times \mathbb{R}^2 \to \mathbb{R}$ a given continuous function, we consider the following Neumann boundary-value problems:

$$\mathcal{M}v = f\left(|x|, v, \frac{dv}{dr}\right) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A} \tag{1.1}$$

and

$$\mathcal{E}v = f\left(|x|, v, \frac{dv}{dr}\right) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A}. \tag{1.2}$$

As usual, we have denoted by $\frac{dv}{dr}$ the radial derivative and by $\frac{\partial v}{\partial \nu}$ the outward normal derivative of v. It should be noticed that for $R_1 = 0$ one has Neumann problems in the ball of radius R_2.

Setting $r = |x|$ and $v(x) = u(r)$, the above problems (1.1) and (1.2) become

$$\left(r^{N-1}\frac{u'}{\sqrt{1-|u'|^2}}\right)' = r^{N-1}f(r,u,u'), \quad u'(R_1) = 0 = u'(R_2), \tag{1.3}$$

respectively,

$$\left(r^{N-1}\frac{u'}{\sqrt{1+|u'|^2}}\right)' = r^{N-1}f(r,u,u'), \quad u'(R_1) = 0 = u'(R_2). \tag{1.4}$$

Clearly, the solutions of (1.3) and (1.4) are classical radial solutions of (1.1), respectively (1.2).

Our approach for problem (1.3) relies upon a Leray-Schauder type continuation theorem, that we recall here for the convenience of the reader (see e.g. [16] and references therein). Let $(X, \|\cdot\|)$ be a real normed space, Ω be a bounded open subset of X and $S : \overline{\Omega} \to X$ be a compact operator such that $0 \notin (I - S)(\partial\Omega)$. The Leray-Schauder degree of $I - S$ with respect to Ω and 0 is denoted by $d_{LS}[I - S, \Omega, 0]$ (see e.g. [11]). We set $B_\rho = \{x \in X : \|x\| < \rho\}$.

Lemma 1.1 *Let $S : \mathbb{R} \times \overline{B}_\rho \to X$ be a compact operator such that*

$$x \neq S(\lambda, x) \quad \textit{for all} \quad (\lambda, x) \in \mathbb{R} \times \partial B_\rho$$

and such that

$$d_{LS}[I - S(\lambda_0, \cdot), B_\rho, 0] \neq 0 \quad \textit{for some} \quad \lambda_0 \in \mathbb{R}.$$

Then the set $\mathcal{S}$ of solutions $(\lambda, x) \in \mathbb{R} \times \overline{B}_\rho$ of the problem

$$x = S(\lambda, x)$$

contains a continuum (closed and connected) $\mathcal{C}$ whose projection on $\mathbb{R}$ is $\mathbb{R}$.

The existence result obtained for (1.3) is then employed, via a cutting method, to derive the existence of solutions for problem (1.4). In particular, we extend the method of (not necessarily ordered) lower and upper solutions to problem of the type (1.3), and give some applications and several examples. In the last section we deal with pendulum-like nonlinearities.

For interesting results concerning radial solutions for Dirichlet boundary value problems associated to some nonlinear perturbations of the operators $\mathcal{E}$ and p-Laplacian we refer the reader to [9, 12–14]. The Neumann problem associated to some nonlinear perturbations of the p-Laplacian is considered for example in papers [19, 22].

2 A class of Neumann boundary-value problems

Consider the Neumann boundary-value problem (BVP)

$$\big(r^{N-1}\phi(u')\big)' = r^{N-1}f(r,u,u'), \quad u'(R_1) = 0 = u'(R_2), \tag{2.1}$$

where ϕ is a homeomorphism such that $\phi(0) = 0$, belonging to one of the following classes ($0 < a < \infty$):

$$\begin{aligned} \phi &: (-a,a) \longrightarrow \mathbb{R} \quad (\textit{singular}),\\ \phi &: \mathbb{R} \longrightarrow \mathbb{R} \quad (\textit{classical}),\\ \phi &: \mathbb{R} \longrightarrow (-a,a) \quad (\textit{bounded}),\end{aligned}$$

and $f : [R_1,R_2] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. By a *solution* of (2.1) we mean a continuously differentiable function u such that $u' \in \mathrm{dom}(\phi)$, $\phi(u')$ is differentiable and (2.1) is satisfied.

We denote by C the Banach space of continuous functions defined on $[R_1,R_2]$ endowed with the usual norm $||\cdot||_\infty$, by C^1 the Banach space of continuously differentiable functions defined on $[R_1,R_2]$ endowed with the norm

$$||u|| = ||u||_\infty + ||u'||_\infty,$$

and by $C^1_\dagger$ the closed subspace of C^1 defined by

$$C^1_\dagger = \{u \in C^1 : u'(R_1) = 0 = u'(R_2)\}.$$

The corresponding open ball with center in 0 and radius ρ is denoted by B_ρ. For any continuous function $w : [R_1,R_2] \to \mathbb{R}$, we write

$$w_L := \min_{[R_1,R_2]} w, \quad w_M := \max_{[R_1,R_2]} w.$$

Let us introduce the continuous projector

$$Q : C \longrightarrow C, \quad Qu = \frac{N}{R_2^N - R_1^N}\int_{R_1}^{R_2} r^{N-1}u(r)\,dr,$$

the continuous function

$$\gamma : (0,\infty) \longrightarrow \mathbb{R}, \quad \gamma(r) = \frac{1}{r^{N-1}},$$

and the linear operators

$$\begin{aligned} L &: C \longrightarrow C, \quad Lu(r) = \gamma(r)\int_{R_1}^{r} t^{N-1}u(t)\,dt \quad (r \in (R_1,R_2]),\\ H &: C \longrightarrow C^1, \quad Hu(r) = \int_{R_1}^{r} u(t)\,dt \quad (r \in [R_1,R_2]).\end{aligned}$$

It is not difficult to prove that L is compact (Arzelà-Ascoli) and H is bounded. Finally, we associate to f its Nemytskii operator

$$N_f : C^1 \longrightarrow C, \quad N_f(u) = f(\cdot,u(\cdot),u'(\cdot)).$$

It is known that N_f is continuous and takes bounded sets into bounded sets.

Let us decompose any function $u \in C^1_\dagger$ in the form

$$u = \overline{u} + \widetilde{u} \quad (\overline{u} = u(R_1), \quad \widetilde{u}(R_1) = 0),$$

and let

$$\widetilde{C}^1_\dagger = \{u \in C^1_\dagger : u(R_1) = 0\}.$$

We first study an associated modified problem.

Lemma 2.1 *If ϕ is singular, the set $\mathcal{S}$ of the solutions $(\overline{u}, \widetilde{u}) \in \mathbb{R} \times \widetilde{C}^1_\dagger$ of problem*

$$\left(r^{N-1}\phi(\widetilde{u}')\right)' = r^{N-1}[N_f(\overline{u}+\widetilde{u}) - Q \circ N_f(\overline{u}+\widetilde{u})] \tag{2.2}$$

contains a continuum $\mathcal{C}$ whose projection on $\mathbb{R}$ is $\mathbb{R}$ and whose projection on $\widetilde{C}^1_\dagger$ is contained in the ball $B_{\rho(a)}$ where $\rho(a) = a(1 + R_2 - R_1)$.

P r o o f. Consider the nonlinear operator

$$\widetilde{M} : \mathbb{R} \times \widetilde{C}^1_\dagger \longrightarrow \widetilde{C}^1_\dagger, \quad \widetilde{M}(\overline{u}, \widetilde{u}) = [H \circ \phi^{-1} \circ L \circ (I - Q) \circ N_f](\overline{u} + \widetilde{u}).$$

Let $(\overline{u}, \widetilde{u}) \in \mathbb{R} \times \widetilde{C}^1_\dagger$ and $\widetilde{v} = \widetilde{M}(\overline{u}, \widetilde{u})$. It follows that $\widetilde{v} \in C^1$, $\widetilde{v}(R_1) = 0$, $||\widetilde{v}'||_\infty < a$ and

$$\phi(\widetilde{v}'(r)) = \gamma(r) \int_{R_1}^{r} t^{N-1}[N_f(\overline{u}+\widetilde{u})(t) - QN_f(\overline{u}+\widetilde{u})]\, dt \quad (r \in (R_1, R_2]). \tag{2.3}$$

Moreover $\phi(\widetilde{v}'(R_1)) = 0$ and

$$\begin{aligned}\phi(\widetilde{v}'(R_2)) &= \gamma(R_2) \int_{R_1}^{R_2} t^{N-1}[N_f(\overline{u}+\widetilde{u})(t) - QN_f(\overline{u}+\widetilde{u})]\, dt \\ &= \gamma(R_2) \left[\int_{R_1}^{R_2} t^{N-1} N_f(\overline{u}+\widetilde{u})(t)\, dt - QN_f(\overline{u}+\widetilde{u}) \int_{R_1}^{R_2} t^{N-1}\, dt \right] = 0.\end{aligned}$$

Hence, $\widetilde{M}$ is well defined and it is clear that $\widetilde{M}$ is compact. Now, using (2.3) we infer that $(\overline{u}, \widetilde{u}) \in \mathbb{R} \times \widetilde{C}^1_\dagger$ is a solution of (2.2) if and only if

$$\widetilde{u} = \widetilde{M}(\overline{u}, \widetilde{u}). \tag{2.4}$$

So, it suffices to prove that the set of solution of the above problem contains a continuum of solutions whose projection on $\mathbb{R}$ is $\mathbb{R}$ and whose projection on $\widetilde{C}^1_\dagger$ is contained in the ball $B_{\rho(a)}$. Note that if $(\overline{u}, \widetilde{u}) \in \mathbb{R} \times \widetilde{C}^1_\dagger$ satisfies (2.4), then

$$||\widetilde{u}'||_\infty < a, \quad ||\widetilde{u}||_\infty < a(R_2 - R_1).$$

We deduce that

$$\widetilde{u} \neq \widetilde{M}(\overline{u}, \widetilde{u}) \quad \text{for all} \quad (\overline{u}, \widetilde{u}) \in \mathbb{R} \times \partial B_{\rho(a)}. \tag{2.5}$$

Consider the compact homotopy

$$\widetilde{\mathcal{M}} : [0,1] \times \widetilde{C}^1_\dagger \longrightarrow \widetilde{C}^1_\dagger, \quad \widetilde{\mathcal{M}}(\lambda, \widetilde{u}) = \left[H \circ \phi^{-1} \circ \lambda L \circ (I - Q) \circ N_f\right](\widetilde{u}).$$

Note that

$$\widetilde{\mathcal{M}}(0, \cdot) = 0, \quad \widetilde{\mathcal{M}}(1, \cdot) = \widetilde{M}(0, \cdot).$$

It is clear that

$$\widetilde{u} \neq \widetilde{\mathcal{M}}(\lambda, \widetilde{u}) \quad \text{for all} \quad (\lambda, \widetilde{u}) \in [0,1] \times \partial B_{\rho(a)}.$$

Hence from the invariance under a homotopy of the Leray-Schauder degree [11] we deduce that

$$d_{LS}\left[I - \widetilde{M}(0, \cdot), B_{\rho(a)}, 0\right] = d_{LS}\left[I, B_{\rho(a)}, 0\right] = 1. \tag{2.6}$$

The result follows now from Lemma 1.1, (2.5) and (2.6). □

Remark 2.2 Assume that ϕ is *classical* or *singular* and let us consider the nonlinear operator

$$\mathcal{N} : C^1_\dagger \longrightarrow C^1_\dagger, \quad \mathcal{N} = P + QN_f + H \circ \phi^{-1} \circ L \circ (I - Q) \circ N_f,$$

where $P : C \to C$ is the continuous projector defined by $Pu = u(R_1)$. Using the same strategy as above, it is not difficult to prove that $\mathcal{N}$ is well defined, compact and for any $u \in C^1_\dagger$ one has that u is a solution of (2.1) iff u is a fixed point of $\mathcal{N}$.

In the *singular* case we have the following existence result.

Theorem 2.3 *Assume that ϕ is singular and there exist $\varepsilon \in \{-1, 1\}$ and $\rho > 0$ such that*

$$\varepsilon\,(sgn\ u)QN_f(u) \geq 0 \tag{2.7}$$

for any $u \in C^1_\dagger$ satisfying $|u|_L \geq \rho$ and $||u'||_\infty < a$. Then the BVP (2.1) *has at least one solution.*

Proof. Let $\mathcal{C}$ be the continuum given in Lemma 2.1 and $\widetilde{u}_1 \in \widetilde{C}^1_\dagger$ be such that $(\rho + \rho(a), \widetilde{u}_1) \in \mathcal{C}$. Taking $u_1 = \rho + \rho(a) + \widetilde{u}_1$, one has that $u_1 \geq 0$, $|u_1|_L > \rho$ and $||u_1'||_\infty < a$. Hence, from (2.7) it follows that $\varepsilon QN_f(u_1) \geq 0$. On the other hand, let $\widetilde{u}_2 \in \widetilde{C}^1_\dagger$ be such that $(-\rho - \rho(a), \widetilde{u}_2) \in \mathcal{C}$. Taking $u_2 = -\rho - \rho(a) + \widetilde{u}_2$, one has that $u_2 \leq 0$, $|u_2|_L > \rho$ and $||u_2'||_\infty < a$. Hence, from (2.7) it follows that $\varepsilon QN_f(u_2) \leq 0$. Using the intermediate value theorem, we infer that there exists $(\overline{u}, \widetilde{u}) \in \mathcal{C}$ such that $QN_f(\overline{u} + \widetilde{u}) = 0$. This implies that $u = \overline{u} + \widetilde{u}$ is a solution of (2.1). □

The following very useful result is a direct consequence of the above theorem.

Corollary 2.4 *Assume that ϕ is singular and let $h : [R_1, R_2] \times \mathbb{R}^2 \to \mathbb{R}$ and $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ be continuous, with h bounded on $[R_1, R_2] \times \mathbb{R} \times (-a, a)$, and g such that*

$$\begin{gathered} \lim_{u \to -\infty} g(r,u) = +\infty, \quad \lim_{u \to +\infty} g(r,u) = -\infty \\ \left(\textit{resp.} \quad \lim_{u \to -\infty} g(r,u) = -\infty, \quad \lim_{u \to +\infty} g(r,u) = +\infty\right) \end{gathered} \tag{2.8}$$

uniformly in $r \in [R_1, R_2]$. Then the BVP

$$\left(r^{N-1}\phi(u')\right)' + r^{N-1}g(r,u) = r^{N-1}h(r,u,u'), \quad u'(R_1) = 0 = u'(R_2)$$

has at least one solution.

In particular, the problem

$$\left(r^{N-1}\phi(u')\right)' + \mu r^{N-1}u = r^{N-1}h(r,u,u'), \quad u'(R_1) = 0 = u'(R_2)$$

has at least one solution for each $\mu \neq 0$.

We now consider the *bounded* and *classical* cases.

Lemma 2.5 *Let $\psi : (-a, a) \to (-b, b)$ be a homeomorphism such that $\psi(0) = 0$ and $0 < a, b \leq \infty$. Assume that there exists a constant $k \geq 0$ such that $\frac{kR_2}{N} < b$ and*

$$|f(r,u,v)| \leq k \quad \textit{for all} \quad (r,u,v) \in [R_1, R_2] \times \mathbb{R}^2. \tag{2.9}$$

If u is a possible solution of the Neumann BVP

$$\left(r^{N-1}\psi(u')\right)' = r^{N-1}f(r,u,u'), \quad u'(R_1) = 0 = u'(R_2), \tag{2.10}$$

then

$$||u'||_\infty \leq \max(|\psi^{-1}(\pm kR_2/N)|) =: \rho_1(\psi). \tag{2.11}$$

P r o o f. If $u \in C^1_\dagger$ solves (2.10) then

$$u'(r) = \psi^{-1}\left(\gamma(r)\int_{R_1}^{r} t^{N-1} f(t,u(t),u'(t))\,dt\right) \quad (r \in (R_1,R_2]). \tag{2.12}$$

Using (2.9) we get

$$\left|\gamma(r)\int_{R_1}^{r} t^{N-1} f(t,u(t),u'(t))\,dt\right| \le \frac{kR_2}{N} \quad (r \in (R_1,R_2]),$$

which, together with (2.12), gives (2.11). □

Theorem 2.6 *Let $\phi : \mathbb{R} \to (-b,b)$ be a homeomorphism such that $\phi(0) = 0$ with $0 < b \le \infty$ and let f be like in Lemma 2.5. Assume that there exist $a \in (\rho_1(\phi),\infty)$, $\varepsilon \in \{-1,1\}$ and $\rho > 0$ such that (2.7) holds for any $u \in C^1_\dagger$ satisfying $|u|_L \ge \rho$ and $||u'||_\infty < a$. Then the BVP (2.1) has at least one solution.*

P r o o f. Let $d \in (\rho_1(\phi),a)$ and $\psi : (-a,a) \to \mathbb{R}$ be a homeomorphism which coincides with ϕ on $[-d,d]$. Then, $\rho_1(\phi) = \rho_1(\psi)$ and using Lemma 2.5 one has that the solutions of (2.10) coincide with the solutions of (2.1). Now the result follows from Theorem 2.3. □

Lemma 2.7 *Let $R_1 > 0$ and $\psi : (-a,a) \to (-b,b)$ be a homeomorphism such that $\psi(0) = 0$ and $0 < a,b \le \infty$. Assume that there exists $c \in C$ such that $2R_1^{1-N}||c^-/\gamma||_{L^1} < b$ and*

$$f(r,u,v) \ge c(r), \quad \text{for all} \quad (r,u,v) \in [R_1,R_2]\times\mathbb{R}^2. \tag{2.13}$$

If u is a possible solution of the Neumann BVP (2.10) then

$$||u'||_\infty \le \max\left(\left|\psi^{-1}\left(\pm 2R_1^{1-N}||c^-/\gamma||_{L^1}\right)\right|\right) =: \rho_2(\psi). \tag{2.14}$$

P r o o f. First of all, let us note that

$$\left|r^{N-1} f(r,u,v)\right| \le r^{N-1} f(r,u,v) + 2\frac{c^-(r)}{\gamma(r)} \tag{2.15}$$

for all $(r,u,v) \in [R_1,R_2]\times\mathbb{R}^2$. If u solves (2.10) then

$$QN_f(u) = 0. \tag{2.16}$$

From (2.15) and (2.16) we get

$$\int_{R_1}^{R_2} \left|r^{N-1} f(r,u(r),u'(r))\right| dr \le 2||c^-/\gamma||_{L^1}. \tag{2.17}$$

Now the result follows from (2.12) and (2.17). □

Theorem 2.8 *Let $R_1 > 0$ and $\phi : \mathbb{R} \to (-b,b)$ be a homeomorphism such that $\phi(0) = 0$ with $0 < b \le \infty$ and f be like in Lemma 2.7. Assume that there exist $a \in (\rho_2(\phi),\infty)$, $\varepsilon \in \{-1,1\}$ and $\rho > 0$ such that (2.7) holds for any $u \in C^1_\dagger$ satisfying $|u|_L \ge \rho$ and $||u'||_\infty < a$. Then the BVP (2.1) has at least one solution.*

P r o o f. See the proof of Theorem 2.6. □

Remark 2.9 In the particular case $N = 1$, Theorem 2.3 was proved in [17] (see also [6] for the periodic case), while Theorems 2.6 and 2.8 were obtained in [5].

3 Existence of radial solutions

The results of the previous section can be used to derive the existence of radial solutions for the Neumann problems (1.1) and (1.2).

Theorem 3.1 *Assume that there exist* $\varepsilon \in \{-1, 1\}$ *and* $\rho > 0$ *such that*

$$\varepsilon(sgn\ u) \int_{R_1}^{R_2} r^{N-1} f(r, u(r), u'(r))\, dr \geq 0 \tag{3.1}$$

for all $u \in C^1_\dagger$ *such that* $|u|_L \geq \rho$ *and* $\|u'\|_\infty < 1$. *Then problem* (1.1) *has at least one classical radial solution.*

P r o o f. Theorem 2.3 applies with

$$\phi : (-1, 1) \longrightarrow \mathbb{R}, \quad \phi(y) = \frac{y}{\sqrt{1 - y^2}}. \tag{3.2}$$

□

Corollary 3.2 *Let* $h : [R_1, R_2] \times \mathbb{R}^2 \to \mathbb{R}$ *and* $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ *be continuous, with* h *bounded on* $[R_1, R_2] \times \mathbb{R} \times (-1, 1)$, *and* g *such that condition* (2.8) *holds. Then the Neumann BVP*

$$\mathcal{M}v + g(|x|, v) = h\left(|x|, v, \frac{dv}{dr}\right) \quad in \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad on \quad \partial\mathcal{A}$$

has at least one radial solution.

Example 3.3 For any $p > 1$ and any $l \in C$, the Neumann problems

$$\mathcal{M}v \pm |v|^{p-1} v = l(|x|) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A}$$

have at least one radial solution.

As another example of application, let us consider the Neumann problem

$$\mathcal{M}v + g(v) = l(|x|) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A}, \tag{3.3}$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous and $l \in C$. It is not difficult to check that Theorem 3.1 with $f(r, u, u') = l(r) - g(u)$ yields the following *Landesman-Lazer-type existence condition*.

Corollary 3.4 *If either*

$$\limsup_{v \to -\infty} g(v) < \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} r^{N-1} l(r)\, dr < \liminf_{v \to +\infty} g(v)$$

or

$$\limsup_{v \to +\infty} g(v) < \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} r^{N-1} l(r)\, dr < \liminf_{v \to -\infty} g(v),$$

then problem (3.3) *has at least one classical radial solution.*

Example 3.5 The Neumann problem

$$\mathcal{M}v + \arctan v + \sin v = l(|x|) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A}$$

has a radial solution if $l \in C$ is such that

$$1 - \frac{\pi}{2} < \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} r^{N-1} l(r)\, dr < \frac{\pi}{2} - 1.$$

Let ϕ be defined by (3.2). For a constant $q \in [0, 1)$, we set

$$\tilde{\rho}(q) := \phi(q).$$

Theorem 3.6 *Assume that there is some $k \geq 0$ such that $kR_2 < N$ and* (2.9) *holds true. If there exist constants $a > \tilde{\rho}(kR_2/N)$, $\varepsilon \in \{-1, 1\}$ and $\rho > 0$ such that the sign conditions* (3.1) *are fulfilled for all $u \in C^1_\dagger$ such that $|u|_L \geq \rho$ and $\|u'\|_\infty < a$, then problem* (1.2) *has at least one classical radial solution.*

P r o o f. Theorem 2.6 applies with $\phi : \mathbb{R} \to (-1, 1)$, $\phi(y) = \frac{y}{\sqrt{1+y^2}}$. □

Theorem 3.7 *Let $R_1 > 0$ and assume that there is some $c \in C$ such that $k := 2R_1^{1-N}||c^-/\gamma||_{L^1} < 1$ and* (2.13) *holds true. If there exist constants $a > \tilde{\rho}(k)$, $\varepsilon \in \{-1, 1\}$ and $\rho > 0$ such that the sign conditions* (3.1) *are fulfilled for all $u \in C^1_\dagger$ such that $|u|_L \geq \rho$ and $\|u'\|_\infty < a$, then problem* (1.2) *has at least one classical radial solution.*

P r o o f. It follows from Theorem 2.8. □

Remark 3.8 It is worth to point out that Theorems 2.6 and 2.8 also can be employed to derive existence results of radial solutions for Neumann problems in an annular domain, associated to p-Laplacian operator.

4 Upper and lower solutions in the singular case

In this section, we extend the method of upper and lower solutions (see e.g. [10]) to the Neumann boundary value problem (2.1).

Definition 4.1 A *lower solution* α (resp. *upper solution* β) of (2.1) is a function $\alpha \in C^1$ such that $\|\alpha'\|_\infty < a$, $r^{N-1}\phi(\alpha') \in C^1$, $\alpha'(R_1) \geq 0 \geq \alpha'(R_2)$ $\big($resp. $\beta \in C^1$, $\|\beta'\|_\infty < a$, $r^{N-1}\phi(\beta') \in C^1$, $\beta'(R_1) \leq 0 \leq \beta'(R_2)\big)$ and

$$\begin{aligned} &\left(r^{N-1}\phi(\alpha'(r))\right)' \geq r^{N-1} f(r, \alpha(r), \alpha'(r)) \\ &\Big(\text{resp.} \quad \left(r^{N-1}\phi(\beta'(r))\right)' \leq r^{N-1} f(r, \beta(r), \beta'(r))\Big) \end{aligned} \tag{4.1}$$

for all $r \in [R_1, R_2]$.

Below we shall invoke the hypothesis:

(H) $R_1 > 0$ and $\phi : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$ with $0 < a < \infty$.

Theorem 4.2 *Assume that (H) holds true. If* (2.1) *has a lower solution α and a upper solution β such that $\alpha(r) \leq \beta(r)$ for all $r \in [R_1, R_2]$, then problem* (2.1) *has a solution u such that $\alpha(r) \leq u(r) \leq \beta(r)$ for all $r \in [R_1, R_2]$.*

P r o o f. Let $\gamma : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ be the continuous function defined by

$$\gamma(r, u) = \begin{cases} \beta(r) & \text{if} \quad u > \beta(r), \\ u & \text{if} \quad \alpha(r) \leq u \leq \beta(r), \\ \alpha(r) & \text{if} \quad u < \alpha(r), \end{cases}$$

and define $F : [R_1, R_2] \times \mathbb{R}^2 \to \mathbb{R}$ by $F(r, u, v) = f(r, \gamma(r, u), v)$. We consider the modified problem

$$\left(r^{N-1}\phi(u')\right)' = r^{N-1}(F(r, u, u') + u - \gamma(r, u)), \quad u'(R_1) = 0 = u'(R_2) \tag{4.2}$$

and first show that if u is a solution of (4.2) then $\alpha(r) \leq u(r) \leq \beta(r)$ for all $r \in [R_1, R_2]$, so that u is a solution of (2.1). Suppose by contradiction that there is some $r_0 \in [R_1, R_2]$ such that $[\alpha - u]_M = \alpha(r_0) - u(r_0) > 0$. If $r_0 \in (R_1, R_2)$ then $\alpha'(r_0) = u'(r_0)$ and there are sequences (r_k) in $[r_0 - \varepsilon, r_0)$ and (r'_k) in $(r_0, t_0 + \varepsilon]$ converging to r_0 such that $\alpha'(r_k) - u'(r_k) \geq 0$ and $\alpha'(r'_k) - u'(r'_k) \leq 0$. As ϕ is an increasing homeomorphism, this implies

$$\begin{aligned} r_k^{N-1}\phi(\alpha'(r_k)) - r_0^{N-1}\phi(\alpha'(r_0)) &\geq r_k^{N-1}\phi(u'(r_k)) - r_0^{N-1}\phi(u'(r_0)), \\ {r'}_k^{N-1}\phi(\alpha'(r'_k)) - r_0^{N-1}\phi(\alpha'(r_0)) &\leq {r'}_k^{N-1}\phi(u'(r'_k)) - r_0^{N-1}\phi(u'(r_0)) \end{aligned}$$

and hence

$$\left(r^{N-1}\phi(\alpha'(r))\right)'_{r=r_0} \leq \left(r^{N-1}\phi(u'(r))\right)'_{r=r_0}.$$

Hence, because α is a lower solution of (2.1) we obtain

$$\begin{aligned}\left(r^{N-1}\phi(\alpha'(r))\right)'_{r=r_0} &\leq \left(r^{N-1}\phi(u'(r))\right)'_{r=r_0}\\ &= r_0^{N-1}(f(r_0,\alpha(r_0),\alpha'(r_0)) + u(r_0) - \alpha(r_0))\\ &< r_0^{N-1} f(r_0,\alpha(r_0),\alpha'(r_0))\\ &\leq \left(r^{N-1}\phi(\alpha'(r))\right)'_{r=r_0},\end{aligned}$$

a contradiction. If $[\alpha - u]_M = \alpha(R_1) - u(R_1) > 0$, then $\alpha'(R_1) - u'(R_1) = \alpha'(R_1) \leq 0$. Using the fact that $\alpha'(R_1) \geq 0$, we deduce that $\alpha'(R_1) = \alpha'(R_1) - u'(R_1) = 0$. This implies that $\phi(\alpha'(R_1)) = \phi(u'(R_1))$. On the other hand, $[\alpha - u]_M = \alpha(R_1) - u(R_1)$ implies, reasoning in a similar way as for $r_0 \in (R_1, R_2)$, that

$$\left(r^{N-1}\phi(\alpha'(r))\right)'_{r=R_1} \leq \left(r^{N-1}\phi(u'(r))\right)'_{r=R_1}.$$

Using the inequality above and $\alpha'(R_1) = u'(R_1)$, since $R_1 > 0$, we can proceed as in the case $r_0 \in (R_1, R_2)$ to obtain again a contradiction. The case where $r_0 = R_2$ is similar. In consequence we have that $\alpha(r) \leq u(r)$ for all $r \in [R_1, R_2]$. Analogously, using the fact that β is a upper solution of (2.1), we can show that $u(r) \leq \beta(r)$ for all $r \in [R_1, R_2]$. We now apply Corollary 2.4 to the modified problem (4.2) to obtain the existence of a solution. □

Remark 4.3 (i) We remark that if in theorem above α, β are strict, then $\alpha(r) < u(r) < \beta(r)$ for all $r \in [R_1, R_2]$. Moreover, let us consider the open set

$$\Omega_{\alpha,\beta} = \left\{u \in C^1_\dagger : \alpha(r) < u(r) < \beta(r) \text{ for all } r \in [R_1, R_2],\ ||u'||_\infty < a\right\}.$$

Then, arguing as in the proof of Lemma 3 [6] one has that

$$d_{LS}[I - \mathcal{N}, \Omega_{\alpha,\beta}, 0] = -1,$$

where $\mathcal{N}$ is the fixed point operator associated to (2.1) introduced in Remark 2.2.

(ii) In contrast to the classical p-Laplacian or Euclidean mean curvature cases, no Nagumo-type condition is required upon f in Theorem 4.2.

We now show, adapting an argument introduced by Amann-Ambrosetti-Mancini [1] in semilinear elliptic problems, that the existence conclusion in Theorem 4.2 also holds when the lower and upper solutions are not ordered. See [6, 17] for the case where $N = 1$.

Theorem 4.4 *Assume that (H) holds true. If* (2.1) *has a lower solution α and an upper solution β, then problem* (2.1) *has at least one solution.*

P r o o f. Let $\mathcal{C}$ be given by Lemma 2.1. If there is some $(\overline{u}, \widetilde{u}) \in \mathcal{C}$ such that

$$\int_{R_1}^{R_2} r^{N-1} f(r, \overline{u} + \widetilde{u}(r), \widetilde{u}'(r))\, dr = 0,$$

then $\overline{u} + \widetilde{u}$ solves (2.1). If

$$\int_{R_1}^{R_2} r^{N-1} f(r, \overline{u} + \widetilde{u}(r), \widetilde{u}'(r))\, dr > 0$$

for all $(\overline{u}, \widetilde{u}) \in \mathcal{C}$, then, using (2.2), $\overline{u} + \widetilde{u}$ is an upper solution for (2.1) for each $(\overline{u}, \widetilde{u}) \in \mathcal{C}$. Then, for $(\alpha_M + a(1 + R_2 - R_1), \widetilde{u}) \in \mathcal{C}$, $\alpha_M + a(1 + R_2 - R_1) + \widetilde{u}(r) \geq \alpha(r)$ for all $r \in [R_1, R_2]$ is an upper solution and the existence of a solution to (2.1) follows from Theorem 4.2. Similarly, if

$$\int_{R_1}^{R_2} r^{N-1} f(r, \overline{u} + \widetilde{u}(r), \widetilde{u}'(r))\, dr < 0$$

for all $(\overline{u}, \widetilde{u}) \in \mathcal{C}$, then $(\beta_L - a(1+R_2-R_1), \widetilde{u}) \in \mathcal{C}$ gives the lower solution $\beta_L - a(1+R_2-R_1) + \widetilde{u}(r) \leq \beta(r)$ for all $r \in [R_1, R_2]$ and the existence of a solution. □

Remark 4.5 Assume that $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$ and the nonlinearity f is bounded from below by a continuous function $c \in C$, that is (2.13) holds. Using Lemma 2.5 and the same strategy as in the proof of Theorem 2.6, it can be shown that Theorem 4.4 also holds for *classical* homeomorphisms under the additional condition (2.13).

The choice of constant lower and upper solutions in Theorems 4.2 and 4.4 leads to the following simple existence condition.

Corollary 4.6 *If (H) holds true then problem* (2.1) *has at least one solution if there exist constants A and B such that*

$$f(r, A, 0) \cdot f(r, B, 0) \leq 0$$

for all $r \in [R_1, R_2]$.

A simple application of Theorem 4.4 provides a necessary and sufficient condition of existence of a solution of (2.1) when $f = f(r, u)$ and $f(r, \cdot)$ is monotone. We adapt an argument first introduced for semilinear Dirichlet problems by Kazdan-Warner [15].

Corollary 4.7 *Assume that (H) holds true. If $f : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ is continuous and $f(r, \cdot)$ is either non decreasing or non increasing for every $r \in [R_1, R_2]$, then problem*

$$\left(r^{N-1}\phi(u')\right)' = r^{N-1} f(r, u), \quad u'(R_1) = 0 = u'(R_2) \tag{4.3}$$

is solvable if and only if there exists $c \in \mathbb{R}$ such that

$$\int_{R_1}^{R_2} r^{N-1} f(r, c)\, dr = 0. \tag{4.4}$$

P r o o f. *Necessity*. If problem (4.3) has a solution u, then, integrating both members of the differential equation in (4.3) and using the boundary condition, it follows that

$$\int_{R_1}^{R_2} r^{N-1} f(r, u(r))\, dr = 0. \tag{4.5}$$

Assuming for example that $f(r, \cdot)$ is non decreasing for every $r \in [R_1, R_2]$, we deduce from (4.5) that

$$\int_{R_1}^{R_2} r^{N-1} f(r, u_L)\, dr \leq 0 \leq \int_{R_1}^{R_2} r^{N-1} f(r, u_M)\, dr,$$

so that, by the intermediate value theorem, there exists some $c \in [u_L, u_M]$ satisfying (4.4). The reasoning is similar when $f(r, \cdot)$ is non decreasing for every $r \in [R_1, R_2]$.

Sufficiency. If $c \in \mathbb{R}$ satisfies (4.4), then the problem

$$\left(r^{N-1}\phi(u')\right)' = r^{N-1} f(r, c), \quad u'(R_1) = 0 = u'(R_2) \tag{4.6}$$

has a one-parameter family of solutions of the form $d + \widetilde{w}(r)$ with $\widetilde{w} \in \widetilde{C}^1_\dagger$. There exists $d_1 \leq d_2$ such that, for all $r \in [R_1, R_2]$,

$$\alpha(r) := d_1 + \widetilde{w}(r) \leq c \leq d_2 + \widetilde{w}(r) =: \beta(r).$$

Hence, if $f(r, \cdot)$ is non decreasing for each $r \in [R_1, R_2]$, then

$$\left(r^{N-1}\phi(\alpha'(r))\right)' = \left(r^{N-1}\phi(\widetilde{w}'(r))\right)' = r^{N-1} f(r, c) \geq r^{N-1} f(r, \alpha(r))$$

and α is a lower solution for (4.3). Similarly β is an upper solution for (4.3). A similar argument shows that, if $f(r, \cdot)$ is non increasing for every $r \in [R_1, R_2]$, α is an upper solution and β a lower solution for (4.3). So the result follows from Theorem 4.4. □

The results above can be applied to classical radial solutions of the Neumann problem (1.1).

Corollary 4.8 *Let $R_1 > 0$. Problem* (1.1) *has at least one classical radial solution if there exist constants A and B such that*

$$f(r, A, 0) \cdot f(r, B, 0) \leq 0$$

for all $r \in [R_1, R_2]$.

Corollary 4.9 *Let $R_1 > 0$. If $f : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ is continuous and $f(r, \cdot)$ is either non decreasing or non increasing for every $r \in [R_1, R_2]$, then problem*

$$\mathcal{M}v = f(|x|, v) \quad in \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad on \quad \partial\mathcal{A}$$

has a classical radial solution if and only if there exists $c \in \mathbb{R}$ such that (4.4) *holds.*

Example 4.10 Let $f(v) = e^v$ or $f(v) = |v|^{p-1}v^+$ with $p > 1$. If $R_1 > 0$ then the Neumann problem

$$\mathcal{M}v + f(v) = l(|x|) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A}$$
$$\left(\text{resp.} \quad \mathcal{M}v - f(v) = l(|x|) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A}\right)$$

has a classical radial solution if and only if $l \in C$ is such that

$$\int_{R_1}^{R_2} r^{N-1} l(r)\, dr > 0 \quad \left(\text{resp.} \quad \int_{R_1}^{R_2} r^{N-1} l(r)\, dr < 0\right).$$

The same result holds true if we replace the operator $\mathcal{M}$ by the p-Laplacian operator.

Remark 4.11 Multiplicity results of the Ambrosetti-Prodi type similar to the ones obtained in [6, 17] for $N = 1$ can be deduced in a similar way from Theorem 4.2 and Remark 4.3 (i).

5 Pendulum-like nonlinearities

Consider the Neumann problem

$$\mathcal{M}v + b \sin v = l(|x|) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A}, \tag{5.1}$$

where $b > 0$ and $l \in C$.

Let us suppose also that $R_1 > 0$. Then, using Theorem 4.2, Remark 4.3 (i) and the method introduced in [18], one can prove that (5.1) has at least one radial solution if $||l||_\infty \leq b$ and at least two radial solutions if $||l||_\infty < b$. The following result shows that one has existence even in the case $R_1 = 0$ for any l with $Ql = 0$, under an additional condition concerning the distance between R_1 and R_2. We adapt to our situation an argument used in [20].

Theorem 5.1 *If $Ql = 0$ and $2(R_2 - R_1) \leq 1$, then* (5.1) *has at least one classical radial solution.*

P r o o f. It is clear that it is sufficient to prove the existence of at least one solution for the Neumann problem

$$\left(r^{N-1}\phi(u')\right)' + r^{N-1} b \sin u = r^{N-1} l(r), \quad u'(R_1) = 0 = u'(R_2), \tag{5.2}$$

with ϕ given in (3.2). Let us make the change of variable

$$u = \arcsin w.$$

Then, we obtain the Neumann problem

$$\left(r^{N-1}\phi\left(\frac{w'}{\sqrt{1-w^2}}\right)\right)' + r^{N-1} bw = r^{N-1} l(r), \quad w'(R_1) = 0 = w'(R_2). \tag{5.3}$$

Consider the closed subspace of C defined by

$$\widehat{C} = \{w \in C : Qw = 0\}$$

and denotes by N the Nemytskii operator N_f associated to $f(r, w) = l(r) - bw$. Consider also the nonlinear operator

$$\mathcal{T} : \widehat{K} \to \widehat{C}, \quad \mathcal{T}(w) = (I - Q) \circ H \circ \sqrt{1 - w^2}\phi^{-1} \circ L \circ N(w),$$

where

$$\widehat{K} = \big\{w \in \widehat{C} : ||w||_\infty \leq 2(R_2 - R_1)\big\}.$$

One has that $\mathcal{T}$ is well defined and compact. It is clear that

$$||\mathcal{T}(w)||_\infty < 2(R_2 - R_1) \leq 1 \quad \text{for all} \quad w \in \widehat{K}.$$

Using Schauder's fixed point theorem, we infer that there exist $w \in \widehat{K}$ such that $w = \mathcal{T}(w)$, and because $||w||_\infty < 1$, it follows that w is a solution of (5.3). Therefore $u = \arcsin w$ is a solution of (5.2). ☐

Acknowledgements Support of C. Bereanu from the Romanian Ministry of Education, Research and Innovation (PN II Program, CNCSIS code RP 3/2008) is gratefully acknowledged.

References

[1] H. Amann, A. Ambrosetti, and G. Mancini, Elliptic equations with noninvertible Fredholm linear part and bounded nonlinearities, Math. Z. **158**, 179–194 (1978).
[2] R. Bartnik and L. Simon, Spacelike hypersurfaces with prescribed boundary values and mean curvature, Comm. Math. Phys. **87**, 131–152 (1982-83).
[3] C. Bereanu, P. Jebelean, and J. Mawhin, Radial solutions for some nonlinear problems involving mean curvature operators in Euclidean and Minkowski spaces, Proc. Amer. Math. Soc. **137**, 171–178 (2009).
[4] C. Bereanu, P. Jebelean, and J. Mawhin, Radial solutions for systems involving mean curvature operators in Euclidean and Minkowski spaces, AIP Conf. Proc., accepted.
[5] C. Bereanu and J. Mawhin, Boundary-value problems with non-surjective ϕ-Laplacian and one-sided bounded nonlinearity, Adv. Differential Equations **11**, 35–60 (2006).
[6] C. Bereanu and J. Mawhin, Existence and multiplicity results for some nonlinear problems with singular ϕ-Laplacian, J. Differential Equations **243**, 536–557 (2007).
[7] E. Calabi, Examples of Bernstein problems for some nonlinear equations, Proc. Sympos. Pure Math. **15**, 223–230 (1970).
[8] S. Y. Cheng and S. T. Yau, Maximal spacelike hypersurfaces in the Lorentz-Minkowski spaces, Ann. of Math. (2) **104**, 407–419 (1976).
[9] P. Clément, R. Manásevich, and E. Mitidieri, On a modified capillary equations, J. Differential Equations **124**, 343–358 (1996).
[10] C. De Coster and P. Habets, Two-point Boundary Value Problems. Lower and Upper Solutions (Elsevier, Amsterdam, 2006).
[11] K. Deimling, Nonlinear Functional Analysis (Springer, Berlin, 1985).
[12] M. García-Huidobro, R. Manásevich, and K. Schmitt, Positive radial solutions of quasilinear elliptic partial differential equations in a ball, Nonlinear Anal. **35**, 175–190 (1999).
[13] M. García-Huidobro, R. Manásevich, and C. S. Yarur, On the structure of positive radial solutions to an equation containing a p-Laplacian with weight, J. Differential Equations **223**, 51–95 (2006).
[14] D. D. Hai and R. Shivaji, Existence and uniqueness for a class of quasilinear elliptic boundary value problems, J. Differential Equations **193**, 500–510 (2003).
[15] J. L. Kazdan and F. W. Warner, Remarks on some quasilinear elliptic equations, Comm. Pure Appl. Math. **28**, 567–597 (1975).
[16] J. Mawhin, Leray-Schauder degree: a half century of extensions and applications, Topol. Methods Nonlinear Anal. **14**, 195–228 (1999).
[17] J. Mawhin, Boundary value problems for nonlinear perturbations of singular ϕ-Laplacians, Progr. Nonlinear Differential Equations Appl. **75**, 247–256 (2007).
[18] J. Mawhin and M. Willem, Multiple solutions of the periodic boundary value problem for some forced pendulum-type equations, J. Differential Equations **52**, 264–287 (1984).
[19] O. S. de Queiroz, A Neumann problem with logarithmic nonlinearity in a ball, Nonlinear Anal. **70**, 1656–1662 (2009).

[20] P. J. Torres, Periodic oscillations of the relativistic pendulum with friction, Phys. Lett. A **372**, 6386–6387 (2008).
[21] A. Treibergs, Entire spacelike hypersurfaces of constant mean curvature in Minkowski space, Invent. Math. **66**, 39–56 (1982).
[22] Q. Zhang, Existence of solutions for weighted $p(r)$-Laplacian systems boundary value problems, J. Math. Anal. Appl. **327**, 127–141 (2007).