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RADIAL SOLUTIONS FOR NEUMANN PROBLEMS WITH ϕ-LAPLACIANS AND PENDULUM-LIKE NONLINEARITIES

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Dedicated, with admiration, to Louis Nirenberg

Abstract. In this paper we study the existence and multiplicity of radial solutions for Neumann problems in a ball and in an annular domain, associated to pendulum-like perturbations of mean curvature operators in Euclidean and Minkowski spaces and of the p-Laplacian operator. Our approach relies on the Leray-Schauder degree and the upper and lower solutions method.

1. **Introduction.** In this paper we present existence and multiplicity results for the Neumann problem

$$\mathcal{T}(v) + \mu \sin v = l(|x|) \quad \text{in} \quad \mathcal{A}, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A}, \tag{1}$$

where $\mathcal{T}$ is in one of the following situations:

$$\mathcal{T}(v) = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) \text{ (\textit{mean extrinsic curvature in Minkowski space})},$$

$$\mathcal{T}(v) = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) \text{ (\textit{mean curvature in Euclidean space})},$$

$$\mathcal{T}(v) = \operatorname{div}(|\nabla v|^{p-2}\nabla v) \text{ (\textit{p-Laplacian})}.$$

Here, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$, $\mu > 0$ is a constant, $\mathcal{A} = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$ $(0 \le R_1, < R_2)$, $l : [R_1, R_2] \to \mathbb{R}$ is a given continuous function and $\dfrac{\partial v}{\partial \nu}$ stands for the outward normal derivative of v.

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Our approach relies upon the idea that setting $r = |x|$ and $v(x) = u(r)$, problem (1) reduces to

$$(r^{N-1}\phi(u'))' + r^{N-1}\mu \sin u = r^{N-1}l(r), \quad u'(R_1) = 0 = u'(R_2), \tag{2}$$

where $\phi(v) = \frac{v}{\sqrt{1-v^2}}$ in the Minkowski case, $\phi(v) = \frac{v}{\sqrt{1+v^2}}$ in the Euclidean case and $\phi(v) = |v|^{p-2}v$ ($p > 1$) in the p-Laplacian case. Actually, in what follows ϕ will be a general increasing homeomorphism with $\phi(0) = 0$ and which is in one of the following situations:

$\phi : (-a, a) \to \mathbb{R}$ (*singular*),
$\phi : \mathbb{R} \to (-a, a)$ (*bounded*),
$\phi : \mathbb{R} \to \mathbb{R}$ (*classical*).

We prove in Corollary 1, using degree arguments, that the problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + \mu \sin v = l(|x|) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A}, \tag{3}$$

has at least two classical radial solutions not differing by a a multiple of 2π if

$$2(R_2 - R_1) < \pi$$

and

$$\left|\frac{N}{R_2^N - R_1^N}\int_{R_1}^{R_2} r^{N-1}l(r)dr\right| < \mu \cos(R_2 - R_1).$$

Moreover, if

$$2(R_2 - R_1) = \pi,$$

then problem (3) has at least one classical radial solution provided that

$$\int_{R_1}^{R_2} r^{N-1}l(r)dr = 0. \tag{4}$$

Notice that in Theorem 5.1 from [4], we have proved that if condition (4) is fulfilled and if $2(R_2 - R_1) \le 1$, then one has existence of at least one classical radial solution. On the other hand for the p-Laplacian, we prove in Corollary 2 that problem

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) + \mu \sin v = l(|x|) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A}, \tag{5}$$

has at least two classical radial solutions not differing by a multiple of 2π if (4) holds and R_2 is sufficiently small (or N sufficiently large). Moreover the same type of result holds true for the Neumann problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) + \mu \sin v = l(|x|) \quad \text{in} \quad \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{A}. \tag{6}$$

In the case where

$$R_1 > 0$$

(i.e., $\mathcal{A}$ is an annular domain) we show in Corollary 3, using again degree arguments and the upper and lower solutions method, that (3) and (5) have at least two classical radial solutions, not differing by a multiple of 2π, if $||l||_\infty < \mu$, and have at least one classical radial solution if $||l||_\infty = \mu$. Moreover, if condition

$$\frac{2\mu R_2}{N} < 1$$

holds, then we prove in Corollary 3 that the same result holds for the Neumann problem (6).

It is worth to point out that corresponding results for the periodic problem and $N = 1$ have been proved in [5, 6]. For existence and multiplicity results concerning periodic solutions of the classical pendulum equation, see for example [14, 16, 17, 19], and for other results concerning boundary value problems associated to singular or bounded ϕ-Laplacians, see [2] - [15], [18, 20, 21].

The remaining of the paper is organized as follows. In Section 2 we introduce the function spaces and the operators which are needed in the sequel. Section 3 presents a fixed point operator and some degree computations in the singular case. Existence and multiplicity results for problem (2) are given in Sections 4 and 5, under conditions upon the radius and the mean value of the forcing term or upon the norm of the forcing term.

2. **Notations, function spaces and operators.** Let $0 \leq R_1 < R_2$. We denote by C the Banach space of continuous functions defined on $[R_1, R_2]$ endowed with the usual norm $||\cdot||_\infty$, by C^1 the Banach space of continuously differentiable functions defined on $[R_1, R_2]$ endowed with the norm

$$||u|| = ||u||_\infty + ||u'||_\infty,$$

and by $C^1_\dagger$ the closed subspace of C^1 defined by

$$C^1_\dagger := \{u \in C^1 : u'(R_1) = 0 = u'(R_2)\}.$$

The corresponding open ball with center in 0 and radius ρ is denoted by B_ρ. For any continuous function $w : [R_1, R_2] \to \mathbb{R}$, we write

$$w_L = \min_{[R_1,R_2]} w, \quad w_M = \max_{[R_1,R_2]} w.$$

Let us introduce the continuous projectors

$$Q : C \to C, \quad \underline{u} = Qu = \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} r^{N-1} u(r) dr,$$
$$P : C \to C, \quad Pu = u(R_1),$$

the continuous function

$$\gamma : (0, \infty) \to \mathbb{R}, \quad \gamma(r) = \frac{1}{r^{N-1}},$$

and the linear operators

$$L \ : \ C \to C, \quad Lu(r) = \gamma(r) \int_{R_1}^{r} t^{N-1} u(t) dt \quad (r \in (R_1, R_2]), \quad Lu(0) = 0,$$
$$H \ : \ C \to C^1, \quad Hu(r) = \int_{R_1}^{r} u(t) dt \quad (r \in [R_1, R_2]).$$

It is not difficult to prove that L is compact (Arzelà-Ascoli's theorem) and H is bounded. Finally, we denote by $\widehat{C}^1_\dagger$ the closed subspace of $C^1_\dagger$ defined by

$$\widehat{C}^1_\dagger := \{u \in C^1_\dagger : \underline{u} = 0\},$$

and notice that

$$C^1_\dagger = \mathbb{R} \oplus \widehat{C}^1_\dagger,$$

so that any $u \in C^1_\dagger$ can be uniquely written as $u = \underline{u} + \widehat{u}$, with $\underline{u} \in \mathbb{R}$, $\widehat{u} \in \widehat{C}^1_\dagger$.

3. **A fixed point operator and degree computations in the singular case.** Throughout this section we assume that ϕ is *singular*. The case where $N=1$ and $R_1=0$ in the results of this section has been considered in [18].

Proposition 1. *Assume that $F: C^1_\dagger \to C$ is continuous and takes bounded sets into bounded sets. The function $u \in C^1_\dagger$ is a solution of the abstract Neumann problem*

$$(r^{N-1}\phi(u'))' = r^{N-1}F(u), \quad u'(R_1) = 0 = u'(R_2) \tag{7}$$

if and only if it is a fixed point of the compact operator M_F defined on $C^1_\dagger$ by

$$M_F = P + QF + H \circ \phi^{-1} \circ L \circ (I-Q) \circ F.$$

Furthermore, one has $\|(M_F(u))'\|_\infty < a$ for all $u \in C^1_\dagger$.

Proof. Let $u \in C^1_\dagger$ and $v = M_F(u)$. One has that $v \in C^1$ and

$$\phi(v') = L \circ (I-Q) \circ F(u).$$

So, $\phi(v'(R_1)) = 0$ and because $QF(u)$ is constant,

$$\phi(v'(R_2)) = \frac{1}{R_2^{N-1}} \int_{R_1}^{R_2} t^{N-1}F(u)(t)dt - \frac{1}{R_2^{N-1}} QF(u) \int_{R_1}^{R_2} t^{N-1}dt = 0.$$

It follows that M_F is well defined. Its compactness follows very easily taking into account the properties of the operators composing M_F. From the above computation and since ϕ is singular, we get $||v'||_\infty < a$. Now, let $u \in C^1_\dagger$ be such that $u = M_F(u)$. It follows

$$QF(u) = 0, \tag{8}$$

implying that

$$u = Pu + H \circ \phi^{-1} \circ L \circ F(u), \quad u' = \phi^{-1} \circ L \circ F(u).$$

Then

$$\phi(u'(r)) = \gamma(r) \int_{R_1}^{r} t^{N-1}F(u)(t)dt \quad (r \in (R_1, R_2]),$$

and u verifies the differential equation in (7).

Conversely, let u be a solution of (7). Then, taking into account the fact that u verifies (8), after two integrations we deduce that u is a fixed point of M_F. $\square$

Lemma 3.1. *Let the continuous function $h: [R_1, R_2] \times \mathbb{R}^2 \to \mathbb{R}$ be bounded on $[R_1, R_2] \times \mathbb{R} \times (-a, a)$, $\mu \neq 0$ and consider the Neumann problem*

$$(r^{N-1}\phi(u'))' + \mu r^{N-1}u = r^{N-1}h(r, u, u'), \quad u'(R_1) = 0 = u'(R_2). \tag{9}$$

If M_μ is the fixed point operator associated to (9), then there exists $\rho > 0$ such that any possible fixed point of M_μ is contained in B_ρ and

$$d_{LS}[I - M_\mu, B_\rho, 0] = sign(\mu).$$

Proof. Let us consider say, the case where $\mu > 0$, the other one being similar. We can find a constant $R > 0$ such that

$$sign(u)[-\mu u + h(r, u, v)] < 0 \tag{10}$$

for all $r \in [R_1, R_2]$, $v \in (-a, a)$ and $|u| \geq R$. One the other hand, consider the compact homotopy $\mathcal{M}: [0,1] \times C^1_\dagger \to C^1_\dagger$ defined by

$$\mathcal{M}(\lambda, \cdot) = P + QF_\mu + H \circ \phi^{-1} \circ \lambda L \circ (I-Q) \circ F_\mu,$$

where

$$F_\mu : C^1_\dagger \to C, \quad F_\mu(u) = -\mu u + h(\cdot, u, u').$$

Let $(\lambda, u) \in [0,1] \times C^1_\dagger$ be such that

$$u = \mathcal{M}(\lambda, u).$$

It follows that

$$u' = \phi^{-1} \circ \lambda L \circ (I - Q) \circ F_\mu(u)$$

and

$$||u'||_\infty < a. \tag{11}$$

Notice also that

$$QF_\mu(u) = 0. \tag{12}$$

Assume that $u_L \geq R$. Then, using (10) and (11) we have

$$F_\mu(u)(r) < 0 \quad \text{for all} \quad r \in [R_1, R_2].$$

This implies that

$$QF_\mu(u) < Q(0) = 0,$$

a contradiction with (12). It follows that $u_L < R$, and analogously $u_M > -R$. Then, from

$$u_M \leq u_L + \int_{R_1}^{R_2} |u'(r)| dr$$

and (11), we deduce that

$$-R - a(R_2 - R_1) < u_L \leq u_M < R + a(R_2 - R_1),$$

which together with (11) gives

$$||u|| < R + a(R_2 - R_1 + 1) =: \rho_0.$$

Since

$$\mathcal{M}(1, \cdot) = M_\mu \quad \text{and} \quad \mathcal{M}(0, \cdot) = P + QF_\mu,$$

the homotopy invariance of the Leray-Schauder degree implies that

$$d_{LS}[I - M_\mu, B_\rho, 0] = d_{LS}[I - (P + QF_\mu), B_\rho, 0],$$

for any $\rho \geq \rho_0$. The range of $P + QF_\mu$ is contained in the subspace of constant functions. Using the reduction property of the Leray Schauder degree we have

$$d_{LS}[I - (P + QF_\mu), B_\rho, 0] = d_B[I - (P + QF_\mu)|_{\mathbb{R}}, (-\rho, \rho), 0],$$

where d_B denotes the Brouwer degree. But,

$$I - (P + QF_\mu)|_{\mathbb{R}} = -QF_\mu|_{\mathbb{R}},$$

and we can take ρ sufficiently large such that

$$QF_\mu(-\rho) > 0 > QF_\mu(\rho),$$

implying that

$$d_B[-QF_\mu|_{\mathbb{R}}, (-\rho, \rho), 0] = 1 = sign(\mu).$$

Consequently,

$$d_{LS}[I - M_\mu, B_\rho, 0] = sign(\mu).$$

□

Now, consider the Neumann boundary-value problem (BVP)

$$(r^{N-1}\phi(u'))' = r^{N-1}f(r,u,u'), \quad u'(R_1) = 0 = u'(R_2), \tag{13}$$

where $f : [R_1, R_2] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

Definition 3.2. A *strict lower solution* α (resp. *strict upper solution* β) of (13) is a function $\alpha \in C^1$ such that $\|\alpha'\|_\infty < a$, $r^{N-1}\phi(\alpha') \in C^1$, $\alpha'(R_1) \geq 0 \geq \alpha'(R_2)$ (resp. $\beta \in C^1$, $\|\beta'\|_\infty < a$, $r^{N-1}\phi(\beta') \in C^1$, $\beta'(R_1) \leq 0 \leq \beta'(R_2)$) and

$$\begin{array}{rrcl} & (r^{N-1}\phi(\alpha'(r)))' & > & r^{N-1}f(r,\alpha(r),\alpha'(r)) \\ (resp. & (r^{N-1}\phi(\beta'(r)))' & < & r^{N-1}f(r,\beta(r),\beta'(r))) \end{array}$$

for all $r \in [R_1, R_2]$.

Lemma 3.3. *Assume that (13) has a strict lower solution α and a strict upper solution β such that*

$$\alpha(r) < \beta(r) \quad \textit{for all} \quad r \in [R_1, R_2],$$

and if $N \geq 2$ assume also that $R_1 > 0$. Then

$$d_{LS}[I - M_f, \Omega_{\alpha,\beta}, 0] = -1,$$

where

$$\Omega_{\alpha,\beta} = \{u \in C^1_\dagger : \alpha(r) < u(r) < \beta(r) \quad \textit{for all} \quad r \in [R_1, R_2], \quad ||u'||_\infty < a\}$$

and M_f is the fixed point operator associated to (13).

Proof. Let $\gamma : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ be the continuous function given by

$$\gamma(r,u) = \begin{cases} \beta(r) & if \quad u > \beta(r) \\ u & if \quad \alpha(r) \leq u \leq \beta(r) \\ \alpha(r) & if \quad u < \alpha(r), \end{cases}$$

and define $f_1 : [R_1, R_2] \times \mathbb{R}^2 \to \mathbb{R}$ by $f_1(r,u,v) = f(r,\gamma(r,u),v)$. We consider the modified problem

$$(r^{N-1}\phi(u'))' = r^{N-1}[f_1(r,u,u') + u - \gamma(r,u)], \quad u'(R_1) = 0 = u'(R_2), \tag{14}$$

and let M_{f_1} be the associated fixed point operator of (14). Then, arguing exactly as in the proof of Theorem 4.2 from [4], one has that if u is a solution of (14) then $\alpha(r) < u(r) < \beta(r)$ for all $r \in [R_1, R_2]$. It follows that any fixed point of M_{f_1} is contained in $\Omega_{\alpha,\beta}$, and using the excision property of the Leray-Schauder degree and Lemma 3.1 we infer that

$$d_{LS}[I - M_{f_1}, \Omega_{\alpha,\beta}, 0] = d_{LS}[I - M_{f_1}, B_\rho, 0] = -1,$$

for any ρ sufficiently large. On the other hand

$$M_f(u) = M_{f_1}(u) \quad \text{for all} \quad u \in \overline{\Omega}_{\alpha,\beta}.$$

Consequently,

$$d_{LS}[I - M_f, \Omega_{\alpha,\beta}, 0] = -1.$$

$\square$

4. **Conditions on the radius and the mean value of the forcing term.** We consider the Neumann boundary value problem

$$(r^{N-1}\phi(u'))' + r^{N-1}g(u) = r^{N-1}l(r), \quad u'(R_1) = 0 = u'(R_2), \tag{15}$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous and $l \in C$.

The idea of the following lemma comes from Theorem 2 in [19].

Lemma 4.1. *Assume that ϕ is singular and that there exist $t < s$ and $A < B$ such that either*

$$Qg(t + \widehat{u}) \leq A \quad \textit{and} \quad Qg(s + \widehat{u}) \geq B \tag{16}$$

or

$$Qg(t + \widehat{u}) \geq B \quad \textit{and} \quad Qg(s + \widehat{u}) \leq A \tag{17}$$

for any $\widehat{u} \in \widehat{C}^1_\dagger$ satisfying $||\widehat{u}||_\infty < a(R_2 - R_1)$. If

$$A < \underline{l} < B, \tag{18}$$

then problem (15) has at least one solution u such that $t < \underline{u} < s$.

Proof. Let us assume that (16) holds true and let $\varepsilon > 0$ be fixed. For any $\lambda \in [0, 1]$, consider the Neumann problem

$$\begin{gathered}(r^{N-1}\phi(u'))' + \lambda r^{N-1}g(u) + (1 - \lambda)\varepsilon r^{N-1}\left(u - \frac{t + s}{2}\right) = \lambda r^{N-1}l(r) \\ u'(R_1) = 0 = u'(R_2).\end{gathered} \tag{19}$$

Let also

$$\mathcal{M}(\lambda, \cdot) : C^1_\dagger \to C^1_\dagger \quad (\lambda \in [0, 1])$$

be the fixed point operator associated to (19) by Proposition 1. We will show that

$$u - \mathcal{M}(\lambda, u) \neq 0 \quad \text{for any} \quad (\lambda, u) \in (0, 1] \times \partial\Omega, \tag{20}$$

and

$$u - \mathcal{M}(0, u) = 0 \quad \text{implies} \quad u \in \Omega,$$

where

$$\Omega = \{u \in C^1_\dagger : \quad t < \underline{u} < s, \quad ||\widehat{u}||_\infty < a(R_2 - R_1), \quad ||u'||_\infty < a\}.$$

Then, using the invariance by homotopy, the excision property of the Leray-Schauder degree and Lemma 3.1, one has that

$$d_{LS}[I - \mathcal{M}(1, \cdot), \Omega, 0] = d_{LS}[I - \mathcal{M}(0, \cdot), \Omega, 0] = 1.$$

Hence, the existence property of the Leray-Schauder degree implies the existence of some $u \in \Omega$ (in particular $t < \underline{u} < s$) with $u = \mathcal{M}(1, u)$ which is also a solution of (15).

So, let us consider $(\lambda, u) \in (0, 1] \times C^1_\dagger$ such that $u = \mathcal{M}(\lambda, u)$. It follows that (11) holds true and $u = \underline{u} + \widehat{u} \in \mathbb{R} \oplus \widehat{C}^1_\dagger$ is a solution of (19). As $Q\widehat{u} = 0$, there exists $r_0 \in [R_1, R_2]$ such that $\widehat{u}(r_0) = 0$, yielding

$$||\widehat{u}||_\infty \leq \int_{r_0}^{R_2} |\widehat{u}'(r)| dr < a(R_2 - R_1). \tag{21}$$

Integrating (19) over $[R_1, R_2]$ we obtain

$$(1 - \lambda)\varepsilon\left(\underline{u} - \frac{t + s}{2}\right) + \lambda(Qg(\underline{u} + \widehat{u}) - \underline{l}) = 0. \tag{22}$$

On the other hand, from (16), (18) and (21) it follows that

$$\begin{aligned} &(1-\lambda)\varepsilon\left(t-\frac{t+s}{2}\right)+\lambda(Qg(t+\widehat{u})-\underline{l})\le(1-\lambda)\varepsilon\frac{t-s}{2}+\lambda(A-\underline{l})<0;\\ &(1-\lambda)\varepsilon\left(s-\frac{t+s}{2}\right)+\lambda(Qg(s+\widehat{u})-\underline{l})\ge(1-\lambda)\varepsilon\frac{s-t}{2}+\lambda(B-\underline{l})>0.\end{aligned} \tag{23}$$

Moreover, if $u\in\partial\Omega$, from (11) and (21) one has $\underline{u}=t$ or $\underline{u}=s$. But $\underline{u}$ verifies (22), contradiction with (23). Consequently, (20) is proved. Now, let $u\in C^1_\dagger$ be such that $u=\mathcal{M}(0,u)$. We deduce that u verifies (11), (21) and (19) with $\lambda=0$. Hence, $\underline{u}=\dfrac{t+s}{2}$ and $u\in\Omega$.

If (17) holds true then one takes $\varepsilon<0$. □

Remark 1. From the proof above it can be seen that if the assumption "$A<B$" is replaced by "$A\le B$" then problem (15) has at least one solution u such that $t\le\underline{u}\le s$, provided that $A\le\underline{l}\le B$.

Theorem 4.2. *If ϕ is singular, $l\in C$, $\mu>0$ and*

$$2a(R_2-R_1)<\pi,$$

then, the Neumann problem (2) has at least two solutions not differing by a multiple of 2π, provided that

$$|\underline{l}|<\mu\cos\left[a(R_2-R_1)\right].$$

Proof. We apply Lemma 4.1 with $g(u)=\mu\sin(u)$ and $A=-\mu\cos\left[a(R_2-R_1)\right]=-B$. Taking $t=-\pi/2$, $s=\pi/2$, condition (16) is fulfilled and so, we get the existence of a solution u_1 which satisfies $-\pi/2<\underline{u}_1<\pi/2$. Then, setting $t=\pi/2$, $s=3\pi/2$, condition (17) is verified and we obtain a second solution u_2 with $\pi/2<\underline{u}_2<3\pi/2$. If we assume that there is some $j\in\mathbb{Z}$ such that $u_2=u_1+2j\pi$ then necessarily one has $0<2j\pi<2\pi$, a contradiction. □

Remark 2. Using Remark 1, if in Theorem 4.2 one has $2a(R_2-R_1)=\pi$, then problem (2) has at least one solution for any $l\in C$ with $\underline{l}=0$.

Corollary 1. *Let $\mu>0$ and $l\in C$. If $2(R_2-R_1)<\pi$, then the Neumann problem (3) has at least two classical radial solutions not differing by a multiple of 2π, provided that $|\underline{l}|<\mu\cos\left(R_2-R_1\right)$. Moreover, if $2(R_2-R_1)=\pi$, then (3) has at least one classical radial solution for any $l\in C$ with $\underline{l}=0$.*

We give now a second proof of Theorem 4.2, and consider also the classical case. The main idea of this proof comes from [1] and has been used for the classical forced pendulum in [14].

Let $f:[R_1,R_2]\times\mathbb{R}^2\to\mathbb{R}$ be a continuous function and $N_f:C^1\to C$ be the Nemytskii operator associated to f. We first consider the modified problem of finding $(\underline{u},\widehat{u})\in\mathbb{R}\times\widehat{C}^1_\dagger$ such that

$$(r^{N-1}\phi(\widehat{u}'))'=r^{N-1}[N_f(\underline{u}+\widehat{u})-Q\circ N_f(\underline{u}+\widehat{u})]. \tag{24}$$

Lemma 4.3. *If ϕ is singular or classical, and if there exists $\alpha>0$ such that*

$$|f(r,u,v)|\le\alpha\quad\textit{for all}\quad(r,u,v)\in[R_1,R_2]\times\mathbb{R}^2,$$

then the set of solutions of problem (24) contains a continuum $\mathcal{C}$ whose projection on $\mathbb{R}$ is $\mathbb{R}$ and whose projection on $\widehat{C}^1_\dagger$ is contained in

$$B_\phi = \{\widehat{u} \in \widehat{C}^1_\dagger : ||\widehat{u}'||_\infty \leq c_\phi, ||\widehat{u}||_\infty \leq c_\phi(R_2 - R_1)\},$$

where $c_\phi = \max(|\phi^{-1}(\pm 2\alpha R_2/N)|)$.

Proof. Let us consider $\widehat{M} : \mathbb{R} \times \widehat{C}^1_\dagger \to \widehat{C}^1_\dagger$ defined by

$$\widehat{M}(\underline{u}, \widehat{u}) = (I - Q) \circ H \circ \phi^{-1} \circ L \circ (I - Q) \circ N_f(\underline{u} + \widehat{u}).$$

It is not difficult to prove that $\widehat{M}$ is well defined and compact. Moreover, if $(\underline{u}, \widehat{u}) \in \mathbb{R} \times \widehat{C}^1_\dagger$ satisfies $\widehat{u} = \widehat{M}(\underline{u}, \widehat{u})$, then $(\underline{u}, \widehat{u})$ is a solution of (24). On the other hand a simple computation shows that the range of $\widehat{M}$ is contained in B_ϕ (in both of the two cases) and the proof follows now exactly like the proof of Lemma 2.1 in [4]. $\square$

Remark 3. The assumption concerning the boundedness of f can be dropped in the singular case but then

$$B_\phi = \{\widehat{u} \in \widehat{C}^1_\dagger : ||\widehat{u}'||_\infty < a, ||\widehat{u}||_\infty < a(R_2 - R_1)\}.$$

Let $\psi : (-b, b) \to (-c, c)$ be a homeomorphism such that $\psi(0) = 0$ and $0 < b, c \leq \infty$. For $l \in C$ and $\mu > 0$ such that $2(||l||_\infty + \mu)R_2/N < c$ we introduce the notation

$$\rho(\psi) = \max\{|\psi^{-1}(\pm 2(||l||_\infty + \mu)R_2/N)|\}.$$

Theorem 4.4. *If ϕ is singular or classical, $l \in C$, $\mu > 0$ and*

$$2\rho(\phi)(R_2 - R_1) < \pi, \tag{25}$$

then, the Neumann problem (2) has at least two solutions not differing by a multiple of 2π, provided that $|\underline{l}| < \mu \cos[\rho(\phi)(R_2 - R_1)]$.

Proof. Consider the continuous function

$$\Gamma : \mathbb{R} \times \mathcal{C} \to \mathbb{R}, \quad \Gamma(\underline{u}, \widehat{u}) = Q \circ N_f(\underline{u} + \widehat{u}).$$

For any $\widehat{u}_1, \widehat{u}_2$ such that $(-\frac{\pi}{2}, \widehat{u}_1), (\frac{\pi}{2}, \widehat{u}_2) \in \mathcal{C}$, one has that

$$\Gamma(-\frac{\pi}{2}, \widehat{u}_1) > 0 > \Gamma(\frac{\pi}{2}, \widehat{u}_2).$$

Hence, using that $\mathcal{C}$ is a continuum and the continuity of Γ, we deduce the existence of $(\underline{u}, \widehat{u}) \in \mathcal{C}$ such that $-\frac{\pi}{2} < \underline{u} < \frac{\pi}{2}$ and $\Gamma(\underline{u}, \widehat{u}) = 0$. Then, $u = \underline{u} + \widehat{u}$ is a solution of (2). Analogously, (2) has a solution w satisfying $\frac{\pi}{2} < \underline{w} < \frac{3\pi}{2}$. Clearly, $u - w$ is not a multiple of 2π. $\square$

Remark 4. (i) If in (25) one has equality, then we have only existence in Theorem 4.4.

(ii) In Theorem 4.4, if ϕ is singular, then $\rho(\phi) < a$. Hence Theorem 4.2 follows from Theorem 4.4.

The following result is a direct consequence of Lemma 2.5 from [4].

Lemma 4.5. *Let $\psi : (-b, b) \to (-c, c)$ be a homeomorphism such that $\psi(0) = 0$ and $0 < b, c \leq \infty$. Let $\mu > 0$ and $l \in C$ be such that $(||l||_\infty + \mu)R_2/N < c$. If u is a possible solution of the Neumann problem*

$$(r^{N-1}\psi(u'))' + r^{N-1}\mu \sin u = r^{N-1}l(r), \quad u'(R_1) = 0 = u'(R_2) \tag{26}$$

then

$$||u'||_\infty \leq \max\{|\psi^{-1}(\pm(||l||_\infty + \mu)R_2/N)|\}.$$

Theorem 4.6. *Assume that $\psi : \mathbb{R} \to (-c, c)$ is a homeomorphism such that $\psi(0) = 0$ and $0 < c \leq \infty$. If $l \in C$, $\mu > 0$, $2(||l||_\infty + \mu)R_2/N < c$ and*

$$2\rho(\psi)(R_2 - R_1) < \pi,$$

then, the Neumann problem (26) has at least two solutions not differing by a multiple of 2π, provided that

$$|\underline{l}| < \mu \cos[\rho(\psi)(R_2 - R_1)]$$

is satisfied.

Proof. Let $d = \rho(\psi) + 1$ and $b = \rho(\psi) + 2$. Consider $\phi : (-b, b) \to \mathbb{R}$ a singular homeomorphism which coincides with ψ on $[-d, d]$. Then $\rho(\psi) = \rho(\phi)$, and, using Lemma 4.5, we infer that the solutions of (2) coincide with the solutions of (26). Now the result follows from Theorem 4.4 (the singular case). $\square$

Corollary 2. *If (25) is satisfied with $\phi(u) = |u|^{p-2}u$ $(p > 1)$, (resp. $\phi(u) = \frac{u}{\sqrt{1+u^2}}$) then the Neumann problem (5) (resp. (6)) has at least two classical radial solutions not differing by a multiple of 2π for any $l \in C$ with $\underline{l} = 0$ (resp. $l \in C$ with $\underline{l} = 0$ and $2(||l||_\infty + \mu)R_2/N < 1$).*

5. **Norm conditions on the forcing term.** In the proof of the following theorem we adapt to our situation a strategy introduced in Theorem 3 from [19].

Theorem 5.1. *Assume that ϕ is singular and let $\mu > 0$, $R_1 > 0$ in the case $N \geq 2$, and assume that $l \in C$ satisfies*

$$||l||_\infty < \mu.$$

Then problem (2) has at least two solutions not differing by a multiple of 2π. Moreover, if

$$||l||_\infty = \mu,$$

then problem (2) has at least one solution.

Proof. Assume that $||l||_\infty \leq \mu$. Then $\alpha = -\frac{3\pi}{2}$ is a constant lower solution for (2) and $\beta = -\frac{\pi}{2}$ is a constant upper solution for (2) such that $\alpha < \beta$. Hence, using Theorem 4.2 from [4], it follows that (2) has a solution u_1 such that $\alpha \leq u_1 \leq \beta$. Notice that if $||l||_\infty < \mu$, then α, β are strict and $\alpha < u_1 < \beta$.

Now, let us assume that $||l||_\infty < \mu$, let M_μ be the fixed point operator associated to (2), and let

$$\Omega = \Omega_{-\frac{3\pi}{2}, \frac{3\pi}{2}} \setminus (\overline{\Omega}_{-\frac{3\pi}{2}, -\frac{\pi}{2}} \cup \overline{\Omega}_{\frac{\pi}{2}, \frac{3\pi}{2}}). \quad \text{(see Lemma 3.3)}$$

Then using the additivity property of the Leray-Schauder degree and Lemma 3.3, we deduce that

$$d_{LS}[I - M_\mu, \Omega, 0] = 1.$$

Hence, the existence property of the Leray-Schauder degree yields the existence of a solution $u_2 \in \Omega$ of (2). If we assume that $u_2 = u_1 + 2j\pi$ for some $j \in \mathbb{Z}$ then, as $-3\pi/2 < u_1 < -\pi/2$, one has

$$-\frac{3\pi}{2} + 2j\pi < u_2 = u_1 + 2j\pi < -\frac{\pi}{2} + 2j\pi.$$

This leads to one of the contradictions: $u_2 \in \Omega_{\frac{\pi}{2}, \frac{3\pi}{2}}$ if $j = 1$ or $u_2 = u_1 \in \Omega_{-\frac{3\pi}{2}, -\frac{\pi}{2}}$ for $j = 0$. $\square$

Using Lemma 4.5, Theorem 5.1 and arguing exactly as in the proof of Theorem 4.6 with $\rho(\psi)$ replaced by $\max\{|\psi^{-1}(\pm 2\mu R_2/N)|\}$ we obtain the following result.

Theorem 5.2. *Let $\psi : \mathbb{R} \to (-c, c)$ be an increasing homeomorphism such that $\psi(0) = 0$ and $0 < c \leq \infty$. Let also $\mu > 0$, $R_1 > 0$ in the case $N \geq 2$ and $l \in C$ be such that $\frac{2\mu R_2}{N} < c$. If $||l||_\infty < \mu$, then (26) has at least two solutions not differing by a multiple of 2π. If $||l||_\infty = \mu$, then (26) has at least one solution.*

Corollary 3. *Let $\mu > 0$, $R_1 > 0$ and $l \in C$ be such that $\frac{2\mu R_2}{N} < 1$. If $||l||_\infty < \mu$, then the Neumann problem (6) has at least two classical radial solutions not differing by a multiple of 2π. Moreover, if $||l||_\infty = \mu$, then (6) has at least one classical radial solution. The same conclusion holds also for (3) and (5) without the assumption $\frac{2\mu R_2}{N} < 1$.*

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