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An Ambrosetti–Prodi-type result for periodic solutions of the telegraph equation

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Using Leray–Schauder degree theory, a theorem of upper and lower solutions and a strong maximum principle for the telegraph equation we prove an Ambrosetti–Prodi-type result for periodic solutions of the telegraph equation.

1. Introduction and the main result

We consider doubly periodic solutions of the nonlinear telegraph equation

$$\left.\begin{aligned} &u_{tt} - u_{xx} + cu_t + f(t,x,u) = s, \\ &u(t+2\pi, x) = u(t, x+2\pi) = u(t,x), \quad (t,x) \in \mathbb{R}^2, \end{aligned}\right\} \tag{1.1}$$

where $c > 0$, $f : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function 2π-periodic in t and x, s is a real parameter.

Let $\mathbb{T}^2$ be the torus defined by

$$\mathbb{T}^2 = \left(\frac{\mathbb{R}}{2\pi\mathbb{Z}}\right) \times \left(\frac{\mathbb{R}}{2\pi\mathbb{Z}}\right).$$

A point of $\mathbb{T}^2$ is denoted by $(\hat{t}, \hat{x})$, where (t,x) is a point of $\mathbb{R}^2$ and $\hat{t} = t + 2\pi\mathbb{Z}$, $\hat{x} = x + 2\pi\mathbb{Z}$. Doubly periodic functions will be identified with functions defined on the torus. In particular,

$$L^p(\mathbb{T}^2), C(\mathbb{T}^2), C^\infty(\mathbb{T}^2), \dots$$

denote the spaces of doubly periodic functions with the indicated degree of regularity. The norm in $L^p(\mathbb{T}^2)$ is denoted by $\|\cdot\|_p$ and the maximum norm in $C(\mathbb{T}^2)$ is denoted by $\|\cdot\|_\infty$. B_r denotes the open ball of centre 0 and radius r in $C(\mathbb{T}^2)$. By a solution of (1.1) we mean a function $u \in C(\mathbb{T}^2)$ satisfying

$$\int_{\mathbb{T}^2} u(\phi_{tt} - \phi_{xx} - c\phi_t) + \int_{\mathbb{T}^2} (f(t,x,u) - s)\phi = 0 \quad \text{for all } \phi \in C^\infty(\mathbb{T}^2).$$

Suppose that the function f satisfies the following assumptions.

(f1) $f(t,x,u_2) - f(t,x,u_1) \leqslant \nu(c)(u_2 - u_1)$ for all $(t,x) \in \mathbb{R}^2$ and every u_1, u_2 with $u_1 \leqslant u_2$. The constant $\nu(c)$ will be specified later.

*Dedicated to Jean Mawhin on his 65th birthday.

(f2) $f(t, x, u) \to \infty$ if $|u| \to \infty$ uniformly in $(t, x) \in \mathbb{R}^2$.

We are now in a position to state the main result of the paper.

THEOREM 1.1. *If f satisfies conditions (f1) and (f2), then there exists $s_1 \in \mathbb{R}$ such that problem (1.1) has zero, at least one or at least two solutions according to $s < s_1$, $s = s_1$ or $s > s_1$.*

An Ambrosetti–Prodi-type result has been also proved in [2]. More precisely, in [2] the function f has the particular form $f(t, x, u) = g(u) - h(t, x)$, where $h \in L^2(\mathbb{T}^2)$ is such that $\int_{\mathbb{T}^2} h = 0$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following conditions.

(g1) There exist $a, b \in \mathbb{R}$ such that

$$|g(u)| \leqslant a|u| + b \quad \text{for all } u \in \mathbb{R}.$$

(g2) There exists $0 < \alpha < 1$ such that

$$|g(u) - g(v)| \leqslant \frac{\alpha}{2\pi C}|u - v| \quad \text{for all } u, v \in \mathbb{R},$$

where C is the norm of some linear operator.

(g3) $g(u) \to \infty$ if $|u| \to \infty$.

The above conditions are essential in order to use a Lyapunov–Schmidt procedure. To prove theorem 1.1, we adapt a method in [1] (see also [4]) to the present situation. The main tool which will be used in this paper is the Leray–Schauder degree together with a theorem of upper and lower solutions and a strong maximum principle for the telegraph equation proved in [5]. For a short history of the Ambrosetti–Prodi problem, see the introduction in [4]. Note that the Leray–Schauder degree theory, a theorem of upper and lower solutions and a strong maximum principle for the telegraph equation have been already used in order to give a multiplicity result for the forced sine-Gordon equation with periodic boundary conditions [6].

2. Auxiliary results

Consider the set

$$\Gamma = \{m^2 : m \in \mathbb{N}\}.$$

The following result plays an important role in the fixed-point reformulation of the problem (1.1).

LEMMA 2.1 (Ortega and Robles-Pérez [5]). *Assume that $\lambda \notin \Gamma$ and $h \in C(\mathbb{T}^2)$. There then exists $u \in C(\mathbb{T}^2)$, a unique solution of the linear problem*

$$\int_{\mathbb{T}^2} u(\phi_{tt} - \phi_{xx} - c\phi_t - \lambda\phi) = \int_{\mathbb{T}^2} h\phi \quad \textit{for all } \phi \in C^\infty(\mathbb{T}^2). \tag{2.1}$$

This solution satisfies the estimate

$$\|u\|_\infty \leqslant C_1\|h\|_1, \tag{2.2}$$

where C_1 is a constant that depends only on c and λ. Moreover, the linear operator $R_\lambda : C(\mathbb{T}^2) \to C(\mathbb{T}^2)$ defined by $R_\lambda(h) = u$ is a compact operator.

If $\lambda = 0$ and $\int_{\mathbb{T}^2} h = 0$, then (2.1) has a unique solution $u \in C(\mathbb{T}^2)$ such that $\int_{\mathbb{T}^2} u = 0$, which satisfies (2.2).

The following result is a strong maximum principle for periodic solutions of the telegraph equation.

LEMMA 2.2 (Ortega and Robles-Pérez [5]). *There exists a function $\nu : (0,\infty) \to (0,\infty)$ such that, for $-\lambda \in (0, \nu(c)]$ and $h \in C(\mathbb{T}^2)$ with*

$$h \geqslant 0, \qquad \int_{\mathbb{T}^2} h > 0,$$

we have

$$R_\lambda(h)(t,x) > 0 \quad \text{for all } (t,x) \in \mathbb{R}^2.$$

Moreover, the function ν satisfies

$$\frac{c^2}{4} < \nu(c) \leqslant \frac{c^2+1}{4}, \qquad \nu(c) \to 0 \text{ as } c \to 0.$$

A function $\alpha \in C(\mathbb{T}^2)$ is a *lower solution* of (1.1) if the following inequality holds

$$\int_{\mathbb{T}^2} \alpha(\phi_{tt} - \phi_{xx} - c\phi_t) + \int_{\mathbb{T}^2} (f(t,x,\alpha) - s)\phi \leqslant 0 \quad \text{for all } \phi \in C_+^\infty(\mathbb{T}^2). \tag{2.3}$$

An *upper solution* $\beta \in C(\mathbb{T}^2)$ is a function that satisfies the reversed inequality.

LEMMA 2.3 (Ortega and Robles-Pérez [5]). *Assume that f satisfies condition (f1) and that (1.1) has a lower solution $\alpha \in C(\mathbb{T}^2)$ and an upper solution $\beta \in C(\mathbb{T}^2)$ satisfying*

$$\alpha(t,x) \leqslant \beta(t,x) \quad \text{for all } (t,x) \in \mathbb{R}^2.$$

Then (1.1) has a solution $u \in C(\mathbb{T}^2)$ such that

$$\alpha(t,x) \leqslant u(t,x) \leqslant \beta(t,x) \quad \text{for all } (t,x) \in \mathbb{R}^2.$$

3. A fixed-point reformulation and *a priori* estimations

Consider the nonlinear operator $N_s : C(\mathbb{T}^2) \to C(\mathbb{T}^2)$ defined by

$$N_s(u)(t,x) = s - f(t,x,u(t,x)) \quad \text{for all } (t,x) \in \mathbb{R}^2.$$

The operator N_s is continuous and takes bounded sets into bounded sets. Let $\mathcal{G}(s,\cdot) : C(\mathbb{T}^2) \to C(\mathbb{T}^2)$ be the operator

$$\mathcal{G}(s,\cdot) = R_{-\nu} \circ [N_s + \nu I].$$

Henceforth, $\nu = \nu(c)$ and $c > 0$ is fixed. Using the compactness of the linear operator $R_{-\nu}$, it is not difficult to see that the homotopy $\mathcal{G}$ is compact on $[a,b] \times \bar{\Omega}$ whenever $a, b \in \mathbb{R}$ and Ω is an open bounded set in $C(\mathbb{T}^2)$. On the other hand, using lemma 2.1 we deduce the following lemma.

Lemma 3.1. *A function $u \in C(\mathbb{T}^2)$ is a solution of (1.1) if and only if u is a fixed point of $\mathcal{G}(s, \cdot)$, that is $u = \mathcal{G}(s, u)$.*

The following lemma gives an *a priori* bound for the possible solutions of (1.1) for s in compact intervals.

Lemma 3.2. *If f satisfies condition (f2), then for each $b > 0$ there exists $\rho = \rho(b)$ such that any possible solution u of (1.1) with $|s| \leqslant b$ satisfies $\|u\|_\infty < \rho$.*

Proof. Let $|s| \leqslant b$ and $u \in C(\mathbb{T}^2)$ be a solution of (1.1). Using (f2) we deduce that f is bounded from below. This implies that there exists $\delta > 0$ such that

$$|f(t, x, u)| \leqslant f(t, x, u) + \delta \quad \text{for all } (t, x, u) \in \mathbb{R}^3,$$

which, together with

$$\int_{\mathbb{T}^2} (f(t, x, u) - s) = 0, \tag{3.1}$$

implies that

$$\int_{\mathbb{T}^2} |f(t, x, u)| \leqslant 4\pi^2(s + \delta). \tag{3.2}$$

Using (3.1) it follows that, for $h = N_s(u)$, (2.1) has a unique solution $\tilde{u} \in C(\mathbb{T}^2)$ such that $\int_{\mathbb{T}^2} \tilde{u} = 0$ and

$$\|\tilde{u}\|_\infty \leqslant C_1 \int_{\mathbb{T}^2} |f(t, x, u) - s|. \tag{3.3}$$

Using (3.2), (3.3) and the fact that $|s| \leqslant b$, we deduce that

$$\|\tilde{u}\|_\infty \leqslant C_1 4\pi^2(2b + \delta). \tag{3.4}$$

On the other hand, it is clear that we have the decomposition

$$u = \bar{u} + \tilde{u} = \left(\frac{1}{4\pi^2} \int_{\mathbb{T}^2} u\right) + \tilde{u}. \tag{3.5}$$

Using (f2), (3.1), (3.4) and (3.5) it follows that there exists a constant C_2 such that

$$|\bar{u}| < C_2. \tag{3.6}$$

Now (3.4)–(3.6) give the conclusion. □

4. Proof of the main result

Let $S_j = \{s \in \mathbb{R} : \text{(1.1) has at least } j \text{ solutions}\}$, $j \geqslant 1$.

(a) $S_1 \neq \varnothing$. Take

$$s^* > \max_{(t,x) \in \mathbb{R}^2} f(t, x, 0)$$

and use (f2) to find $R_+^* > 0$ such that

$$\min_{(t,x) \in \mathbb{R}^2} f(t, x, R_+^*) > s^*.$$

Then $\alpha \equiv 0$ is a lower solution and $\beta \equiv R_+^*$ is an upper solution of (1.1) with $s = s^*$ such that $\alpha < \beta$. Using lemma 2.3 it follows that $s^* \in S_1$.

(b) If $\tilde{s} \in S_1$ and $s > \tilde{s}$, then $s \in S_1$. Let $\tilde{u}$ be a solution of (1.1) with $s = \tilde{s}$, and let $s > \tilde{s}$. Then $\tilde{u}$ is a lower solution for (1.1). Take $R_+ > \max_{\mathbb{R}^2} \tilde{u}$ such that

$$\min_{(t,x)\in\mathbb{R}^2} f(t,x,R_+) > s.$$

Then $\alpha = \tilde{u}$ is a lower solution and $\beta \equiv R_+$ is an upper solution of (1.1) such that $\alpha < \beta$. From lemma 2.3, $s \in S_1$.

(c) $s_1 = \inf S_1$ is finite and $S_1 \supset (s_1, \infty)$. Let $s \in \mathbb{R}$ and suppose that (1.1) has a solution u. Then (3.1) holds, implying that $s \geqslant \inf_{\mathbb{R}^3} f > -\infty$. To obtain the second part of claim (c), we apply (b).

(d) $S_2 \supset (s_1, \infty)$. Let $s_3 < s_1 < s_2$. For each $s \in \mathbb{R}$, let $\mathcal{G}(s,\cdot)$ be the fixed-point operator in $C(\mathbb{T}^2)$ associated with problem (1.1) and defined in lemma 3.1. Using lemma 3.2 we find ρ such that each possible zero of $I - \mathcal{G}(s,\cdot)$ with $s \in [s_3, s_2]$ satisfies $\|u\|_\infty < \rho$. Consequently, the invariance property of the Leray–Schauder degree implies that

$$d_{\mathrm{LS}}[I - \mathcal{G}(s,\cdot), B_\rho, 0]$$

is well defined and does not depend upon $s \in [s_3, s_2]$ (see [3]).

However, using (c), we see that $u - \mathcal{G}(s_3, u) \neq 0$ for all $u \in C(\mathbb{T}^2)$. This implies that $d_{\mathrm{LS}}[I - \mathcal{G}(s_3,\cdot), B_\rho, 0] = 0$, so that $d_{\mathrm{LS}}[I - \mathcal{G}(s_2,\cdot), B_\rho, 0] = 0$ and, by the excision property of Leray–Schauder degree [3],

$$d_{\mathrm{LS}}[I - \mathcal{G}(s_2,\cdot), B_{\rho'}, 0] = 0 \quad \text{if } \rho' > \rho. \tag{4.1}$$

Let $\hat{s} \in (s_1, s_2)$ and $\hat{u}$ be a solution of (1.1) with $s = \hat{s}$ (using (c)). Using (f2), it follows that there exists a constant $\beta > \max_{\mathbb{R}^2} \hat{u}$ such that

$$\min_{(t,x)\in\mathbb{R}^2} f(t,x,\beta) > s_2. \tag{4.2}$$

Consider the open bounded convex set in $C(\mathbb{T}^2)$ defined by

$$\Omega_{\hat{u},\beta} = \{u \in C(\mathbb{T}^2) : \hat{u} < u < \beta\}.$$

Let $u \in \bar{\Omega}_{\hat{u},\beta}$ and $v = \mathcal{G}(s_2, u)$. Consider $w = \beta - v$. Hence, $w = R_{-\nu}(h)$, where $h \in C(\mathbb{T}^2)$ is defined as

$$h(t,x) := \nu\beta - \nu u(t,x) - s_2 + f(t,x,u(t,x)) \quad \text{for all } (t,x) \in \mathbb{R}^2.$$

Note that (f1), (4.2) and $u \leqslant \beta$ imply that $h > 0$ on $\mathbb{R}^2$. Using lemma 2.2, we deduce that $w > 0$ on $\mathbb{R}^2$, that is $v < \beta$ on $\mathbb{R}^2$. Analogously, we can prove that $\hat{u} < v$ on $\mathbb{R}^2$. Consequently, $v \in \Omega_{\hat{u},\beta}$ and

$$\mathcal{G}(s_2, \bar{\Omega}_{\hat{u},\beta}) \subset \Omega_{\hat{u},\beta}.$$

This implies that

$$d_{\mathrm{LS}}[I - \mathcal{G}(s_2,\cdot), \Omega_{\hat{u},\beta}, 0] = 1. \tag{4.3}$$

Hence, the existence property of Leray–Schauder degree [3] implies that $\mathcal{G}(s_2,\cdot)$ has a fixed point in $\Omega_{\hat{u},\beta}$ which is also a solution of (1.1) with $s = s_2$ (lemma 3.1).

On the other hand, the additivity property of Leray–Schauder degree [3], (4.1) and (4.3) imply, for ρ' sufficiently large, that

$$\begin{gathered} d_{\mathrm{LS}}[I - \mathcal{G}(s_2, \cdot), B_{\rho'} \setminus \bar{\Omega}_{\hat{u},\beta}, 0] = d_{\mathrm{LS}}[I - \mathcal{G}(s_2, \cdot), B_{\rho'}, 0], \\ -d_{\mathrm{LS}}[I - \mathcal{G}(s_2, \cdot), \Omega_{\hat{u},\beta}, 0] = -d_{\mathrm{LS}}[I - \mathcal{G}(s_2, \cdot), \Omega_{\hat{u},\beta}, 0] = -1, \end{gathered}$$

and (1.1) with $s = s_2$ has a second solution in $B_{\rho'} \setminus \bar{\Omega}_{\hat{u},\beta}$.

(e) $s_1 \in S_1$. Let (τ_k) be a sequence in $(s_1, +\infty)$ converging to s_1, and let u_k be a solution of (1.1) with $s = \tau_k$ given by (c). Using lemma 3.1, we deduce that

$$u_k = \mathcal{G}(\tau_k, u_k). \tag{4.4}$$

From lemma 3.2, there exists $\rho > 0$ such that $\|u_k\|_\infty < \rho$ for all $k \geqslant 1$. The compactness of $\mathcal{G}$ implies that, up to a subsequence, the right-hand member of (4.4) converges in $C(\mathbb{T}^2)$, and hence (u_k) converges to some $u \in C(\mathbb{T}^2)$ such that $u = \mathcal{G}(s_1, u)$, i.e. to a solution of (1.1) with $s = s_1$.

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