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Nonlinear Differential Equations
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Multiple periodic solutions of ordinary differential equations with bounded nonlinearities and ϕ-Laplacian

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Abstract. The existence, non-existence and multiplicity of solutions to periodic boundary value problems

$$(\phi(u'))' + g(u) = e(t) + s, \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

is discussed, where $\phi : (-a, a) \to \mathbb{R}$ or $\phi : \mathbb{R} \to (-a, a)$ is an increasing homeomorphism such that $\phi(0) = 0$ and $0 < a \leq \infty$. The nonlinear term g is assumed to be bounded, positive and $g(\pm\infty) = 0$.

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1 Introduction and main results

Consider nonlinear second order differential equations with periodic boundary conditions of the form

$$u'' + g(u) = e(t) + s, \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{1}$$

where $g : \mathbb{R} \to \mathbb{R}$, $e : [0, T] \to \mathbb{R}$ are continuous functions and $s \in \mathbb{R}$ is a parameter. Assume that the following assumptions are satisfied.

(H1) $\int_0^T e(t)dt = 0$.

(H2) $g(u) > 0$ for all $u \in \mathbb{R}$.

(H3) $\lim_{u\to\pm\infty} g(u) = 0$.

(H4) There is $M > 0$ such that $|G(u)| \leq M$ for all $u \in \mathbb{R}$, where $G(u) = \int_0^u g(t)dt$.

The main result in [8] is the following one, proved using critical point theory and the method of upper and lower solutions.

Theorem *If conditions (H1)-(H4) hold, there exists $s^*(e) > 0$ such that problem (1) has at least one solution if and only if $s \in (0, s^*(e)]$.*

In this paper we consider periodic boundary value problems of the form

$$(\phi(u'))' + g(u) = e(t) + s, \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{2}$$

where $\phi : (-a, a) \to \mathbb{R}$ or $\phi : \mathbb{R} \to (-a, a)$ is an increasing homeomorphism such that $\phi(0) = 0$ and $0 < a \leq +\infty$. By solution of (2) we mean a function $u \in C^1([0, T])$ such that $\phi \circ u' \in C^1([0, T])$ and which verifies (2).

We first consider the case where $\phi : (-a, a) \to \mathbb{R}$. When $a = +\infty$, a typical example is the p-Laplacian operator for which $\phi(v) = |v|^{p-2}v$ for some $p > 1$. Ward's situation corresponds to $p = 2$. When $a < +\infty$, a typical example is $\phi(v) = \frac{v}{\sqrt{1-v^2}}$, associated to the acceleration in special relativity. We prove the following result.

Theorem 1 *If $\phi : (-a, a) \to \mathbb{R}$ with $0 < a \leq +\infty$ and conditions (H1)-(H3) hold, there exists $s^*(e) \in (0, \sup_R g]$ such that problem (2) has zero, at least one or at least two solutions according to $s \notin (0, s^*(e)]$, $s = s^*(e)$ or $s \in (0, s^*(e))$.*

This theorem not only extends Ward's one to a more general class of problems, but, even in the classical case, improves it by suppressing Assumption (H4) and refining the conclusion.

In the case where $\phi : \mathbb{R} \to (-a, a)$ with $0 < a < +\infty$, for which a typical example is the curvature operator associated to $\phi(v) = \frac{v}{\sqrt{1+v^2}}$, Theorem 1 together with some a priori estimates imply the following result, where the usual maximum norm is denoted by $||\cdot||_\infty$.

Theorem 2 *If $\phi : \mathbb{R} \to (-a, a)$ with $0 < a < +\infty$, if conditions (H1)-(H3) hold and if*

$$||e||_\infty \leq ||g||_\infty < \frac{a}{2T},$$

there exists $s^*(e) \in (0, \sup_R g]$ such that problem (2) has zero, at least one or at least two solutions according to $s \notin (0, s^*(e)]$, $s = s^*(e)$ or $s \in (0, s^*(e))$.

In contrast to Ward's approach, we use Leray-Schauder degree techniques [4]. The existence of a second solution is proved using an idea from [5], which combines degree and lower and upper solutions. We use also the existence of a continuum of solutions for a periodic boundary value problem associated to (2). The idea of such a continuum comes from [1].

2 Notation and auxiliary results

In this section $\phi : (-a, a) \to \mathbb{R}$ denotes an increasing homeomorphism such that $\phi(0) = 0$ and $0 < a \leq +\infty$.

Let C denote the Banach space of continuous functions on $[0, T]$ endowed with the uniform norm $\| \cdot \|_\infty$, C^1 denote the Banach space of continuously differentiable functions on $[0, T]$, equipped with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$. We consider its closed subspace

$$C^1_\# \ = \ \{u \in C^1 : u(0) = u(T), u'(0) = u'(T)\},$$

and denote corresponding open balls of center 0 and radius r by B_r. We denote by $P, Q : C \to C$ the continuous projectors defined by

$$P, Q : C \to C, \quad Pu(t) = u(0), \quad Qu(t) = \frac{1}{T} \int_0^T u(\tau)\, d\tau \quad (t \in [0, T]),$$

and define the continuous linear operator $H : C \to C^1$ by

$$Hu(t) = \int_0^t u(\tau)\, d\tau \quad (t \in [0, T]).$$

If $u \in C$, we write

$$u^+ = \max\{u, 0\}, \quad u^- = \max\{-u, 0\}.$$

A technical result from [2, 3] is needed for the construction of the equivalent fixed point problems.

Proposition 1 *For each $h \in C$, there exists a unique $\alpha := Q_\phi(h) \in$ Range h such that*

$$\int_0^T \phi^{-1}(h(t) - \alpha)\, dt = 0.$$

Moreover, the function $Q_\phi : C \to \mathbb{R}$ is continuous.

The following fixed point reformulation of periodic boundary value problems like (2) is taken from [2, 3].

Proposition 2 *Assume that $F : C^1 \to C$ is continuous and takes bounded sets into bounded sets. Then u is a solution of the abstract periodic problem*

$$(\phi(u'))' = F(u), \quad u(0) - u(T) = 0 = u'(0) - u'(T)$$

if and only if $u \in C^1_\#$ is a fixed point of the operator $M_\#$ defined on $C^1_\#$ by

$$M_\#(u) = Pu + QF(u) + H \circ \phi^{-1} \circ (I - Q_\phi) \circ [H(I - Q)F](u).$$

Furthermore, $M_\#$ is completely continuous on $C^1_\#$.

Let us decompose any $u \in C^1_\#$ in the form

$$u = \overline{u} + \widetilde{u} \quad (\overline{u} = u(0), \quad \widetilde{u}(0) = 0),$$

and let

$$\widetilde{C^1_\#} = \{u \in C^1_\# : u(0) = 0\}.$$

Lemma 1 *Assume that $F : C^1 \to C$ is continuous and has bounded range. Then, $(\overline{u}, \widetilde{u}) \in \mathbb{R} \times \widetilde{C^1_\#}$ is a solution of the problem*

$$(\phi(\widetilde{u}'))' = F(\overline{u} + \widetilde{u}) - \frac{1}{T}\int_0^T F(\overline{u} + \widetilde{u})(t)\, dt \tag{3}$$

if and only if

$$\widetilde{u} = H \circ \phi^{-1} \circ (I - Q_\phi) \circ [H(I - Q)F](\overline{u} + \widetilde{u}) =: \widetilde{F}(\overline{u}, \widetilde{u}).$$

Furthermore, the set of the solutions $(\overline{u}, \widetilde{u}) \in \mathbb{R} \times \widetilde{C^1_\#}$ of problem (3) contains a continuum $\mathcal{C}$ whose projection on $\mathbb{R}$ is $\mathbb{R}$ and projection on $\widetilde{C^1_\#}$ is contained in an open ball B_ρ.

Proof. The fixed point reformulation of (3) follows from Proposition 2. Let $m \in \mathbb{R}$ be a constant such that

$$||F(u)||_\infty \leq m \quad \text{for all} \quad u \in C^1.$$

It follows that there is $\rho > 0$ such that for each $(\overline{u}, \widetilde{u}) \in \mathbb{R} \times \widetilde{C^1_\#}$, we have

$$\|\widetilde{F}(\overline{u}, \widetilde{u})\| = \|\widetilde{F}(\overline{u}, \widetilde{u})\|_\infty + \|(\widetilde{F}(\overline{u}, \widetilde{u}))'\|_\infty < \rho. \tag{4}$$

It follows from (4) that, for each $\overline{u} \in \mathbb{R}$, any possible fixed point $\widetilde{u}$ of $\widetilde{F}(\overline{u}, \cdot)$ is such that

$$\|\widetilde{u}\| < \rho. \tag{5}$$

Furthermore, for each $\lambda \in [0, 1]$, each possible fixed point $\widetilde{u}$ of

$$\widetilde{\mathcal{F}}(\lambda, \widetilde{u}) := H \circ \phi^{-1} \circ (I - Q_\phi) \circ [\lambda H(I - Q)F](\widetilde{u})$$

satisfies, for the same reasons, inequality (5), which implies that

$$\begin{aligned} d_{LS}[I - \widetilde{F}(0, \cdot), B_\rho, 0] &= d_{LS}[I - \widetilde{\mathcal{F}}(1, \cdot), B_\rho, 0] \\ &= d_{LS}[I - \widetilde{\mathcal{F}}(0, \cdot), B_\rho, 0] = d_{LS}[I, B_\rho, 0] = 1. \end{aligned} \tag{6}$$

Relations (5), (6) and Theorem 1.2 in [6] or Lemma 2.3 in [7] then imply the existence of $\mathcal{C}$. $\square$

We finally need some concepts and results from the method of lower and upper solutions taken from [2, 3]. Consider the periodic boundary value problem

$$(\phi(u'))' = f(t, u), \quad u(0) - u(T) = 0 = u'(0) - u'(T). \tag{7}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Definition 1 A strict lower solution α (resp. *strict upper solution* β) of (7) is a function $\alpha \in C^1$ such that $\|\alpha'\|_\infty < a$, $\phi(\alpha') \in C^1$, $\alpha(0) = \alpha(T)$, $\alpha'(0) \geq \alpha'(T)$ (resp. $\beta \in C^1$, $\|\beta'\|_\infty < a$, $\phi(\beta') \in C^1$, $\beta(0) = \beta(T)$, $\beta'(0) \leq \beta'(T)$) and

$$(\phi(\alpha'(t)))' > f(t, \alpha(t)) \quad (resp. \quad (\phi(\beta'(t)))' < f(t, \beta(t)))$$

for all $t \in [0, T]$.

Proposition 3 *If (7) has a strict lower solution α and a strict upper solution β such that $\alpha(t) \leq \beta(t)$ for all $t \in [0, T]$, then problem (7) has a solution u such that $\alpha(t) < u(t) < \beta(t)$ for all $t \in [0, T]$. Moreover,*

$$d_{LS}[I - M^f_\#, \Omega^r_{\alpha,\beta}, 0] = -1,$$

where

$$\Omega^r_{\alpha,\beta} = \{u \in C^1_\# : \alpha(t) < u(t) < \beta(t) \quad \text{for all} \quad t \in [0, T], \quad \|u'\|_\infty < r\},$$

$M^f_\#$ is the fixed point operator associated to (7) and r is sufficiently large.

3 Proof of Theorem 1

Let us return to problem (2). Consider

$$S_j = \{s \in \mathbb{R} : (2) \text{ has at least j solutions } \} \quad (j \geq 1).$$

Lemma 2 *If $s \in S_1$, then $0 < s \leq ||g||_\infty$.*

Proof. Assumptions (H2) and (H3) imply that g is bounded and $0 < g(u) \leq ||g||_\infty$ for all $u \in \mathbb{R}$. Hence, if u is a solution of (2) then, using (H1), it follows that

$$\frac{1}{T} \int_0^T g(u(t))dt = s, \tag{8}$$

and $0 < s \leq ||g||_\infty$. □

For $s \in \mathbb{R}$, we define the continuous nonlinear operator $N_s : C^1 \to C$ by

$$N_s(u)(t) = e(t) + s - g(u(t)) \quad (t \in [0, T]).$$

Using Proposition 2, it follows that $u \in C^1_\#$ is a solution of (2) if and only if

$$u = Pu + QN_s(u) + H \circ \phi^{-1} \circ (I - Q_\phi) \circ [H(I-Q)N_s](u) =: \mathcal{G}(s,u),$$

and the nonlinear operator $\mathcal{G}(s,\cdot) : C^1_\# \to C^1_\#$ is completely continuous.

Let $M : C^1 \to C$ be the continuous mapping with bounded range defined by

$$M(u)(t) = e(t) - g(u(t)) \quad (t \in [0,T]),$$

and $\widetilde{M} : \mathbb{R} \times \widetilde{C^1_\#} \to \widetilde{C^1_\#}$ be the completely continuous operator defined by

$$\widetilde{M}(\overline{u}, \widetilde{u}) = H \circ \phi^{-1} \circ (I - Q_\phi) \circ [H(I-Q)M](\overline{u} + \widetilde{u}).$$

If u is a solution of (2), then (8) holds and $\widetilde{u} = \widetilde{M}(\overline{u}, \widetilde{u})$. Reciprocally, if $(\overline{u}, \widetilde{u}) \in \mathbb{R} \times \widetilde{C^1_\#}$ is such that $\widetilde{u} = \widetilde{M}(\overline{u}, \widetilde{u})$, then $u = \overline{u} + \widetilde{u}$ is a solution of (2) with $s = \frac{1}{T}\int_0^T g(u(t))dt$.

Using Lemma 1, we deduce the following useful result.

Lemma 3 *The set of the solutions $(\overline{u}, \widetilde{u}) \in \mathbb{R} \times \widetilde{C^1_\#}$ of problem*

$$(\phi(\widetilde{u}'))' + g(\overline{u} + \widetilde{u}) = e(t) + \frac{1}{T}\int_0^T g(\overline{u} + \widetilde{u}(t))\,dt \tag{9}$$

contains a continuum $\mathcal{C}$ whose projection on $\mathbb{R}$ is $\mathbb{R}$ and projection on $\widetilde{C^1_\#}$ is contained in an open ball B_ρ.

Let $\gamma : \mathbb{R} \times \widetilde{C^1_\#} \to \mathbb{R}$ be the continuous function defined by

$$\gamma(\overline{u}, \widetilde{u}) = \frac{1}{T}\int_0^T g(\overline{u} + \widetilde{u}(t))\,dt.$$

Lemma 4 $S_1 \neq \varnothing$.

Proof. Let $(\overline{u}, \widetilde{u}) \in \mathcal{C}$. Then $u = \overline{u} + \widetilde{u}$ is a solution of (2) with $s = \gamma(\overline{u}, \widetilde{u})$. □

Let us consider

$$s^*(e) = \sup S_1.$$

Lemma 5 *We have that* $0 < s^*(e) \leq ||g||_\infty$ *and* $s^*(e) \in S_1$.

Proof. The first assertion follows from Lemma 2. Let $\{s_n\}$ be a sequence belonging to S_1 which converges to $s^*(e)$. Let $u_n = \overline{u}_n + \widetilde{u}_n$ be a solution of (2) with $s = s_n = \gamma(\overline{u}_n, \widetilde{u}_n)$. It follows that $\widetilde{u}_n = \widetilde{M}(\overline{u}_n, \widetilde{u}_n)$ and $\{\widetilde{u}_n\}$ belongs to B_ρ. Hence, if up to a subsequence $\overline{u}_n \to \pm\infty$, then using (H3) it follows that $\gamma(\overline{u}_n, \widetilde{u}_n) \to 0$, which means that $s^*(e) = 0$, contradiction. We have proved that $\{(\overline{u}_n, \widetilde{u}_n)\}$ is a bounded sequence in $\mathbb{R} \times \widetilde{C^1_\#}$. Because $\widetilde{M}$ is completely continuous, we can assume, passing to a subsequence, that $\widetilde{M}(\overline{u}_n, \widetilde{u}_n) \to \widetilde{u}$ and $\overline{u}_n \to \overline{u}$. We deduce that $\widetilde{u} = \widetilde{M}(\overline{u}, \widetilde{u})$, $\gamma(\overline{u}, \widetilde{u}) = s^*(e)$ and u is a solution of (2) with $s = s^*(e)$. □

Arguing as in the proof of Lemma 5 we deduce the following a priori estimate result.

Lemma 6 *Let* $0 < s_1 < s^*(e)$. *Then, there is* $\rho' > 0$ *such that any possible solution* u *of (2) with* $s \in [s_1, s^*(e)]$ *belongs to* $B_{\rho'}$.

Lemma 7 *We have* $(0, s^*(e)) \subset S_2$.

Proof. Let $s_1, s_2 \in \mathbb{R}$ such that $0 < s_1 < s^*(e) < s_2$. Using Lemma 2, Lemma 6 and the invariance property of Leray-Schauder degree, it follows that there is $\rho' > 0$ sufficiently large such that $d_{LS}[I - \mathcal{G}(s, \cdot), B_{\rho'}, 0]$ is well defined and independent of $s \in [s_1, s_2]$. However, using Lemma 2 we deduce that $u - \mathcal{G}(s_2, u) \neq 0$ for all $u \in C^1_\#$. This implies that $d_{LS}[I - \mathcal{G}(s_2, \cdot), B_{\rho'}, 0] = 0$, so that $d_{LS}[I - \mathcal{G}(s_1, \cdot), B_{\rho'}, 0] = 0$ and, by excision property of Leray-Schauder degree,

$$d_{LS}[I - \mathcal{G}(s_1, \cdot), B_{\rho''}, 0] = 0 \quad \text{if} \quad \rho'' \geq \rho'. \tag{10}$$

Let u_* be a solution of (2) with $s = s^*(e)$ (using Lemma 5). Then, u_* is a strict lower solution of (2) with $s = s_1$. Using Lemma 3 and (H3), there is $(\overline{u}^*, \widetilde{u}^*) \in \mathcal{C}$ such that $u^* = \overline{u}^* + \widetilde{u}^* > u_*$ on $[0, T]$ and $\gamma(\overline{u}^*, \widetilde{u}^*) < s_1$. It follows that u^* is an upper solution of (2) with $s = s_1$. So, using Proposition 3, we have that

$$d_{LS}[I - \mathcal{G}(s_1, \cdot), \Omega^r_{u_*, u^*}, 0] = -1, \tag{11}$$

for some $r > 0$, and (2) has a solution in $\Omega^r_{u_*, u^*}$. Taking ρ'' sufficiently large and using (10) and (11), we deduce from the additivity property of Leray-Schauder degree that

$$\begin{aligned} d_{LS}[I - \mathcal{G}(s_1, \cdot), B_{\rho''} \backslash \overline{\Omega^r}_{u_*, u^*}, 0] &= d_{LS}[I - \mathcal{G}(s_1, \cdot), B_{\rho''}, 0] \\ -d_{LS}[I - \mathcal{G}(s_1, \cdot), \Omega^r_{u_*, u^*}, 0] &= -d_{LS}[I - \mathcal{G}(s_1, \cdot), \Omega^r_{u_*, u^*}, 0] = 1, \end{aligned}$$

and (2) with $s = s_1$ has a second solution in $B_{\rho''} \setminus \overline{\Omega^r}_{u_*, u^*}$. □

Proof of Theorem 1. The conclusion of Theorem 1 follows from Lemmas 2, 5 and 7. □

Remark 1 When $\phi : (-a, a) \to \mathbb{R}$ and $0 < a < +\infty$, it follows from the version of the method of lower and upper solutions given in [3] that the proof given here remains valid for the more general problem

$$(\phi(u'))' + g(u, u') = e(t) + s, \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

when (H2) and (H3) are replaced by

(H2') $g(u,v) > 0$ for all $(u,v) \in \mathbb{R}^2$.

(H3') $\lim_{u \to \pm\infty} g(u,v) = 0$ uniformly in $v \in (-a,a)$.

Example 1 Let e be a continuous function on $[0,T]$ such that $\int_0^T e(t)dt = 0$. If $b > 0$ then, from Theorem 1, there is $s^* > 0$ such that the periodic boundary value problem

$$(|u'|^{p-2}u')' + \frac{b}{1+|u|} = e(t) + s, \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has zero, at least one or at least two solutions according to $s \notin (0, s^*]$, $s = s^*$ or $s \in (0, s^*)$.

Notice that, even for $p = 2$, Ward result does not apply because (H4) is not satisfied.

Example 2 Let e be a continuous function on $[0,T]$ such that $\int_0^T e(t)dt = 0$. If $b > 0$ and $c \geq 0$ then, from Remark 1, there is $s^* > 0$ such that the periodic boundary value problem

$$\left(\frac{u'}{\sqrt{1-u^2}}\right)' + \frac{cu'^4 + b}{1+|u|} = e(t) + s, \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has zero, at least one or at least two solutions according to $s \notin (0, s^*]$, $s = s^*$ or $s \in (0, s^*)$.

4 Proof of Theorem 2

To deal with the case where $\phi : \mathbb{R} \to (-a,a)$ $(0 < a < +\infty)$ is an increasing homeomorphism such that $\phi(0) = 0$ and prove Theorem 2, we use a modification argument introduced in [2], which requires the obtention of some a priori bounds for the possible solutions.

Lemma 8 *Let $\psi : \mathbb{R} \to (-a,a)$ be an increasing homeomorphism such that $\psi(0) = 0$ and $0 < a \leq \infty$. Consider the periodic boundary value problem*

$$(\psi(u'))' + g(u) = e(t) + s, \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{12}$$

where $g : \mathbb{R} \to \mathbb{R}$, $e : [0,T] \to \mathbb{R}$ are continuous functions and $s \in \mathbb{R}$ is a parameter. Assume that (H1)-(H3) hold and that

$$||e||_\infty \leq ||g||_\infty < \frac{a}{2T}. \tag{13}$$

Then, any possible solution of (12) satisfies the a priori estimate

$$||u'||_\infty \leq m := \max\{|\psi^{-1}(\pm 2T||g||_\infty)|\}.$$

Proof. Let u be a solution of (12). As in Lemma 2 we have that (8) holds and $0 < s \leq ||g||_\infty$. It follows that we have

$$e(t) - g(u) + s \leq e(t) + ||g||_\infty \quad \text{for all} \quad (t,u) \in [0,T] \times \mathbb{R},$$

which together with (13) imply that

$$\begin{aligned} |e(t) - g(u) + s| &\leq e(t) - g(u) + s + 2[e(t) + ||g||_\infty]^+ \\ &= 3e(t) - g(u) + s + 2||g||_\infty \end{aligned} \tag{14}$$

for all $(t,u) \in [0,T] \times \mathbb{R}$. Using (8), (12), (H1) and (14) it follows that

$$\int_0^T |(\psi(u'(t)))'|dt \leq 2T||g||_\infty. \tag{15}$$

Because $u \in C^1$ is such that $u(0) = u(T)$, there exists $\xi \in [0,T]$ such that $u'(\xi) = 0$, which implies $\psi(u'(\xi)) = 0$ and

$$\psi(u'(t)) = \int_\xi^t (\psi(u'(t)))'\, dt \quad (t \in [0,T]).$$

Using the equality above and (15) we have that

$$|\psi(u'(t))| \leq 2T||g||_\infty \quad (t \in [0,T]),$$

and hence $\|u'\|_\infty \leq m$. □

Proof of Theorem 2. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a homeomorphism which coincides with ϕ on $[-(m+1), m+1]$. Using Lemma 8 it follows that u is a solution of (2) if and only if u is a solution of (12). Now the result follows from Theorem 1. □

Example 3 Let e be a continuous function on $[0,T]$ such that $\int_0^T e(t)dt = 0$. If $||e||_\infty \leq b < \frac{1}{2T}$ then, from Theorem 2, there is $s^* > 0$ such that the periodic boundary value problem

$$\left(\frac{u'}{\sqrt{1+u^2}}\right)' + b\exp(-u^2) = e(t) + s, \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has zero, at least one or at least two solutions according to $s \notin (0, s^*]$, $s = s^*$ or $s \in (0, s^*)$.

Added in proof. When this paper was under printing, we learned that the multiplicity result for the special case of equation (1) had already been proved by J. Angel Cid and L. Sanchez in *J. Math. Anal. Appl.* **288** (2003), 349–364, using a combinatin of Ward's existence result and lower-upper solutions technique of De Coster-Tarallo.

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