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Boundary value problems for some nonlinear systems with singular ϕ-laplacian

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Cordially dedicated to Felix Browder for his eightieth birthday anniversary

Abstract. Systems of differential equations of the form

$$(\phi(u'))' = f(t, u, u')$$

with ϕ a homeomorphism of the ball $B_a \subset \mathbb{R}^n$ onto $\mathbb{R}^n$ are considered, under various boundary conditions on a compact interval $[0, T]$. For non-homogeneous Cauchy, terminal and some Sturm–Liouville boundary conditions including in particular the Dirichlet–Neumann and Neumann–Dirichlet conditions, existence of a solution is proved for arbitrary continuous right-hand sides f. For Neumann boundary conditions, some restrictions upon f are required, although, for Dirichlet boundary conditions, the restrictions are only upon ϕ and the boundary values. For periodic boundary conditions, both ϕ and f have to be suitably restricted. All the boundary value problems considered are reduced to finding a fixed point for a suitable operator in a space of functions, and the Schauder fixed point theorem or Leray–Schauder degree are used. Applications are given to the relativistic motion of a charged particle in some exterior electromagnetic field.

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1. Introduction

In this note, we extend some existence results given in [2] for Dirichlet and Neumann problems for scalar quasilinear equations of the form

$$(\phi(u'))' = f(t, u, u'),$$

when $\phi :]-a, a[\to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$, to the case of systems of such equations, and other boundary conditions.

For the *Cauchy boundary conditions on* $[0, T]$

$$u(0) = A, \quad \phi[u'(0)] = B,$$

the *terminal boundary conditions on* $[0, T]$

$$u(T) = A, \quad \phi[u'(T)] = B,$$

and the mixed *Sturm–Liouville boundary conditions*

$$\mathcal{C}u(0) - \mathcal{D}\phi[u'(0)] = A, \quad \mathcal{E}u(T) - \mathcal{F}\phi[u'(T)] = B,$$

with the $(n \times n)$-matrices $\mathcal{C} = \mathcal{I}$, $\mathcal{D} = \mathcal{O}$ and $\mathcal{F}$ invertible, or $\mathcal{E} = \mathcal{I}$, $\mathcal{F} = \mathcal{O}$ and $\mathcal{D}$ invertible, we prove the existence of at least one solution for a *general homeomorphism* ϕ and *any* right-hand member (f, A, B) (Theorems 1–4). This contains as special cases the *Dirichlet–Neumann boundary conditions* ($\mathcal{C} = -\mathcal{F} = \mathcal{I}$, $\mathcal{D} = \mathcal{E} = \mathcal{O}$),

$$u(0) = A, \quad \phi[u'(T)] = B$$

and the *Neumann–Dirichlet boundary conditions* ($\mathcal{C} = \mathcal{F} = \mathcal{O}$, $\mathcal{E} = -\mathcal{D} = \mathcal{I}$),

$$\phi[u'(0)] = A, \quad u(T) = B$$

(Corollaries 1–4).

For the *Neumann boundary conditions*

$$\phi[u'(0)] = A, \quad \phi[u'(T)] = B,$$

we prove the existence of at least one solution for a *general homeomorphism* ϕ and some conditions upon the right-hand member (f, A, B) (Theorem 5).

For the *Dirichlet boundary conditions*

$$u(0) = A, \quad u(T) = B,$$

we prove the existence of at least one solution for *homeomorphisms* $\phi = \nabla\Phi$, with $\Phi : B_a \to \,]-\infty, 0]$ *strictly convex*, *any* f and A, B such that $|A - B| < aT$ (Theorem 6).

Finally, for the *periodic boundary conditions*

$$u(0) - u(T) = 0 = u'(0) - u'(T),$$

we prove the existence of at least one solution for the class of homeomorphisms $\phi = \nabla\Phi$ of the Dirichlet case, and the class of f introduced in the homogeneous Neumann case (Theorem 7).

In all cases, fixed point problems equivalent to the various boundary value problems are constructed, and studied using the Schauder fixed point theorem or Leray–Schauder degree. The reason in the difference of assumptions for ϕ between the Dirichlet or periodic boundary conditions and the other ones lies in the fact that the construction of the fixed point operator in those cases requires the unique solvability of some finite-dimensional system, which is proved using convex analysis (Lemma 2). It is likely that the class of admissible ϕ could be increased by using monotone operator theory [4].

Applications are given to 3-dimensional systems of the form

$$\left(\frac{\mathbf{r}'(t)}{\sqrt{1-|\mathbf{r}'(t)|^2}}\right)' = -\frac{e^2}{\hbar c}[\mathbf{E}^{\rm ext}(t,\mathbf{r}(t)) + \mathbf{r}'(t)\times\mathbf{B}^{\rm ext}(t,\mathbf{r}(t))],$$

which is a good approximation for the motion of a charged particle with high speed but gentle acceleration (Theorem 8). See [3, 6], and also [7] for the case of constant $\mathbf{E}^{\rm ext}$ and $\mathbf{H}^{\rm ext}$.

2. Notations and general assumptions

In $\mathbb{R}^n$, we denote the usual inner product by $\langle\cdot,\cdot\rangle$ and the corresponding Euclidian norm by $|\cdot|$. We denote the usual norm in $L^1[(0,T),\mathbb{R}^n]$ by $\|\cdot\|_1$. Let C denote the Banach space of continuous functions from $[0,T]$ into $\mathbb{R}^n$, endowed with the uniform norm $\|\cdot\|_\infty$, C^1 the Banach space of continuously differentiable functions from $[0,T]$ into $\mathbb{R}^n$, equipped with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$, and B_r the open ball of centre 0 and radius r in any normed space.

We introduce the continuous linear operator $H : C \to C^1$ defined by

$$Hu(t) = \int_0^t u(s)\,ds \quad (t\in[0,T]), \tag{1}$$

and the continuous linear operator $K : C \to C^1$ defined by

$$Ku(t) = -\int_t^T u(s)\,ds \quad (t\in[0,T]). \tag{2}$$

We will use the following general assumptions:

(H_ϕ) *ϕ is a homeomorphism from $B_a\subset\mathbb{R}^n$ onto $\mathbb{R}^n$.*
(H_F) *$F : C^1 \to C$ is continuous and takes bounded sets into bounded sets.*
(H_f) *$f : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous.*

To such a continuous function f, we associate its *Nemytskiĭ operator* $N_f : C^1 \to C$ defined by

$$N_f(u)(t) = f(t,u(t),u'(t)) \quad (t\in[0,T]). \tag{3}$$

It is easy to show that N_f is continuous and takes bounded sets into bounded sets.

3. Cauchy and terminal boundary conditions

Given $A, B\in\mathbb{R}^n$, let us consider the *Cauchy problem on* $[0,T]$

$$(\phi(u'))' = f(t,u,u') \quad (t\in[0,T]), \qquad u(0)=A, \quad \phi[u'(0)] = B. \tag{4}$$

Notice that the initial conditions can be written in the more classical form

$$u(0)=A, \quad u'(0) = C\in B_a,$$

with $C=\phi^{-1}(B)$.

The following result shows that the Cauchy problem on $[0,T]$ is solvable for *any* f, A and B.

Theorem 1. *If Assumptions* (H_ϕ) *and* (H_f) *hold, then problem* (4) *has at least one solution.*

Proof. Problem (4) is equivalent to

$$\phi(u') = B + HN_f(u), \quad u(0) = A,$$

i.e. to

$$u' = \phi^{-1} \circ [B + HN_f(u)], \quad u(0) = A,$$

i.e. to the fixed point problem

$$u = A + H \circ \phi^{-1} \circ [B + HN_f(u)] =: M_C(u).$$

It is easy to see that M_C is a completely continuous mapping of C^1 into itself, and that, for each $u \in C^1$,

$$\|(M_C(u))'\|_\infty = \|\phi^{-1} \circ [B + HN_f(u)]\|_\infty < a.$$

Hence,

$$\|M_C(u)\|_\infty \leq |A| + T\|M_C'(u)\|_\infty < |A| + aT,$$

and M_C maps C^1 into its ball $B_{|A|+a(T+1)}$. The existence of a fixed point follows from the Schauder fixed point theorem. □

Theorem 1 implies in particular that if f is defined for all $t \in \mathbb{R}$, the solution of the Cauchy problem never explodes in finite time.

Remark 1. Problem (4) can be written in the equivalent normal form

$$u' = \phi^{-1}(v), \quad v' = f[t, u, \phi^{-1}(v)], \quad u(0) = A, \quad v(0) = B, \tag{5}$$

from which one can deduce in a standard way that *the Cauchy problem* (4) *is locally uniquely solvable if* ϕ^{-1} *and* $f(t,\cdot,\cdot)$ *are locally Lipschitzian.*

A similar existence result holds for the *terminal problem on* $[0,T]$

$$(\phi(u'))' = f(t,u,u') \quad (t \in [0,T]), \quad u(T) = A, \quad \phi[u'(T)] = B. \tag{6}$$

Theorem 2. *If Assumptions* (H_ϕ) *and* (H_f) *hold, then problem* (6) *has at least one solution.*

Proof. Problem (6) is equivalent to

$$\phi(u') = B + KN_f(u), \quad u(T) = A,$$

i.e. to

$$u' = \phi^{-1} \circ [B + KN_f(u)], \quad u(T) = A,$$

i.e. to the fixed point problem

$$u = A + K \circ \phi^{-1} \circ [B + KN_f(u)] =: M_T(u).$$

Again M_T is a completely continuous mapping of C^1 into itself, which maps C^1 into $B_{|A|+a(T+1)}$. The existence of a fixed point follows from the Schauder fixed point theorem. □

4. Some Sturm–Liouville boundary conditions

Given $A, B \in \mathbb{R}^n$, and $(n \times n)$-matrices $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$, let us consider the *Sturm–Liouville problem*

$$\begin{aligned} &(\phi(u'))' = f(t, u, u') \quad (t \in [0, T]), \\ &\mathcal{C}u(0) - \mathcal{D}\phi[u'(0)] = A, \quad \mathcal{E}u(T) - \mathcal{F}\phi[u'(T)] = B. \end{aligned} \tag{7}$$

Let us introduce the following assumptions:

(SL_1) *$\mathcal{C}$ and $\mathcal{F}$ are invertible,*
(SL_2) *$\mathcal{D}$ and $\mathcal{E}$ are invertible,*

and define $\Omega_{j+1} := \{u \in C^1 : |u'(jT)| < a\}$ $(j = 0, 1)$.

Proposition 1. *If Assumptions (H_ϕ), (H_f) and (SL_1) hold, then u is a solution to problem (7) if and only if $u \in \Omega_1$ is a fixed point of the operator $M_1 : \Omega_1 \to C^1$ defined by*

$$M_1(u) := \mathcal{C}^{-1}\{\mathcal{D}\phi[u'(0)] + A\} + H \circ \phi^{-1} \circ \{\mathcal{F}^{-1}[\mathcal{E}u(T) - B] + KN_f(u)\}.$$

If Assumptions (H_ϕ), (H_f) and (SL_2) hold, then u is a solution to problem (7) if and only if $u \in \Omega_2$ is a fixed point of the operator $M_2 : \Omega_2 \to C^1$ defined by

$$M_2(u) := \mathcal{E}^{-1}\{\mathcal{F}\phi[u'(T)] + B\} + K \circ \phi^{-1} \circ \{\mathcal{D}^{-1}[\mathcal{C}u(0) - A] + HN_f(u)\}.$$

Furthermore, M_1 and M_2 are completely continuous, and

$$\|(M_j)'(u)\|_\infty < a \tag{8}$$

for all $u \in \Omega_j$ $(j = 1, 2)$.

Proof. Assume first that Assumption (SL_1) holds. Problem (7) is equivalent to

$$\phi(u') = \mathcal{F}^{-1}[\mathcal{E}u(T) - B] + KN_f(u), \quad \mathcal{C}u(0) - \mathcal{D}\phi[u'(0)] = A,$$

i.e. to

$$u' = \phi^{-1} \circ [\mathcal{F}^{-1}[\mathcal{E}u(T) - B] + KN_f(u)], \quad \mathcal{C}u(0) - \mathcal{D}\phi[u'(0)] = A,$$

i.e. to

$$\begin{aligned} u &= \mathcal{C}^{-1}\{\mathcal{D}\phi[u'(0)] + A\} + H \circ \phi^{-1} \circ [\mathcal{F}^{-1}[\mathcal{E}u(T) - B] + KN_f(u)] \\ &=: M_1(u). \end{aligned}$$

It is easy to see that M_1 is a completely continuous mapping of Ω_1 into itself, and that, for each $u \in \Omega_1$,

$$\|(M_1(u))'\|_\infty = \|\phi^{-1} \circ [\mathcal{F}^{-1}[\mathcal{E}u(T) - B] + KN_f(u)]\|_\infty < a.$$

The case where Assumption (SL_2) holds is similar and left to the reader. □

Proposition 1 allows us to prove existence theorems for *any* (f, A, B) in the case of some particular Sturm–Liouville boundary conditions.

Theorem 3. *If Assumptions* (H_ϕ) *and* (H_f) *hold, and if* $\mathcal{F}$ *is invertible, then the problem*

$$\begin{aligned}(\phi(u'))' &= f(t,u,u') \quad (t \in [0,T]),\\ u(0) = A, &\quad \mathcal{E}u(T) - \mathcal{F}\phi[u'(T)] = B,\end{aligned} \tag{9}$$

has at least one solution.

Proof. Assumption (SL_1) holds and the corresponding fixed point operator M_1 reduces here to the operator M_{DSL} defined by

$$M_{DSL}(u) := A + H \circ \phi^{-1} \circ \{\mathcal{F}^{-1}[\mathcal{E}u(T) - B] + KN_f(u)\}. \tag{10}$$

It follows from Proposition 1 that M_{DSL} maps C^1 into the ball $B_{|A|+(1+T)a}$, and the existence of a fixed point follows from Schauder's fixed point theorem. □

The special case where $\mathcal{E} = \mathcal{O}$ and $\mathcal{F} = -\mathcal{I}$ corresponds to the Dirichlet–Neumann conditions.

Corollary 1. *If Assumptions* (H_ϕ) *and* (H_F) *hold, then the problem*

$$(\phi(u'))' = F(u), \quad u(0) = A, \quad \phi[u'(T)] = B, \tag{11}$$

has at least one solution.

Corollary 2. *If Assumptions* (H_ϕ) *and* (H_f) *hold, then the problem*

$$(\phi(u'))' = f(t,u,u'), \quad u(0) = A, \quad \phi[u'(T)] = B,$$

has at least one solution.

Theorem 4. *If Assumptions* (H_ϕ) *and* (H_f) *hold, and if* $\mathcal{D}$ *is invertible, then the problem*

$$\begin{aligned}(\phi(u'))' &= f(t,u,u') \quad (t \in [0,T]),\\ \mathcal{C}u(0) - \mathcal{D}\phi[u'(0)] &= A, \quad u(T) = B,\end{aligned} \tag{12}$$

has at least one solution.

Proof. It is similar to that of Theorem 3 and left to the reader. The corresponding fixed point operator is given by

$$M_{SLD}(u) := B + K \circ \phi^{-1} \circ \{\mathcal{D}^{-1}[\mathcal{C}u(0) - A] + HN_f(u)\}.$$ □

The special case where $\mathcal{C} = \mathcal{O}$ and $\mathcal{D} = -\mathcal{I}$ corresponds to the Neumann–Dirichlet conditions.

Corollary 3. *If Assumptions* (H_ϕ) *and* (H_F) *hold, then the problem*

$$(\phi(u'))' = F(u), \quad \phi[u'(0)] = A, \quad u(T) = B, \tag{13}$$

has at least one solution.

Corollary 4. *If Assumptions* (H_ϕ) *and* (H_f) *hold, then the problem*

$$(\phi(u'))' = f(t,u,u'), \quad \phi[u'(0)] = A, \quad u(T) = B,$$

has at least one solution.

5. Neumann boundary conditions

The counterexample (with $n = 1$)

$$(\phi(u'))' = 0, \quad \phi[u'(0)] = 0, \quad \phi[u'(T)] = 1,$$

which has no solution, shows that no result corresponding to Corollary 1 or 3 exists for the Neumann boundary conditions

$$\phi[u'(0)] = A, \quad \phi[u'(T)] = B.$$

To construct the equivalent fixed point problem, we first study the simple system

$$(\phi(u'))' = e(t), \quad \phi[u'(0)] = A, \quad \phi[u'(T)] = B. \tag{14}$$

with $e \in C$ and ϕ satisfying Assumption (H_ϕ). We define the linear projectors P and Q on C by

$$Pu = u(0), \quad Qu = \frac{1}{T}\int_0^T u(t)\,dt. \tag{15}$$

We also introduce the linear mapping

$$\widehat{Q} : C \times \mathbb{R}^2 \to C \times \mathbb{R}^2, \quad (u, A, B) \mapsto (Qu - T^{-1}(B - A), 0, 0). \tag{16}$$

We have

$$\begin{aligned}\widehat{Q}^2(u, A, B) &= \widehat{Q}(Qu - T^{-1}(B - A), 0, 0) = (Q(Qu - T^{-1}(B - A)), 0, 0)\\ &= (Qu - T^{-1}(B - A), 0, 0) = \widehat{Q}(u, A, B),\end{aligned}$$

and hence $\widehat{Q}$ is a projector.

The elementary proof of the following proposition is entirely similar to the one given for the scalar case in [2], and will not be repeated here.

Proposition 2. *If Assumption (H_ϕ) holds, then problem* (14) *has a solution if and only if*

$$Qe = T^{-1}(B - A), \tag{17}$$

i.e. if and only if

$$\widehat{Q}(e, A, B) = 0, \tag{18}$$

in which case the solutions of (14) *are given by the family*

$$u = Pu + H \circ \phi^{-1} \circ [A + He]. \tag{19}$$

Remark 2. Proposition 2 means that (e, A, B) belongs to the range of the nonlinear mapping $u \mapsto [(\phi(u'))', \phi[u'(0)], \phi[u'(T)]]$ if and only if $\widehat{Q}(e, A, B) = 0$.

Proposition 3. *If Assumptions (H_ϕ) and (H_F) hold, then u is a solution of the problem*

$$(\phi(u'))' = F(u), \quad \phi[u'(0)] = A, \quad \phi[u'(T)] = B, \tag{20}$$

if and only if $u \in C^1$ is a fixed point of the operator M_N defined on C^1 by

$$M_N(u) := Pu + Q[F(u) - T^{-1}(B - A)] + H \circ \phi^{-1} \circ [H(I - Q)F(u) + c_{A,B}],$$

where

$$c_{A,B}(t) := \left(1 - \frac{t}{T}\right)A + \frac{t}{T}B \quad (t \in [0,T]). \tag{21}$$

Furthermore, $\|(M_N(u))'\|_\infty < a$ *for all* $u \in C^1$*, and* M_N *is completely continuous.*

Proof. Problem (20) can be written in the equivalent form

$$(\phi(u'))' = F(u) - Q[F(u) - T^{-1}(B-A)], \tag{22}$$

$$Q[F(u) - T^{-1}(B-A)] = 0. \tag{23}$$

Now,

$$Q\{F(u) - Q[F(u) - T^{-1}(B-A)]\} = T^{-1}(B-A),$$

so that, by Proposition 2, equation (22) is equivalent to

$$u = Pu + H \circ \phi^{-1} \circ \{A + H[F(u) - QF(u) + T^{-1}(B-A)]\},$$

which can be written as

$$u - Pu - H \circ \phi^{-1} \circ [H(I-Q)F(u) + c_{A,B}] = 0. \tag{24}$$

As the left-hand members of (23) and (24) take values in direct summands of C^1, they can be written as the single equation

$$u - Pu - Q[F(u) - T^{-1}(B-A)] - H \circ \phi^{-1} \circ [H(I-Q)F(u) + c_{A,B}] = 0. \qquad \square$$

Consider now the Neumann boundary value problems

$$(\phi(u'))' = f(t,u,u'), \quad \phi[u'(0)] = A, \quad \phi[u'(T)] = B. \tag{25}$$

In order to apply Leray–Schauder degree to the equivalent fixed point operator, we introduce, for $\lambda \in [0,1]$, the family of abstract nonlinear Neumann boundary value problems

$$[(\phi(u'))', \phi[u'(0)], \phi[u'(T)]] = \lambda[N_f(u), A, B] + (1-\lambda)\widehat{Q}[N_f(u), A, B], \tag{26}$$

where $\widehat{Q}$ is defined in (16), or, in a more explicit form,

$$(\phi(u'))' = \lambda N_f(u) + (1-\lambda)[QN_f(u) - T^{-1}(B-A)],$$
$$\phi[u'(0)] = \lambda A, \quad \phi[u'(T)] = \lambda B.$$

Notice that (26) coincides with (25) for $\lambda = 1$. Furthermore, if u is a solution of (26), then, applying $\widehat{Q}$ to both members and using Remark 2, we see that

$$0 = \lambda\widehat{Q}[N_f(u), A, B] + (1-\lambda)\widehat{Q}[N_f(u), A, B] = \widehat{Q}[N_f(u), A, B]$$

and hence (26) can be written equivalently as

$$\begin{gathered}[(\phi(u'))', \phi[u'(0)], \phi[u'(T)]] = \lambda[N_f(u), A, B], \\ \widehat{Q}[N_f(u), A, B] = 0,\end{gathered} \tag{27}$$

or, in a more explicit way,

$$(\phi(u'))' = \lambda N_f(u), \tag{28}$$
$$\phi[u'(0)] = \lambda A, \quad \phi[u'(T)] = \lambda B,$$
$$0 = QN_f(u) - T^{-1}(B - A). \tag{29}$$

For each $\lambda \in [0,1]$, the nonlinear operator M_N on C^1 associated to (26) by Proposition 3 is the operator $\mathcal{M}(\lambda, \cdot)$, where $\mathcal{M}$ is defined on $[0,1] \times C^1$ by

$$\begin{aligned}\mathcal{M}(\lambda, u) &= Pu + Q\{\lambda N_f(u) + (1-\lambda)QN_f(u) - (1-\lambda)T^{-1}(B-A)\}\\ &\quad - T^{-1}(\lambda B - \lambda A)\\ &\quad + H \circ \phi^{-1} \circ \{H(I-Q)[\lambda N_f(u) + (1-\lambda)QN_f(u)\\ &\quad - (1-\lambda)T^{-1}(B-A)] + \lambda c_{A,B}\}\\ &= Pu + QN_f(u) - T^{-1}(B-A)\\ &\quad + H \circ \phi^{-1} \circ [\lambda H(I-Q)N_f(u) + \lambda c_{A,B}],\end{aligned} \tag{30}$$

where $c_{A,B}$ is defined in (21). Using Arzelà–Ascoli's theorem it is not difficult to see that $\mathcal{M} : [0,1] \times C^1 \to C^1$ is completely continuous.

The first lemma gives *a priori* bounds for the possible fixed points. We introduce the following assumption:

$(H_{f,A,B})$ *There exist* $R > 0$ *such that*

$$\int_0^T f(t, u(t), u'(t))\, dt \neq B - A \tag{31}$$

for all $u \in C^1$ *satisfying* $\min_{t\in[0,T]} |u(t)| \geq R$ *and* $\|u'\|_\infty < a$.

Lemma 1. *If Assumptions* (H_ϕ), (H_f) *and* $(H_{f,A,B})$ *hold, and if* $(\lambda, u) \in [0,1] \times C^1$ *is such that* $u = \mathcal{M}(\lambda, u)$, *then*

$$\|u\| < R + a(T+1). \tag{32}$$

Proof. Let $(\lambda, u) \in [0,1] \times C^1$ be such that $u = \mathcal{M}(\lambda, u)$. Then

$$u' = [\mathcal{M}(\lambda, u)]' = \phi^{-1} \circ [\lambda H(I-Q)N_f(u) + \lambda c_{A,B}],$$

so that

$$\|u'\|_\infty < a. \tag{33}$$

Then taking $t = 0$ we get $QN_f(u) - T^{-1}(B - A) = 0$, i.e.

$$\int_0^T f(t, u(t), u'(t))\, dt = B - A. \tag{34}$$

From (33), (34) and (31), it follows that

$$\min_{t\in[0,T]} |u(t)| < R, \tag{35}$$

and hence, if $|u(t_0)| = \min_{t\in[0,T]} |u(t)|$,

$$|u(t)| \leq |u(t_0)| + \left| \int_{t_0}^{t} |u'(s)|\, ds \right| < R + aT \qquad (t \in [0,T]),$$

which, together with (33), gives (32). □

We can now prove an existence theorem for (25). We denote by d_B the Brouwer degree and by d_{LS} the Leray–Schauder degree [5, 8], and define the mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ by

$$F : \mathbb{R}^n \to \mathbb{R}^n, \qquad c \mapsto \int_0^T f(t,c,0)\, dt. \tag{36}$$

Theorem 5. *If Assumptions* (H_ϕ), (H_f) *and* $(H_{f,A,B})$ *hold, then for all sufficiently large* $\rho > 0$,

$$d_{LS}[I - \mathcal{M}(1,\cdot), B_\rho, 0] = (-1)^n d_B[F, B_R, B - A].$$

If furthermore

$$d_B[F, B_R, B - A] \neq 0, \tag{37}$$

then problem (25) *has at least one solution.*

Proof. It follows from Assumption $(H_{f,A,B})$ applied to constant functions c that $F(c) \neq B - A$ for $|c| \geq R$, and hence the Brouwer degree $d_B[F, B_\rho, B - A]$ is well defined for any $\rho \geq R$. Let $\mathcal{M}$ be the operator given by (30) and let $\rho > R + a(T+1)$. Lemma 1 and the homotopy invariance of Leray–Schauder degree imply that

$$d_{LS}[I - \mathcal{M}(0,\cdot), B_\rho, 0] = d_{LS}[I - \mathcal{M}(1,\cdot), B_\rho, 0]. \tag{38}$$

On the other hand, we have

$$d_{LS}[I - \mathcal{M}(0,\cdot), B_\rho, 0] = d_{LS}[I - (P + QN_f - T^{-1}(B - A)), B_\rho, 0]. \tag{39}$$

But the range of the mapping

$$u \mapsto Pu + QN_f(u) - T^{-1}(B - A)$$

is contained in the subspace of constant functions, isomorphic to $\mathbb{R}^n$, so, using a reduction property of Leray–Schauder degree and excision [5, 8], we obtain

$$\begin{aligned} d_{LS}&[I - (P + QN_f - T^{-1}(B - A)), B_\rho, 0] \\ &= d_B[I - (P + QN_f - T^{-1}(B - A))|_{\mathbb{R}^n},\, B_\rho, 0] \\ &= d_B[-QN_f + T^{-1}(B - A), B_\rho, 0] \\ &= (-1)^n d_B[F, B_\rho, B - A] = (-1)^n d_B[F, B_R, B - A] \neq 0. \end{aligned}$$

Then, from the existence property of Leray–Schauder degree, there exists $u \in B_\rho$ such that $u = \mathcal{M}(1,u)$, which is a solution for (25). □

Corollary 5. *If $n \geq 2$ and Assumptions (H_ϕ) and (H_f) hold, and if there exists $R > 0$ such that*

$$\langle f(t, u + w, v) - T^{-1}(B - A), u \rangle \neq 0 \tag{40}$$

for all $t \in [0, T]$, $|u| \geq R$, $|w| < aT$ and $|v| < a$, then problem (25) has at least one solution.

Proof. Elementary considerations show that (40) implies $(H_{f,A,B})$ and

$$d_B[F, B_R, B - A] = d_B[\pm I, B_R, B - A] = \pm 1. \qquad \square$$

Example 1. If Assumption (H_ϕ) holds, $e \in C$, $c, d \in \mathbb{R}$, $p > 1$, $q \geq 0$, then the Neumann problem

$$(\phi(u'))' = (c + d|u'|^q)|u|^{p-2}u + e(t), \quad \phi[u'(0)] = A, \quad \phi[u'(T)] = B,$$

has at least one solution if $|c| > |d|a^q$.

6. A class of homeomorphisms

As we shall see, the case of Dirichlet conditions is more delicate to treat and requires a technical lemma proved here for a special class of homeomorphisms ϕ. Let us assume that

(H_Φ) *ϕ is a homeomorphism from B_a onto $\mathbb{R}^n$ such that $\phi(0) = 0$, $\phi = \nabla\Phi$, with $\Phi : B_a \subset \mathbb{R}^n \to \,]-\infty, 0]$ of class C^1, and strictly convex.*

So, ϕ is strictly monotone on B_a.

If $\Phi^* : \mathbb{R}^n \to \mathbb{R}$ is the Legendre–Fenchel transform of Φ [9] defined by

$$\Phi^*(v) = \langle \phi^{-1}(v), v \rangle - \Phi[\phi^{-1}(v)] = \sup_{u \in B_a} \{\langle u, v \rangle - \Phi(u)\},$$

then Φ^* is also strictly convex,

$$\Phi^*(v) \leq a|v| - \inf_{|v|<a} \Phi \circ \phi^{-1} =: a|v| + d, \tag{41}$$

and, using the nonnegativity of Φ,

$$\Phi^*(v) \geq \sup_{u \in B_a} \langle v, u \rangle = a|v|, \tag{42}$$

so that Φ^* is coercive on $\mathbb{R}^n$. Adapting the reasoning of Proposition 2.4 in [9], we find that Φ^* is of class C^1. Hence

$$\phi^{-1} = \nabla\Phi^*,$$

so that

$$v = \nabla\Phi(u) = \phi(u),\ u \in B_a \ \Leftrightarrow\ u = \phi^{-1}(v) = \nabla\Phi^*(v),\ v \in \mathbb{R}^n.$$

Given $h \in C$ and $b \in \mathbb{R}^n$, define

$$F(b;h) = \int_0^T \phi^{-1}[h(t) - b]\,dt = \int_0^T \nabla_b \Phi^*[h(t) - b]\,dt$$
$$= \nabla_b \int_0^T \Phi^*[h(t) - b]\,dt = \nabla_b f(b;h),$$

where

$$f(b;h) = \int_0^T \Phi^*[h(t) - b]\,dt.$$

Lemma 2. *If $\phi = \nabla\Phi$, with Φ satisfying Assumption (H_Φ), then, for each $h \in C$ and each $e \in B_{aT} \subset \mathbb{R}^n$, the system*

$$\int_0^T \phi^{-1}[h(t) - b]\,dt = e \tag{43}$$

has a unique solution $b := Q_\phi(h,e)$. Moreover, $Q_\phi : C \times B_{aT} \to \mathbb{R}^n$ is continuous, and, for each fixed $e \in B_{aT}$, $Q_\phi(\cdot, e)$ takes bounded subsets of C into bounded subsets of $\mathbb{R}^n$.

Proof. For each $b, c \in \mathbb{R}^n$ and any $\lambda \in]0,1[$, we have

$$f[(1-\lambda)b + \lambda c] = \int_0^T \Phi^*[(1-\lambda)(b - h(t)) + \lambda(c - h(t))]\,dt$$
$$< \int_0^T \{(1-\lambda)\Phi^*(b - h(t)) + \lambda\Phi^*(c - h(t))\}\,dt$$
$$\leq (1-\lambda)f(b;h) + \lambda f(c;h),$$

so that $f(\cdot;h)$ is strictly convex on $\mathbb{R}^n$ for each $h \in C$. Hence, $F(\cdot;h) = \nabla_b f(\cdot;h)$ is strictly monotone on $\mathbb{R}^n$ for each $h \in C$. On the other hand, using (41) and (42), we get

$$aT|b| - \|h\|_1 \leq f(b;h) \leq T(a|b| + d) + \|h\|_1, \tag{44}$$

so that, for each $h \in C$ and each $e \in B_{aT}$,

$$f(b;h) - \langle e, b\rangle \geq aT|b| - \|h\|_1 - |e|\,|b| = (aT - |e|)|b| - \|h\|_1$$

is coercive. Consequently, for each $h \in C$, $f(\cdot;h) - \langle e,\cdot\rangle$ admits a unique minimum $b := Q_\phi(h,e)$, which is the unique critical point of $f(\cdot;h) - \langle e,\cdot\rangle$. This implies that, for each $h \in C$ and each $e \in B_{aT}$, the system

$$F(b;h) = e \tag{45}$$

has a unique solution $b = Q_\phi(h,e)$.

Let us now show that Q_ϕ is continuous. Let (h_n, e_n) be a sequence converging in $C \times \mathbb{R}^n$ to $(h,e) \in C \times B_{aT}$. Then (h_n, e_n) is bounded. Without loss of generality, we can assume that $e_n \in B_{a(|d|+T)/2}$. Let $b_n = Q_\phi(h_n, e_n)$. Then, by convexity,

$$f(0;h_n) \geq f(b_n;h_n) - \langle \nabla_b f(b_n;h_n), b_n\rangle = f(b_n;h_n) - \langle e_n, b_n\rangle$$
$$\geq (aT - |e_n|)|b_n| - \|h_n\|_1 \geq [(aT - |e|)/2]|b_n| - \|h_n\|_1,$$

so that

$$|b_n| \leq [2/(aT - |e|)][\|h_n\|_1 + f(0; h_n)]$$

which shows that (b_n) is bounded. Passing to a subsequence if necessary, we can assume that (b_n) converges to β. From the relations

$$\int_0^T \phi^{-1}[h_n(t) - b_n]\,dt = e_n \quad (n \in \mathbb{N}),$$

and the dominated convergence theorem, we deduce that

$$\int_0^T \phi^{-1}[h(t) - \beta]\,dt = e,$$

i.e. by the uniqueness of the solutions, $\beta = Q_\phi(h, e)$, a limit independent of the subsequence. Hence

$$Q_\phi(h, e) = \lim_{n\to\infty} Q_\phi(h_n, e_n),$$

and Q_ϕ is continuous. Notice also that $Q_\phi(0, 0) = 0$.

Finally, given $e \in B_{aT}$, to show that $Q_\phi(\cdot, e)$ takes bounded subsets of C into bounded subsets of $\mathbb{R}^n$, we use again convexity and (44) to obtain

$$f(0; h) \geq f(b; h) - \langle \nabla_b f(b; h), b\rangle = f(b, h) - \langle e, b\rangle \geq (aT - |e|)|b| - \|h\|_1,$$

and hence, using again (44),

$$|Q_\phi(h, e)| = |b| \leq (aT - |e|)^{-1}[2T(\|h\|_\infty + d)]. \qquad \square$$

Example 2. Let us consider the C^∞-mapping

$$\Phi : B_1 \subset \mathbb{R}^n \to \mathbb{R}, \quad u \mapsto -\sqrt{1 - |u|^2}, \tag{46}$$

so that

$$-1 \leq \Phi(u) \leq 0 \quad (u \in B_1),$$

and

$$\phi(u) = \nabla\Phi(u) = \frac{u}{\sqrt{1 - |u|^2}} \quad (u \in B_1). \tag{47}$$

As $u \mapsto |u|^2$ is strictly convex on $\mathbb{R}^n$, it follows that Φ is strictly convex on B_1. Furthermore, $\phi : B_1 \to \mathbb{R}^n$ is a homeomorphism such that, for any $v \in \mathbb{R}^n$.

$$\phi^{-1}(v) = \frac{v}{\sqrt{1 + |v|^2}} = \nabla\Phi^*(v), \tag{48}$$

where

$$\Phi^*(v) = \sqrt{1 + |v|^2} \tag{49}$$

is strictly convex and of class C^∞. Hence, $u \mapsto u/\sqrt{1 - |u|^2}$ satisfies Assumption (H_Φ) with $a = 1$.

7. Dirichlet boundary conditions

Let ϕ satisfy Assumption (H_Φ). To construct the fixed point operator associated to Dirichlet boundary conditions, we follow the approach in [1], and we first study the solvability of the forced system

$$(\phi(u'))' = e(t), \quad u(0) = A, \quad u(T) = B, \tag{50}$$

with $e \in C,\ A, B \in \mathbb{R}^n$. We introduce the assumption

(H_{ABT}) $|B - A| < aT$.

Proposition 4. *If ϕ satisfies Assumption (H_Φ), then problem* (50) *has a unique solution if and only if Assumption (H_{ABT}) holds, in which case the solution of* (50) *is given by*

$$u = A + H \circ \phi^{-1} \circ [Hf - Q_\phi(He, B - A)], \tag{51}$$

where Q_ϕ is defined in Lemma 2.

Proof. Problem (50) is equivalent to

$$\phi(u') = c + He, \quad u(0) = A, \quad u(T) = B,$$

i.e. to

$$u' = \phi^{-1}(c + He), \quad u(0) = A, \quad u(T) = B,$$

i.e. to

$$u = A + H \circ \phi^{-1}(c + He), \quad u(T) = B. \tag{52}$$

Hence one must have

$$A + \int_0^T \phi^{-1}[c + He(t)]\, dt = B,$$

i.e., using Lemma 2,

$$c = -Q_\phi(He, B - A),$$

which inserted in (52) gives (51). □

Remark 3. Assumption (H_{ABT}) has a physical interpretation, if we consider the differential system in (50) as an equation of motion where the speed is limited by a. The maximum distance travelled in time T is aT, and hence one can only go from A to B in time T if $|B - A| < aT$.

For $F : C^1 \to C$ satisfying Assumption (H_F), we now construct a nonlinear operator on C^1 whose fixed points are the solutions of

$$(\phi(u'))' = F(u), \quad u(0) = A, \quad u(T) = B. \tag{53}$$

Proposition 5. *Assume that Assumptions* (H_Φ), (H_F) *and* (H_{ABT}) *hold. Then* u *is a solution of problem* (53) *if and only if* $u \in C^1$ *is a fixed point of the operator* M_D *defined on* C^1 *by*

$$M_D(u) := A + H \circ \phi^{-1} \circ [HF(u) - Q_\phi(HF(u), B - A)]. \tag{54}$$

Furthermore,

$$\|(M_D(u))'\|_\infty < a \tag{55}$$

for all $u \in C^1$, *and* M_D *is completely continuous.*

Proof. It is an easy consequence of Proposition 4. □

A consequence of Proposition 5 is the solvability, for *any* $F : C^1 \to C$ satisfying (H_F), of the nonlinear Dirichlet problem (53). However, in contrast to the Cauchy, terminal, Dirichlet–Neumann and Neumann–Dirichlet cases, some restriction upon A and B is required, namely Assumption (H_{ABT}).

Theorem 6. *If Assumptions* (H_Φ), (H_F) *and* (H_{ABT}) *hold, then problem* (53) *has at least one solution.*

Proof. It suffices to prove that the operator M_D defined in (54) has a fixed point. Using (55) we also have

$$\|M_D(u)\|_\infty < |A| + aT \quad (u \in C^1). \tag{56}$$

Hence M_D maps C^1 into $B_{|A|+(a+1)T} \subset C^1$, and has at least one fixed point by the Schauder fixed point theorem. □

Corollary 6. *If Assumptions* (H_Φ), (H_f) *and* (H_{ABT}) *hold, then the problem*

$$(\phi(u'))' = f(t, u, u'), \quad u(0) = A, \quad u(T) = B, \tag{57}$$

has at least one solution.

8. Periodic boundary conditions

We now deal with the case of periodic boundary conditions, which, in some sense, cumulates the difficulties of Dirichlet and of Neumann problems. The proofs of the following two propositions are analogous to those given for the scalar case in [2], and are not repeated here. We assume that ϕ satisfies Assumption (H_Φ) and define $\widetilde{Q}_\phi : C \to C$ by

$$\widetilde{Q}_\phi(h) = Q_\phi(h, 0), \tag{58}$$

where $\mathbb{R}^n$ is identified with the subset of C consisting of constant functions.

Proposition 6. *If* ϕ *satisfies Assumption* (H_Φ), *then the problem*

$$(\phi(u'))' = e(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{59}$$

has a solution if and only if the condition

$$\int_0^T e(t)\,ds = 0 \tag{60}$$

holds, in which case the solutions are given by the family

$$u(t) = Pu + \int_0^t [\phi^{-1} \circ (I - \widetilde{Q}_\phi) \circ He(s)]\,ds \qquad (t \in [0,T]).$$

Let $C^1_\# = \{u \in C^1 : u(0) - u(T) = 0 = u'(0) - u'(T)\}$.

Proposition 7. *If Assumptions* (H_Φ) *and* (H_F) *hold, then* u *is a solution of the problem*

$$(\phi(u'))' = F(u), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

if and only if $u \in C^1_\#$ *is a fixed point of the operator* $M_\#$ *defined on* $C^1_\#$ *by*

$$M_\#(u) = Pu + QF(u) + H \circ \phi^{-1} \circ (I - \widetilde{Q}_\phi) \circ [H(I-Q)F](u).$$

Furthermore, $\|(M_\#(u))'\|_\infty < a$ *for all* $u \in C^1_\#$*, and* $M_\#$ *is completely continuous.*

The counterexample

$$(\phi(u'))' = 1, \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

which has no solution, shows that no result corresponding to Theorem 6 exists for periodic boundary conditions.

Consider the periodic boundary value problem

$$(\phi(u'))' = f(t,u,u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{61}$$

when Assumptions (H_Φ) and (H_f) hold. In order to apply Leray–Schauder degree to the equivalent fixed point operator $M_\#$, we introduce, for $\lambda \in [0,1]$, the family of periodic boundary value problems

$$(\phi(u'))' = \lambda N_f(u) + (1-\lambda)QN_f(u), \quad u(0) - u(T) = 0 = u'(0) - u'(T). \tag{62}$$

Notice that (62) coincides with (61) for $\lambda = 1$. For each $\lambda \in [0,1]$, the nonlinear operator $M_\#$ on $C^1_\#$ associated to (62) by Proposition 7 is the operator $\mathcal{M}(\lambda,\cdot)$, where $\mathcal{M}$ is defined on $[0,1] \times C^1_\#$ by

$$\mathcal{M}(\lambda,u) = Pu + QN_f(u) + H \circ \phi^{-1} \circ (I - \widetilde{Q}_\phi) \circ [\lambda H(I-Q)N_f](u). \tag{63}$$

Using Lemma 2 and Arzelà–Ascoli's theorem it is not difficult to see that $\mathcal{M} : [0,1] \times C^1_\# \to C^1_\#$ is completely continuous.

The first lemma gives *a priori* bounds for the possible fixed points. Its proof is entirely analogous to that of Lemma 1 and is omitted. Let us introduce the assumption

$(H_{f,\#})$ *There exist* $R > 0$ *such that*

$$\int_0^T f(t,u(t),u'(t))\,dt \neq 0 \tag{64}$$

for all $u \in C^1_\#$ *satisfying* $\min_{t\in[0,T]} |u(t)| \geq R$ *and* $\|u'\|_\infty < a$.

Lemma 3. *Assume that Assumptions* (H_Φ), (H_f) *and* $(H_{f,\#})$ *hold. If* $(\lambda, u) \in [0,1] \times C^1_\#$ *is such that* $u = \mathcal{M}(\lambda, u)$, *then*

$$\|u\| < R + a(T+1).$$

We can now prove an existence theorem for (61).

Theorem 7. *If Assumptions* (H_Φ), (H_f) *and* $(H_{f,\#})$ *hold, then, for all sufficiently large* $\rho > 0$,

$$d_{LS}[I - \mathcal{M}(1,\cdot), B_\rho, 0] = (-1)^n d_B[F, B_R, 0],$$

where F *is defined in* (36). *If we assume furthermore that*

$$d_B[F, B_R, 0] \neq 0, \tag{65}$$

then problem (61) *has at least one solution.*

Corollary 7. *If* $n \geq 2$ *and Assumptions* (H_Φ) *and* (H_f) *hold, and if there exists* $R > 0$ *such that*

$$\langle f(t, u+w, v), u\rangle \neq 0$$

for all $t \in [0,T]$, $|u| \geq R$, $|w| < aT$ *and* $|v| < a$, *then problem* (61) *has at least one solution.*

Example 3. If Assumption (H_Φ) holds, $e \in C$, $c, d \in \mathbb{R}$, $p > 1$, $q \geq 0$, then the periodic problem

$$(\phi(u'))' = (c + d|u'|^q)|u|^{p-2}u + e(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has at least one solution if $|c| > |d|a^q$.

9. An application

When an electron moves at high speed but is only gently accelerated by the radiation field and the static Coulomb field, the asymptotically evolutionary law of the point electron is Newton's law of radiation-reaction-free motion, equating the rate of change of kinematical particle momentum to a Lorenz force in which only the external (incoming radiation and static Coulomb) fields enter, while the kinematical particle momentum and particle velocity are related by the special relativity formula (see e.g. [3, p. 118]). In suitable dimensionless units this becomes (see e.g. [6, p. 1106])

$$\mathbf{r}'(t) = \frac{\mathbf{p}(t)}{\sqrt{1+|\mathbf{p}(t)|^2}}, \quad \mathbf{p}'(t) = -\frac{e^2}{\hbar c}[\mathbf{E}(t,\mathbf{r}(t)) + \mathbf{r}'(t) \times \mathbf{B}(t,\mathbf{r}(t))],$$

where $\mathbf{r}, \mathbf{p} : \mathbb{R} \to \mathbb{R}^3$ denote respectively the position and momentum of the particle, $\mathbf{E}, \mathbf{B} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ are the exterior electrical and magnetic fields, $\times$ denotes the vector product in $\mathbb{R}^3$ and $e, \hbar, c$ are the usual physical constants. As

$$\mathbf{r}'(t) = \frac{\mathbf{p}(t)}{\sqrt{1+|\mathbf{p}(t)|^2}} \ \Leftrightarrow \ \mathbf{p}(t) = \frac{\mathbf{r}'(t)}{\sqrt{1-|\mathbf{r}'(t)|^2}},$$

this system can be written

$$\left(\frac{\mathbf{r}'(t)}{\sqrt{1-|\mathbf{r}'(t)|^2}}\right)' = -\frac{e^2}{\hbar c}[\mathbf{E}(t,\mathbf{r}(t)) + \mathbf{r}'(t)\times\mathbf{B}(t,\mathbf{r}(t))]. \qquad (66)$$

Now system (66) is of the form $(\phi(\mathbf{r}'))' = f(t,\mathbf{r},\mathbf{r}')$ where

$$\phi(\mathbf{u}) = \frac{\mathbf{u}}{\sqrt{1-|\mathbf{u}|^2}} = \nabla[-\sqrt{1-|\mathbf{u}|^2}] \quad (\mathbf{u}\in\mathbb{R}^3),$$

so that we can take

$$\Phi(\mathbf{u}) = -\sqrt{1-|\mathbf{u}|^2} \quad (\mathbf{u}\in\mathbb{R}^3),$$

which is the 3-dimensional case of Example 2. Using Theorems 1, 2, 3, 4, 6, and Corollaries 5, 7, we immediately obtain the following result.

Theorem 8. *For any continuous* $\mathbf{E},\mathbf{B}:\mathbb{R}\times\mathbb{R}^3\to\mathbb{R}^3$, *and any* $T>0$ *and* $\mathbf{A},\mathbf{B}\in\mathbb{R}^3$ *system* (66) *has at least one solution over* $[0,T]$ *satisfying the Cauchy conditions*

$$\mathbf{r}(0)=\mathbf{A}, \qquad \frac{\mathbf{r}'(0)}{\sqrt{1-|\mathbf{r}'(0)|^2}} = \mathbf{B},$$

or the terminal conditions

$$\mathbf{r}(T)=\mathbf{A}, \qquad \frac{\mathbf{r}'(T)}{\sqrt{1-|\mathbf{r}'(T)|^2}} = \mathbf{B},$$

or the Sturm–Liouville boundary conditions

$$\mathbf{r}(0)=\mathbf{A}, \qquad \mathcal{E}\mathbf{r}(T) - \mathcal{F}\frac{\mathbf{r}'(T)}{\sqrt{1-|\mathbf{r}'(T)|^2}} = \mathbf{B},$$

with $\mathcal{F}$ *invertible, or the Sturm–Liouville boundary conditions*

$$\mathcal{C}\mathbf{r}(0) - \mathcal{D}\frac{\mathbf{r}'(0)}{\sqrt{1-|\mathbf{r}'(0)|^2}} = \mathbf{A}, \qquad \mathbf{r}(T)=\mathbf{B},$$

with $\mathcal{D}$ *invertible. If*

$$|\mathbf{B}-\mathbf{A}| < T,$$

then system (66) *has at least one solution satisfying the Dirichlet boundary conditions*

$$\mathbf{r}(0)=\mathbf{A}, \qquad \mathbf{r}(T)=\mathbf{B}.$$

If there exists $R>0$ *such that*

$$\langle \mathbf{E}(t,\mathbf{u}+\mathbf{w}) + \mathbf{v}\times\mathbf{B}(t,\mathbf{u}+\mathbf{w}) - T^{-1}(B-A), \mathbf{u}\rangle \neq 0$$

for all $t\in[0,T]$, $|\mathbf{u}|\geq R$, $|\mathbf{w}|<T$ *and* $|\mathbf{v}|<1$, *then system* (66) *has at least one solution satisfying the Neumann boundary conditions*

$$\frac{\mathbf{r}'(0)}{\sqrt{1-|\mathbf{r}'(0)|^2}} = \mathbf{A}, \qquad \frac{\mathbf{r}'(T)}{\sqrt{1-|\mathbf{r}'(T)|^2}} = \mathbf{B}.$$

If there exists $R>0$ *such that*

$$\langle \mathbf{E}(t,\mathbf{u}+\mathbf{w}) + \mathbf{v}\times\mathbf{B}(t,\mathbf{u}+\mathbf{w}), \mathbf{u}\rangle \neq 0$$

for all $t \in [0,T]$, $|\mathbf{u}| \geq R$, $|\mathbf{w}| < T$ *and* $|\mathbf{v}| < 1$, *then system* (66) *has at least one solution satisfying the periodic boundary conditions*

$$\mathbf{r}(0) - \mathbf{r}(T) = \mathbf{0} = \mathbf{r}'(0) - \mathbf{r}'(T).$$

Example 4. If there exist $\beta > 0$, $\gamma \geq 1$, and $R > 0$ such that, for all $t \in [0,T]$, $|\mathbf{w}| < T$ and $|\mathbf{u}| \geq R$, one has

$$\langle \mathbf{E}(t, \mathbf{u}+\mathbf{w}), \mathbf{u} \rangle \geq \beta |\mathbf{u}|^{\gamma}, \qquad |\mathbf{B}(t, \mathbf{u}+\mathbf{w})| \leq \beta |\mathbf{u}|^{\gamma-1},$$

then, for all $t \in [0,T]$, $|\mathbf{u}| \geq R$, $|\mathbf{w}| < T$ and $|\mathbf{v}| < 1$, one has

$$\langle \mathbf{E}(t, \mathbf{u}+\mathbf{w}) + \mathbf{v} \times \mathbf{B}(t, \mathbf{u}+\mathbf{w}), \mathbf{u} \rangle \geq \beta |\mathbf{u}|^{\gamma} - \beta |\mathbf{v}|\, |\mathbf{u}|^{\gamma} \geq \beta (1 - |\mathbf{v}|) R^{\gamma} > 0.$$

Consequently, system (66) has at least one solution satisfying the homogeneous Neumann boundary conditions

$$\frac{\mathbf{r}'(0)}{\sqrt{1 - |\mathbf{r}'(0)|^2}} = \mathbf{0} = \frac{\mathbf{r}'(T)}{\sqrt{1 - |\mathbf{r}'(T)|^2}},$$

and at least one solution satisfying the periodic boundary conditions

$$\mathbf{r}(0) - \mathbf{r}(T) = \mathbf{0} = \mathbf{r}'(0) - \mathbf{r}'(T).$$

References

[1] C. Bereanu and J. Mawhin, *Existence and multiplicity results for some nonlinear problems with singular ϕ-laplacian.* J. Differential Equations **243** (2007), 536–557.

[2] C. Bereanu and J. Mawhin, *Non-homogeneous boundary value problems for some nonlinear equations with singular ϕ-laplacian.* J. Math. Anal. Appl., to appear.

[3] P. G. Bergmann, *Introduction to the Theory of Relativity.* Dover, New York, 1976.

[4] F. E. Browder, *Nonlinear Operators and Nonlinear Equations of Evolution in Banach Spaces.* Proc. Sympos. Pure Math. 18, Part 2, Amer. Math. Soc., Providence, 1976.

[5] K. Deimling, *Nonlinear Functional Analysis.* Springer, Berlin, 1985.

[6] M. K. H. Kiessling, *Electromagnetic field theory without divergence problems. 1. The Born legacy.* J. Statist. Phys. **116** (2004), 1057–1122.

[7] L. Landau and E. Lifschitz, *The Classical Theory of Fields.* Addison-Wesley, Cambridge, MA, 1951.

[8] J. Mawhin, *Topological Degree Methods in Nonlinear Boundary Value Problems.* CBMS Reg. Conf. Ser. Math. 40, Amer. Math. Soc., Providence, 1979.

[9] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems.* Springer, New York, 1989.

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