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Multiple positive radial solutions for a Dirichlet problem involving the mean curvature operator in Minkowski space ☆

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Abstract

We study the Dirichlet problem with mean curvature operator in Minkowski space

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right)+\lambda\left[\mu\left(|x|\right)v^q\right]=0 \quad \text{in } \mathcal{B}(R), \qquad v=0 \quad \text{on } \partial\mathcal{B}(R),$$

where $\lambda > 0$ is a parameter, $q > 1$, $R > 0$, $\mu : [0, \infty) \to \mathbb{R}$ is continuous, strictly positive on $(0, \infty)$ and $\mathcal{B}(R) = \{x \in \mathbb{R}^N: |x| < R\}$. Using upper and lower solutions and Leray–Schauder degree type arguments, we prove that there exists $\Lambda > 0$ such that the problem has zero, at least one or at least two positive radial solutions according to $\lambda \in (0, \Lambda)$, $\lambda = \Lambda$ or $\lambda > \Lambda$. Moreover, Λ is strictly decreasing with respect to R.

Keywords: Dirichlet problem; Positive radial solutions; Mean curvature operator; Minkowski space; Leray–Schauder degree; Upper and lower solutions

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1. Introduction

In this paper we present some non-existence, existence and multiplicity results for radial solutions of Dirichlet problems in a ball, associated to the mean curvature operator in the flat Minkowski space

$$\mathbb{L}^{N+1} := \{(x,t)\colon\ x \in \mathbb{R}^N,\ t \in \mathbb{R}\}$$

endowed with the Lorentzian metric

$$\sum_{j=1}^{N}(dx_j)^2 - (dt)^2,$$

where (x,t) are the canonical coordinates in $\mathbb{R}^{N+1}$.

These problems are originated in the study – in differential geometry or special relativity, of maximal or constant mean curvature hypersurfaces, i.e., spacelike submanifolds of codimension one in $\mathbb{L}^{N+1}$, having the property that their mean extrinsic curvature (trace of its second fundamental form) is respectively zero or constant (see e.g. [1,9,21]). More specifically, let M be a spacelike hypersurface of codimension one in $\mathbb{L}^{N+1}$ and assume that M is the graph of a smooth function $v : \Omega \to \mathbb{R}$ with Ω a domain in $\{(x,t)\colon\ x \in \mathbb{R}^N,\ t = 0\} \simeq \mathbb{R}^N$. The spacelike condition implies $|\nabla v| < 1$ and the mean curvature H satisfies the equation

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = NH(x,v) \quad \text{in } \Omega.$$

If H is bounded, then it has been shown in [3] that the above equation has at least one solution $u \in C^1(\Omega) \cap W^{2,2}(\Omega)$ with $u = 0$ on $\partial\Omega$.

In this paper we consider the Dirichlet boundary value problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + \lambda\left[\mu(|x|)v^q\right] = 0 \quad \text{in } \mathcal{B}(R), \qquad v = 0 \quad \text{on } \partial\mathcal{B}(R), \tag{1}$$

where $\lambda > 0$ is a parameter, $q > 1$, $R > 0$, $\mu : [0,\infty) \to \mathbb{R}$ is continuous, strictly positive on $(0,\infty)$ and $\mathcal{B}(R) = \{x \in \mathbb{R}^N\colon\ |x| < R\}$.

Using a variational type argument, in [8] it is shown that if

$$(q+1)R^N < \lambda N \int_0^R r^{N-1}\mu(r)(R-r)^{q+1}\,dr,$$

then problem (1) has at least one positive classical radial solution. In particular, it is clear that the above condition is satisfied provided that λ is sufficiently large. On account of the main result of this paper (Theorem 1), this result becomes more precise. Namely, we prove (Corollary 1) that

- *there exists $\Lambda > 0$ such that (1) has zero, at least one or at least two positive classical radial solutions according to $\lambda \in (0,\Lambda)$, $\lambda = \Lambda$ or $\lambda > \Lambda$. Moreover, Λ is strictly decreasing with respect to R.*

Up to our knowledge, such bifurcation scheme is completely new and has not been described before in related problems. If we compare with known results for classical elliptic equations with convex-concave nonlinearities (see for instance [2]), the bifurcation diagram is reversed in some sense. In particular, the non-existence of solutions for small values of the bifurcation parameter is a striking effect and a genuine consequence of the Minkowski mean curvature operator.

In the case $\mu = 1$, it is interesting to compare (1) with the analogous problem in the Euclidean context:

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) + \lambda v^q = 0 \quad \text{in } \mathcal{B}(R), \qquad v = 0 \quad \text{on } \partial\mathcal{B}(R), \tag{2}$$

with $1 < q < \frac{N+2}{N-2}$. The assumption on q is natural because, from [19] it follows that (2) has no nontrivial solutions if $q \geqslant \frac{N+2}{N-2}$. Notice also that, according to [13], all positive solutions of (2) have radial symmetry. Using critical point theory, in [11] it is proved that (2) has at least one positive radial solution for λ sufficiently large. One the other hand, in [10] it is shown that if $\lambda = 1$ then there exists a non-negative number R^* such that (2) has at least one positive radial solution for every $R > R^*$; this is done by means of a generalization of a Liouville type theorem concerning ground states due to Ni and Serrin. Also, notice that in [20] it has been shown that there exists $R_* > 0$ such that (2) has no positive radial solution when $R < R_*$. The case $q = 1$ is considered in [17] for λ in a left neighborhood of the principal eigenvalue of $-\Delta$ in H_0^1. In dimension one for $R = 1$, in [14] it is given a complete description of the exact number of positive solutions of (2).

For $\mu(r) \equiv r^m$, the analogous semilinear problem in which the mean curvature operator is replaced by the Laplacian is

$$\Delta v + |x|^m v^q = 0 \quad \text{in } \mathcal{B}(1), \qquad v = 0 \quad \text{on } \partial\mathcal{B}(1),$$

and we point out that, as shown in [18], the above problem has a positive radial solution provided that $1 < q < \frac{N+2m+2}{N-2}$ and $N \geqslant 3$, $m > 0$.

Setting, as usual, $r = |x|$ and $v(x) = u(r)$, we reduce the Dirichlet problem (1) to the mixed boundary value problem

$$\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)' + r^{N-1}\left[\lambda\mu(r)u^q\right] = 0, \qquad u'(0) = 0 = u(R). \tag{3}$$

The rest of the paper is organized as follows. In Section 2 we associate to a larger class of problems of type (3) a fixed point operator and we prove a lower and upper solution result (Proposition 1). A Cauchy problem associated to the differential equation in (3) is studied in Section 3. The main result of this section (Proposition 2) will be employed to prove the monotonicity of Λ with respect to R. By means of a degree computation inspired in the proof of the cone compression–expansion theorem by Krasnosel'skii (see [15]), in Section 4 we show that the Leray–Schauder index in zero of the fixed point operator introduced in Section 2 is 1. Section 5 is devoted to the proof of the main result.

For other results concerning the Neumann problem associated to prescribed mean curvature operator in Minkowski space we refer the reader to [5–7,16].

2. A fixed point operator, lower and upper solutions and degree

In this section we consider problems of the type

$$\big(r^{N-1}\phi\big(u'\big)\big)' + r^{N-1}g(r,u) = 0, \qquad u'(0) = 0 = u(R), \tag{4}$$

where $N \geqslant 2$ is an integer, $R > 0$ and the following main hypotheses hold true:

(H_ϕ) $\phi : (-a, a) \to \mathbb{R}$ $(0 < a < \infty)$ *is an odd, increasing homeomorphism*;
(H_g) $g : [0, R] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function.*

In the sequel, the space $C := C[0, R]$ will be endowed with the usual sup-norm $\|\cdot\|_\infty$ and $C^1 := C^1[0, R]$ will be considered with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$. Also, we shall use the closed subspace of C^1 defined by

$$C^1_M = \big\{u \in C^1\colon\ u'(0) = 0 = u(R)\big\}.$$

For $u_0 \in C^1_M$, we set $B(u_0, \rho) := \{u \in C^1_M\colon\ \|u\| < \rho\}$ $(\rho > 0)$ and, for shortness, we shall write B_ρ instead $B(0, \rho)$.

Recall, by *a solution* of (4) we mean a function $u \in C^1$ with $\|u'\|_\infty < a$, such that $r^{N-1}\phi(u') \in C^1$ and (4) is satisfied.

Setting

$$\sigma(r) := 1/r^{N-1} \quad (r > 0),$$

we introduce the linear operators

$$S : C \to C, \quad Su(r) = \sigma(r)\int_0^r t^{N-1}u(t)\,dt \quad \big(r \in (0, R]\big), \qquad Su(0) = 0;$$

$$K : C \to C^1, \quad Ku(r) = \int_r^R u(t)\,dt \quad \big(r \in [0, R]\big).$$

It is easy to see that K is bounded and standard arguments, invoking the Arzela–Ascoli theorem, show that S is compact. This implies that the nonlinear operator $K \circ \phi^{-1} \circ S : C \to C^1$ is compact. On the other hand, an easy computation shows that, for a given function $h \in C$, the mixed problem

$$\big(r^{N-1}\phi\big(u'\big)\big)' + r^{N-1}h(r) = 0, \qquad u'(0) = 0 = u(R),$$

has an unique solution u given by

$$u = K \circ \phi^{-1} \circ S \circ h.$$

Next, let N_g be the Nemytskii operator associated to g, i.e.,

$$N_g : C \to C, \quad N_g(u) = g\big(\cdot, u(\cdot)\big).$$

Noticing that N_g is continuous and takes bounded sets into bounded sets, we have the following fixed point reformulation of problem (4).

Lemma 1. *A function $u \in C^1_M$ is a solution of* (4) *if and only if it is a fixed point of the compact nonlinear operator*

$$\mathcal{N}_g : C^1_M \to C^1_M, \quad \mathcal{N}_g = K \circ \phi^{-1} \circ S \circ N_g.$$

Moreover, any fixed point u of $\mathcal{N}_g$ satisfies

$$\|u'\|_\infty < a, \qquad \|u\|_\infty < aR \tag{5}$$

and

$$d_{LS}[I - \mathcal{N}_g, B_\rho, 0] = 1 \quad \textit{for all } \rho \geqslant a(R+1).$$

Proof. Inequalities in (5) follow immediately from the fact that the range of ϕ^{-1} is $(-a, a)$. Next, consider the compact homotopy

$$\mathcal{H} : [0,1] \times C^1_M \to C^1_M, \quad \mathcal{H}(\tau, \cdot) = \tau \mathcal{N}_g(\cdot).$$

One has that

$$\mathcal{H}\big([0,1] \times C^1_M\big) \subset B_{a(R+1)},$$

which together with the invariance under homotopy of the Leray–Schauder degree, imply that

$$d_{LS}\big[I - \mathcal{H}(0, \cdot), B_\rho, 0\big] = d_{LS}\big[I - \mathcal{H}(1, \cdot), B_\rho, 0\big],$$

for all $\rho \geqslant a(R+1)$. The result follows from $\mathcal{H}(0, \cdot) = 0$, $\mathcal{H}(1, \cdot) = \mathcal{N}_g$ and $d_{LS}[I, B_\rho, 0] = 1$. $\square$

A *lower solution* of (4) is a function $\alpha \in C^1$ such that $\|\alpha'\|_\infty < a$, $r^{N-1}\phi(\alpha') \in C^1$ and

$$\big(r^{N-1}\phi\big(\alpha'(r)\big)\big)' + r^{N-1} g\big(r, \alpha(r)\big) \geqslant 0 \quad \big(r \in [0, R]\big), \qquad \alpha(R) \leqslant 0.$$

Similarly, an *upper solution* of (4) is defined by reversing the above inequalities.

Proposition 1. *If* (4) *has a lower solution α and an upper solution β such that $\alpha(r) \leqslant \beta(r)$ for all $r \in [0, R]$, then* (4) *has a solution u such that $\alpha(r) \leqslant u(r) \leqslant \beta(r)$ for all $r \in [0, R]$.*

Proof. Let $\gamma : [0, R] \times \mathbb{R} \to \mathbb{R}$ be the continuous function defined by

$$\gamma(r,u) = \begin{cases} \alpha(r), & \text{if } u < \alpha(r), \\ u, & \text{if } \alpha(r) \leqslant u \leqslant \beta(r), \\ \beta(r), & \text{if } u > \beta(r), \end{cases}$$

and define $G : [0, R] \times \mathbb{R} \to \mathbb{R}$ by $G(r,u) = g(r, \gamma(r,u))$. We consider the modified problem

$$\left(r^{N-1}\phi\left(u'\right)\right)' + r^{N-1}\left[G(r,u) - u + \gamma(r,u)\right] = 0, \qquad u'(0) = 0 = u(R). \tag{6}$$

It follows from [4] that problem (6) has at least one solution.

We show that if u is a solution of (6), then $\alpha(r) \leqslant u(r) \leqslant \beta(r)$ for all $r \in [0, R]$. This will conclude the proof.

Suppose that there exists some $r_0 \in [0, R]$ such that

$$\max_{[0,R]}(\alpha - u) = \alpha(r_0) - u(r_0) > 0.$$

If $r_0 \in (0, R)$ then $\alpha'(r_0) = u'(r_0)$ and there is a sequence $\{r_k\}$ in $(0, r_0)$ converging to r_0 such that $\alpha'(r_k) - u'(r_k) \geqslant 0$. As ϕ is an increasing homeomorphism, this implies

$$r_k^{N-1}\phi\left(\alpha'(r_k)\right) - r_0^{N-1}\phi\left(\alpha'(r_0)\right) \geqslant r_k^{N-1}\phi\left(u'(r_k)\right) - r_0^{N-1}\phi\left(u'(r_0)\right),$$

implying that

$$\left(r^{N-1}\phi\left(\alpha'(r)\right)\right)'_{r=r_0} \leqslant \left(r^{N-1}\phi\left(u'(r)\right)\right)'_{r=r_0}.$$

Hence, because α is a lower solution of (4), we obtain

$$\begin{aligned} \left(r^{N-1}\phi\left(\alpha'(r)\right)\right)'_{r=r_0} &\leqslant \left(r^{N-1}\phi\left(u'(r)\right)\right)'_{r=r_0} \\ &= r_0^{N-1}\left[-g\left(r_0, \alpha(r_0)\right) + u(r_0) - \alpha(r_0)\right] \\ &< r_0^{N-1}\left[-g\left(r_0, \alpha(r_0)\right)\right] \\ &\leqslant \left(r^{N-1}\phi\left(\alpha'(r)\right)\right)'_{r=r_0}, \end{aligned}$$

a contradiction. If $r_0 = R$ then $\alpha(R) - u(R) > 0$. But $u(R) = 0$ and $\alpha(R) \leqslant 0$, obtaining again a contradiction. Finally, if $r_0 = 0$ then there exists $r_1 \in (0, R]$ such that $\alpha(r) - u(r) > 0$ for all $r \in [0, r_1]$ and $\alpha'(r_1) - u'(r_1) \leqslant 0$. It follows that

$$r_1^{N-1}\phi\left(\alpha'(r_1)\right) \leqslant r_1^{N-1}\phi\left(u'(r_1)\right).$$

On the other hand, integrating (6) from 0 to r_1 and using that α is a lower solution of (4) we obtain

$$r_1^{N-1}\phi\big(u'(r_1)\big) = \int_0^{r_1} r^{N-1}\big[-g\big(r,\alpha(r)\big)+u(r)-\alpha(r)\big]\,dr$$
$$< \int_0^{r_1} \big(r^{N-1}\phi\big(\alpha'(r)\big)\big)'\,dr$$
$$= r_1^{N-1}\phi\big(u'(r_1)\big),$$

a contradiction. Consequently, $\alpha(r) \leqslant u(r)$ for all $r \in [0, R]$. Analogously, it follows that $u(r) \leqslant \beta(r)$ for all $r \in [0, R]$. The proof is completed. □

Lemma 2. *Assume that* (4) *has a lower solution* α *and an upper solution* β *such that* $\alpha(r) \leqslant \beta(r)$ *for all* $r \in [0, R]$, *and let* $\Omega_{\alpha,\beta} := \{u \in C^1_M\colon\ \alpha \leqslant u \leqslant \beta\}$. *Assume also that problem* (4) *has an unique solution* u_0 *in* $\Omega_{\alpha,\beta}$ *and there exists* $\rho_0 > 0$ *such that* $\overline{B}(u_0, \rho_0) \subset \Omega_{\alpha,\beta}$. *Then,*

$$d_{LS}\big[I - \mathcal{N}_g, B(u_0,\rho), 0\big] = 1 \quad \textit{for all } 0 < \rho \leqslant \rho_0,$$

where $\mathcal{N}_g$ *is the fixed point operator associated to* (4).

Proof. Let $\mathcal{N}_\gamma$ be the fixed point operator associated to the modified problem (6). From the proof of Proposition 1 it follows that any fixed point u of $\mathcal{N}_\gamma$ is contained in $\Omega_{\alpha,\beta}$ and u is also a fixed point of $\mathcal{N}_g$. It follows that u_0 is the unique fixed point of $\mathcal{N}_\gamma$. Then, from Lemma 1 and the excision property of the Leray–Schauder degree one has that

$$d_{LS}\big[I - \mathcal{N}_\gamma, B(u_0,\rho), 0\big] = 1 \quad \text{for all } \rho > 0.$$

The result follows from the fact that

$$\mathcal{N}_\gamma(u) = \mathcal{N}_g(u) \quad \text{for all } u \in \overline{B}(u_0, \rho_0). \qquad \square$$

3. A Cauchy problem

In this section we consider the Cauchy problem

$$\big(r^{N-1}\phi\big(u'(r)\big)\big)' + r^{N-1}\big[\lambda\mu(r)p\big(u(r)\big)\big] = 0 \quad \big(r \in [0, R]\big),$$
$$u(0) = \xi, \qquad u'(0) = 0, \tag{7}$$

where $\lambda, \xi > 0$ and

- $\mu : [0, R] \to \mathbb{R}$ *is continuous*;
- $p : \mathbb{R} \to \mathbb{R}$ *is Lipschitz continuous on bounded sets.*

We denote $\mu_M := \max_{[0,R]} |\mu|$. In the proof of the next result we use some ideas from the last section in [12].

Proposition 2. *Assume* (H_ϕ) *and that* ϕ *is of class* C^1, $\phi' > 0$. *Then, problem* (7) *has an unique solution* $u(\lambda, \xi; \cdot)$ *and the mapping* $(\lambda, \xi) \mapsto u(\lambda, \xi; \cdot)$ *is continuous from* $(0, \infty) \times (0, \infty)$ *to* C^1.

Proof. We divide the proof in three steps.

1. *Existence*. Consider the nonlinear compact operator

$$\mathcal{C} : C \to C, \quad \mathcal{C}u(r) \equiv \xi - \int_0^r \phi^{-1}\left(\frac{1}{t^{N-1}} \int_0^t s^{N-1}\big[\lambda\mu(s) p\big(u(s)\big)\big]\, ds\right) dt.$$

One has that $u \in C$ is solution of (7) if and only if $u = \mathcal{C}u$. Using that $\|\mathcal{C}u\|_\infty < \xi + aR$ for all $u \in C$, it follows from Schauder's fixed point theorem that $\mathcal{C}$ has at least one fixed point u which is a solution of (7). Notice that

$$\|u\|_\infty < \xi + aR. \tag{8}$$

2. *Uniqueness*. Let u and v be solutions of (7) and

$$\omega = \phi\big(u'\big) - \phi\big(v'\big), \qquad \psi = \lambda\mu\big[p(v) - p(u)\big].$$

It follows that, for all $r \in [0, R]$, one has

$$\big|\omega(r)\big| = \left|\frac{1}{r^{N-1}} \int_0^r t^{N-1}\psi(t)\, dt\right| \leqslant \frac{R}{N} \sup_{[0,r]} |\psi|.$$

On the other hand, from (8) we have

$$\big|\psi(r)\big| \leqslant M\big|u(r) - v(r)\big| \quad \big(r \in [0, R]\big),$$

where $M = \lambda L \mu_M$ and L is the Lipschitz constant of p corresponding to the interval $[-(\xi + aR), \xi + aR]$. Hence, using that $u(0) = v(0)$, we infer that for all $r \in [0, R]$,

$$\big|\psi(r)\big| \leqslant M \int_0^r \big|u'(t) - v'(t)\big|\, dt \leqslant \frac{M}{m} \int_0^r \big|\omega(t)\big|\, dt,$$

where m is the minimum of ϕ' on the interval $[0, \max\{\|u'\|_\infty, \|v'\|_\infty\}]$. It follows that

$$\big|\omega(r)\big| \leqslant \frac{MR}{mN} \int_0^r \big|\omega(t)\big|\, dt \quad \big(r \in [0, R]\big),$$

which together with Gronwall's inequality imply $\omega = 0$, hence $u = v$.

3. *Continuous dependence on* (λ, ξ). Let $u(\lambda, \xi; \cdot)$ be the unique solution of (7). For $l, h \in \mathbb{R}$ sufficiently small, we set

$$u := u(\lambda, \xi; \cdot), \qquad v := u(\lambda + l, \xi + h; \cdot).$$

From (8) we may assume that

$$\|v\|_\infty < \xi + 1 + aR.$$

This and

$$-v'(r) = \phi^{-1}\left(\frac{1}{r^{N-1}} \int_0^r s^{N-1}\big[(\lambda + l)\mu(s) p\big(v(s)\big)\big]\, ds\right) \tag{9}$$

imply that there exists $\delta > 0$, which is independent on l and h, such that

$$\|v'\|_\infty \leqslant \delta < a.$$

Let ω, ψ be as in Step 2. Using (9), for all $r \in [0, R]$, one has

$$|\omega(r)| = \left|\frac{1}{r^{N-1}} \int_0^r t^{N-1}\big[\psi(t) - l\mu(t) p\big(v(t)\big)\big]\, dt\right| \leqslant \frac{R}{N}\Big[\sup_{[0,r]} |\psi| + |l| c\Big],$$

where $c = \mu_M \max_{[-(\xi+1+aR), \xi+1+aR]} |p|$. On the other hand, arguing as above we infer that for all $r \in [0, R]$,

$$|\psi(r)| \leqslant \frac{M}{k} \int_0^r |\omega(t)|\, dt + M|h|,$$

where $M = \lambda L \mu_M$ and L is the Lipschitz constant of p corresponding to the interval $[-(\xi + 1 + aR), \xi + 1 + aR]$, and k is the minimum of ϕ' on the interval $[0, \delta]$. It follows

$$|\omega(r)| \leqslant \frac{cR|l| + MR|h|}{N} + \frac{MR}{kN} \int_0^r |\omega(t)|\, dt \quad \big(r \in [0, R]\big),$$

which together with Gronwall's inequality imply that

$$|\omega(r)| \leqslant \left(\frac{cR|l| + MR|h|}{N}\right) \exp\left(\frac{MR^2}{kN}\right) \quad \big(r \in [0, R]\big).$$

So, $\|u' - v'\|_\infty \to 0$ as $l, h \to 0$, implying also that $\|u - v\|_\infty \to 0$. $\square$

4. Non-negative nonlinearities, positive solutions and degree around zero

Here, we consider mixed boundary value problems of the type

$$\big(r^{N-1}\phi\big(u'\big)\big)' + r^{N-1} f(r,u) = 0, \qquad u'(0) = 0 = u(R), \tag{10}$$

where $N \geqslant 2$ is an integer, $R > 0$ under hypotheses (H_ϕ) and

(H_f) $f : [0, R] \times [0, \infty) \to [0, \infty)$ *is continuous and* $f(r,s) > 0$ *for all* $(r,s) \in (0, R] \times (0, \infty)$.

We need the following elementary result, which is proved in [8].

Lemma 3. *Assume* (H_ϕ), (H_f) *and let* u *be a nontrivial solution of*

$$\big(r^{N-1}\phi\big(u'\big)\big)' + r^{N-1} f\big(r,|u|\big) = 0, \qquad u'(0) = 0 = u(R). \tag{11}$$

Then $u > 0$ *on* $[0, R)$ *and* u *is strictly decreasing.*

Notice that, by virtue of Lemma 3, u is a nontrivial solution of the mixed boundary value problem (11) if and only if u is a positive solution of (10). In this case, u is strictly decreasing.

Let $\mathcal{N}_f$ be the fixed point operator associated to (11). In the next lemma we assume that f is sublinear with respect to ϕ at zero.

Lemma 4. *Assume* (H_ϕ), (H_f),

$$\lim_{s\to 0+} \frac{f(r,s)}{\phi(s)} = 0 \quad \textit{uniformly for } r \in [0, R] \tag{12}$$

and

$$\liminf_{s\to 0} \frac{\phi(\sigma s)}{\phi(s)} > 0 \quad \textit{for all } \sigma > 0. \tag{13}$$

Then there exists $\rho_0 > 0$ *such that*

$$d_{LS}[I - \mathcal{N}_f, B_\rho, 0] = 1 \quad \textit{for all } 0 < \rho \leqslant \rho_0.$$

Proof. Using (13) we can find $\varepsilon > 0$ such that

$$R\varepsilon/N < \liminf_{s\to 0} \frac{\phi(s/R)}{\phi(s)}. \tag{14}$$

From (12) it follows that there exists $s_\varepsilon > 0$ such that

$$f(r,s) \leqslant \varepsilon\phi(s) \quad \text{for all } (r,s) \in [0, R] \times [0, s_\varepsilon]. \tag{15}$$

Let us consider the compact homotopy

$$\mathcal{H} : [0,1] \times C_M^1 \to C_M^1, \quad \mathcal{H}(\tau, u) = \tau \mathcal{N}_f(u).$$

We will show that there exists $\rho_0 > 0$ such that

$$u \neq \mathcal{H}(\tau, u) \quad \text{for all } (\tau, u) \in [0,1] \times \left(\overline{B}_{\rho_0} \setminus \{0\}\right). \tag{16}$$

By contradiction, assume that one has

$$u_k = \tau_k \mathcal{N}_f(u_k)$$

with $\tau_k \in [0,1]$, $u_k \in C_M^1 \setminus \{0\}$ for all $k \in \mathbb{N}$ and $\|u_k\| \to 0$. Using Lemma 3 it follows that u_k are strictly decreasing functions which are also strictly positive on $[0, R)$. Passing if necessary to a subsequence, we may assume that $\|u_k\| \leqslant s_\varepsilon$ for all $k \in \mathbb{N}$, and then using (15) it follows

$$f\left(r, u_k(r)\right) \leqslant \varepsilon \phi\left(\|u_k\|_\infty\right) \quad \text{for all } r \in [0, R],\ k \in \mathbb{N}.$$

This implies that, for any $k \in \mathbb{N}$,

$$\begin{aligned} \|u_k\|_\infty &\leqslant \int_0^R \phi^{-1}\left(\sigma(t) \int_0^t r^{N-1} f\left(r, u_k(r)\right) dr\right) dt \\ &\leqslant R\phi^{-1}\left(\frac{\varepsilon R}{N} \phi\left(\|u_k\|_\infty\right)\right). \end{aligned}$$

It follows

$$\frac{\phi(\frac{1}{R}\|u_k\|_\infty)}{\phi(\|u_k\|_\infty)} \leqslant \frac{\varepsilon R}{N} \quad (k \in \mathbb{N}),$$

which together with $\|u_k\|_\infty \to 0$ contradict (14). Hence, (16) holds true. So, for any $\rho \in (0, \rho_0]$ one has

$$d_{LS}\left[I - \mathcal{H}(1, \cdot), B_\rho, 0\right] = d_{LS}\left[I - \mathcal{H}(0, \cdot), B_\rho, 0\right],$$

implying that

$$d_{LS}[I - \mathcal{N}_f, B_\rho, 0] = d_{LS}[I, B_\rho, 0] = 1,$$

and the proof is complete. □

5. Main result

Now, we come to study the one-parameter problem (3) under the hypothesis

(H) *$N \geqslant 2$ is an integer, $R > 0$, $q > 1$ and $\mu : [0, \infty) \to \mathbb{R}$ is continuous, $\mu(r) > 0$ for all $r > 0$.*

As the results in the previous sections apply with

$$\phi(s) = \frac{s}{\sqrt{1 - s^2}} \quad \big(s \in (-1, 1)\big),$$

note that $u \in C^1$ is a positive solution of (3) if and only if u is a nontrivial solution of

$$\left(r^{N-1} \frac{u'}{\sqrt{1 - u'^2}}\right)' + r^{N-1}\big[\lambda\mu(r)|u|^q\big] = 0, \qquad u'(0) = 0 = u(R); \tag{17}$$

in this case, u is strictly decreasing.

The main result of the paper is the following one. Notice that $\mu_M = \max_{[0,R]} \mu$.

Theorem 1. *Under hypothesis (H), there exists $\Lambda > 2N/(\mu_M R^{q+1})$ such that problem (3) has zero, at least one or at least two positive solutions according to $\lambda \in (0, \Lambda)$, $\lambda = \Lambda$ or $\lambda > \Lambda$. Moreover, Λ is strictly decreasing with respect to R.*

Proof. We denote

$$\begin{aligned} S_j &:= \big\{\lambda > 0\text{: (3) has at least } j \text{ positive solutions}\big\} \\ &= \big\{\lambda > 0\text{: (17) has at least } j \text{ non-trivial solutions}\big\} \quad (j = 1, 2) \end{aligned}$$

and divide the proof in three steps.

1. *Finding Λ.* Let $\lambda > 0$ and u be a positive solution of (3). Integrating (3) on $[0, r]$, it follows

$$-r^{N-1} \frac{u'(r)}{\sqrt{1 - u'^2(r)}} = \lambda \int_0^r t^{N-1}\mu(t)u^q(t)\,dt \quad \text{for all } r \in [0, R].$$

Using that u is strictly decreasing on $[0, R]$, we deduce that, for all $r \in [0, R]$, one has

$$\begin{aligned} -r^{N-1}u'(r) &\leqslant -r^{N-1} \frac{u'(r)}{\sqrt{1 - u'^2(r)}} \\ &\leqslant \lambda u^q(0)\mu_M r^N/N \end{aligned}$$

and integrating over $[0, R]$, we obtain

$$u(0) \leqslant \lambda u^q(0)\mu_M R^2/(2N). \tag{18}$$

This, together with $0 < u(0) < R$ (see (5)) and $q > 1$ imply

$$\lambda > 2N/\big(\mu_M R^{q+1}\big).$$

From [8, Corollary 2] we know that (3) has a least one positive solution for $\lambda > 0$, sufficiently large. In particular, $S_1 \neq \emptyset$ and we can define

$$\Lambda = \Lambda(R) := \inf S_1.$$

Clearly, we have $\Lambda \geqslant 2N/(\mu_M R^{q+1})$. We *claim* that $\Lambda \in S_1$. Indeed, let $\{\lambda_k\} \subset S_1$ be a sequence converging to Λ, and $u_k \in C^1_M$ be positive on $[0, R)$ such that

$$u_k = K \circ \phi^{-1} \circ S \circ \big(\lambda_k \mu u_k^q\big).$$

Then, from (5) and the Arzela–Ascoli theorem, we infer that there exists $u \in C$ such that, passing eventually to a subsequence, $\{u_k\}$ converges to u in C. So, it follows that $u \geqslant 0$ and

$$u = K \circ \phi^{-1} \circ S \circ \big(\Lambda \mu u^q\big).$$

Using (18) we deduce that there is a constant $c_1 > 0$ such that $u_k(0) > c_1$, for all $k \in \mathbb{N}$. This ensures that $u(0) \geqslant c_1$, hence $u > 0$ on $[0, R)$ (by Lemma 3) and the claim is proved. Also, it is clear that $\Lambda > 2N/(\mu_M R^{q+1})$.

Next, let $\lambda_0 > \Lambda$ be arbitrarily chosen. We shall apply Proposition 1 to show that $\lambda_0 \in S_1$. In this view, let u_1 be a positive solution for (3) corresponding to $\lambda = \Lambda$. It is easy to see that u_1 is a lower solution for (17) with $\lambda = \lambda_0$. To construct an upper solution, let $H > 0$, $\widetilde{R} > R$ and consider the mixed problem

$$\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)' + r^{N-1}H = 0, \qquad u'(0) = 0 = u(\widetilde{R}). \tag{19}$$

Then, by a simple integration, one has that the unique (positive) solution of (19) is given by

$$u(r) = \frac{N}{H}\left[\sqrt{1+\frac{H^2}{N^2}\widetilde{R}^2} - \sqrt{1+\frac{H^2}{N^2}r^2}\right] \quad \big(r \in [0, \widetilde{R}]\big).$$

For fixed $\lambda_2 > \lambda_0$, let u_2 be the solution of (19) corresponding to $H = \lambda_2 \mu_M \widetilde{R}^q$. Using that $u_2(R) > 0$ and

$$\lambda_0 \mu(r) u_2^q(r) \leqslant \lambda_2 \mu_M \widetilde{R}^q \quad \big(r \in [0, R]\big),$$

it follows that u_2 is an upper solution for (17) with $\lambda = \lambda_0$. Since

$$u_2(R) = N\left[\sqrt{\frac{1}{(\lambda_2\mu_M)^2\widetilde{R}^{2q}} + \frac{\widetilde{R}^2}{N^2}} - \sqrt{\frac{1}{(\lambda_2\mu_M)^2\widetilde{R}^{2q}} + \frac{R^2}{N^2}}\right],$$

we can find $\widetilde{R}$ sufficiently large, such that $u_1(0) < u_2(R)$. Then, taking into account that u_1, u_2 are strictly decreasing, we infer that $u_1 < u_2$ on $[0, R]$. By virtue of Proposition 1, we get $\lambda_0 \in S_1$. Therefore, we have

$$S_1 = [\Lambda, \infty).$$

2. *Multiplicity.* We use some ideas from the proof of Theorem 3.10 in [2]. Let $\lambda_0 > \Lambda$. We shall apply Lemmas 1, 2, 4 to show that $\lambda_0 \in S_2$. With this aim, let u_1, u_2 be constructed as in Step 1 and u_0 be a solution of (17) with $\lambda = \lambda_0$ such that $u_1 \leqslant u_0 \leqslant u_2$, i.e., $u_0 \in \Omega_{u_1,u_2}$ (see Lemma 2).

First, we *claim* that there exists $\varepsilon > 0$ with $\overline{B}(u_0, \varepsilon) \subset \Omega_{u_1,u_2}$. Notice that, for all $r \in [0, R]$, one has

$$u_2(r) = \int_r^{\widetilde{R}} \phi^{-1}\left(\sigma(t)\int_0^t s^{N-1}\left[\lambda_2\mu_M\widetilde{R}^q\right]ds\right)dt,$$

implying that

$$\begin{aligned} u_2(r) &> \int_r^{R} \phi^{-1}\left(\sigma(t)\int_0^t s^{N-1}\left[\lambda_2\mu(s)u_2^q(s)\right]ds\right)dt \\ &\geqslant \int_r^{R} \phi^{-1}\left(\sigma(t)\int_0^t s^{N-1}\left[\lambda_0\mu(s)u_0^q(s)\right]ds\right)dt \\ &= u_0(r), \end{aligned}$$

so, there exists $\varepsilon_2 > 0$ such that $v \leqslant u_2$ for all $v \in \overline{B}(u_0, \varepsilon_2)$. Similar arguments show that $u_1 < u_0$ on $[0, R/2]$. Thus, we can find $\varepsilon_1' > 0$ so that

$$v \in C_M^1 \quad \text{and} \quad \|v - u_0\|_\infty \leqslant \varepsilon_1' \quad \Rightarrow \quad v \geqslant u_1 \quad \text{on } [0, R/2]. \tag{20}$$

On the other hand, we have

$$-u_0' = \phi^{-1} \circ S \circ \left[\lambda_0\mu u_0^q\right] \quad \text{and} \quad -u_1' = \phi^{-1} \circ S \circ \left[\Lambda\mu u_1^q\right],$$

yielding ${u'}_0 < {u'}_1$ on $[R/2, R]$. So, we can find $\varepsilon_1 \in (0, \varepsilon_1')$ sufficiently small, such that $v' < {u'}_1$ on $[R/2, R]$ whenever $v \in \overline{B}(u_0, \varepsilon_1)$. Then, using $u_0(R) = 0 = v(R)$, we deduce that $v > u_1$ on $[R/2, R)$, for all $v \in \overline{B}(u_0, \varepsilon_1)$. Now, on account of (20), the claim follows with any $\varepsilon \in (0, \min\{\varepsilon_1, \varepsilon_2\})$.

Next, if (17) has a second solution contained in Ω_{u_1,u_2}, this solution is nontrivial and the proof of the multiplicity part is completed. If not, using Lemma 2 we deduce that

$$d_{LS}\left[I - \mathcal{N}_{\lambda_0}, B(u_0, \rho), 0\right] = 1 \quad \text{for all } 0 < \rho \leqslant \varepsilon,$$

where $\mathcal{N}_{\lambda_0}$ is the fixed point operator associated to (17) with $\lambda = \lambda_0$. On the other hand, from Lemma 1 one has

$$d_{LS}[I - \mathcal{N}_{\lambda_0}, B_\rho, 0] = 1 \quad \text{for all } \rho \geqslant R + 1,$$

and from Lemma 4 we have

$$d_{LS}[I - \mathcal{N}_{\lambda_0}, B_\rho, 0] = 1 \quad \text{for all } \rho \text{ sufficiently small.}$$

Now, consider $\rho_1, \rho_2 > 0$ sufficiently small and $\rho_3 \geqslant R + 1$ such that $\overline{B}(u_0, \rho_1) \cap \overline{B}_{\rho_2} = \emptyset$ and $\overline{B}(u_0, \rho_1) \cup \overline{B}_{\rho_2} \subset B_{\rho_3}$. Then, from the additivity-excision property of the Leray–Schauder degree it follows that

$$d_{LS}\big[I - \mathcal{N}_{\lambda_0}, B_{\rho_3} \setminus \big[\overline{B}(u_0, \rho_1) \cup \overline{B}_{\rho_2}\big], 0\big] = -1,$$

which, together with the existence property of the Leray–Schauder degree, imply that $\mathcal{N}_{\lambda_0}$ has a fixed point $\widetilde{u}_0 \in B_{\rho_3} \setminus [\overline{B}(u_0, \rho_1) \cup \overline{B}_{\rho_2}]$. We infer that (3) has a second positive solution.

3. *Monotonicity of Λ.* Let u_0 be a nontrivial solution of (17) with $\lambda = \lambda_0 := \Lambda(R_0)$ and $R = R_0$. We fix $R > R_0$. Then, setting $\xi_0 = u_0(0)$, from Proposition 2 with $p(s) = |s|^q$, one has that $u(\lambda_0, \xi_0; \cdot)|_{[0,R_0]} = u_0$. Since $u(\lambda_0, \xi_0; \cdot)$ is strictly decreasing on $[0, R]$ (this is easily seen) and $u(\lambda_0, \xi_0; R_0) = 0$, it follows that $u(\lambda_0, \xi_0; R) < 0$. Using again Proposition 2, we infer that there exists $\varepsilon > 0$ such that $u(\lambda, \xi_0; R) < 0$ for all $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$; in particular, $u(\lambda, \xi_0; \cdot)$ is a lower solution of (17). Arguing exactly as in Step 1, we can show that (17) has an upper solution β_λ such that $u(\lambda, \xi_0, \cdot) \leqslant \beta_\lambda$ on $[0, R]$. Then, applying Proposition 1 we deduce that (17) has at least one nonzero solution which is also a strictly positive solution of (3). Consequently, $\Lambda(R_0) > \Lambda(R)$ and the proof is complete. □

Corollary 1. *Under hypothesis (H), there exists $\Lambda > 2N/(\mu_M R^{q+1})$ such that problem (1) has zero, at least one or at least two positive classical radial solutions according to $\lambda \in (0, \Lambda)$, $\lambda = \Lambda$ or $\lambda > \Lambda$. Also, Λ is strictly decreasing with respect to R.*

Example 1. If $N \geqslant 2$ is an integer and $q > 1$, $m \geqslant 0$, $R > 0$ are real numbers, then there exists $\Lambda > 2N/R^{m+q+1}$ such that the problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}}\right) + \lambda |x|^m v^q = 0 \quad \text{in } \mathcal{B}(R), \qquad v = 0 \quad \text{on } \partial\mathcal{B}(R),$$

has zero, at least one or at least two positive classical radial solutions according to $\lambda \in (0, \Lambda)$, $\lambda = \Lambda$ or $\lambda > \Lambda$. In addition, Λ is strictly decreasing with respect to R.

Remark 1. The reader will emphasize that, excepting the part concerning the monotonicity of Λ as function of R, the statements of Theorem 1 and Corollary 1 still remain true if the continuous weight function μ is defined only on $[0, R]$ instead of $[0, \infty)$ and positive on $(0, R]$.

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