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Multiple solutions for Neumann and periodic problems with singular ϕ-Laplacian

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Abstract

We use the critical point theory for convex, lower semicontinuous perturbations of C^1-functionals to establish existence of multiple radial solutions for some one parameter Neumann problems involving the operator $v \mapsto \operatorname{div}(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}})$. Similar results for periodic problems are also provided.

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1. Introduction

This paper is motivated by the existence of nontrivial solutions for the Neumann problems:

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$$-\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right)+\alpha|v|^{p-2}v=f\big(|x|,v\big)+\lambda b\big(|x|\big)|v|^{q-2}v \quad \text{in } \mathcal{A},$$
$$\frac{\partial v}{\partial \nu}=0 \quad \text{on } \partial\mathcal{A}, \tag{1}$$

$$-\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right)+\lambda|v|^{m-2}v=f\big(|x|,v\big)+h\big(|x|\big) \quad \text{in } \mathcal{A},$$
$$\frac{\partial v}{\partial \nu}=0 \quad \text{on } \partial\mathcal{A} \tag{2}$$

and for the periodic problems:

$$-\left(\frac{u'}{\sqrt{1-|u'|^2}}\right)'+\alpha|u|^{p-2}u=f(r,u)+\lambda b(r)|u|^{q-2}u \quad \text{in } [R_1,R_2],$$
$$u(R_1)-u(R_2)=0=u'(R_1)-u'(R_2), \tag{3}$$

$$-\left(\frac{u'}{\sqrt{1-|u'|^2}}\right)'+\lambda|u|^{m-2}u=f(r,u)+h(r) \quad \text{in } [R_1,R_2],$$
$$u(R_1)-u(R_2)=0=u'(R_1)-u'(R_2), \tag{4}$$

where $0\leqslant R_1<R_2$ and $\mathcal{A}=\{x\in\mathbb{R}^N\colon\ R_1\leqslant|x|\leqslant R_2\}$.

We assume the following hypothesis on the data.

(H_f) *The functions* $f:[R_1,R_2]\times\mathbb{R}\to\mathbb{R}$, $b,h:[R_1,R_2]\to\mathbb{R}$ *are continuous; the constants* $\alpha>0$, $p>q\geqslant 2$, $m\geqslant 2$ *are fixed and* λ *is a real positive parameter.*

Viewing the radial symmetry, we shall look for radial solutions of problems (1) and (2). So, letting $r=|x|$ and $v(x)=u(r)$, we reduce (1) and (2) to the one-dimensional Neumann problems

$$\big[r^{N-1}\phi\big(u'\big)\big]'=r^{N-1}\big[\alpha|u|^{p-2}u-f(r,u)-\lambda b(r)|u|^{q-2}u\big] \quad \text{in } [R_1,R_2],$$
$$u'(R_1)=0=u'(R_2), \tag{5}$$

and

$$\big[r^{N-1}\phi\big(u'\big)\big]'=r^{N-1}\big[\lambda|u|^{m-2}u-f(r,u)-h(r)\big] \quad \text{in } [R_1,R_2],$$
$$u'(R_1)=0=u'(R_2), \tag{6}$$

where $\phi(y)=\frac{y}{\sqrt{1-y^2}}$, $\forall y\in(-1,1)$. Also, it is clear that problems (3) and (4) can be rewritten as

$$\big[\phi\big(u'\big)\big]'=\alpha|u|^{p-2}u-f(r,u)-\lambda b(r)|u|^{q-2}u \quad \text{in } [R_1,R_2],$$
$$u(R_1)-u(R_2)=0=u'(R_1)-u'(R_2), \tag{7}$$

and

$$\begin{aligned} \left[\phi\left(u'\right)\right]' &= \lambda|u|^{m-2}u - f(r,u) - h(r) \quad \text{in } [R_1, R_2], \\ u(R_1) - u(R_2) &= 0 = u'(R_1) - u'(R_2), \end{aligned} \tag{8}$$

with the same choice of ϕ.

More generally, in this paper the mapping $\phi : (-a, a) \to \mathbb{R}$ entering in the above boundary value problems will be an increasing homeomorphism with $\phi(0) = 0$. Following [7], this type of ϕ is called *singular*. Precisely, we assume the following hypothesis on ϕ introduced in [8] (see also [5,6,9]):

(H_Φ) *$\Phi : [-a, a] \to \mathbb{R}$ is continuous, of class C^1 on $(-a, a)$, $\Phi(0) = 0$ and $\phi := \Phi' : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$.*

Denoting by F the indefinite integral of f with respect to the second variable, it is easy to see that if F satisfies

$$\limsup_{|x|\to 0} \frac{pF(r,x)}{|x|^p} < \alpha \quad \text{uniformly in } r \in [R_1, R_2], \tag{9}$$

then $f(r, 0) = 0$ for all $r \in [R_1, R_2]$, meaning that problems (5)–(8) admit the trivial solution $u = 0$ provided that $h \equiv 0$. If, in addition, F satisfies the Ambrosetti–Rabinowitz type condition [4]:

(AR) *there exists $\theta > p$ and $x_0 > 0$ such that*

$$0 < \theta F(r,x) \leqslant x f(r,x) \quad \text{for all } r \in [R_1, R_2] \text{ and } |x| \geqslant x_0, \tag{10}$$

then problems (5) and (7) with $\lambda = 0$ or problems (6) and (8) with $h \equiv 0$ have at least one nontrivial solution (see [5]).

We prove in Theorem 1 and Theorem 2 that if, in addition to (9) and (10) we assume:

(i) *either*

$$\liminf_{x\to 0-} \frac{F(r,x)}{|x|^p} \geqslant 0 \quad \textit{uniformly in } r \in [R_1, R_2] \tag{11}$$

or

$$\liminf_{x\to 0+} \frac{F(r,x)}{x^p} \geqslant 0 \quad \textit{uniformly in } r \in [R_1, R_2]; \tag{12}$$

(ii) *it holds*

$$\int_{R_1}^{R_2} r^{N-1} b(r)\, dr > 0,$$

then problems (5) and (7) have at least two nontrivial solutions for sufficiently small values of the parameter λ. It is easy to see that those assumptions correspond to problems with *convex–concave nonlinearities* initiated in 1994 for semilinear Dirichlet problems by Ambrosetti, Brezis and Cerami [2], extended to quasilinear Dirichlet problems involving the p-Laplacian by Ambrosetti, Garcia Azorero, Peral [3] and Garcia Azorero, Peral, Manfredi [13]. Radial solutions with Dirichlet conditions have been considered independently by Kormann [16], using bifurcation theory and by Tang [31] using ordinary differential equations methods.

One the other hand, under the hypotheses:

(i)′ *there exists* $k_1, k_2 > 0$ *and* $0 < \sigma < m$ *such that*

$$-l(r) \leqslant F(r,x) \leqslant k_1|x|^{\sigma} + k_2, \quad \text{for all } (r,x) \in [R_1, R_2] \times \mathbb{R}, \tag{13}$$

where $l \geqslant 0$ *is measurable and* $\int_{R_1}^{R_2} r^{N-1} l(r)\,dr < +\infty$;

(ii)′ *one has that either*

$$\lim_{|x|\to\infty} \int_{R_1}^{R_2} r^{N-1} F(r,x)\,dr = +\infty, \tag{14}$$

or the limits $F_{\pm}(r) = \lim_{x\to\pm\infty} F(r,x)$ *exist for all* $r \in [R_1, R_2]$ *and*

$$\begin{aligned} &F(r,x) < F_+(r), \quad \forall r \in [R_1, R_2],\ x \geqslant 0, \\ &F(r,x) < F_-(r), \quad \forall r \in [R_1, R_2],\ x \leqslant 0; \end{aligned} \tag{15}$$

(iii)′ *it holds*

$$\int_{R_1}^{R_2} r^{N-1} h(r)\,dr = 0,$$

we prove in Theorem 3 (see also Theorem 4 for the periodic case) that problem (6) has at least three solutions for sufficiently small values of the parameter λ. Results of this type in the classical case, called *multiplicity results near resonance*, have been initiated in [23] (for $N = 1$), using bifurcation from infinity and Leray–Schauder degree theory. A variational approach was introduced by Sanchez in [28] to attack such multiplicity problems, and conditions of type (i)′ and (ii)′ were introduced by Ma, Ramos and Sanchez in [27,20] for semilinear and quasilinear Dirichlet problems involving the p-Laplacian. See also [21,19,24,10,26] for a similar variational treatment of various semilinear or quasilinear equations, systems or inequalities with Dirichlet conditions, [25] for perturbations of p-Laplacian with Neumann boundary conditions, and [18] for periodic solutions of perturbations of the one-dimensional p-Laplacian. The existence of at least two solutions near resonance at a nonprincipal eigenvalue have been first obtained in [22] using a topological approach and then for semilinear or quasilinear problems using critical point theory in [11,15,29], but this question seems to be meaningless for the singular ϕ considered here because resonance only occurs at 0.

The main used tools are some abstract local minimization results combined with mountain pass techniques in the frame of the Szulkin's critical point theory [30]. The rest of the paper is organized as follows. In Section 2 we give some abstract results (Proposition 1 and Proposition 2) which we need in the sequel. The concrete functional framework and the variational setting, employed in the treatment of the above problems, are described in Section 3. Sections 4 and 5 are devoted to the proofs of the main multiplicity results.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space and I be a functional of the type

$$I = \mathcal{F} + \psi,$$

where $\psi : X \to (-\infty, +\infty]$ is proper (i.e., $D(\psi) := \{v \in X:\ \psi(v) < +\infty\} \neq \emptyset$), convex, lower semicontinuous (in short, l.s.c.) and $\mathcal{F} \in C^1(X; \mathbb{R})$.

According to Szulkin [30], a point $u \in X$ is said to be *a critical point* of I if it satisfies the inequality

$$\left\langle \mathcal{F}'(u), v - u \right\rangle + \psi(v) - \psi(u) \geqslant 0, \quad \forall v \in X.$$

A number $c \in \mathbb{R}$ such that $I^{-1}(c)$ contains a critical point is called *a critical value* of I.

The functional I is said *to satisfy the Palais–Smale* (in short, (PS)) *condition* if every sequence $\{u_n\} \subset X$ for which $I(u_n) \to c \in \mathbb{R}$ and

$$\left\langle \mathcal{F}'(u_n), v - u_n \right\rangle + \psi(v) - \psi(u_n) \geqslant -\varepsilon_n \|v - u_n\|, \quad \forall v \in X,$$

where $\varepsilon_n \to 0$ (called (PS)-*sequence*), possesses a convergent subsequence.

Proposition 1. *Suppose that I satisfies the (PS) condition and there exists an open set U such that*

$$-\infty < \inf_{\overline{U}} I < \inf_{\partial U} I. \tag{16}$$

Then I has at least one critical point $u \in U$ such that $I(u) = \inf_{\overline{U}} I$.

Proof. Let

$$c_0 = \inf_{\overline{U}} I \tag{17}$$

and $\{\varepsilon_n\}$ be a sequence with $\varepsilon_n \to 0$ and

$$0 < \varepsilon_n < \inf_{\partial U} I - c_0 \quad \text{for all } n \in \mathbb{N}. \tag{18}$$

Using Ekeland's variational principle, applied to $I|_{\overline{U}}$, for each $n \in \mathbb{N}$, we can find $v_n \in \overline{U}$ such that

$$I(v_n) \leqslant c_0 + \varepsilon_n \tag{19}$$

and

$$I(v) \geqslant I(v_n) - \varepsilon_n \|v - v_n\| \quad \text{for all } v \in \overline{U}. \tag{20}$$

From (18) and (19) it follows $I(v_n) < \inf_{\partial U} I$, which ensures that $v_n \in U$, for all $n \in \mathbb{N}$. Let $v \in X$, $n \in \mathbb{N}$ be arbitrarily chosen and $t_0 := t_0(v, n) \in (0, 1)$ be so that $v_n + t(v - v_n) \in U$, for all $t \in (0, t_0)$. Using (20) and the convexity of ψ, we get

$$\frac{\mathcal{F}(v_n + t(v - v_n)) - \mathcal{F}(v_n)}{t} + \psi(v) - \psi(v_n) \geqslant -\varepsilon_n \|v - v_n\|$$

and, letting $t \to 0_+$, one obtains

$$\langle \mathcal{F}'(v_n), v - v_n \rangle + \psi(v) - \psi(v_n) \geqslant -\varepsilon_n \|v - v_n\| \quad \text{for all } v \in X. \tag{21}$$

On the other hand, from (19) it is clear that

$$I(v_n) \to c_0. \tag{22}$$

Since I satisfies the (PS) condition, (21) and (22) ensure that $\{v_n\}$ contains a subsequence, still denoted by $\{v_n\}$, convergent to some $u \in \overline{U}$.

By the lower semicontinuity of ψ it holds

$$\psi(u) \leqslant \liminf_{n\to\infty} \psi(v_n) \tag{23}$$

and, on account of $\mathcal{F} \in C^1(X; \mathbb{R})$, one obtains

$$\lim_{n\to\infty} \langle \mathcal{F}'(v_n), v - v_n \rangle = \langle \mathcal{F}'(u), v - u \rangle \quad \text{for all } v \in X. \tag{24}$$

From (21), (23) and (24) we deduce

$$\langle \mathcal{F}'(u), v - u \rangle + \psi(v) - \psi(u) \geqslant 0 \quad \text{for all } v \in X. \tag{25}$$

Also, from (17), (22) and (23) we have

$$c_0 \leqslant I(u) \leqslant \lim_{n\to\infty} \mathcal{F}(v_n) + \liminf_{n\to\infty} \psi(v_n) = \liminf_{n\to\infty} I(v_n) = c_0,$$

hence $I(u) = c_0$ and from (16), $u \in U$. This together with (25) shows that c_0 is a critical value of I. □

For $\sigma > 0$, we shall denote $B_\sigma = \{v \in X: \|v\| < \sigma\}$ and by $\overline{B}_\sigma$ its closure.

Proposition 2. *Suppose that I satisfies the (PS) condition together with*

(i) $I(0) = 0$ *and there exists* $\rho > 0$ *such that*

$$-\infty < \inf_{\overline{B}_\rho} I < 0 < \inf_{\partial B_\rho} I; \tag{26}$$

(ii) $I(e) \leqslant 0$ *for some* $e \in X \setminus \overline{B}_\rho$.

Then I has at least two nontrivial critical points.

Proof. From the Mountain Pass Theorem [30, Theorem 3.2] there exists a first nontrivial critical point $u_0 \in X$ with $I(u_0) > 0$. On the other hand, using Proposition 1 with $U = B_\rho$ and (26), it follows that $\inf_{\overline{B}_\rho} I$ is a critical value of I. This implies the existence of a second critical point u_1 with $I(u_1) < 0$. We have that u_1 is nontrivial and different from u_0 because $I(0) = 0$ and $I(u_0) > 0$. □

Remark 1.

(i) It is a simple matter to check that if, in addition ψ and $\mathcal{F}$ are even, then I has at least four nontrivial critical points.
(ii) If the operator $\mathcal{F}' : X \to X^*$ maps bounded sets into bounded sets, then condition $-\infty < \inf_{\overline{B}_\rho} I$ in (26) is automatically satisfied. Indeed, in this case, by the mean value theorem one has

$$\left|\mathcal{F}(u) - \mathcal{F}(0)\right| \leqslant \rho \sup_{v \in \overline{B}_\rho} \left\|\mathcal{F}'(v)\right\| \quad \text{for all } u \in \overline{B}_\rho,$$

showing that $\mathcal{F}$ is bounded on $\overline{B}_\rho$. On the other hand, we know that the proper, convex and l.s.c. function ψ is bounded from below by a continuous affine function.
(iii) Proposition 2 is implicitly employed in [17] to derive the existence of at least two nontrivial solutions for a variational inequality on the half line.

3. Hypotheses and the functional framework

Throughout this paper we assume that *hypotheses* (H_f) *and* (H_Φ) *from Section* 1 *hold true*. Clearly, from (H_Φ) we have that Φ is strictly convex and $\Phi(x) \geqslant 0$ for all $x \in [-a, a]$. Also, it is worth noticing that choosing $\Phi(y) = 1 - \sqrt{1 - y^2}$, $\forall y \in [-1, 1]$, one has $\phi(y) = \frac{y}{\sqrt{1-y^2}}$, $\forall y \in (-1, 1)$, as it is particularly involved when dealing with problems (1)–(4).

The approaches for problems (5) and (7) (resp. (6) and (8)) are based on the Szulkin's critical point theory and are quite similar. That is why we shall treat in detail problem (5) (resp. (6)) and we restrict ourselves to only point out the corresponding adaptations for the treatment of problem (7) (resp. (8)).

We set $C := C[R_1, R_2]$, $L^1 := L^1(R_1, R_2)$, $L^\infty := L^\infty(R_1, R_2)$ and $W^{1,\infty} := W^{1,\infty}(R_1, R_2)$. The usual norm $\|\cdot\|_\infty$ is considered on C and L^∞. The space $W^{1,\infty}$ is endowed with the norm

$$\|v\| = \|v\|_\infty + \|v'\|_\infty, \quad v \in W^{1,\infty}.$$

Denoting

$$L^1_{N-1} := \left\{ v : (R_1, R_2) \to \mathbb{R} \text{ measurable: } \int_{R_1}^{R_2} r^{N-1} |v(r)|\, dr < +\infty \right\},$$

each $v \in L^1_{N-1}$ can be written $v(r) = \bar{v} + \tilde{v}(r)$, with

$$\bar{v} := \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} v(r) r^{N-1}\, dr, \qquad \int_{R_1}^{R_2} \tilde{v}(r) r^{N-1}\, dr = 0.$$

If $v \in W^{1,\infty}$ then $\tilde{v}$ vanishes at some $r_0 \in (R_1, R_2)$ and

$$|\tilde{v}(r)| = |\tilde{v}(r) - \tilde{v}(r_0)| \leqslant \int_{R_1}^{R_2} |v'(t)|\, dt \leqslant (R_2 - R_1)\|v'\|_\infty,$$

so, one has that

$$\|\tilde{v}\|_\infty \leqslant (R_2 - R_1)\|v'\|_\infty. \tag{27}$$

Putting

$$K := \{ v \in W^{1,\infty} : \|v'\|_\infty \leqslant a \},$$

it is clear that K is a convex subset of $W^{1,\infty}$.

Let $\Psi : C \to (-\infty, +\infty]$ be defined by

$$\Psi(v) = \begin{cases} \int_{R_1}^{R_2} r^{N-1} \Phi(v')\, dr, & \text{if } v \in K, \\ +\infty, & \text{otherwise.} \end{cases}$$

Obviously, Ψ is proper and convex. On the other hand, as shown in [6] (also, see [5]), $K \subset C$ is closed and Ψ is lower semicontinuous on C.

Next, denoting by $F : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ the primitive of f, i.e.,

$$F(r, x) := \int_0^x f(r, \xi)\, d\xi, \quad (r, x) \in [R_1, R_2] \times \mathbb{R},$$

we define $\mathcal{F}_\lambda : C \to \mathbb{R}$ by

$$\mathcal{F}_\lambda(u) = \int_{R_1}^{R_2} r^{N-1}\left[\frac{\alpha}{p}|u|^p - F(r,u) - \frac{\lambda}{q}b(r)|u|^q\right]dr, \quad u \in C$$

and $\widehat{\mathcal{F}}_\lambda : C \to \mathbb{R}$ by

$$\widehat{\mathcal{F}}_\lambda(u) = \int_{R_1}^{R_2} r^{N-1}\left[\frac{\lambda}{m}|u|^m - F(r,u) - h(r)u\right]dr, \quad u \in C.$$

A standard reasoning (also see [14, Remark 2.7]) shows that $\mathcal{F}_\lambda$ and $\widehat{\mathcal{F}}_\lambda$ are of class C^1 on C and

$$\left\langle \mathcal{F}'_\lambda(u), v\right\rangle = \int_{R_1}^{R_2} r^{N-1}\left[\alpha|u|^{p-2}u - f(r,u) - \lambda b(r)|u|^{q-2}u\right]v\,dr, \quad u, v \in C,$$

$$\left\langle \widehat{\mathcal{F}}'_\lambda(u), v\right\rangle = \int_{R_1}^{R_2} r^{N-1}\left[\lambda|u|^{m-2}u - f(r,u) - h(r)\right]v\,dr, \quad u, v \in C.$$

Then it is clear that $I_\lambda, \widehat{I}_\lambda : C \to (-\infty, +\infty]$ defined by

$$I_\lambda = \mathcal{F}_\lambda + \Psi, \qquad \widehat{I}_\lambda = \widehat{\mathcal{F}}_\lambda + \Psi$$

have the structure required by Szulkin's critical point theory. At this stage, the search of solutions of problem (5) (resp. (6)) reduces to finding critical points of the energy functional I_λ (resp. $\widehat{I}_\lambda$) by the following Proposition which is proved in [5, Proposition 1].

Proposition 3. *If $u \in C$ is a critical point of I_λ (resp. $\widehat{I}_\lambda$), then u is a solution of* (5) (*resp.* (6)).

In the case of the periodic problems (7) and (8), taking $N = 1$, one works with

$$K_P := \left\{v \in W^{1,\infty}\colon\ \left\|v'\right\|_\infty \leqslant a,\ v(R_1) = v(R_2)\right\}$$

instead of K, and $\Psi_P : C \to (-\infty, +\infty]$ given by

$$\Psi_P(v) = \begin{cases} \int_{R_1}^{R_2} \Phi(v'), & \text{if } v \in K_P, \\ +\infty, & \text{otherwise} \end{cases}$$

instead of Ψ. With $\mathcal{F}_{P,\lambda}, \widehat{\mathcal{F}}_{P,\lambda} : C \to \mathbb{R}$ defined by

$$\mathcal{F}_{P,\lambda}(u) = \int_{R_1}^{R_2}\left[\frac{\alpha}{p}|u|^p - F(r,u) - \frac{\lambda}{q}b(r)|u|^q\right]dr, \quad u \in C,$$

$$\widehat{\mathcal{F}}_{P,\lambda}(u) = \int_{R_1}^{R_2} \left[\frac{\lambda}{m} |u|^m - F(r,u) - h(r)u \right] dr, \quad u \in C,$$

the energy functionals $I_{P,\lambda}, \widehat{I}_{P,\lambda} : C \to (-\infty, +\infty]$ will be now $I_{P,\lambda} = \Psi_P + \mathcal{F}_{P,\lambda}$ and $\widehat{I}_{P,\lambda} = \Psi_P + \widehat{\mathcal{F}}_{P,\lambda}$. We have (see [5, Proposition 2]) the following

Proposition 4. *If $u \in C$ is a critical point of $I_{P,\lambda}$ (resp. $\widehat{I}_{P,\lambda}$), then u is a solution of* (7) (*resp.* (8)).

4. Nontrivial solutions for problems (5) and (7)

4.1. The Neumann problem (5)

Toward the application of Proposition 2, we have to know that the energy functional satisfies the (PS) condition. In this respect, we need the following inequalities which are proved in [5, Lemma 4].

Lemma 1. *Let $s \geqslant 1$ be a real number. Then*

$$\left|u(r)\right|^s \geqslant |\overline{u}|^s - sa(R_2 - R_1)|\overline{u}|^{s-1}, \quad \forall u \in K,\ \forall r \in [R_1, R_2] \tag{28}$$

and there are constants $k_1, k_2 \geqslant 0$ such that

$$\left|u(r)\right|^s \leqslant |\overline{u}|^s + k_1|\overline{u}|^{s-1} + k_2, \quad \forall u \in K,\ \forall r \in [R_1, R_2]. \tag{29}$$

The following lemma states that under the hypothesis (AR) the functional I_λ satisfies the (PS) condition and is anticoercive on the subspace of constant functions.

Lemma 2. *If* (10) *holds, then I_λ satisfies the* (*PS*) *condition and*

$$I_\lambda(c) \to -\infty \quad \text{as } |c| \to \infty,\ c \in \mathbb{R}, \tag{30}$$

for any $\lambda > 0$.

Proof. We shall denote by c_i a generic constant, which may depend on λ. Also, we shall invoke the positive constant

$$A = \frac{\alpha(R_2^N - R_1^N)}{pN}. \tag{31}$$

Let $\{u_n\} \subset K$ be a sequence for which $I_\lambda(u_n) \to c \in \mathbb{R}$ and

$$\left\langle \mathcal{F}'_\lambda(u_n), v - u_n \right\rangle + \Psi(v) - \Psi(u_n) \geqslant -\varepsilon_n \|v - u_n\|_\infty, \quad \forall v \in C, \tag{32}$$

where $\varepsilon_n \to 0$.

We *claim* that $\{\overline{u}_n\}$ is bounded.

To see this, let $j \in (\max\{p-1, q\}, p)$ be fixed. From (28), (29) we infer

$$\int_{R_1}^{R_2} r^{N-1}\left[\frac{\alpha}{p}|u_n|^p - \frac{\lambda}{q}b(r)|u_n|^q\right] dr \geqslant A|\overline{u}_n|^p - c_1|\overline{u}_n|^j - c_2$$

and, since $\{I_\lambda(u_n)\}$ and Φ are bounded, it follows

$$A|\overline{u}_n|^p - c_1|\overline{u}_n|^j - \int_{R_1}^{R_2} r^{N-1}F(r,u_n) \leqslant c_3 \quad \text{for all } n \in \mathbb{N}. \tag{33}$$

Letting $v = u_n \pm 1$ in (32), as $\varepsilon_n \to 0$, we may assume that

$$-1 \leqslant \int_{R_1}^{R_2} r^{N-1}\big[\alpha|u_n|^{p-2}u_n - f(r,u_n) - \lambda b(r)|u_n|^{q-2}u_n\big] dr \leqslant 1,$$

for all $n \in \mathbb{N}$, hence, setting

$$\beta(u_n) := \int_{R_1}^{R_2} r^{N-1}\big[\alpha|u_n|^{p-2}u_n - \lambda b(r)|u_n|^{q-2}u_n\big] dr,$$

we have

$$-1 - \beta(u_n) \leqslant -\int_{R_1}^{R_2} r^{N-1}f(r,u_n)\,dr \leqslant 1 - \beta(u_n) \quad \text{for all } n \in \mathbb{N}. \tag{34}$$

Using (29) and taking into account that $j-1 \in (\max\{p-2, q-1\}, p-1)$ we obtain the estimate

$$\big|\beta(u_n)\big| \leqslant pA|\overline{u}_n|^{p-1} + c_4|\overline{u}_n|^{j-1} + c_5,$$

which, by virtue of (34), gives

$$\left|\int_{R_1}^{R_2} r^{N-1}f(r,u_n)\,dr\right| \leqslant pA|\overline{u}_n|^{p-1} + c_4|\overline{u}_n|^{j-1} + c_6 \quad \text{for all } n \in \mathbb{N}. \tag{35}$$

Clearly, we have

$$\left|\frac{1}{\theta}\int_{R_1}^{R_2} r^{N-1}f(r,u_n)\overline{u}_n\,dr\right| \leqslant \frac{pA}{\theta}|\overline{u}_n|^p + c_7|\overline{u}_n|^j + c_8|\overline{u}_n| \quad \text{for all } n \in \mathbb{N}. \tag{36}$$

Now, suppose, by contradiction, that $\{|\overline{u}_n|\}$ is not bounded. Then, there is a subsequence of $\{|\overline{u}_n|\}$, still denoted by $\{|\overline{u}_n|\}$, with $|\overline{u}_n| \to \infty$. Let $n_0 \in \mathbb{N}$ be such that $|\overline{u}_n| \geqslant x_0 + a(R_2 - R_1)$ for all $n \geqslant n_0$. Condition (10) ensures that

$$\operatorname{sign} \overline{u}_n = \operatorname{sign} u_n(r) = \operatorname{sign} f\big(r, u_n(r)\big) \quad \text{for all } r \in [R_1, R_2],\ n \geqslant n_0.$$

As $\{u_n\} \subset K$, using (27) and (35) we obtain

$$\left| \frac{1}{\theta} \int_{R_1}^{R_2} r^{N-1} f(r, u_n) \tilde{u}_n \, dr \right| \leqslant c_9 |\overline{u}_n|^{p-1} + c_{10} |\overline{u}_n|^{j-1} + c_{11} \quad \text{for all } n \geqslant n_0. \tag{37}$$

From (10) it holds

$$-\int_{R_1}^{R_2} r^{N-1} F(r, u_n) \, dr \geqslant -\frac{1}{\theta} \int_{R_1}^{R_2} r^{N-1} f(r, u_n) \overline{u}_n \, dr - \frac{1}{\theta} \int_{R_1}^{R_2} r^{N-1} f(r, u_n) \widetilde{u}_n \, dr,$$

for all $n \geqslant n_0$. Then, on account of (36) and (37), we get

$$-\int_{R_1}^{R_2} r^{N-1} F(r, u_n) \, dr \geqslant -\frac{pA}{\theta} |\overline{u}_n|^p - \gamma(|\overline{u}_n|) \quad \text{for all } n \geqslant n_0,$$

where

$$\gamma\big(|\overline{u}_n|\big) := c_9 |\overline{u}_n|^{p-1} + c_7 |\overline{u}_n|^j + c_{10} |\overline{u}_n|^{j-1} + c_8 |\overline{u}_n| + c_{11}.$$

This together with $\theta > p$ imply

$$\begin{aligned} &A|\overline{u}_n|^p - c_1 |\overline{u}_n|^j - \int_{R_1}^{R_2} r^{N-1} F(r, u_n) \\ &\quad \geqslant A \frac{\theta - p}{p} |\overline{u}_n|^p - c_1 |\overline{u}_n|^j - \gamma\big(|\overline{u}_n|\big) \to +\infty \quad \text{as } n \to \infty, \end{aligned}$$

contradicting (33). Consequently, $\{\overline{u}_n\}$ is bounded, as claimed.

Since $\{u_n\} \subset K$, the sequence $\{u_n\}$ is bounded in $W^{1,\infty}$. By the compactness of the embedding $W^{1,\infty} \subset C$, we deduce that $\{u_n\}$ has a convergent subsequence in C. Therefore, I_λ satisfies the (PS) condition.

Condition (10) implies (see [12]) that there exists $\gamma \in C$, $\gamma > 0$, such that

$$F(r, x) \geqslant \gamma(r) |x|^\theta \quad \text{for all } r \in [R_1, R_2] \text{ and } |x| \geqslant x_0.$$

We infer (see (31)):

$$I_\lambda(c) = A|c|^p - \int_{R_1}^{R_2} r^{N-1} F(r,c)\,dr - \frac{\lambda}{q}|c|^q \int_{R_1}^{R_2} r^{N-1} b(r)\,dr$$
$$\leqslant A|c|^p - |c|^\theta \int_{R_1}^{R_2} r^{N-1}\gamma(r) - \frac{\lambda}{q}|c|^q \int_{R_1}^{R_2} r^{N-1} b(r)\,dr,$$

for all $c \in \mathbb{R}$ with $|c| \geqslant x_0$. Then, (30) follows from $\theta > p > q$ and $\gamma > 0$. □

Lemma 3. *If $\bar{b} > 0$ and either condition* (11) *or condition* (12) *holds true, then*

$$\inf_{B_\eta} I_\lambda < 0, \tag{38}$$

for all η, $\lambda > 0$.

Proof. Let us suppose that (11) holds true. A similar argument works under assumption (12). Condition (11) means that

$$\lim_{\varepsilon\to 0+} \inf_{x\in(-\varepsilon,0)} \frac{F(r,x)}{|x|^p} = h(r), \quad \text{uniformly in } r \in [R_1, R_2] \quad \text{and} \quad h \geqslant 0 \quad \text{on } [R_1, R_2].$$

This yields the existence of some $\varepsilon_1 > 0$ so that

$$F(r,x) \geqslant -|x|^p \quad \text{for all } r \in [R_1, R_2],\ x \in (-\varepsilon_1, 0]. \tag{39}$$

Clearly, we may assume that $\eta < \varepsilon_1$. For $c \in (-\eta, 0) \subset (-\varepsilon_1, 0]$, using (39) and $\bar{b} > 0$, we estimate $I_\lambda(c)$ as follows (see (31)):

$$I_\lambda(c) = A|c|^p - \int_{R_1}^{R_2} r^{N-1} F(r,c) - \frac{\lambda}{q}\left(\int_{R_1}^{R_2} r^{N-1} b(r)\,dr\right)|c|^q$$
$$\leqslant A\left(1+\frac{p}{\alpha}\right)|c|^p - \frac{\lambda}{q}\left(\int_{R_1}^{R_2} r^{N-1} b(r)\,dr\right)|c|^q$$
$$= |c|^q\left[A\left(1+\frac{p}{\alpha}\right)|c|^{p-q} - \frac{\lambda}{q}\left(\int_{R_1}^{R_2} r^{N-1} b(r)\,dr\right)\right] < 0,$$

provided that $|c| > 0$ is small enough. Obviously, this implies (38) and the proof is complete. □

Lemma 4. *If $\bar{b} > 0$ and* (9) *holds true, then there exist ρ, $\lambda_0 > 0$ such that*

$$\inf_{\partial B_\rho} I_\lambda > 0, \tag{40}$$

for all $\lambda \in (0, \lambda_0)$.

Proof. Assumption (9) ensures that there are constants ε, $\rho > 0$ such that

$$F(r,x) \leqslant \frac{\alpha-\varepsilon}{p}|x|^p \quad \text{for all } r \in [R_1,R_2] \text{ and } |x| \leqslant \rho. \tag{41}$$

We know (see the proof of Lemma 7 in [5]) that

$$\beta_0 := \inf_{u\in K\cap\partial B_\rho} \int_{R_1}^{R_2} r^{N-1}|u|^p\,dr > 0.$$

Also, from $\overline{b} > 0$ it follows

$$\beta_1 := \int_{R_1}^{R_2} r^{N-1} b^+(r)\,dr > 0.$$

We set

$$\lambda_0 := \frac{\varepsilon p^{-1}\beta_0}{\rho^q q^{-1}\beta_1} \quad (>0).$$

Using (41), for arbitrary $\lambda \in (0,\lambda_0)$ and $u \in K\cap\partial B_\rho$ one obtains

$$\begin{aligned} I_\lambda(u) &\geqslant \frac{\alpha}{p}\int_{R_1}^{R_2} r^{N-1}|u|^p\,dr - \int_{R_1}^{R_2} r^{N-1}F(r,u)\,dr - \frac{\lambda}{q}\int_{R_1}^{R_2} r^{N-1}b^+(r)|u|^q\,dr \\ &\geqslant \frac{\varepsilon}{p}\int_{R_1}^{R_2} r^{N-1}|u|^p\,dr - \lambda\frac{\rho^q}{q}\int_{R_1}^{R_2} r^{N-1}b^+(r)\,dr \\ &\geqslant \frac{\varepsilon}{p}\beta_0 - \lambda\frac{\rho^q}{q}\beta_1 = \frac{\rho^q}{q}\beta_1(\lambda_0-\lambda) =: c_\lambda > 0. \end{aligned}$$

Then (40) follows from

$$\inf_{\partial B_\rho} I_\lambda = \inf_{u\in K\cap\partial B_\rho} I_\lambda(u) \geqslant c_\lambda. \qquad \square$$

Theorem 1. *Assume* (10)*,* (9) *and that* $\overline{b} > 0$*. If either* (11) *or* (12) *holds true, then there exists* $\lambda_0 > 0$ *such that problem* (5) *has at least two nontrivial solutions for any* $\lambda \in (0,\lambda_0)$*.*

Proof. It is clear that I_λ is bounded from below on bounded subsets of C. Then, the conclusion follows from Proposition 2, Lemmas 2–4 and Proposition 3. $\square$

Remark 2. On account of Remark 1(i) it is easy to see that under the hypotheses of Theorem 1, if, in addition, Φ is even and $f(r,\cdot)$ is odd for all $r \in [R_1, R_2]$, then (5) has at least four nontrivial solutions for any $\lambda \in (0, \lambda_0)$.

Corollary 1. *Assume* (10) *and that* $\overline{b} > 0$. *If*

$$0 \leqslant \lim_{x\to 0} \frac{F(r,x)}{|x|^p} < \frac{\alpha}{p} \quad \text{uniformly in } r \in [R_1, R_2], \tag{42}$$

then there exists $\lambda_0 > 0$ *such that problem* (1) *has at least two nontrivial solutions for any* $\lambda \in (0, \lambda_0)$. *If, in addition,* $f(r,\cdot)$ *is odd for all* $r \in [R_1, R_2]$, *then* (1) *has at least four nontrivial radial solutions for any* $\lambda \in (0, \lambda_0)$.

Example 1. If $\alpha > 0$, $\theta > p > q \geqslant 2$ are constants and $\gamma, b \in C$, $\gamma > 0$, $\overline{b} > 0$, then there exists $\lambda_0 > 0$ such that the Neumann problem

$$-\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + \alpha|v|^{p-2}v = \gamma\big(|x|\big)|v|^{\theta-2}v + \lambda b\big(|x|\big)|v|^{q-2}v \quad \text{in } \mathcal{A},$$
$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\mathcal{A}$$

has at least four nontrivial radial solutions for any $\lambda \in (0, \lambda_0)$.

4.2. The periodic problem (7)

It is easy to check that Lemma 2 remains valid with $I_{P,\lambda}$ instead of I_λ. Also, if condition "$\overline{b} > 0$" is replaced by

$$\int_{R_1}^{R_2} b(r)\,dr > 0 \tag{43}$$

then Lemmas 3 and 4 remain true with $I_{P,\lambda}$ instead of I_λ. Thus, we obtain the following.

Theorem 2. *Assume* (10), (9) *and* (43). *If either* (11) *or* (12) *holds true, then there exists* $\lambda_0 > 0$ *such that problem* (7) *has at least two nontrivial solutions for any* $\lambda \in (0, \lambda_0)$.

Corollary 2. *Assume* (10) *and* (43). *If* (42) *holds true, then there exists* $\lambda_0 > 0$ *such that problem* (3) *has at least two nontrivial solutions for any* $\lambda \in (0, \lambda_0)$. *If, in addition,* $f(r,\cdot)$ *is odd for all* $r \in [R_1, R_2]$, *then* (3) *has at least four nontrivial solutions for any* $\lambda \in (0, \lambda_0)$.

Example 2. Let $\alpha > 0$, $\theta > p > q \geqslant 2$ be constants and $\gamma, b \in C$, $\gamma > 0$ and b satisfying (43). Then there exists $\lambda_0 > 0$ such that the periodic problem

$$-\left(\frac{u'}{\sqrt{1-|u'|^2}}\right)' + \alpha|u|^{p-2}v = \gamma(r)|u|^{\theta-2}u + \lambda b(r)|u|^{q-2}u \quad \text{in } [R_1, R_2],$$
$$u(R_1) - u(R_2) = 0 = u'(R_1) - u'(R_2)$$

has at least four nontrivial solutions for any $\lambda \in (0, \lambda_0)$.

5. Multiple solutions for problems (6) and (8)

5.1. The Neumann problem (6)

The following existence result, inspired from [27,20] provides a useful tool in obtaining multiple solutions.

Lemma 5. *We assume* $\overline{h} = 0$ *and there exists* $k_1, k_2 > 0$ *and* $0 < \sigma < m$ *such that*

$$-l(r) \leqslant F(r,x) \leqslant k_1|x|^\sigma + k_2, \quad \text{for all } (r,x) \in [R_1, R_2] \times \mathbb{R}_+, \tag{44}$$

with some $l \in L^1_{N-1}$, $l \geqslant 0$, *together with either*

$$\lim_{x\to+\infty} \int_{R_1}^{R_2} r^{N-1} F(r,x)\, dr = +\infty, \tag{45}$$

or the limit $F_+(r) = \lim_{x\to+\infty} F(r,x)$ *exists for all* $r \in [R_1, R_2]$ *and*

$$F(r,x) < F_+(r), \quad \forall r \in [R_1, R_2],\ x \geqslant 0. \tag{46}$$

Then there exists $\lambda_+ > 0$ *such that problem* (6) *has at least one solution* $u_\lambda > 0$ *for any* $0 < \lambda < \lambda_+$ *which minimize* $\widehat{I}_\lambda$ *on* $C^+ = \{v \in C:\ v \geqslant 0\}$. *Moreover,* u_λ *is a local minimum for* $\widehat{I}_\lambda$.

Proof. First, notice that from (27) it holds

$$\|\widetilde{u}\|_\infty \leqslant a(R_2 - R_1) \quad \text{for all } u \in K. \tag{47}$$

This implies that

$$\overline{u} - a(R_2 - R_1) \leqslant u(r) \leqslant \overline{u} + a(R_2 - R_1) \quad \text{for all } u \in K, \tag{48}$$

hence

$$\overline{u} \to +\infty \quad \text{as } \|u\|_\infty \to \infty,\ u \in C^+ \cap K. \tag{49}$$

Also, it is clear that

$$\big|u(r)\big| \leqslant |\overline{u}| + a(R_2 - R_1) \quad \text{for all } u \in K,\ r \in [R_1, R_2]. \tag{50}$$

From (44) it follows that

$$\widehat{I}_\lambda(u) \geqslant \int_{R_1}^{R_2} r^{N-1} \left[\frac{\lambda}{m}|u|^m - k_1|u|^\sigma - k_2 - \|h\|_\infty |u| \right] dr,$$

for all $u \in C^+$. Hence, using (28), (50), (49) and $\sigma < m$, we deduce immediately that

$$\widehat{I}_\lambda(u) \to +\infty \quad \text{whenever } \|u\|_\infty \to \infty,\ u \in C^+, \tag{51}$$

that is $\widehat{I}_\lambda$ is coercive on C^+. This immediately implies that $\widehat{I}_\lambda$ is bounded from below on C^+. Now, let $\{u_n\} \subset C^+ \cap K$ be a minimizing sequence, $\widehat{I}_\lambda(u_n) \to \inf_{C^+} \widehat{I}_\lambda$ as $n \to \infty$. Then, from (51) it follows that $\{u_n\}$ is bounded in C, and using that $\{u_n\} \subset K$, we infer that $\{u_n\}$ is bounded in $W^{1,\infty}$. But $W^{1,\infty}$ is compactly embedded in C, hence $\{u_n\}$ has a convergent subsequence in C to some $u_\lambda \in C^+ \cap K$. By the lower semicontinuity of $\widehat{I}_\lambda$ it follows

$$\widehat{I}_\lambda(u_\lambda) = \inf_{C^+} \widehat{I}_\lambda.$$

We *claim* that

$$\overline{u}_\lambda \to +\infty \quad \text{as } \lambda \to 0. \tag{52}$$

Assuming this for the moment, it follows from (48) and (52) that there exists $\lambda_+ > 0$ such that $u_\lambda > 0$ for any $0 < \lambda < \lambda_+$, implying that u_λ is a local minimum for $\widehat{I}_\lambda$. Consequently, from Proposition 1.1 in [30], u_λ is a critical point of $\widehat{I}_\lambda$, and hence a solution of (6) (by Proposition 3) for any $0 < \lambda < \lambda_+$.

In order to prove the claim, assume first that (45) holds true. Then, consider $M > 0$ and $x_M > 0$ such that

$$\int_{R_1}^{R_2} r^{N-1} F(r, x_M)\, dr > 2M. \tag{53}$$

On the other hand, as $\overline{h} = 0$, one has that for all $\lambda > 0$,

$$\widehat{I}_\lambda(x) = \frac{\lambda(R_2^N - R_1^N)}{Nm}|x|^m - \int_{R_1}^{R_2} r^{N-1} F(r, x)\, dr \quad (x \in \mathbb{R}). \tag{54}$$

So, choosing $\lambda_M > 0$ such that

$$\frac{\lambda_M(R_2^N - R_1^N)}{Nm} x_M^m < M,$$

and using (53), (54), it follows that

$$\widehat{I}_\lambda(x_M) < -M \quad \text{for all } 0 < \lambda < \lambda_M.$$

Consequently, one has that

$$\inf_{C^+} \widehat{I}_\lambda \to -\infty \quad \text{as } \lambda \to 0,$$

which, together with (48) imply (52), as claimed.

Now, let (46) hold true, and assume also by contradiction that there exists $\lambda_n \to 0$ such that $\{\overline{u}_{\lambda_n}\}$ is bounded. On account of (48) and of the compactness of the embedding in $W^{1,\infty} \subset C$, one can assume, passing if necessary to a subsequence, that $\{u_{\lambda_n}\}$ is convergent in C to some $u \in C^+$. Using (46) and Fatou's lemma it follows that

$$\int_{R_1}^{R_2} r^{N-1} F(r,u)\,dr < \int_{R_1}^{R_2} r^{-1} F_+(r)\,dr \leqslant \liminf_{s\to\infty} \int_{R_1}^{R_2} r^{N-1} F(r, s+\widetilde{u})\,dr,$$

which imply that there exists $s_0 > 0$ sufficiently large, with $s_0 + \widetilde{v} \in C^+$ for all $v \in K$, and $\rho > 0$ such that

$$\int_{R_1}^{R_2} r^{N-1}\big[F(r,u) - F(r, s_0+\widetilde{u})\big]\,dr < -\rho.$$

So, for n sufficiently large, we have

$$\int_{R_1}^{R_2} r^{N-1}\big[F(r,u_{\lambda_n}) - F(r, s_0+\widetilde{u}_{\lambda_n})\big]\,dr < -\rho. \tag{55}$$

On the other hand, using (47) and (48) it follows

$$\int_{R_1}^{R_2} r^{N-1}\frac{\lambda_n}{m}\big[|s_0+\widetilde{u}_{\lambda_n}|^m - |u_{\lambda_n}|^m\big]\,dr \to 0 \quad \text{as } n\to\infty. \tag{56}$$

Notice that, as $\overline{h} = 0$, for all $\lambda > 0$ and $s \in \mathbb{R}$, one has

$$\begin{aligned}\widehat{I}_\lambda(s+\widetilde{u}_\lambda) = {} & \int_{R_1}^{R_2} r^{N-1}\Phi\big(u'_\lambda\big)\,dr + \int_{R_1}^{R_2} r^{N-1}\frac{\lambda}{m}|s+\widetilde{u}_\lambda|^m\,dr \\ & - \int_{R_1}^{R_2} r^{N-1}F(r,s+\widetilde{u}_\lambda)\,dr - \int_{R_1}^{R_2} r^{N-1}h(r)\widetilde{u}_\lambda\,dr.\end{aligned}$$

Then, by (55) and (56) we obtain

$$\widehat{I}_{\lambda_n}(s_0+\widetilde{u}_{\lambda_n}) < \widehat{I}_{\lambda_n}(u_{\lambda_n}),$$

for n sufficiently large, contradicting the definition of u_{λ_n}. This proves the claim and the proof is complete. □

Theorem 3. *Assume that conditions* $\overline{h} = 0$, (13) *and either* (14) *or* (15) *hold true. Then there exists* $\lambda_0 > 0$ *such that problem* (6) *has at least three solutions for any* $\lambda \in (0, \lambda_0)$.

Proof. From Lemma 5, it follows that there exists $\lambda_+ > 0$ such that $\widehat{I}_\lambda$ has a local minimum at some $u_{\lambda,1} > 0$ for any $0 < \lambda < \lambda_+$. Using exactly the same strategy, we can find $\lambda_- > 0$ such that $\widehat{I}_\lambda$ has a local minimum at some $u_{\lambda,2} < 0$ for any $0 < \lambda < \lambda_-$. Taking $\lambda_0 = \min\{\lambda_-, \lambda_+\}$ it follows that $\widehat{I}_\lambda$ has two local minima for any $\lambda \in (0, \lambda_0)$. On the other hand, from the proof of Lemma 5, it is easy to see that $\widehat{I}_\lambda$ is coercive on C, implying that $\widehat{I}_\lambda$ satisfies the (PS) condition for any $\lambda > 0$. Hence, from Corollary 3.3 in [30] we infer that $\widehat{I}_\lambda$ has at least three critical points for all $\lambda \in (0, \lambda_0)$ which are solutions of (6) (Proposition 3). □

Corollary 3. *Under the assumptions of Theorem* 3, *there exists* $\lambda_0 > 0$ *such that problem* (2) *has at least three radial solutions for any* $\lambda \in (0, \lambda_0)$.

Remark 3.

(i) When f is bounded, it is well known [1] that the Ahmad–Lazer–Paul condition (14) generalizes the Landesman–Lazer condition

$$\int_{R_1}^{R_2} r^{N-1} f^-(r)\,dr < 0 < \int_{R_1}^{R_2} r^{N-1} f_+(r)\,dr,$$

where $f^-(r) = \limsup_{x\to-\infty} f(r,x)$ and $f_+(r) = \liminf_{x\to+\infty} f(r,x)$.

(ii) Condition (15) holds true whenever one has the sign condition

$$xf(r,x) > 0 \quad \text{for all } r \in [R_1, R_2] \text{ and } x \neq 0.$$

(iii) The condition:
there exists $0 < \theta < m$ *such that*

$$xf(r,x) - \theta F(r,x) \to -\infty \quad \text{as } |x| \to \infty, \text{ uniformly in } r \in [R_1, R_2],$$

introduced in [28,21], together with the sign condition

$$xf(r,x) > 0 \quad \text{for all } r \in [R_1, R_2] \text{ and } |x| \geqslant x_0$$

for some $x_0 > 0$, imply (13) and (14).

Example 3. Let $m \in \mathbb{N}$ be even and $h \in C$ be with $\overline{h} = 0$. Then, using Corollary 3 and Remark 3(iii), it follows that there exists $\lambda_0 > 0$ such that the Neumann problem

$$-\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + \lambda |v|^{m-2} v = \frac{v^{m-1}}{1+v^m} + h\big(|x|\big) \quad \text{in } \mathcal{A},$$

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\mathcal{A},$$

has at least three radial solutions for all $\lambda \in (0, \lambda_0)$.

5.2. The periodic problem (8)

Using exactly the same strategy as above, but with $\widehat{I}_{P,\lambda}$ instead of $\widehat{I}_\lambda$, we have the following

Theorem 4. *Assume that conditions*

$$\int_{R_1}^{R_2} h(r)\,dr = 0, \tag{57}$$

(13)*, and either* (14) *or* (15) *hold true for* $N = 1$*. Then there exists* $\lambda_0 > 0$ *such that problem* (8) *has at least three solutions for any* $\lambda \in (0, \lambda_0)$.

Corollary 4. *Under the assumptions of Theorem* 4*, there exists* $\lambda_0 > 0$ *such that problem* (4) *has at least three solutions for any* $\lambda \in (0, \lambda_0)$.

Example 4. Let $m \in \mathbb{N}$ be even and $h \in C$ satisfying (57). Then there exists $\lambda_0 > 0$ such that the periodic problem

$$-\left(\frac{u'}{\sqrt{1-|u'|^2}}\right)' + \lambda|u|^{m-2}u = \frac{u^{m-1}}{1+u^m} + h(r),$$
$$u(R_1) - u(R_2) = 0 = u'(R_1) - u'(R_2)$$

has at least three solutions for all $\lambda \in (0, \lambda_0)$.

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