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Periodic Solutions of Pendulum-Like Perturbations of Singular and Bounded ϕ-Laplacians

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Abstract In this paper we study the existence and multiplicity of periodic solutions of pendulum-like perturbations of bounded or singular ϕ-Laplacians. Our approach relies on the Leray-Schauder degree and the upper and lower solutions method.

Keywords Curvature operators · Periodic problem · Pendulum-like non linearities · Leray-Schauder degree · Upper and lower solutions

Mathematics Subject Classification (2000) 34B15

1 Introduction

In this paper we present existence and multiplicity results for the periodic problem

$$\left(\frac{u'}{\sqrt{1 \pm u'^2}}\right)' + \mu \sin u = h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{1}$$

where $\mu > 0$ and h is continuous on $[0, T]$. We denote by $\overline{h}$ the mean value of h over $[0, T]$.

This manuscript is dedicated to Professor Jack Hale on the occasion of his 80th birthday.

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Considering first the case where the minus sign is taken, which occurs in questions of special relativity, we prove in Theorem 1, using degree arguments, that problem (1) has at least two solutions not differing by a multiple of 2π if

$$T < \pi\sqrt{3}, \quad |\overline{h}| < \mu \cos\left(\frac{T}{2\sqrt{3}}\right),$$

Moreover, if

$$T = \pi\sqrt{3},$$

then problem (1) has at least one solution, provided that $\overline{h} = 0$. Theorem 1, which deals with the more general equation

$$(\phi(u'))' + f(u)u' + \mu \sin u = h(t) \tag{2}$$

where $\phi : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$ (so-called singular ϕ), and f is an arbitrary continuous function, improves some very recent results from [14] and [15] (see Remark 3).

Then we show in Theorem 2, using again degree arguments and the upper and lower solutions method developed in [2], that equation (2) has at least two solutions not differing by a multiple of 2π if $||h||_\infty < \mu$, and at least one solution exists if $||h||_\infty = \mu$.

Finally, in the case where the plus sign is taken in problem (1), which deals with curvature problems, a result corresponding to Theorem 2 is obtained for problem (1) under further restrictions upon μ (Corollary 1).

For existence and multiplicity results concerning periodic solutions of the classical pendulum equation, see for example [10,11,13]. Although the literature devoted to problems involving the 1-curvature operator with Dirichlet boundary conditions is quite vast, the one devoted to the periodic solutions of problems associated to singular or bounded ϕ-Laplacians is much more limited. See [1–9,12,14,15].

2 Notations, Operators and Function Spaces

Let C denote the Banach space of continuous functions on $[0, T]$ endowed with the uniform norm $\|\cdot\|_\infty$, C^1 denote the Banach space of continuously differentiable functions on $[0, T]$, equipped with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$. The usual norm in $L^1(0, T)$ will be denoted by $\|\cdot\|_1$.

Let $P, Q : C \to C$ be the continuous projectors defined by

$$Pu(t) = u(0), \quad \overline{u} = Qu(t) = \frac{1}{T}\int_0^T u(s)\,ds \quad (t \in [0, T]),$$

and define the continuous linear operator $H : C \to C^1$ by

$$Hu(t) = \int_0^t u(s)\,ds \quad (t \in [0, T]).$$

If $u \in C$, we write

$$u^+ = \max\{u, 0\}, \quad u^- = \max\{-u, 0\}, \quad \widetilde{u} = u - \overline{u},$$

and we shall consider the following closed subspaces of C^1:

$$C^1_\# = \{u \in C^1 : u(0) = u(T), u'(0) = u'(T)\},$$
$$\widetilde{C}^1_\# = \{u \in C^1_\# : \overline{u} = 0\}.$$

3 The Case of a Singular ϕ

We deal with the periodic boundary value problem

$$(\phi(u'))' + f(u)u' + g(u) = h(t) \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{3}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are continuous and $h \in C$. Throughout this section we assume that $0 < a < +\infty$ and the mapping ϕ satisfies the hypothesis :

(H_ϕ) *$\phi : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$.*

We first recall the following technical result given as Lemma 1 from [2].

Lemma 1 *For each $h \in C$, there exists a unique $\alpha := Q_\phi(h) \in$ Range h such that*

$$\int_0^T \phi^{-1}(h(t) - \alpha)\, dt = 0.$$

Moreover, the function $Q_\phi : C \to \mathbb{R}$ is continuous.

We also need the following fixed point reduction given in Proposition 2 from [2].

Lemma 2 *Assume that $F : C^1 \to C$ is continuous and takes bounded sets into bounded sets. The function u is a solution of the abstract periodic problem*

$$(\phi(u'))' = F(u), \quad u(0) - u(T) = 0 = u'(0) - u'(T)$$

if and only if $u \in C^1_\#$ is a fixed point of the operator $M_\#$ defined on $C^1_\#$ by

$$M_\#(u) = Pu + QF(u) + H \circ \phi^{-1} \circ (I - Q_\phi) \circ [H(I - Q)F](u).$$

Furthermore, $M_\#$ is completely continuous on $C^1_\#$ and $\|(M_\#(u))'\|_\infty < a$ for all $u \in C^1_\#$.

Combining Lemma 2 with Leray-Schauder degree, we obtain the following existence result.

Lemma 3 *Assume that there exists $r < s$ and $A < B$ such that*

$$\frac{1}{T}\int_0^T g(r + \widetilde{u}(t))dt \le A \quad \text{and} \quad \frac{1}{T}\int_0^T g(s + \widetilde{u}(t))dt \ge B \tag{4}$$

for any $\widetilde{u} \in \widetilde{C}^1_\#$ satisfying $||\widetilde{u}||_\infty < \frac{aT}{2\sqrt{3}}$. If

$$A < \overline{h} < B, \tag{5}$$

then (3) has at least one solution u such that $r < \overline{u} < s$.

Proof Let us fix $\varepsilon > 0$ and, for any $\lambda \in [0, 1]$, consider the periodic problem

$$(\phi(u'))' + \lambda f(u)u' + \lambda g(u) + (1-\lambda)\varepsilon\left(u - \frac{r+s}{2}\right) = \lambda h(t)$$
$$u(0) - u(T) = 0 = u'(0) - u'(T). \tag{6}$$

Let also

$$\mathcal{M}(\lambda, \cdot) : C^1_\# \to C^1_\# \quad (\lambda \in [0, 1])$$

be the fixed point operator associated to (6) (see Lemma 2). Setting

$$\Omega = \left\{ u \in C^1_\# : \quad r < \overline{u} < s, \quad ||\widetilde{u}||_\infty < \frac{aT}{2\sqrt{3}}, \quad ||u'||_\infty < a \right\},$$

we will show that

$$u - \mathcal{M}(\lambda, u) \neq 0 \quad \text{for any} \quad (\lambda, u) \in (0, 1] \times \partial\Omega, \tag{7}$$

and

$$u - \mathcal{M}(0, u) = 0 \quad \text{implies} \quad u \in \Omega.$$

Then, using the invariance by homotopy, the excision property of the Leray-Schauder degree and Lemma 3 from [2], one has that

$$d_{LS}[I - \mathcal{M}(1, \cdot), \Omega, 0] = d_{LS}[I - \mathcal{M}(0, \cdot), \Omega, 0] = 1.$$

Hence, the existence property of the Leray-Schauder degree implies the existence of $u \in \Omega$ (in particular $r < \overline{u} < s$) such that $u = \mathcal{M}(1, u)$ which is also a solution of (3).

So, let $(\lambda, u) \in (0, 1] \times C^1_\#$ be such that

$$u = \mathcal{M}(\lambda, u).$$

It follows that u is a solution of (6) and

$$||u'||_\infty < a. \tag{8}$$

Using (8) and the Sobolev inequality

$$||\widetilde{u}||^2_\infty \leq \frac{T}{12} \int_0^T |u'(t)|^2 dt,$$

we infer that

$$||\widetilde{u}||_\infty < \frac{aT}{2\sqrt{3}}. \tag{9}$$

Integrating (6) over $[0, T]$ we obtain

$$(1-\lambda)\varepsilon\left(\overline{u} - \frac{r+s}{2}\right) + \lambda\left(\frac{1}{T}\int_0^T g(\overline{u} + \widetilde{u}(t))dt - \overline{h}\right) = 0. \tag{10}$$

On the other hand, from (4) and (5) it follows that

$$\begin{aligned}(1-\lambda)\varepsilon\left(r-\frac{r+s}{2}\right)&+\lambda\left(\frac{1}{T}\int_0^T g(r+\widetilde{u}(t))dt-\overline{h}\right)\\ &\leq(1-\lambda)\varepsilon\frac{r-s}{2}+\lambda(A-\overline{h})<0;\\ (1-\lambda)\varepsilon\left(s-\frac{r+s}{2}\right)&+\lambda\left(\frac{1}{T}\int_0^T g(s+\widetilde{u}(t))dt-\overline{h}\right)\\ &\geq(1-\lambda)\varepsilon\frac{s-r}{2}+\lambda(B-\overline{h})>0.\end{aligned} \tag{11}$$

Then, if $u\in\partial\Omega$, from (8) and (9) it follows that either $\overline{u}=r$ or $\overline{u}=s$. But $\overline{u}$ verifies (10), contradiction with (11). Consequently, (7) is proved.

Now, let $u\in C^1_\#$ be such that

$$u=\mathcal{M}(0,u).$$

We deduce that u verifies (8), (9) and (6) with $\lambda=0$. Hence, $\overline{u}=\frac{r+s}{2}$ and $u\in\Omega$. □

Remark 1 From the proof above it follows that:
(i) Relation (4) can be replaced by the following one

$$\frac{1}{T}\int_0^T g(r+\widetilde{u}(t))dt\geq B \quad\text{and}\quad \frac{1}{T}\int_0^T g(s+\widetilde{u}(t))dt\leq A$$

for any $\widetilde{u}\in\widetilde{C}^1_\#$ satisfying $||\widetilde{u}||_\infty<\frac{aT}{2\sqrt{3}}$. Note that in this case one takes $\varepsilon<0$ in the above proof;
(ii) If we assume that $A\leq B$, then (3) has at least one solution u such that $r\leq\overline{u}\leq s$, provided that $A\leq\overline{h}\leq B$.

An immediate consequence of Lemma 3 and Remark 1 is the following

Theorem 1 *Let $f:\mathbb{R}\to\mathbb{R}$ be continuous, $h\in C$ and $\mu>0$. If*

$$aT<\pi\sqrt{3},\quad |\overline{h}|<\mu\cos\left(\frac{aT}{2\sqrt{3}}\right),$$

then the periodic problem

$$(\phi(u'))'+f(u)u'+\mu\sin u=h(t),\quad u(0)-u(T)=0=u'(0)-u'(T), \tag{12}$$

has at least two solutions u_1,u_2 such that $-\frac{\pi}{2}<\overline{u}_1<\frac{\pi}{2}<\overline{u}_2<\frac{3\pi}{2}$.

Proof A simple computation shows that we can apply Lemma 3 and Remark 1 (i) with

$$r=-\frac{\pi}{2},\quad s=\frac{\pi}{2}\quad\text{and}\quad A=\mu\sin\left(-\frac{\pi}{2}+\frac{aT}{2\sqrt{3}}\right)=-B;$$

$$r=\frac{\pi}{2},\quad s=\frac{3\pi}{2}\quad\text{and}\quad A=\mu\sin\left(-\frac{\pi}{2}+\frac{aT}{2\sqrt{3}}\right)=\mu\sin\left(\frac{3\pi}{2}+\frac{aT}{2\sqrt{3}}\right)=-B.$$

□

Remark 2 If in Theorem 1 one assumes

$$aT = \pi\sqrt{3},$$

then problem (12) has at least one solution for any $h \in C$ with $\overline{h} = 0$.

Remark 3 Using Schauder fixed point theorem P.J. Torres [15] has proved Theorem 1 under the more restrictive assumptions

$$aT < 2\sqrt{3}, \quad |\overline{h}| < \mu\left(1 - \frac{aT}{2\sqrt{3}}\right).$$

We now prove an existence result where the condition upon $\overline{h}$ is replaced by a condition upon $\|h\|_\infty$. In this view we need a result (Lemma 4 below) which is proved in [2, Theorem 3] and concerns the more general periodic problem:

$$(\phi(u'))' = f(t, u, u') \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{13}$$

where $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function.

Recall that a *lower solution* α (resp. *upper solution* β) of (13) is a function $\alpha \in C^1$ such that $\|\alpha'\|_\infty < a, \phi(\alpha') \in C^1, \alpha(0) = \alpha(T), \alpha'(0) \geq \alpha'(T)$ (resp. $\beta \in C^1$, $\|\beta'\|_\infty < a, \phi(\beta') \in C^1, \beta(0) = \beta(T), \beta'(0) \leq \beta'(T)$) and

$$\begin{aligned} (\phi(\alpha'(t)))' &\geq f(t, \alpha(t), \alpha'(t)) \\ (resp. \quad (\phi(\beta'(t)))' &\leq f(t, \beta(t), \beta'(t))) \end{aligned} \tag{14}$$

for all $t \in [0, T]$. Such a lower or upper solution is called *strict* if the inequality (14) is strict for all $t \in [0, T]$.

Lemma 4 *If* (13) *has a lower solution* α *and an upper solution* β *such that* $\alpha \leq \beta$ *on* $[0, T]$, *then problem* (13) *has a solution* u *such that* $\alpha(t) \leq u(t) \leq \beta(t)$ *for all* $t \in [0, T]$. *Moreover, if* α *and* β *are strict and* $\alpha < \beta$ *on* $[0, T]$, *then* $\alpha(t) < u(t) < \beta(t)$ *for all* $t \in [0, T]$ *and* $d_{LS}[I - M_f, \Omega_{\alpha,\beta}, 0] = -1$, *where*

$$\Omega_{\alpha,\beta} = \{u \in C^1_\# : \alpha(t) < u(r) < \beta(t) \quad \text{for all} \quad t \in [0, T], \quad ||u'||_\infty < a\}$$

and M_f *is the fixed point operator associated to (13).*

Theorem 2 *Let* $\mu > 0$ *and assume that* $h \in C$ *satisfies*

$$||h||_\infty \leq \mu.$$

Then the periodic problem (12) *has at least one solution. Moreover, if*

$$||h||_\infty < \mu,$$

then (12) *has at least two solutions not differing by a multiple of* 2π.

Proof Assume that $||h||_\infty \leq \mu$. Then $\alpha = -\frac{3\pi}{2}$ is a constant lower solution for (12) and $\beta = -\frac{\pi}{2}$ is a constant upper solution for (12) such that $\alpha < \beta$. Hence, using Lemma 4, it follows that (12) has a solution u_1 such that $\alpha \leq u_1 \leq \beta$. Note that if $||h||_\infty < \mu$, then α, β are strict and $\alpha < u_1 < \beta$.

Now, let us assume that $||h||_\infty < \mu$ and let M_μ be the fixed point operator associated to (12) and let

$$\Omega = \Omega_{-\frac{3\pi}{2},\frac{3\pi}{2}} \setminus (\overline{\Omega}_{-\frac{3\pi}{2},-\frac{\pi}{2}} \cup \overline{\Omega}_{\frac{\pi}{2},\frac{3\pi}{2}}).$$

(see Lemma 4). Then using the additivity property of the Leray-Schauder degree and Lemma 4, we deduce that

$$d_{LS}[I - M_\mu, \Omega, 0] = 1.$$

Hence, the existence property of the Leray-Schauder degree yields the existence of a solution $u_2 \in \Omega$ of (12). If we assume that $u_2 = u_1 + 2j\pi$ for some $j \in \mathbb{Z}$ then, as $-3\pi/2 < u_1 < -\pi/2$, one has

$$-\frac{3\pi}{2} + 2j\pi < u_2 = u_1 + 2j\pi < -\frac{\pi}{2} + 2j\pi.$$

This leads to one of the contradictions : $u_2 \in \Omega_{\frac{\pi}{2},\frac{3\pi}{2}}$ if $j = 1$ or $u_2 = u_1 \in \Omega_{-\frac{3\pi}{2},-\frac{\pi}{2}}$ for $j = 0$. □

4 The Case of a Bounded ϕ

To obtain the same type of result for the case of bounded ϕ-Laplacians, i.e. when $\phi : \mathbb{R} \to (-a, a)$ is an increasing homeomorphism such that $\phi(0) = 0$, we need the following a priori estimate result.

Lemma 5 *Let* $0 < b, c \leq \infty$ *and* $\psi : (-b, b) \to (-c, c)$ *be a homeomorphism such that* $\psi(0) = 0$ *and* $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ *be a continuous function. Assume that there exists* $e \in C$ *such that* $2\|e^-\|_1 < c$ *and*

$$f(t, u, v) \geq e(t) \quad \text{for all} \quad (t, u, v) \in [0, T] \times \mathbb{R}^2. \tag{15}$$

If u *is a possible solution of problem*

$$(\psi(u'))' = f(t, u, u'), \quad u(0) - u(T) = 0 = U'(0) - u'(T), \tag{16}$$

then $\|u'\|_\infty \leq M_\psi$, *where* $M_\psi = \max(|\psi^{-1}(\pm 2\|e^-\|_1)|)$.

Proof Let us denote by $N_f : C^1 \to C$ the Nemytskii operator defined by

$$N_f(u)(\cdot) = f(\cdot, u(\cdot), u'(\cdot)).$$

Let u be a solution of (16). This implies that

$$QN_f(u) = 0. \tag{17}$$

Using the fact that f is bounded from below by e, we deduce the inequality

$$|f(t, u, v)| \leq f(t, u, v) + 2e^-(t) \quad \text{for all} \quad (t, u, v) \in [0, T] \times \mathbb{R}^2. \tag{18}$$

From (16), (17) and (18) it follows that

$$\|(\psi(u'))'\|_1 = \|N_f(u)\|_1 \leq \int_0^T N_f(u)(s)\, ds + 2\|e^-\|_1 = 2\|e^-\|_1. \tag{19}$$

Because $u \in C^1$ is such that $u(0) = u(T)$, there exists $\xi \in [0, T]$ such that $u'(\xi) = 0$, which implies $\psi(u'(\xi)) = 0$ and

$$\psi(u'(t)) = \int_\xi^t (\psi(u'(s)))'\, ds \quad (t \in [0, T]).$$

Using the equality above and (19) we have that

$$|\psi(u'(t))| \leq 2\|e^-\|_1 \quad (t \in [0, T]),$$

and hence $\|u'\|_\infty \leq M_\psi$. □

Corollary 1 *Assume that* $\psi : \mathbb{R} \to (-c, c)(0 < c \leq \infty)$ *is an increasing homeomorphism such that* $\psi(0) = 0$ *and*

$$||[h - \mu]^-||_1 < c/2.$$

If $||h||_\infty \leq \mu$ *then*

$$(\psi(u'))' + \mu \sin u = h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{20}$$

has at least one solution. Moreover, if $||h||_\infty < \mu$, *then (20) has at least two solutions not differing by a multiple of* 2π.

Proof Let M_ψ be the constant given in Lemma 5 with $e = h - \mu$. Let $b = M_\psi + 1$ and consider an increasing homeomorphism $\phi : (-b, b) \to \mathbb{R}$ such that $\phi = \psi$ on $[-M_\psi, M_\psi]$. It follows that $M_\psi = M_\phi$ and applying Lemma 5, we deduce that u is a solution of (12) with $f = 0$ if and only if u is a solution of (20). Now the result follows from Theorem 2. □

Example 1 If $h \in C$ is such that

$$\overline{h} = 0, \quad ||h||_\infty < \mu < 1/2T,$$

then the periodic problem

$$\left(\frac{u'}{\sqrt{1 + u'^2}}\right)' + \mu \sin u = h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has at least two solutions not differing by a multiple of 2π.

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References

1. Benevieri, P., do Ó, J.M., de Medeiros, E.S.: Periodic solutions for nonlinear systems with mean curvature-like operators. Nonlinear Anal. **65**, 1462–1475 (2006)
2. Bereanu, C., Mawhin, J.: Existence and multiplicity results for some nonlinear problems with singular ϕ-Laplacian. J. Differ. Equ. **243**, 536–557 (2007)
3. Bereanu, C., Mawhin, J.: Multiple periodic solutions of ordinary differential equations with bounded nonlinearities and ϕ-Laplacian. NoDEA Nonlinear Differ. Equ. Appl. **15**, 159–168 (2008)
4. Bereanu, C., Mawhin, J.: Periodic solutions of nonlinear perturbations of ϕ-Laplacians with possibly bounded ϕ. Nonlinear Anal. **68**, 1668–1681 (2008)
5. Bereanu, C., Mawhin, J.: Boundary value problems for some nonlinear systems with singular ϕ-Laplacian. J. Fixed Point Theory Appl. **4**, 57–75 (2008)
6. Bereanu, C., Mawhin, J.: Nonhomogeneous boundary value problems for some nonlinear equations with singular ϕ-Laplacian. J. Math. Anal. Appl. **352**, 218–233 (2009)
7. Cid, J.A., Torres, P.J.: Solvability for some boundary value problems with ϕ-Laplacian operators. Discret. Contin. Dynam. Syst. A **23**, 727–732 (2009)
8. Chu, J., Lei, J., Zhang, M.: The stability of the equilibrium of a nonlinear planar system and application to the relativistic oscillator. J. Differ. Equ. **247**, 530–542 (2009)
9. Ferracuti, L., Papalini, F.: Boundary-value problems for strongly nonlinear multivalued equations involving different Φ-Laplacians. Adv. Differ. Equ. **14**, 541–566 (2009)

10. Fournier, G., Mawhin, J.: On periodic solutions of forced pendulum-like equations. J. Differ. Equ. **60**, 381–395 (1985)
11. Kannan, R., Ortega, R.: Periodic solutions of pendulum-type equations. J. Differ. Equ. **59**, 123–144 (1985)
12. Mawhin, J.: Boundary value problems for nonlinear perturbations of singular ϕ-Laplacians. In: Staicu, V.: (ed.) Differential Equations, Chaos and Variational Problems. Progr. Nonlin. Diff. Eq. Appl., Birkhauser **75**, 247–256 (2007)
13. Mawhin, J., Willem, M.: Multiple solutions of the periodic boundary value problem for some forced pendulum-type equations. J. Differ. Equ. **52**, 264–287 (1984)
14. Torres, P.J.: Periodic oscillations of the relativistic pendulum with friction. Phys. Lett. A **372**, 6386–6387 (2008)
15. Torres, P.J.: Nondegeneracy of the periodically forced Liénard differential equation with ϕ-Laplacian, Commun. Contemporary Math., to appear