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MULTIPLE CRITICAL POINTS FOR A CLASS OF PERIODIC LOWER SEMICONTINUOUS FUNCTIONALS

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Dedicated to Jean Mawhin with great esteem for his 70th anniversary

Abstract. We deal with a class of functionals I on a Banach space X, having the structure $I = \Psi + \mathcal{G}$, with $\Psi : X \to (-\infty, +\infty]$ proper, convex, lower semicontinuous and $\mathcal{G} : X \to \mathbb{R}$ of class C^1. Also, I is G-invariant with respect to a discrete subgroup $G \subset X$ with dim (span G) $= N$. Under some appropriate additional assumptions we prove that I has at least $N + 1$ critical orbits. As a consequence, we obtain that the periodically perturbed N-dimensional relativistic pendulum equation has at least $N + 1$ geometrically distinct periodic solutions.

1. **Introduction.** In the recent paper [4], Brezis and Mawhin show that the forced pendulum like problem

$$(\phi(u'))' = f(t, u) + h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{1}$$

has at least one solution, provided that $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function for which there exists $\omega > 0$ such that

$$F(t, u) = F(t, u + \omega), \qquad \forall (t, u) \in [0, T] \times \mathbb{R},$$

where $F : [0, T] \times \mathbb{R} \to \mathbb{R}$ is defined by

$$F(t, u) = \int_0^u f(t, \xi) d\xi, \qquad \forall (t, u) \in [0, T] \times \mathbb{R},$$

$h : [0, T] \to \mathbb{R}$ is a continuous function satisfying

$$\int_0^T h(t) dt = 0,$$

and $\phi : (-a, a) \to \mathbb{R}$ $(0 < a < \infty)$ is an increasing homeomorphism with $\phi(0) = 0$ and there exists $\Phi : [-a, a] \to \mathbb{R}$ a continuous function with $\Phi(0) = 0$, Φ of class C^1

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on $(-a, a)$ and $\Phi' = \phi$. They consider the action functional $\mathcal{I} : K_\# \to \mathbb{R}$ associated to (1), given by

$$\mathcal{I}(u) = \int_0^T \{\Phi(u') + F(t,u) + hu\}dt, \qquad (u \in K_\#),$$

where

$$K_\# = \{u \in Lip(\mathbb{R}) : \ |u'(t)| \le a \text{ for a.e. } t \in \mathbb{R}, \ u \text{ is } T-\text{periodic}\},$$

and prove that $\mathcal{I}$ has at least one minimizer u in $K_\#$ satisfying the variational inequality

$$\int_0^T [\Phi(v') - \Phi(u')] + \int_0^T [f(t,u) + h][v - u] \ge 0, \qquad \forall v \in K_\#. \tag{2}$$

Then, using (2) and a topological result from [1], they show that any minimizer of $\mathcal{I}$ on $K_\#$ is a solution of (1). Hence, (1) has at least one solution. Notice that the corresponding classical result ($\phi = id_{\mathbb{R}}$) was proved by Hamel [13] and rediscovered independently by Dancer [7] and Willem [26]. Also, Brezis and Mawhin extend their result from [4] to systems in their subsequent paper [5].

In [2] it is emphasized that Szulkin's critical point theory [24] is an appropriate functional framework for problems of this type. More precisely, set

$$\widehat{K} = \{u \in W^{1,\infty}(0,T) \ : \ \|u'\|_\infty \le a, \ u(0) = u(T)\}$$

and let $\Psi : C[0,T] \to (-\infty, +\infty]$,

$$\Psi(u) = \int_0^T \Phi(u') \text{ if } u \in \widehat{K}, \qquad \Psi(u) = +\infty \text{ if } u \in C[0,T] \setminus \widehat{K},$$

and $\mathcal{F} : C[0,T] \to \mathbb{R}$,

$$\mathcal{F}(u) = \int_0^T \{F(t,u) + hu\}dt, \qquad (u \in C[0,T]).$$

Then, Ψ is a lower semicontinuous, convex functional and $\mathcal{F}$ is of class C^1. Hence, the action $\widehat{\mathcal{I}} : C[0,T] \to (-\infty, +\infty]$ defined by $\widehat{\mathcal{I}} = \Psi + \mathcal{F}$, has the structure required by Szulkin's critical point theory. In this context, a critical point of $\widehat{\mathcal{I}}$ means a function $u \in \widehat{K}$ such that (2) holds true. Then, using some ideas from [4], it is shown that any critical point of $\widehat{\mathcal{I}}$ is a solution of (1). Note that $C[0,T]$ is not reflexive, so the direct method in the calculus of variations cannot be applied. Nevertheless, a substitute for this is provided, namely, it is shown that if there exists $\rho > 0$ such that $\inf_{\widehat{K}_\rho} \widehat{\mathcal{I}} = \inf_{\widehat{K}} \widehat{\mathcal{I}}$, where

$$\widehat{K}_\rho = \{u \in \widehat{K} : \ \left| \int_0^T u \right| \le \rho\},$$

then $\widehat{\mathcal{I}}$ is bounded from below on $C[0,T]$ and attains its infimum at some $u \in \widehat{K}_\rho$ which solves (1). The Brezis-Mawhin result is an immediate consequence of this result.

Another proof of Brezis-Mawhin result is given by Manásevich and Ward in [15]. The main idea is to introduce the change of variable $\phi(u') = v$. Then, problem (1) becomes

$$u' = \phi^{-1}(v), \quad v' = f(t,u) + h(t), \quad u(0) - u(T) - 0 = v(0) - v(T). \tag{3}$$

Letting

$$\widehat{\phi}(v) = \int_0^v \phi^{-1}(s)ds, \quad w = (u, v),$$

with the Hamiltonian function $H(t, w) = -\widehat{\phi}(v) + F(t, u) + h(t)u$, system (3) takes the Hamiltonian form

$$w' = J\nabla_w H(t, w), \qquad w(0) = w(T),$$

where J is the standard symplectic matrix. The classical saddle point theorem of Rabinowitz is then applied to a sequence of approximating problems, obtaining a sequence of critical points. A subsequence of these critical points converges to a solution. Notice that the action functional associated to the above Hamiltonian system is strongly indefinite and the classical saddle point theorem does not apply to it.

A second geometrically distinct solution of problem (1) is obtained in [3] using the functional framework introduced in [2] and a mountain pass type argument (Corollary 3.3 from [24]). We note that the corresponding classical result was proved by Mawhin and Willem in [21] using a modified version of the Mountain Pass Theorem. Another proof of the Mawhin-Willem result was given by Franks [12] using a generalization of the Poincaré -Birkhoff theorem. Very recently Fonda and Toader [10] prove the results from [3, 21] in a unified way, using Ding's version of the Poincaré -Birkhoff theorem (see [8]). Using Franks's generalization of the Poincaré -Birkhoff theorem, Maró [16] give another proof of the main result from [3].

In the very recent paper [19], Mawhin obtains multiplicity of solutions for the N-dimensional analogous of (1):

$$(\phi(u'))' = \nabla_u F(t, u) + h(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{4}$$

under the following hypotheses:

(H_ϕ) ϕ is a homeomorphism from $B(a) \subset \mathbb{R}^N$ onto $\mathbb{R}^N$ such that $\phi(0) = 0$, $\phi = \nabla\Phi$, with $\Phi : \overline{B(a)} \to \mathbb{R}$ of class C^1 on $B(a)$, continuous, strictly convex on $\overline{B(a)}$, and such that $\Phi(0) = 0$;

(H_F) $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is continuous, ω_i-periodic ($\omega_i > 0$) with respect to each u_i ($1 \leq i \leq N$) and $\nabla_u F$ exists and is continuous on $[0, T] \times \mathbb{R}^N$;

(H_h) $h : [0, T] \to \mathbb{R}^N$ is continuous and

$$\int_0^T h(t)dt = 0.$$

Under the above assumptions, Mawhin shows that (4) has a Hamiltonian formulation, then applies a generalized saddle point theorem for strongly indefinite functionals due to Szulkin [25] (see also [9, 14]) in order to prove that (4) has at least $N + 1$ geometrically distinct solutions. The corresponding classical result has been proved independently, using Lusternik-Schnirelman theory in Hilbert manifolds or variants of it, by Chang [6], Mawhin [17] and Rabinowitz [23]. The case $N = 2$ has been discussed by Fournier and Willem in [11]. It is interesting to note that the Hamiltonian system associated to (4) is spatially periodic like in [9], but the results in [9] cannot be applied to it because the superlinearity condition (H_3) in [9] with respect to the spatial variable is not satisfied in this relativistic case.

For a nice presentation of the classical forced pendulum equation we refer the reader to the paper [18].

The aim of this paper is to give a different proof of Mawhin's result in [19], based upon a Lusternik-Schnirelman type approach for Szulkin functionals. More precisely, we will consider functionals $I : X \to (-\infty, +\infty]$ in a Banach space X such that $I = \Psi + \mathcal{G}$, Ψ is proper, convex, lower semicontinuous and $\mathcal{G}$ is of class C^1. Also, I will be G-invariant with respect to a discrete subgroup G with dim (span G) $= N$ and bounded from below. Under some additional assumptions, which are automatically satisfied by the Lagrangian action associated to (4), we prove that I has $N+1$ critical orbits (Theorem 7.1). With this aim, we use a Deformation Lemma (Proposition 5.2) together with Ekeland's variational principle and the classical Lusternik-Schnirelman category in order to prove that one has critical value at the levels (introduced in [23] for C^1-functionals),

$$c_j = \inf_{A \in \mathcal{A}_j} \sup_A I \qquad (1 \le j \le N+1),$$

where

$$\mathcal{A}_j = \{A \subset X : \ A \text{ is compact and } \operatorname{cat}_{\pi(X)}(\pi(A)) \ge j\},$$

and $\pi : X \to X/G$ denotes the canonical projection. The corresponding abstract result for C^1-functionals is proved in [22]. We point out that we use also some ideas from the proof of Theorem 4.3 in [24], but the deformation obtained in Proposition 2.3 from [24] can not be employed in our case because it is not "G-invariant" (see Proposition 5.2 (ii)).

The paper is organized as follows. In Section 2 we show that the action functional associated to problem (4) has the structure required by Szulkin's critical point theory and present the main properties involved in the proof of the existence of at least $N+1$ geometrically distinct solutions for (4). In Section 3 we introduce some notations and the hypotheses. In Section 4 we prove a technical result (Proposition 4.9); this is the key ingredient in the proof of the deformation lemma (Proposition 5.2) which is given in Section 5. The next Section is a resume of the main tools of the proof of the main result: Ekeland's variational principle and the classical Lusternik-Schnirelman category. In the last Section we prove the main result of the paper (Theorem 7.1).

2. **A nonsmooth variational approach for problem (4).** Consider the periodic boundary value problem (4) under the hypotheses (H_ϕ), (H_F) and (H_h). The following variational setting is taken from [2] when $N = 1$ and [20] in the general case.

We set $C = C([0,T], \mathbb{R}^N)$ and $W^{1,\infty} = W^{1,\infty}([0,T], \mathbb{R}^N)$. The usual norm $||\cdot||_\infty$ is considered on C and L^∞. Setting

$$\widetilde{C} := \{u \in C : \int_0^T u(t)dt = 0\},$$

we can split

$$C = \mathbb{R}^N \oplus \widetilde{C}$$

and each $v \in C$ can be uniquely written as

$$u = \overline{u} + \widetilde{u}, \qquad \text{with } \overline{u} \in \mathbb{R}^N, \ \widetilde{u} \in \widetilde{C}.$$

Also, note that setting

$$G_p := \left\{ \sum_{k=1}^{N} k_i \omega_i e_i \ : \ k_i \in \mathbb{Z},\ 1 \le i \le N \right\},$$

one has that $G_p \simeq \mathbb{Z}^N$ and span $G_p = \mathbb{R}^N$. Putting

$$\widehat{K} = \{v \in W^{1,\infty} :\ ||v'||_\infty \le a,\ v(0) = v(T)\},$$

we have that $\widehat{K}$ is a convex and closed set in C.

Let $\Psi_p : C \to (-\infty, +\infty]$ be defined by

$$\Psi_p(v) = \int_0^T \Phi(v') \text{ if } v \in \widehat{K}, \qquad \Psi_p(v) = +\infty \text{ if } v \in C \setminus \widehat{K}.$$

and $\mathcal{G}_p : C \to \mathbb{R}$ be defined by

$$\mathcal{G}_p(v) = \int_0^T F(t,u)dt + \int_0^T h(t)u \qquad (u \in C).$$

The following hold true.

(p_1) $\mathcal{G}_p \in C^1(C, \mathbb{R})$ and $\mathcal{G}_p'$ takes bounded sets into bounded sets; Ψ_p is convex, lower semicontinuous and $D(\Psi_p) = \{u \in C :\ \Psi_p(u) < +\infty\} = \widehat{K}$ is a closed set in C. Note also that

$$\mathcal{G}_p(u+g) = \mathcal{G}_p(u) \ \text{ and } \ \Psi_p(u+g) = \Psi_p(u) \quad \forall u \in C,\ g \in G_p.$$

(p_2) One has that $\Psi_p(0) = 0$ and

$$\Psi_p(u) = \Psi_p(\widetilde{u}) \quad \text{for all} \quad u \in C.$$

(p_3) There exists $\rho > 0$ such that

$$||\widetilde{u}||_\infty \le \rho, \quad |\Psi(u)| \le \rho \quad \text{for all} \quad u \in \widehat{K}.$$

(p_4) Any sequence $\{u_n\} \subset \widehat{K}$ with $\{\overline{u}_n\}$ bounded, has a convergent subsequence.

With Ψ_p and $\mathcal{G}_p$ as above, we define $I_p := \Psi_p + \mathcal{G}_p$.

Proposition 2.1. *If $u \in C$ is a critical point of I_p, i.e.,*

$$\langle \mathcal{G}_p'(u), v - u\rangle + \Psi_p(v) - \Psi_p(u) \ge 0, \qquad \forall v \in C,$$

then u is a solution of problem (4).

3. **Notations and hypotheses.** The space $\mathbb{R}^N$ $(N \ge 1)$ will be endowed with the norm

$$|u| = \max_{i=1}^{N} |u_i| \quad \text{for all} \quad u = (u_1, ..., u_N) \in \mathbb{R}^N.$$

Let $(X, ||\cdot||_X)$ be a real Banach space with the dual denoted by X^* and G be a discrete subgroup of X. We denote by $\pi : X \to X/G$ the canonical projection. The following definitions are taken from [22] and are classical. A set $A \subset X$ is said to be *G-invariant* if

$$A = \pi^{-1}(\pi(A)).$$

Notice that a set A is G-invariant if and only if $u + g \in A$ for all $u \in A$ and $g \in G$. If M is an arbitrary set and $f : X \to M$ is a function, then f is called *G-invariant* if

$$f(u+g) = f(u) \quad \text{for all} \quad u \in X, g \in G.$$

For any G-invariant functional $\mathcal{G} \in C^1(X, \mathbb{R})$, one has that $\mathcal{G}' : X \to X^*$ is G-invariant. In what follows we assume that

$$\dim\,(\text{span } G) = N.$$

Then, we have

$$G \simeq \mathbb{Z}^N, \qquad X \simeq \mathbb{R}^N \oplus Y,$$

where Y is a closed subspace of X. So, any $u \in X$ can be uniquely decomposed as

$$u = \overline{u} + \widetilde{u}, \quad \text{ with } \overline{u} \in \mathbb{R}^N,\ \widetilde{u} \in Y,$$

and the mappings $u \mapsto \overline{u}$, $u \mapsto \widetilde{u}$ are bounded linear projections. We will consider on X the equivalent norm

$$||u|| = |\overline{u}| + ||\widetilde{u}||_X \quad (u \in X).$$

In the sequel we assume the following hypotheses.

(H_1) The functional $\mathcal{G} \in C^1(X, \mathbb{R})$ is G-invariant and $\mathcal{G}'$ takes bounded sets into bounded sets. On the other hand, $\Psi : X \to (-\infty, +\infty]$ is G-invariant, convex, lower semicontinuous and $D(\Psi) = \{u \in X : \ \Psi(u) < +\infty\}$ is a closed nonempty set.

(H_2) One has that $\Psi(0) = 0$ and

$$\Psi(u) = \Psi(\widetilde{u}) \quad \text{for all} \quad u \in X.$$

(H_3) There exists $\rho > 0$ such that

$$||\widetilde{u}|| \le \rho, \quad |\Psi(u)| \le \rho \quad \text{for all} \quad u \in D(\Psi).$$

(H_4) Any sequence $\{u_n\} \subset D(\Psi)$ with $\{\overline{u}_n\}$ bounded, has a convergent subsequence.

Note that from (H_2) it follows that Ψ is G-invariant and

$$\Psi(\overline{u}) = 0 \quad \text{for all} \quad \overline{u} \in \mathbb{R}^N.$$

With Ψ and $\mathcal{G}$ as above, we shall consider the functional

$$I = \Psi + \mathcal{G}. \tag{5}$$

According to Szulkin [24], a point $u \in X$ is said to be *a critical point* of I if $u \in D(\Psi)$ and it holds

$$\langle \mathcal{G}'(u), v - u\rangle + \Psi(v) - \Psi(u) \ge 0 \quad \text{for all} \quad v \in X. \tag{6}$$

For any $c \in \mathbb{R}$, we shall use the notations:

$$K = \{u \in X : \ u \text{ is a critical point}\}, \quad K_c = \{u \in K : \ I(u) = c\}.$$

Since $\mathcal{G}'$ and Ψ are G-invariant, it follows immediately that if $u \in K$, then $\pi^{-1}(\pi(u)) \subset K$. In this case the set $\pi^{-1}(\pi(u))$ is called *a critical orbit* of I. Moreover, using that I is G-invariant, it follows that if $u \in K_c$, then $\pi^{-1}(\pi(u)) \subset K_c$.

If $\mathcal{N}$ is an open neighborhood of K_c and $\epsilon > 0$, we denote

$$\mathcal{N}_\epsilon = \{u \in X \setminus \mathcal{N} : \ |\overline{u}| \le 2,\ I(u) \le c + \epsilon\}.$$

If $K_c = \emptyset$, then we will consider $\mathcal{N} = \emptyset$. Notice that $\mathcal{N}_\epsilon$ is a compact set. Indeed, using that I is lower semicontinuous, it follows that $\mathcal{N}_\epsilon$ is closed. If $\{u_n\}$ is a sequence in $\mathcal{N}_\epsilon$, then $\{u_n\} \subset D(\Psi)$ and $\{\overline{u}_n\}$ is bounded. Hence from (H_4) it follows that $\{u_n\}$ has a convergent subsequence. So, if $\mathcal{N}_\epsilon \neq \emptyset$, we can define

$$\alpha = \max_{u \in \mathcal{N}_\epsilon} |\langle \mathcal{G}'(u), \overline{u}\rangle|. \tag{7}$$

4. **Some auxiliary results.** Below, all the neighborhoods will be assumed to be open.

Lemma 4.1. *Let $c \in \mathbb{R}$ and $\mathcal{N}$ be a G-invariant neighborhood of K_c. Then, for each $\bar{\epsilon} > 0$, there exists $\epsilon \in (0, \bar{\epsilon}]$ such that for any $u_0 \in X \setminus \mathcal{N}$ with $c - \epsilon \leq I(u_0) \leq c + \epsilon$, there exists $v_0 \in X$ satisfying*

$$\langle \mathcal{G}'(u_0), v_0 - u_0 \rangle + \Psi(v_0) - \Psi(u_0) < -3\epsilon.$$

Proof. By contradiction, assume that for any positive integer n there exists $u_n \in X \setminus \mathcal{N}$ with

$$c - 1/n \leq I(u_n) \leq c + 1/n,$$

and

$$\langle \mathcal{G}'(u_n), v - u_n \rangle + \Psi(v) - \Psi(u_n) \geq -3/n, \quad \forall v \in X. \tag{8}$$

Clearly, one has that $\{u_n\} \subset D(\Psi)$. On the other hand, using that $\mathcal{G}', \Psi$ and $\mathcal{N}$ are G-invariant, we may assume that $\{\overline{u}_n\} \subset [0,1)^N$. So, using (H_4), passing if necessary to a subsequence, it follows that $\{u_n\}$ converges to some $u \in D(\Psi)$. We deduce that

$$\mathcal{G}(u_n) \to \mathcal{G}(u) \quad \text{and} \quad \Psi(u_n) \to c - \mathcal{G}(u).$$

As Ψ is lower semicontinuous, it follows that

$$c - \mathcal{G}(u) = \liminf_{n\to\infty} \Psi(u_n) \geq \Psi(u).$$

On the other hand, taking in (8) $v = u$ we obtain

$$\limsup_{n\to\infty} \Psi(u_n) \leq \Psi(u).$$

Hence,

$$\Psi(u_n) \to \Psi(u)$$

and using (8), we infer that $u \in K$. But, $I(u_n) \to I(u)$ and $I(u_n) \to c$, hence $I(u) = c$ and $u \in K_c$. This is in contradiction with $u_n \to u$, $\{u_n\} \subset X \setminus \mathcal{N}$ and $\mathcal{N}$ is a neighborhood of K_c. □

Lemma 4.2. *Let $c \in \mathbb{R}$ and $\mathcal{N}$ be a G-invariant neighborhood of K_c. Then, for each $\bar{\epsilon} > 0$, there exists $\epsilon \in (0, \bar{\epsilon}]$ such that for any $u_0 \in X \setminus \mathcal{N}$ with $I(u_0) \leq c + \epsilon$, there are $\epsilon_0 \in (0, \epsilon]$, $v_0 \in X$ and U_0 a neighborhood of u_0, satisfying*

$$\langle \mathcal{G}'(u), v_0 - u \rangle + \Psi(v_0) - \Psi(u) \leq 1, \quad \forall u \in U_0, \tag{9}$$

and

$$\langle \mathcal{G}'(u), v_0 - u \rangle + \Psi(v_0) - \Psi(u) \leq -2\epsilon_0, \quad \forall u \in U_0 \text{ with } I(u) \geq c - \epsilon. \tag{10}$$

Proof. Let $\bar{\epsilon} > 0$ and the corresponding $\epsilon \in (0, \bar{\epsilon}]$ be given in Lemma 4.1. We have to consider the following three cases.

Case 1: $\underline{u_0 \in K.}$ In this case, we shall prove the assertions with $v_0 = u_0$. We have

$$\langle \mathcal{G}'(u_0), u - u_0 \rangle + \Psi(u) - \Psi(u_0) \geq 0 \quad \text{for all} \quad u \in X.$$

Then, from the continuity of $\mathcal{G}'$, we infer that

$$\begin{aligned} \langle \mathcal{G}'(u), u_0 - u \rangle + \Psi(u_0) - \Psi(u) &\leq \langle \mathcal{G}'(u) - \mathcal{G}'(u_0), u_0 - u \rangle \\ &\leq ||\mathcal{G}'(u) - \mathcal{G}'(u_0)|| \; ||u - u_0|| \\ &\leq 1, \end{aligned}$$

for all $u \in U_1$, where U_1 is a sufficiently small neighborhood of u_0. On the other hand, using Lemma 4.1, it follows

$$[u \in K,\ c - \epsilon \leq I(u) \leq c + \epsilon] \Rightarrow u \in \mathcal{N},$$

which ensures that

$$I(u_0) < c - \epsilon.$$

Next, we prove that there exists U_2 a neighborhood of u_0 and $\epsilon_0 \in (0, \epsilon]$ such that

$$\Psi(u) - \Psi(u_0) > 3\epsilon_0, \qquad \forall u \in U_2,\ I(u) \geq c - \epsilon.$$

Assume by contradiction that there exists a sequence $\{u_n\}$ converging to u_0, with

$$I(u_n) \geq c - \epsilon, \qquad \Psi(u_n) - \Psi(u_0) \leq 1/n$$

for all $n \geq 1$. This, together with the lower semicontinuity of Ψ imply that $\Psi(u_n) \to \Psi(u_0)$, hence $I(u_n) \to I(u_0)$. But $I(u_0) < c - \epsilon$ and $I(u_n) \geq c - \epsilon$, which give a contradiction. Note that, as $\mathcal{G}'$ takes bounded sets into bounded sets, we may assume

$$||\mathcal{G}'(u)||\ ||u - u_0|| \leq \epsilon_0, \qquad \forall u \in U_2.$$

It follows that

$$\langle \mathcal{G}'(u), u_0 - u\rangle + \Psi(u_0) - \Psi(u) \leq ||\mathcal{G}'(u)||\ ||u - u_0|| - 3\epsilon_0 \leq -2\epsilon_0,$$

for all $u \in U_2$ with $I(u) \geq c - \epsilon$. So, in this case we take $U_0 = U_1 \cap U_2$.

Case 2: $\underline{u_0 \notin K,\ I(u_0) < c - \epsilon}$. Let $v_0 \in D(\Psi)$ be with the property

$$\langle \mathcal{G}'(u_0), v_0 - u_0\rangle + \Psi(v_0) - \Psi(u_0) < 0. \tag{11}$$

We may assume that v_0 is arbitrarily close to u_0. Indeed, consider $t \in (0, 1)$ and $w_0 = tv_0 + (1 - t)u_0$. Then, from (11) and the convexity of Ψ, it follows that

$$\langle \mathcal{G}'(u_0), w_0 - u_0\rangle + \Psi(w_0) - \Psi(u_0) < 0.$$

The assertion follows by taking $t \to 0_+$.

First, we deal with (10). Using that $I(u_0) < c - \epsilon$ and arguing exactly as in the previous case, there exists U_3 a neighborhood of u_0 and $\epsilon_0 \in (0, \epsilon]$ such that

$$\Psi(u) - \Psi(u_0) > 4\epsilon_0, \qquad \forall u \in U_3,\ I(u) \geq c - \epsilon.$$

Using that $\mathcal{G}'$ takes bounded sets into bounded sets, it follows that there exists $M_0 > \epsilon_0/2$ with

$$||\mathcal{G}'(u)|| < M_0, \qquad \forall u \in X,\ ||u - u_0|| \leq 1.$$

Now, let us consider v_0 satisfying (11) and $||v_0 - u_0|| < \epsilon_0/(2M_0)$. From the choice of M_0, it follows

$$||\mathcal{G}'(u)||\ ||v_0 - u_0|| \leq \frac{\epsilon_0}{2} \quad \text{and} \quad ||\mathcal{G}'(u)||\ ||u - u_0|| \leq \frac{\epsilon_0}{2},$$

for all $u \in X$ with $||u - u_0|| \leq \epsilon_0/(2M_0)$. Set $U_4 = U_3 \cap B(u_0, \epsilon_0/(2M_0))$. One has the following estimates:

$$\begin{aligned}
\Psi(v_0) - \Psi(u) &= (\Psi(v_0) - \Psi(u_0)) + (\Psi(u_0) - \Psi(u)) \\
&< \langle \mathcal{G}'(u_0), u_0 - v_0\rangle + (\Psi(u_0) - \Psi(u)) \\
&\leq ||\mathcal{G}'(u_0)||\ ||v_0 - u_0|| + (\Psi(u_0) - \Psi(u)) \\
&\leq \epsilon_0/2 - 4\epsilon_0 < -3\epsilon_0,
\end{aligned}$$

for all $u \in U_4$ with $I(u) \geq c - \epsilon$. We infer

$$\begin{aligned}\langle \mathcal{G}'(u), v_0 - u\rangle + \Psi(v_0) - \Psi(u) &\leq ||\mathcal{G}'(u)||\,(||v_0 - u_0|| + ||u_0 - u||) - 3\epsilon_0 \\ &\leq \epsilon_0/2 + \epsilon_0/2 - 3\epsilon_0 = -2\epsilon_0,\end{aligned}$$

for all $u \in U_4,\ I(u) \geq c - \epsilon$, and (10) is proved.

Next, we have in view (9). Let $\delta_0 > 0$ be such that

$$\langle \mathcal{G}'(u_0), v_0 - u_0\rangle + \Psi(v_0) - \Psi(u_0) = -2\delta_0.$$

Using the continuity of $\mathcal{G}'$, it follows that there exists a neighborhood of u_0 denoted by U_5 such that

$$||\mathcal{G}'(u) - \mathcal{G}'(u_0)||\,||v_0 - u|| < \delta_0/4 \quad \text{and} \quad ||\mathcal{G}'(u_0)||\,||u - u_0|| < \delta_0/4,$$

for all $u \in U_5$. We get

$$\begin{aligned}\langle \mathcal{G}'(u), v_0 - u\rangle &= \langle \mathcal{G}'(u) - \mathcal{G}'(u_0), v_0 - u\rangle \\ &+ \langle \mathcal{G}'(u_0), v_0 - u_0\rangle + \langle \mathcal{G}'(u_0), u_0 - u\rangle \\ &\leq ||\mathcal{G}'(u) - \mathcal{G}'(u_0)||\,||v_0 - u|| \\ &+ \langle \mathcal{G}'(u_0), v_0 - u_0\rangle + ||\mathcal{G}'(u_0)||\,||u - u_0|| \\ &\leq \delta_0/2 + \langle \mathcal{G}'(u_0), v_0 - u_0\rangle,\end{aligned}$$

for all $u \in U_5$. On the other hand, by the lower semicontinuity of Ψ, there exists U_6 a neighborhood of u_0 such that

$$\Psi(u_0) - \Psi(u) \leq \delta_0/2, \qquad \forall u \in U_6.$$

Consequently, taking $U_7 = U_5 \cap U_6$, one has

$$\begin{aligned}\langle \mathcal{G}'(u), v_0 - u\rangle + \Psi(v_0) - \Psi(u) \leq \delta_0/2 + \langle \mathcal{G}'(u_0), v_0 - u_0\rangle + \Psi(v_0) - \Psi(u_0) \\ +\Psi(u_0) - \Psi(u) \leq -\delta_0,\end{aligned}$$

for all $u \in U_7$, and (9) is proved. Therefore, in this case U_0 will be $U_4 \cap U_7$.

Case 3: $\underline{u_0 \notin K,\ I(u_0) \geq c - \epsilon}$. From Lemma 4.1, there exists $v_0 \in X$ satisfying

$$\langle \mathcal{G}'(u_0), v_0 - u_0\rangle + \Psi(v_0) - \Psi(u_0) < -3\epsilon.$$

Now, arguing exactly as in the proof of (9) in Case 2, it follows that there exists U_0 a neighborhood of u_0 such that

$$\langle \mathcal{G}'(u), v_0 - u\rangle \leq \epsilon/2 + \langle \mathcal{G}'(u_0), v_0 - u_0\rangle \quad \text{and} \quad \Psi(u_0) - \Psi(u) \leq \epsilon/2,$$

for all $u \in U_0$. Also, an argument similar to that used in the proof of (9) in Case 2 yields

$$\langle \mathcal{G}'(u), v_0 - u\rangle + \Psi(v_0) - \Psi(u) \leq -2\epsilon \qquad \forall u \in U_0.$$

□

Lemma 4.3. *Let $c \in \mathbb{R}$ and $\mathcal{N}$ be a G-invariant neighborhood of K_c. Then, for each $\overline{\epsilon} > 0$, there exist $\epsilon \in (0, \overline{\epsilon}]$, $M_\epsilon > 0$, $\epsilon' \in (0, \epsilon]$ such that: $\forall u_0 \in \mathcal{N}_\epsilon$, $\exists v_0 \in X$ with $||v_0|| \leq M_\epsilon$, $\exists U_0$ a neighborhood of u_0 satisfying (9) and*

$$\langle \mathcal{G}'(u), v_0 - u\rangle + \Psi(v_0) - \Psi(u) \leq -2\epsilon', \quad \forall u \in U_0 \text{ with } I(u) \geq c - \epsilon. \tag{12}$$

Proof. Let $\overline{\epsilon} > 0$ and the corresponding $\epsilon \in (0, \overline{\epsilon}]$ be given by Lemma 4.2. For each $u_0 \in \mathcal{N}_\epsilon$, let ϵ_0, v_0 and U_0 constructed in Lemma 4.2. The sets U_0 cover $\mathcal{N}_\epsilon$. Using that $\mathcal{N}_\epsilon$ is compact, it follows that there exists $(U_j)_{j=1}^l$ a finite subcovering. Let u_j, ϵ_j, v_j be related to U_j in the same way as u_0, ϵ_0, v_0 are related to U_0. We set

$$M_\epsilon = \max_{j=1}^{l} ||v_j|| \quad \text{and} \quad \epsilon' = \min_{j=1}^{l} \epsilon_j.$$

Then, for $u_0 \in \mathcal{N}_\epsilon$, there exists U_{j_0} such that $u_0 \in U_{j_0}$. We take $v_0 = v_{j_0}$ and $U_0 = U_{j_0}$. The proof follows now from Lemma 4.2. □

Lemma 4.4. *Let $u_0 \in D(\Psi)$ be such that*

$$\mathcal{G}'(u_0)|_{\mathbb{R}^N} \neq 0. \tag{13}$$

Then, for any $r > 0$, there exists $v_r \in X$ and U_0 a neighborhood of u_0 such that

$$\langle \mathcal{G}'(u+g), v_r - (u+g)\rangle + \Psi(v_r) - \Psi(u+g) \leq -r, \tag{14}$$

for all $g \in G$ with $|g| \leq 6$ and $u \in U_0$.

Proof. Since $\mathcal{G}'$ is bounded on bounded subsets of X, we can fix some $\rho_0 > 0$ such that

$$|\langle \mathcal{G}'(u), g\rangle| \leq \rho_0, \quad \forall u \in B(u_0, 1) \text{ and } g \in G \text{ with } |g| \leq 6.$$

On the other hand, one has that there exists some $e_j = (0, ..., 1, ..., 0) \in \mathbb{R}^N$ with

$$\langle \mathcal{G}'(u_0), e_j\rangle \neq 0.$$

We may assume that

$$\langle \mathcal{G}'(u_0), e_j\rangle \ > 0.$$

Let $r > 0$ and consider $v_r = u_0 + \overline{v}_r \in D(\Psi)$, where $\overline{v}_r = (0, ..., w_r, ..., 0)$, $(w_r \in \mathbb{R})$. We have

$$\langle \mathcal{G}'(u_0), v_r - u_0\rangle = \langle \mathcal{G}'(u_0), \overline{v}_r\rangle = w_r \langle \mathcal{G}'(u_0), e_j\rangle.$$

It follows that there is some $w_r < 0$ such that

$$\langle \mathcal{G}'(u_0), v_r - u_0\rangle \ < -r - 2(\rho_0 + 2\rho),$$

with ρ entering in (H_3). Then, for $u \in X$, we write

$$\begin{aligned} \langle \mathcal{G}'(u), v_r - u\rangle \ &\leq \ ||\mathcal{G}'(u) - \mathcal{G}'(u_0)|| \ ||v_r - u|| + ||\mathcal{G}'(u_0)|| \ ||u_0 - u|| \\ &+ \ \langle \mathcal{G}'(u_0), v_r - u_0\rangle. \end{aligned}$$

Using the continuity of $\mathcal{G}'$, it follows that there exists $U_r \subset B(u_0, 1)$ a neighborhood of u_0 such that

$$||\mathcal{G}'(u) - \mathcal{G}'(u_0)|| \leq \frac{\rho_0 + 2\rho}{2(|w_r| + 1)}, \quad \forall u \in U_r.$$

Also, we may assume that

$$||\mathcal{G}'(u_0)|| \ ||u_0 - u|| \leq \frac{\rho_0 + 2\rho}{2}, \quad \forall u \in U_r.$$

Then, from

$$||v_r - u|| \leq |w_r| + 1$$

we obtain

$$||\mathcal{G}'(u) - \mathcal{G}'(u_0)|| \ ||v_r - u|| \leq \frac{\rho_0 + 2\rho}{2}, \quad \forall u \in U_r.$$

Hence,

$$\langle \mathcal{G}'(u), v_r - u\rangle \leq -r - (\rho_0 + 2\rho), \quad \forall u \in U_r.$$

Now, the result follows immediately from the G-invariance of $\mathcal{G}'$ and Ψ and from (H_3). $\square$

Lemma 4.5. *Let $c \in \mathbb{R}$ and $\mathcal{N}$ be a neighborhood of K_c. Then, for any $\epsilon, r > 0$, there exists $M_{\epsilon,r} > 0$ such that $\forall u_0 \in \mathcal{N}_\epsilon$, satisfying (13), $\exists v_0 \in X$ with $||v_0|| \le M_{\epsilon,r}$, $\exists U_0$ a neighborhood of u_0 such that (14) holds true, for all $g \in G$ with $|g| \le 6$ and $u \in U_0$.*

Proof. Using that the set $\mathcal{N}_\epsilon \subset D(\Psi)$ is compact, the argument is similar to that employed in the proof of Lemma 4.3, but with Lemma 4.4 instead of Lemma 4.2. $\square$

Remark 4.6. Let U be an open subset of X $u_0 \in U$. Using that G is discrete, there exists $\mu_0 \in (0,1]$ such that the square

$$D(u_0, \mu_0) = \{u \in X : \ |\overline{u} - \overline{u}_0| < \mu_0, \ ||\widetilde{u} - \widetilde{u}_0|| < \mu_0\}$$

satisfies $D(u_0, \mu_0) \subset U$ and

$$u \in D(u_0, \mu_0) \Rightarrow u + g \notin D(u_0, \mu_0) \quad \forall g \in G \setminus \{0\}. \tag{15}$$

It follows that U_0 in the above Lemmas 4.2 - 4.5 can be supposed to be such a square.

Remark 4.7. (i) Let $u_0 \in X$ be such that

$$\mathcal{G}'(u_0)|_{\mathbb{R}^N} = 0. \tag{16}$$

From the continuity of $\mathcal{G}'$ in u_0, we infer that for any $\eta > 0$, there exists $\delta_\eta > 0$ so that

$$|\langle \mathcal{G}'(u), v\rangle| \le \eta |v|, \quad \forall v \in \mathbb{R}^N, \ \forall u \in X \text{ with } ||u - u_0|| \le \delta_\eta.$$

(ii) Let $U_0 = D(u_0, \mu_0)$ be as in Lemma 4.3 (see also Remark 4.6) with $\overline{\epsilon} \le 1$. Assume that u_0 is such that (16) holds true. Let $\eta = \epsilon'/12$ and $\delta_\eta > 0$ be the corresponding number associated to η by (i). Consider also $\nu_0 \in (0, \min\{\mu_0, \delta_\eta/2\})$, and note that $D(u_0, \nu_0) \subset B(u_0, \delta_\eta) \cap D(u_0, \mu_0)$. It is clear that

$$|\langle \mathcal{G}'(u + g), v\rangle| \le (\epsilon'/12)|v|, \tag{17}$$

for all $g \in G$, $v \in \mathbb{R}^N$ and $u \in D(u_0, \nu_0)$. Then, for all $g \in G$ with $|g| \le 6$ and $u \in D(u_0, \nu_0)$, one has

$$|\langle \mathcal{G}'(u + g), -g\rangle| \le \epsilon'/2.$$

This, together with Lemma 4.3 and the G-invariance of $\mathcal{G}'$ and Ψ, imply that

$$\langle \mathcal{G}'(u + g), v_0 - (u + g)\rangle + \Psi(v_0) - \Psi(u + g) \le 2, \tag{18}$$

for all $g \in G$ with $|g| \le 6$ and $u \in D(u_0, \nu_0)$. If, moreover, $I(u) \ge c - \epsilon$, then

$$\langle \mathcal{G}'(u + g), v_0 - (u + g)\rangle + \Psi(v_0) - \Psi(u + g) \le -\epsilon'. \tag{19}$$

Remark 4.8. Let α be defined in (7) and $\epsilon > 0$. Then, taking in Lemma 4.5, $r = \epsilon + \alpha$, we obtain that there exists $M'_\epsilon := M_{\epsilon,\epsilon+\alpha} > 0$ such that for any $u_0 \in \mathcal{N}_\epsilon$ satisfying (13), $\exists v_0 \in X$ with $||v_0|| \le M'_\epsilon$, $\exists D(u_0, \mu_0)$, such that

$$\langle \mathcal{G}'(u + g), v_0 - (u + g)\rangle + \Psi(v_0) - \Psi(u + g) \le -(\epsilon + \alpha), \tag{20}$$

for all $g \in G$ with $|g| \le 6$ and $u \in D(u_0, \mu_0)$.

The main result of this Section is the following

Proposition 4.9. *Let $c \in \mathbb{R}$ and $\mathcal{N}$ be a G-invariant neighborhood of K_c. Then, for each $\overline{\epsilon} \in (0,1]$ there exist $\epsilon \in (0, \overline{\epsilon}]$, $m_\epsilon > 0$ and $\epsilon' \in (0, \epsilon]$ with the following properties.*

1^0 *For any* $u_0 \in \mathcal{N}_\epsilon$ *with* $\mathcal{G}'(u_0)|_{\mathbb{R}^N} = 0$, $\exists v_0 \in X$ *with* $||v_0|| \le m_\epsilon$, $\exists \mu_0 > 0$, *such that*

(i) (17) holds true for all $g \in G$, $v \in \mathbb{R}^N$ *and* $u \in D(u_0, \mu_0)$;

(ii) (18) holds true for all $g \in G$ *with* $|g| \le 6$ *and* $u \in D(u_0, \mu_0)$;

(iii) (19) holds true for all $g \in G$ *with* $|g| \le 6$ *and* $u \in D(u_0, \mu_0)$ *with* $I(u) \ge c - \epsilon$.

2^0 *For any* $u_0 \in \mathcal{N}_\epsilon$ *with* $\mathcal{G}'(u_0)|_{\mathbb{R}^N} \ne 0$, $\exists v_0 \in X$ *with* $||v_0|| \le m_\epsilon$, $\exists \mu_0 > 0$, *such that (20) holds true for all* $g \in G$ *with* $|g| \le 6$ *and* $u \in D(u_0, \mu_0)$.

Note that μ_0 *above is taken such that (15) holds true.*

Proof. For 1^0 one applies Lemma 4.3 and Remark 4.7, while 2^0 follows from Lemma 4.5 and Remark 4.8; one takes $m_\epsilon = \max\{M_\epsilon, M'_\epsilon\}$. □

5. **A deformation lemma.**

Lemma 5.1. *Let* $c \in \mathbb{R}$ *and* $\mathcal{N}$ *be a* G*-invariant neighborhood of* K_c. *Then, for each* $\bar{\epsilon} \in (0, 1]$ *there exist* $\epsilon \in (0, \bar{\epsilon}]$, $d_\epsilon > 0$, $\epsilon' \in (0, \epsilon]$ *and* $\eta : [0, \bar{t}] \times \mathcal{N}_\epsilon \to X$ *a continuous function, with the following properties.*

(i) $\eta(0, \cdot) = id_{\mathcal{N}_\epsilon}$.

(ii) $\eta(t, u + g) = \eta(t, u) + g, \quad \forall (t, u) \in [0, \bar{t}] \times \mathcal{N}_\epsilon, \ \forall g \in G$ *with* $u + g \in \mathcal{N}_\epsilon$.

(iii) $||\eta(t, u) - u|| \le d_\epsilon t, \quad \forall (t, u) \in [0, \bar{t}] \times \mathcal{N}_\epsilon$.

(iv) $I(\eta(t, u)) - I(u) \le d_\epsilon t, \quad \forall (t, u) \in [0, \bar{t}] \times \mathcal{N}_\epsilon$.

(v) $I(\eta(t, u)) - I(u) \le -\epsilon' t/2, \quad \forall (t, u) \in [0, \bar{t}] \times \mathcal{N}_\epsilon$ *with* $I(u) \ge c - \epsilon$.

(vi) If A *is a closed subset of* $\mathcal{N}_\epsilon$ *with* $c \le \sup_A I$, *then*

$$\sup_{u \in A} I(\eta(t, u)) - \sup_{u \in A} I(u) \le -\epsilon' t/2, \qquad \forall t \in [0, \bar{t}].$$

Proof. *Covering.* Let $\bar{\epsilon} \in (0, 1]$ and the corresponding $\epsilon \in (0, \bar{\epsilon}]$, $m_\epsilon > 0$ and $\epsilon' \in (0, \epsilon]$ be given in Proposition 4.9. Also, for each $u_0 \in \mathcal{N}_\epsilon$, let v_0, μ_0 and $D(u_0, \mu_0)$ be as in Proposition 4.9. Since the sets $D(u_0, \mu_0)$ cover the compact set $\mathcal{N}_\epsilon$, it follows that there exists $(D_j)_{j=1}^l$ a finite subcovering. Below, u_j, v_j, μ_j will be related to D_j in the same way as u_0, v_0, μ_0 are related to $D(u_0, \mu_0)$.

Partition of unity. Let $\rho_i^1 : X \to [0, \infty)$ be a continuous function (we can take the distance function $d(\cdot, X \setminus D_i)$) such that

$$\rho_i^1(u) > 0, \ \ \forall u \in D_i \ \text{ and } \ \rho_i^1(u) = 0, \ \ \forall u \in X \setminus D_i.$$

Consider the G-invariant set

$$V_i = \bigcup_{g \in G} (D_i + g).$$

Note that, from the choice of the squares D_i (see (15)), one has that the sets $D_i + g$ ($g \in G$) are mutually disjoint. It follows that, the function $\rho_i^2 : X \to [0, \infty)$ given by $\rho_i^2(u + g) = \rho_i^1(u)$ for all $u \in D_i$, $g \in G$, and $\rho_i^2(u) = 0$ for all $u \in X \setminus V_i$ is correctly defined, continuous and G-invariant.

Now, let us define

$$D = \bigcup_{i=1}^{l} D_i$$

and

$$\sigma_i : D \to [0, 1], \quad \sigma_i = \frac{\rho_i^2}{\sum_{j=1}^{l} \rho_j^2}.$$

One has that σ_i is correctly defined, continuous and G-invariant in the sense that

$$\sigma_i(u+g) = \sigma_i(u), \quad \forall u \in D,\ g \in G \text{ with } u+g \in D.$$

Also, we have

$$\sum_{i=1}^{l} \sigma_i = 1$$

and

$$\sigma_i(w) \neq 0 \ \Leftrightarrow\ w = w_i + g_i \text{ with some } w_i \in D_i,\ g_i \in G. \tag{21}$$

Deformation. Consider the function $\eta : [0,1] \times \mathcal{N}_\epsilon \to X$ given by

$$\eta(t,u) = (1-t)u + t\sum_{i=1}^{l} \sigma_i(u)v_i + t\overline{u} \qquad ((t,u) \in [0,1] \times \mathcal{N}_\epsilon).$$

It is clear that η is continuous and $\eta(0,\cdot) = id_{\mathcal{N}_\epsilon}$.

To prove (ii), let $(t,u) \in [0,1] \times \mathcal{N}_\epsilon$ and $g \in G$ be with $u+g \in \mathcal{N}_\epsilon$. Then, we have

$$\begin{aligned} \eta(t,u+g) &= (1-t)(u+g) + t\sum_{i=1}^{l} \sigma_i(u+g)v_i + t\overline{u+g} \\ &= (1-t)u + (1-t)g + t\sum_{i=1}^{l} \sigma_i(u)v_i + t\overline{u} + tg \\ &= \eta(t,u) + g. \end{aligned}$$

In order to prove (iii), let us consider $(t,u) \in [0,1] \times \mathcal{N}_\epsilon$. Using (H_3) and denoting $d_\epsilon^1 = m_\epsilon + \rho$, one has:

$$\begin{aligned} ||\eta(t,u) - u|| &= t||\sum_{i=1}^{l} \sigma_i(u)v_i - \widetilde{u}|| \\ &\leq t\left[\sum_{i=1}^{l} \sigma_i(u)||v_i|| + ||\widetilde{u}||\right] \\ &\leq t\left[m_\epsilon \sum_{i=1}^{l} \sigma_i(u) + \rho\right] = td_\epsilon^1. \end{aligned}$$

Estimations. Let us consider $(t,u) \in [0,1] \times \mathcal{N}_\epsilon$. Setting

$$w := \sum_{i=1}^{l} \sigma_i(u)v_i - \widetilde{u},$$

we have $\|w\| \leq d_\epsilon^1$ and

$$\eta(t,u) = u + tw.$$

By the mean value theorem, we can write

$$\mathcal{G}(u+tw) - \mathcal{G}(u) = t\langle \mathcal{G}'(u+\theta tw), w\rangle,$$

with some $\theta \in (0,1)$. Hence,

$$I(\eta(t,u)) = \mathcal{G}(u) + t\langle \mathcal{G}'(u+\theta tw), w\rangle + \Psi(u+tw). \tag{22}$$

On the other hand, from (H_2) and the convexity of Ψ we get

$$\begin{aligned}\Psi(u+tw) &= \Psi\left((1-t)u + t\sum_{i=1}^{l}\sigma_i(u)v_i + t\overline{u}\right)\\ &= \Psi\left((1-t)u + t\sum_{i=1}^{l}\sigma_i(u)v_i\right)\\ &\leq (1-t)\Psi(u) + t\sum_{i=1}^{l}\sigma_i(u)\Psi(v_i).\end{aligned}$$

Then, using (22), it follows

$$\begin{aligned}I(\eta(t,u)) - I(u) &\leq t\sum_{i=1}^{l}\sigma_i(u)\left[\Psi(v_i) - \Psi(u)\right] + t\langle \mathcal{G}'(u+\theta tw), w\rangle\\ &= t\sum_{i=1}^{l}\sigma_i(u)\left[\langle \mathcal{G}'(u), v_i - u\rangle + \Psi(v_i) - \Psi(u)\right]\\ &+ t\left[\langle \mathcal{G}'(u+\theta tw), w\rangle - \langle \mathcal{G}'(u), w\rangle + \langle \mathcal{G}'(u), \overline{u}\rangle\right]\\ &\leq t\sum_{i=1}^{l}\sigma_i(u)\left[\langle \mathcal{G}'(u), v_i - u\rangle + \Psi(v_i) - \Psi(u)\right]\\ &+ t\left[(\|\mathcal{G}'(u+\theta tw) - \mathcal{G}'(u)\|)d_\epsilon^1 + \langle \mathcal{G}'(u), \overline{u}\rangle\right]\end{aligned}$$

Next, as $\mathcal{G}'$ is continuous and $\mathcal{N}_\epsilon$ is compact, there exists $\delta = \delta(\epsilon, \epsilon') > 0$ such that

$$||\mathcal{G}'(v) - \mathcal{G}'(u)|| \leq \epsilon'/(4d_\epsilon^1), \quad \forall u \in \mathcal{N}_\epsilon, v \in X \text{ with } ||v-u|| \leq \delta.$$

Then, denoting

$$\bar{t}_1 := \delta/d_\epsilon^1,$$

it follows

$$||\mathcal{G}'(u+\theta tw) - \mathcal{G}'(u)|| \leq \epsilon'/(4d_\epsilon^1), \quad \forall t \in [0, \bar{t}_1], \quad \forall u \in \mathcal{N}_\epsilon.$$

So, we obtain

$$\begin{aligned}I(\eta(t,u)) - I(u) &\leq t\sum_{i=1}^{l}\sigma_i(u)\left[\langle \mathcal{G}'(u), v_i - u\rangle + \Psi(v_i) - \Psi(u)\right]\\ &+ t\left[\epsilon'/4 + \langle \mathcal{G}'(u), \overline{u}\rangle\right] \qquad (23)\end{aligned}$$

for all $t \in [0, \bar{t}_1]$ and $u \in \mathcal{N}_\epsilon$.

Let us prove (iv). Consider $(t,u) \in [0, \bar{t}_1] \times \mathcal{N}_\epsilon$. From (21), if $\sigma_i(u) \neq 0$ then $u = u_i' + g_i$, with $u_i' \in D_i$ and $g_i \in G$. In this situation we have

$$|g_i| \leq |\overline{u}| + |\overline{u_i'}| \leq |\overline{u}| + |\overline{u_i'} - \overline{u_i}| + |\overline{u_i}| \leq 2 + \mu_i + 2 \leq 6$$

and, from Proposition 4.9 (ii) it follows

$$\langle \mathcal{G}'(u), v_i - u\rangle + \Psi(v_i) - \Psi(u) \leq 2.$$

This, together with (7) and (23) yield

$$I(\eta(t,u)) - I(u) \leq t(\alpha + 3).$$

To prove (v), let $(t,u) \in [0, \bar{t}_1] \times \mathcal{N}_\epsilon$ be such that $I(u) \geq c - \epsilon$. We rewrite (23) as follows

$$I(\eta(t,u)) - I(u) \leq \tag{24}$$
$$t\sum_{i=1}^{l}\sigma_i(u)\left[\langle \mathcal{G}'(u), v_i - u\rangle + \Psi(v_i) - \Psi(u) + \langle \mathcal{G}'(u), \overline{u}\rangle\right] + t\epsilon'/4.$$

As above, if $\sigma_i(u) \neq 0$ then $u = u_i' + g_i$, with $u_i' \in D_i$, $g_i \in G$ and $|g_i| \leq 6$. From Proposition 4.9, if $\mathcal{G}'(u_i)|_{\mathbb{R}^N} = 0$, then

$$\langle \mathcal{G}'(u), v_i - u\rangle + \Psi(v_i) - \Psi(u) \leq -\epsilon',$$

and

$$|\langle \mathcal{G}'(u), \overline{u}\rangle| \leq \epsilon'/6,$$

while, if $\mathcal{G}'(u_i)|_{\mathbb{R}^N} \neq 0$, then

$$\langle \mathcal{G}'(u), v_i - u\rangle + \Psi(v_i) - \Psi(u) \leq -\epsilon - \alpha.$$

In both cases, one has that

$$\langle \mathcal{G}'(u), v_i - u\rangle + \Psi(v_i) - \Psi(u) + \langle \mathcal{G}'(u), \overline{u}\rangle \leq -\epsilon' + (\epsilon'/6).$$

This, together with (24) give

$$I(\eta(t,u)) - I(u) \leq t\sum_{i=1}^{l}\sigma_i(u)[-\epsilon' + (\epsilon'/6)] + t\epsilon'/4 < -\epsilon' t/2.$$

In order to prove (vi), we set $\overline{t} := \min\{\overline{t}_1, 1/2, \frac{\epsilon}{2(\alpha+3)}\}$ and let $A \subset \mathcal{N}_\epsilon$ be closed such that $c \leq \sup_A I$. For $t \in [0, \overline{t}]$, we have two cases.

If

$$\sup_{u\in A} I(\eta(t,u)) \leq c - (\epsilon/2),$$

then, using that $t \leq 1/2$, it follows

$$\sup_{u\in A} I(\eta(t,u)) - \sup_{u\in A} I(u) \leq -\epsilon t \leq -\epsilon' t.$$

If

$$\sup_{u\in A} I(\eta(t,u)) > c - (\epsilon/2),$$

then, putting

$$B := \{u \in A : \ I(u) \geq c - \epsilon\},$$

it follows

$$I(\eta(t,u)) \leq I(u) + (\alpha+3)t < c - \epsilon + (\alpha+3)t \leq c - (\epsilon/2),$$

for all $u \in A \setminus B$. We infer that

$$\sup_{u\in A} I(\eta(t,u)) = \sup_{u\in B} I(\eta(t,u)).$$

Consequently,

$$\begin{aligned}\sup_{u\in A} I(\eta(t,u)) - \sup_{u\in A} I(u) &\leq \sup_{u\in B} I(\eta(t,u)) - \sup_{u\in B} I(u)\\ &\leq \sup_{u\in B}[I(\eta(t,u)) - I(u)]\\ &\leq -\epsilon' t/2.\end{aligned}$$

Now, to finish the proof it suffices to take $d_\epsilon := \max\{d_\epsilon^1, \alpha+3\}$. □

The main result of this Section is the following

Proposition 5.2. *Let $c \in \mathbb{R}$ and $\mathcal{N}$ be a G-invariant neighborhood of K_c. Then, for each $\overline{\epsilon} > 0$ there exist $d > 0$, $\epsilon'' \in (0, \overline{\epsilon}]$ with $2d\epsilon'' < \overline{\epsilon}$, and $\eta : [0, \overline{t}] \times \mathcal{N}_{\epsilon''} \to X$ a continuous function, with the following properties.*

(i) $\eta(0, \cdot) = id_{\mathcal{N}_{\epsilon''}}$.
(ii) $\eta(t, u + g) = \eta(t, u) + g, \qquad \forall (t, u) \in [0, \overline{t}] \times \mathcal{N}_{\epsilon''}, \ \forall g \in G$ *with* $u + g \in \mathcal{N}_{\epsilon''}$.
(iii) $||\eta(t, u) - u|| \le dt, \qquad \forall (t, u) \in [0, \overline{t}] \times \mathcal{N}_{\epsilon''}$.
(iv) If A is a closed subset of $\mathcal{N}_{\epsilon''}$ with $c \le \sup_A I$, then

$$\sup_{u \in A} I(\eta(t, u)) - \sup_{u \in A} I(u) \le -\epsilon'' t, \qquad \forall t \in [0, \overline{t}].$$

Proof. The result follows immediately from Lemma 5.1 taking

$$0 < \epsilon'' < \min\{\epsilon'/2, \overline{\epsilon}/2d_\epsilon\}.$$

Note that $\mathcal{N}_{\epsilon''} \subset \mathcal{N}_\epsilon$. □

Lemma 5.3. *Let η be as in Proposition 5.2. Then $\widehat{\eta} : [0, \overline{t}] \times \pi(\mathcal{N}_{\epsilon''}) \to \pi(X)$ defined by*

$$\widehat{\eta}(t, \Gamma) = \pi(\eta(t, v)), \quad \textit{for } v \in \mathcal{N}_{\epsilon''} \textit{ with } \pi(v) = \Gamma \quad (t \in [0, \overline{t}])$$

is well defined and continuous.

Proof. Let $(t, \Gamma) \in [0, \overline{t}] \times \pi(\mathcal{N}_{\epsilon''})$. It follows that there exists $u \in \mathcal{N}_{\epsilon''}$ such that $\pi(u) = \Gamma$. Assume that $u_1, u_2 \in \mathcal{N}_{\epsilon''}$ are such that $\pi(u_1) = \Gamma = \pi(u_2)$. It follows that $u_2 = u_1 + g$, for some $g \in G$. Then, using Proposition 5.2 (ii), we get

$$\eta(t, u_2) = \eta(t, u_1 + g) = \eta(t, u_1) + g,$$

which means that $\pi(\eta(t, u_1)) = \pi(\eta(t, u_2))$ and $\widehat{\eta}$ is well defined.

For the continuity of $\widehat{\eta}$, consider a sequence $\{(t_k, \Gamma_k)\} \subset [0, \overline{t}] \times \pi(\mathcal{N}_{\epsilon''})$ converging to some $(t, \Gamma) \in [0, \overline{t}] \times \pi(\mathcal{N}_{\epsilon''})$. It follows that there exists $\{u_k\} \subset X$ with $\pi(u_k) = \Gamma_k$ such that $u_k \to u \in X$ and $\pi(u) = \Gamma$. Note that $\widetilde{u}_k \to \widetilde{u}$ and $\overline{u}_k \to \overline{u}$. On the other hand, $u_k = v_k + g_k$ with some $v_k \in \mathcal{N}_{\epsilon''}$ and $g_k \in G$. So, using that I is G-invariant, we deduce $I(u_k) \le c + \epsilon''$. Similarly, $u = v + g$ with $v \in \mathcal{N}_{\epsilon''}$, $g \in G$ and $I(u) \le c + \epsilon''$. Consider $g' \in G$ with $\overline{u} + g' \in [0, 1)^N$. Then, we may assume that $|\overline{u}_k + g'| \le 2$ for all $k \in \mathbb{N}$. Using that $\mathcal{N}$ and I are G-invariant, it follows that $w_k := u_k + g' \in \mathcal{N}_{\epsilon''}$ and $w := u + g' \in \mathcal{N}_{\epsilon''}$. By the continuity of η and π, we have

$$\widehat{\eta}(t_k, \Gamma_k) = \pi(\eta(t_k, w_k)) \to \pi(\eta(t, w)) = \widehat{\eta}(t, \Gamma)$$

and the proof is complete. □

Remark 5.4. If $A \subset [0, 1)^N + Y$ is compact, $b \in X$ and $\inf_{a \in A} ||b - a|| \le 1$, then $|\overline{b}| \le 2$. Indeed, using the compactness of A, it follows that there exists $a_0 \in A$ such that $||b - a_0|| = \inf_{a \in A} ||b - a||$. As $||b - a_0|| = |\overline{b} - \overline{a}_0| + ||\widetilde{b} - \widetilde{a}_0||_X$, one has that $|\overline{b} - \overline{a}_0| \le 1$. It follows that $|\overline{b}| \le |\overline{b} - \overline{a}_0| + |\overline{a}_0| \le 2$.

6. **Main tools.** The results in this section are proved in [22].

1. Lusternik-Schnirelman category. Recall, a subset C of a topological spaces E is called *contractible* in E if there exists a continuous function $h : [0, 1] \times C \to E$ and $e \in E$ such that

$$h(0, \cdot) = id_C, \qquad h(1, \cdot) = e.$$

A subset A of a topological space E is said to has *category* k in E if k is the least integer such that A can be covered by k closed sets contractible in E. The category of A in E is denoted by $\mathrm{cat}_E(A)$.

The main properties of the Lusternik-Schnirelman category are given in the following

Lemma 6.1. *Let E be a topological space and let $A, B \subset E$.*

(i) If $A \subset B$, then $cat_E(A) \leq cat_E(B)$.
(ii) $cat_E(A \cup B) \leq cat_E(A) + cat_E(B)$.
(iii) If A is closed and $B = \eta(\bar{t}, A)$, where $\eta : [0, \bar{t}] \times A \to E$ is a continuous function such that $\eta(0, \cdot) = id_A$, then $cat_E(A) \leq cat_E(B)$.

Remark 6.2. In the functional framework from the previous section, if $A = [0,1]^N + \{0\}$ $(\subset X = \mathbb{R}^N \oplus Y)$, then $\mathrm{cat}_{\pi(X)}(\pi(A)) = N + 1$.

2. Ekeland variational principle. *Let (E, d) be a complete metric space and $\gamma : E \to (-\infty, +\infty]$ a proper, lower semi-continuous function bounded from below. Given $\delta, \lambda > 0$, and $x \in E$ with*

$$\gamma(x) \leq \inf_E \gamma + \delta,$$

there exists $y \in E$ such that

$$\begin{aligned} \gamma(y) &\leq \gamma(x), \\ d(x, y) &\leq 1/\lambda, \\ \gamma(z) - \gamma(y) &\geq -\delta\lambda d(y, z), \qquad \forall z \in E. \end{aligned}$$

3. Hausdorff distance and a complete metric space. On account of Remark 6.2, it follows that, for $1 \leq j \leq N + 1$, the set

$$\mathcal{A}_j = \{A \subset X : \ A \text{ is compact and } \mathrm{cat}_{\pi(X)}(\pi(A)) \geq j\}$$

is nonempty. In order to apply Ekeland's variational principle, we need the following

Lemma 6.3. *Let $1 \leq j \leq N + 1$ be fixed.*

(i) The space $\mathcal{A}_j$ with the Hausdorff distance

$$\delta(A, B) = \max\{\sup_{a \in A} dist(a, B), \sup_{b \in B} dist(b, A)\}$$

is a complete metric space.

(ii) If $I : X \to (-\infty, +\infty]$ is lower semicontinuous, then the function $\gamma : \mathcal{A}_j \to (-\infty, +\infty]$, defined by

$$\gamma(A) = \sup_A I \quad (A \in \mathcal{A}_j) \tag{25}$$

is lower semicontinuous.

7. **Main result.** The main abstract result of the paper is the following

Theorem 7.1. *Under the assumptions $(H_1) - (H_4)$, the functional I defined in (5) is bounded from below and has at least $N + 1$ critical orbits.*

Proof. First, let us note that, $D(\Psi)$ closed and (H_4) imply that $\{u \in D(\Psi) : \ |\overline{u}| \leq 1\}$ is a compact set. This, together with the G-invariance and the continuity of $\mathcal{G}$, imply that $\mathcal{G}$ is bounded on $D(\Psi)$. So, from (H_3), we deduce that I is bounded from below on X.

For $1 \leq j \leq N + 1$, let $\gamma : \mathcal{A}_j \to (-\infty, +\infty]$ be defined by (25) and

$$c_j := \inf_{A \in \mathcal{A}_j} \gamma(A).$$

Using also that $\mathcal{A}_{j+1} \subset \mathcal{A}_j$, one has that

$$-\infty < \inf_X I \leq c_1 \leq \ldots \leq c_{N+1}.$$

Moreover, from Remark 6.2 one has $A = [0,1]^N + \{0\} \in \mathcal{A}_{N+1}$, and using (H_2), we have that $I(u) = \mathcal{G}(u)$ for all $u \in A$. This together with the continuity of $\mathcal{G}$ and the compactness of A imply that

$$c_{N+1} < \infty.$$

We will show that $K_{c_j} \neq \emptyset$. By contradiction, assume that $K_{c_j} = \emptyset$. Then, let d, ϵ'', η be given by Proposition 5.2 with $\mathcal{N} = \emptyset$ and $\bar{\epsilon} = 1/2$. Consider $B \in \mathcal{A}_j$ with

$$\gamma(B) \leq c_j + \epsilon''^2.$$

Using the G-invariance of I, we may assume that $B \subset [0,1)^N + Y$. Using Ekeland's variational principle (see Lemma 6.3) with $\delta = \epsilon''^2$ and $\lambda = 1/2\epsilon'' d$, it follows that there exists $C_B \in \mathcal{A}_j$ such that

$$\gamma(C_B) \leq \gamma(B) \leq c_j + \epsilon''^2, \tag{26}$$

$$\delta(B, C_B) \leq 2\epsilon'' d < 1/2,$$

$$\gamma(D) - \gamma(C_B) \geq -\frac{\epsilon''}{2d}\delta(C_B, D), \qquad \forall D \in \mathcal{A}_j. \tag{27}$$

In particular, one has that $\delta(B, C_B) < 1$ and $\gamma(C_B) \leq c_j + \epsilon''$, which together with $B \subset [0,1)^N + Y$ and Remark 5.4 imply $C_B \subset \mathcal{N}_{\epsilon''}$. So, we can consider the compact set $D_B := \eta(\bar{t}, C_B)$. Then, with $\widehat{\eta}$ introduced in Lemma 5.3, we have

$$\pi(D_B) = \widehat{\eta}(\bar{t}, \pi(C_B))$$

and from Lemma 6.1 (iii) it follows

$$\operatorname{cat}_{\pi(X)}(\pi(D_B)) \geq \operatorname{cat}_{\pi(X)}(\pi(C_B)) \geq j,$$

showing that $D_B \in \mathcal{A}_j$. So, $\gamma(D_B) \geq c_j$. On the other hand, from Proposition 5.2, one has

$$\delta(C_B, D_B) \leq d\bar{t}, \qquad \gamma(D_B) - \gamma(C_B) \leq -\epsilon''\bar{t}.$$

Consequently,

$$-\epsilon''\bar{t} \geq \gamma(D_B) - \gamma(C_B) \geq -\frac{\epsilon''}{2d}\delta(C_B, D_B) \geq -\frac{\epsilon''}{2d} d\bar{t},$$

giving $1 \leq 1/2$, a contradiction.

It suffices to prove that, if $c_k = c_j = c$ for some $1 \leq j < k \leq N+1$, then K_c contains at least $k-j+1$ critical orbits. By contradiction, assume that K_c contains at most $n \leq k-j$ critical orbits denoted by $\pi^{-1}(\pi(u_1)), \ldots, \pi^{-1}(\pi(u_n))$. Note that, from the above step it follows that $n \geq 1$. Let $\rho \in (0,1)$ be such that π restricted to $\overline{B(u_m, \rho)}$ is injective. We introduce the G-invariant set

$$\mathcal{M}_\rho := \bigcup_{m=1}^{n} \bigcup_{g \in G} B(u_m + g, \rho),$$

which, clearly is an open neighborhood of K_c.

Let d, ϵ'', and η be given by Proposition 5.2 with $\mathcal{N} = \mathcal{M}_{\rho/2}$ and $\bar{\epsilon} = \rho/2$. Pick $A \in \mathcal{A}_k$ such that

$$\gamma(A) \leq c + \epsilon''^2.$$

Using the G-invariance of I, we may assume that $A \subset [0,1)^N + Y$. Setting $B = A \setminus \mathcal{M}_\rho$ and using Lemma 6.1, we have

$$\begin{aligned} k &\leq \operatorname{cat}_{\pi(X)}(\pi(A)) \\ &\leq \operatorname{cat}_{\pi(X)}(\pi(B) \cup \pi(\mathcal{M}_\rho)) \\ &\leq \operatorname{cat}_{\pi(X)}(\pi(B)) + \operatorname{cat}_{\pi(X)}(\pi(\mathcal{M}_\rho)). \end{aligned}$$

Since from the injectivity of π on $\overline{B(u_m, \rho)}$ and Lemma 6.1 (ii), one has that $\operatorname{cat}_{\pi(X)}(\pi(\mathcal{M}_\rho)) \leq n$, it follows

$$k \leq \operatorname{cat}_{\pi(X)}(\pi(B)) + n \leq \operatorname{cat}_{\pi(X)}(\pi(B)) + k - j,$$

hence $B \in \mathcal{A}_j$. It is clear that

$$\gamma(B) \leq \gamma(A) \leq c + \epsilon''^2.$$

By Ekeland's variational principle with $\delta = \epsilon''^2$ and $\lambda = 1/2\epsilon'' d$, there exists $C_B \in \mathcal{A}_j$ such that (26), (27) hold true and

$$\delta(B, C_B) \leq 2\epsilon'' d < \rho/2.$$

Note that $B \cap \mathcal{M}_\rho = \emptyset$ and $\delta(B, C_B) < \rho/2$ imply $C_B \cap \mathcal{M}_{\rho/2} = \emptyset$. Then $C_B \subset \mathcal{N}_{\epsilon''}$, and reasoning as above we arrive at the same contradiction ($1 \leq 1/2$), and the proof is completed. □

Corollary 7.2. *Under the hypothesis* (H_ϕ), (H_F) *and* (H_h), *the differential system* (4) *has at least* $N+1$ *geometrically distinct solutions.*

Proof. It follows immediately from the Theorem 7.1 and the results of Section 2. □

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