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# Periodic Solutions for Delay Competition Systems and Delay Prey-Predator Systems

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**Abstract**

Using Mawhin's continuation theorem, sufficient conditions are obtained for the existence of positive periodic solutions for periodic delay-Lotka-Volterra systems.

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## 1 Introduction

The purpose of this paper is to study the periodic solutions of some generalizations of competition systems, in particular the May-Leonard model, and of prey-predator systems.

In [9], Zanolin has studied the delay-Lotka-Volterra system

$$\dot{x}_i(t) = x_i(t)\left[r_i(t) - a_{ii}(t)x_i(t) - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}(t)x_j(t-\tau_j)\right] \qquad (1.1)$$

$$(i = 1, 2, ..., n),$$

where $r_i, a_{ii} > 0, a_{ij} \geq 0$ $(j \neq i), (i, j = 1, \ldots, n)$ are T-periodic continuous functions,

$\tau_j \in \mathbb{R}$ $(j = 1, 2, \dots, n)$. If the condition

$$\overline{r}_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} \overline{a}_{ij} \left| \frac{r_j}{a_{jj}} \right|_0 > 0 \quad (i = 1, 2, ..., n) \tag{1.2}$$

is satisfied, where $|f|_0 = \sup_{t \in \mathbb{R}} |f(t)|$ denotes the maximum norm and $\overline{f} = \frac{1}{T} \int_0^T f$ the mean value of the T-periodic continuous function f, then it is proved that system (1.1) has at least one T-periodic, positive solution.

The system (1.1) is generalized by Y. Li in [5] to the delay-Lotka-Volterra system

$$\begin{aligned} &\dot{x}_i(t) \\ = \quad & x_i(t) \left[ r_i(t) - a_{ii}(t)x_i(t) - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}(t)x_j(t - \tau_j(t, x_1(t), \dots, x_n(t))) \right] \\ &(i = 1, 2, \dots, n), \end{aligned} \tag{1.3}$$

where $\tau_j \in C(\mathbb{R}^{n+1}, \mathbb{R})$ and $\tau_j$ $(j = 1, 2, \dots, n)$ are T-periodic with respect to their first argument. It is shown that, if condition (1.2) is satisfied and system

$$\sum_{j=1}^{n} \overline{a}_{ij} \exp(y_j) = \overline{r}_i, \; i = 1, 2, \dots, n \tag{1.4}$$

has only one solution, then system (1.3) has at least one T-periodic positive solution.

In Section 2 we prove that the same conclusion holds if condition (1.2) only is satisfied. A particular case of system (1.3) is the May-Leonard-type system

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[1 - x_1(t) - \alpha_1(t)x_2(t - \tau_2(t, x_1(t), \dots, x_3(t))) \\ &- \beta_1(t)x_3(t - \tau_3(t, x_1(t), \dots, x_3(t)))] \\ \dot{x}_2(t) &= x_2(t)[1 - \beta_2(t)x_1(t - \tau_1(t, x_1(t), \dots, x_3(t))) \\ &- x_2(t) - \alpha_2(t)x_3(t - \tau_3(t, x_1(t), \dots, x_3(t)))] \\ \dot{x}_3(t) &= x_3(t)[1 - \alpha_3(t)x_1(t - \tau_1(t, x_1(t), \dots, x_3(t))) \\ &- \beta_3(t)x_2(t - \tau_2(t, x_1(t), \dots, x_3(t))) - x_3(t)] \end{aligned} \tag{1.5}$$

where $\alpha_i, \beta_i \geq 0$ $(i = 1, 2, 3)$ are continuous T-periodic functions, $\tau_j \in C(\mathbb{R}^4, \mathbb{R})$ and $\tau_j$ $(j = 1, 2, 3)$ are T-periodic with respect to their first argument. In this case condition (1.2) becomes

$$\overline{\alpha_i} + \overline{\beta_i} < 1 \quad (i = 1, 2, 3).$$

In [3] (see also [7]) it is shown that (1.5) has at least one non constant periodic positive solution if

$$0 < \alpha_i < 1 < \beta_i \quad (i = 1, 2, 3),$$

where $\alpha_i, \beta_i$ $(i = 1, 2, 3)$ are constants and $\tau_i \equiv 0$ $(i = 1, 2, 3)$. It is shown in [2] that (1.3) (for $n = 3$) has at least one $T-$periodic positive solution if $\tau_j \equiv 0 \quad (j = 1, 2, 3)$ and

$$\left[\frac{r_i}{r_j}\right]_L > \max\left\{\left[\frac{a_{ii}}{a_{ji}}\right]_M, \left[\frac{a_{ij}}{a_{jj}}\right]_M\right\} \quad (i, j) \in \{(1, 2), (2, 3), (3, 1)\}, \tag{1.6}$$

where $[f]_L$ denotes the minimum of f and $[f]_M$ denotes the maximum of $f$. In Section 3 we prove that (1.3) has at least one $T-$periodic positive solution if condition (1.6) is satisfied.

In Section 4 we study the system

$$\begin{aligned} \dot{u}(t) &= u(t)[a(t) - b(t)u(t) - c(t)v(t - \beta(t, u(t), v(t)))] \\ \dot{v}(t) &= v(t)[d(t) + f(t)u(t - \alpha(t, u(t), v(t))) - g(t)v(t)] \end{aligned} \tag{1.7}$$

where a,b,c,d,f,g are continuous T-periodic functions and $\alpha, \beta \in C(\mathbb{R}^3, \mathbb{R})$ are T-periodic with respect to their first variable. It is also assumed that a,b,c,f and g are strictly positive. We prove that if the functions a,b,…, g,$\alpha, \beta$ are like above and Gopalsamy's condition

$$-\frac{[f]_L}{[b]_M} < \min\left\{\frac{[d]_L}{[a]_M}, \frac{[d]_L}{[a]_L}\right\} \leq \max\left\{\frac{[d]_M}{[a]_M}, \frac{[d]_M}{[a]_L}\right\} < \frac{[g]_L}{[c]_M} \tag{1.8}$$

is satisfied, then system (1.7) has at least one positive T-periodic solution.

On the other hand, in the same section, we study the system

$$\begin{aligned} \dot{u}(t) &= u(t)[a(t) - b(t)u(t) - c(t)v(t)] \\ \dot{v}(t) &= \tau(t)v(t)[u(t - \alpha(t, u(t), v(t))) - \sigma(t)] \end{aligned} \tag{1.9}$$

where a,b,c,$\tau, \sigma$ are continuous T -periodic strictly positive functions and $\alpha \in C(\mathbb{R}^3, \mathbb{R})$ is T-periodic with respect to its first variable. We show that system (1.9) has at least one T-periodic positive solution if condition

$$[a]_L - [b]_M[\sigma]_M > 0 \tag{1.10}$$

is satisfied. The autonomous case has been considered in [4].

The main tool used in this paper is Mawhin's continuation theorem [8], with which we end this Introduction. Let $X, Y$ be real Banach spaces, let $L : D(L) \subset X \to Y$ be a Fredholm mapping of index zero, and let $P : X \to X$, and $Q : Y \to Y$ be continuous projectors such that $ImP = kerL$, $kerQ = ImL$ and $X = kerL \oplus kerP$, $Y = ImL \oplus ImQ$. Let $J : ImQ \to kerL$ an isomorphism.

**Theorem 1.1** *Let $\Omega \subset X$ be an open bounded set and let $N : X \to Y$ be a continuous operator which is L-compact on $\overline{\Omega}$. Assume*

*i) For each $\lambda \in ]0, 1[, x \in \partial\Omega \cap D(L), Lx \neq \lambda Nx$*

*ii) For each $x \in \partial\Omega \cap kerL, QNx \neq 0$ and $deg(JQN, \Omega \cap kerL, 0) \neq 0$*

*Then, $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap D(L)$.*

## 2 Delay-competition systems with Zanolin type condition

Let $T > 0$ and

$$C_T = \{x : \mathbb{R} \longrightarrow \mathbb{R}^n | x \text{ is a continuous T-periodic function}\}$$

with the norm $|x|_0 = \sup_{t\in\mathbb{R}} |x(t)|$. $(C_T, |\cdot|_0)$ is a Banach space.

We search a positive function $x \in C_T$ which is a solution of (1.3). To find such a function, it is sufficient to show that the following system has T-periodic solutions

$$\begin{aligned} \dot{x}_i(t) &= r_i(t) - a_{ii}(t)\exp(x_i(t)) \\ &- \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}(t)\exp\Big[x_j(t-\tau_j(t,\exp x_1(t),\ldots,\exp x_n(t)))\Big] \\ &(i = 1, 2, ..., n) \end{aligned} \tag{2.1}$$

We reformulate problem (2.1) to use the continuation theorem. Let $(L, D(L))$ be the operator defined by

$$D(L) = C_T \cap C^1(\mathbb{R}, \mathbb{R}^n), Lx = \dot{x}$$

and $N : C_T \to C_T, Nx = y$ where

$$\begin{aligned} y_i(t) &= r_i(t) - a_{ii}(t)\exp(x_i(t)) \\ &- \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}(t)\exp\Big[x_j(t-\tau_j(t,\exp x_1(t),\ldots,\exp x_n(t)))\Big] \\ &(i = 1, 2, \ldots, n). \end{aligned}$$

It is obvious that $x \in C_T$ is a solution of (2.1) if and only if $x \in D(L)$ and $Lx = Nx$. Define the continuous projectors $P, Q$ as

$$\begin{aligned} Q &: C_T \to C_T,\ Qx = \frac{1}{T}\int_0^T x(t)dt = \overline{x}, \\ P &: C_T \to C_T,\ Px = x(0). \end{aligned}$$

We know that

$$\begin{aligned} ImP &= kerL,\ kerQ = ImL, \\ C_T &= kerL \oplus kerP = ImL \oplus ImQ, \\ kerL &= ImQ \simeq \mathbb{R}^n. \end{aligned}$$

Consequently, $L$ is a Fredholm operator of index zero (see [8]). It is easy to prove that $N$ is an $L$-compact operator(see [5]). The following lemma is proved in [5].

**Lemma 2.1** *Suppose that condition (1.2) holds. Then there is a bounded, open set* $\Omega \subset C_T$ *such that*

$$\bigcup_{\lambda \in ]0,1]} \{x \in D(L) : Lx = \lambda N x\} \subset \Omega.$$

**Lemma 2.2** *Let* $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \varphi(x) = y,$ *where*

$$y_i = \overline{r_i} - \sum_{j=1}^{n} \overline{a_{ij}} \exp(x_j) \quad (i = 1, 2, \cdots, n).$$

*If relation (1.2) holds, then there exists an open, bounded set* $\Omega_1 \subset \mathbb{R}^n$ *such that*

$$\{x \in \mathbb{R}^n ; \varphi(x) = 0\} \subset \Omega_1$$

*and* $\deg(\varphi, \Omega, 0) = (-1)^n$.

*Proof.* Let $H : [0,1] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n,\ H(\lambda, x) = y$, where

$$y_i = \overline{r}_i - \overline{a_{ii}} \exp(x_i) - \lambda \sum_{\substack{j=1 \\ j \neq i}}^{n} \overline{a}_{ij} \exp(x_j) \quad (i = 1, 2, ..., n).$$

Let $\lambda \in [0,1]$ and $x \in \mathbb{R}^n$ such that $H(\lambda, x) = 0$. It follows that

$$\overline{r}_i - \overline{a_{ii}} \exp(x_i) - \lambda \sum_{\substack{j=1 \\ j \neq i}}^{n} \overline{a}_{ij} \exp(x_j) = 0 \quad (i = 1, 2, ..., n). \tag{2.2}$$

We deduce that $0 \leq \overline{r}_i - \overline{a_{ii}} \exp(x_i)$, so

$$\exp(x_i) \leq \frac{\overline{r_i}}{\overline{a_{ii}}} \leq \left| \frac{r_i}{a_{ii}} \right|_0, \quad x_i \leq \ln \left| \frac{r_i}{a_{ii}} \right|_0 \quad (i = 1, 2, ..., n). \tag{2.3}$$

On the other hand, from (2.2) , (2.3) and (1.2) we get

$$\overline{a_{ii}} \exp(x_i) \geq \overline{r}_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} \overline{a}_{ij} \exp(x_j) \geq \overline{r}_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} \overline{a}_{ij} \left| \frac{r_i}{a_{ii}} \right|_0 > 0$$
$$(i = 1, 2, , \cdots, n).$$

which implies that there exists a constant $M \in \mathbb{R}$ such that

$$M \leq x_i \quad (i = 1, 2, , \cdots, n). \tag{2.4}$$

From (2.3) and (2.4) we obtain the existence of an open, bounded set $\Omega_1 \subseteq \mathbb{R}^n$ such that

$$\bigcup_{\lambda \in [0,1]} \{x \in \mathbb{R}^n : H(\lambda, x) = 0\} \subset \Omega_1.$$

Furthermore, $H(0, x) = 0$ has the unique solution $x^0$ with

$$x_i^0 = \ln \frac{\overline{r_i}}{\overline{a_{ii}}} \quad (i = 1, 2, \cdots, n)$$

for which

$$J_{H(0,\cdot)}(x^0) = (-1)^n \prod_{i=1}^{n} (\exp x_i^0) \overline{a_{ii}}.$$

Consequently, we have that

$$\deg(\varphi, \Omega_1, 0) = \deg(H(1, \cdot), \Omega_1, 0) = \deg(H(0, \cdot), \Omega_1, 0) = (-1)^n.$$

**Theorem 2.1** *Assume that relation (1.2) holds. Then system (1.3) has at least one T-periodic positive solution.*

*Proof.* We have noticed that it is enough to show the existence of an element $x \in D(L)$ such that $Lx = Nx$. We see that the restriction of $QN$ to $\mathbb{R}^n$ is $\varphi$, the function defined in Lemma 2.2. Using Lemma 2.1, Lemma 2.2 and the continuation theorem, we find an element $x \in D(L)$ such that $Lx = Nx$.

**Theorem 2.2** *If*

$$\overline{\alpha}_i + \overline{\beta}_i < 1 \quad (i = 1, 2, 3) \tag{2.5}$$

*then system (1.5) has at least one T-periodic positive solution.*

*Proof.* In this particular case, condition (1.2) is exactly condition (2.5). We apply Theorem 2.1 for:

$$\begin{aligned} r_i &\equiv 1 \equiv a_{ii} \quad (i = 1, 2, 3) \\ a_{12} &= \alpha_1;\ a_{13} = \beta_1;\ a_{21} = \beta_2;\ a_{23} = \alpha_2; a_{31} = \alpha_3;\ a_{32} = \beta_3. \end{aligned}$$

# 3 Delay-competition systems with May-Leonard-type condition

We keep the notations of Section 2 with $n = 3$. The proof of the following lemma uses some techniques of Ahmad [1].

**Lemma 3.1** *Suppose that condition (1.6) holds. Then there is a bounded and open set* $\Omega \subset C_T$ *such that*

$$\bigcup_{\lambda\in]0,1]} \{x \in D(L) : \ Lx = \lambda Nx\} \subset \Omega.$$

*Proof.* Let $\lambda \in ]0,1]$ and $x \in D(L)$ such that

$$Lx = \lambda Nx. \tag{3.1}$$

In what follows $C_i$ denotes a fixed constant independent of $\lambda$ and $x$. Integrating (3.1) we obtain

$$\begin{aligned} T\overline{r}_i &= \int_0^T a_{ii}(t)\exp x_i(t)dt \\ &+ \sum_{\substack{j=1\\ i\neq j}}^{3} \int_0^T a_{ij}(t)\exp[x_j(t-\tau_j(t,\exp x_1(t),\ldots,\exp x_3(t)))]dt \\ &(i=1,2,3). \end{aligned} \tag{3.2}$$

On the other hand, from (3.1) we have

$$\begin{aligned} |\dot{x}_i(t)| &\leq r_i(t) + a_{ii}(t)\exp x_i(t) \\ &+ \sum_{\substack{j=1\\ i\neq j}}^{3} a_{ij}(t)\exp[x_j(t-\tau_j(t,\exp x_1(t),\ldots,\exp x_3(t)))] \\ &(i=1,2,3). \end{aligned} \tag{3.3}$$

Using (3.2) and (3.3) we deduce that

$$\|\dot{x}_i\|_{L^1(0,T)} \leq 2T \max_{1\leq i\leq 3} \overline{r_i} := C_2 \quad (i=1,2,3). \tag{3.4}$$

Using again (3.2) we obtain $\overline{r}_1 \geq \sum_{j=1}^{3} \overline{a}_{1j}\exp[x_j]_L$ which implies the existence of $C_3 > 0$, such that

$$[x_i]_L \leq C_3 \quad (i=1,2,3). \tag{3.5}$$

By (3.4) and (3.5) we have

$$[x_i]_M \leq [x_i]_L + \|\dot{x}_i\|_{L^1(0,T)} \leq C_3 + C_2 := C_4 \quad (i=1,2,3). \tag{3.6}$$

We prove the existence of a constant $C_5$ such that

$$[x_i]_M \geq C_5 \quad (i=1,2,3). \tag{3.7}$$

Assume, by contradiction, that (3.7) is not true. Then there exists $(\lambda_n)_n \subset ]0,1]$, $(x^n)_n \subset D(L)$ with $Lx^n = \lambda_n Nx^n$ such that one of the following three possible situations holds :

I. $[x_i^n]_M \to -\infty$ for all $i \in \{1,2,3\}$

II. $[x_i^n]_M \to -\infty$ for all $i \in I \subset \{1,2,3\},\ |I| = 2$

III. $[x_i^n]_M \to -\infty$ for all $i \in I \subset \{1,2,3\},\ |I| = 1$.

*Let us first deal with situation I.* Using (3.2) we obtain that

$$\overline{r}_1 \leq \sum_{j=1}^{3} \overline{a}_{1j} \exp[x_j^n]_M \underset{n}{\longrightarrow} 0,$$

which implies that $\overline{r}_1 \leq 0$. But $[r_1]_L > 0$, so we have obtained the desired contradiction.

*Consider now situation II.* Suppose that $I = \{2,3\}$, the treatment of the other ones being completely similar. Let $C_6$ be a constant such that

$$[x_1^n]_M \geq C_6, \ \ (n \in \mathbb{N}) \tag{3.8}$$

and

$$[x_i^n]_M \longrightarrow -\infty \quad (i = 2,3). \tag{3.9}$$

In view of (3.4) and (3.8) we have that

$$[x_1^n]_L \geq [x_1^n]_M - \|\dot{x}_1^n\|_{L^1(0,T)} \geq C_6 - C_2 = C_7 \quad (n \in \mathbb{N}). \tag{3.10}$$

Integrating the relation $Lx^n = \lambda_n N x^n$ $(n \in \mathbb{N})$ we obtain

$$\begin{aligned} T\overline{r}_i &= \int_0^T a_{ii}(t) \exp x_i^n(t) dt \qquad (3.11) \\ &+ \sum_{\substack{j=1 \\ i \neq j}}^{3} \int_0^T a_{ij}(t) \exp[x_j^n(t - \tau_j(t, \exp x_1^n(t), \ldots, \exp x_3^n(t)))] dt \\ & (n \in \mathbb{N},\ i = 2,3). \end{aligned}$$

Using (3.9)and (3.11) we deduce the existence of two sequences $(A_i^n)_n,\ A_i^n \underset{n}{\longrightarrow} 0$ such that

$$\begin{aligned} T\overline{r}_i &= \int_0^T a_{i1}(t) \exp[x_1^n(t - \tau_1(t, \exp x_1^n(t), \ldots, \exp x_3^n(t)))] dt + A_i^n \\ & (n \in \mathbb{N}, \quad i = 2,3). \end{aligned} \tag{3.12}$$

By (1.6) for $(i,j) = (1,2)$ we have

$$a_{21}(t) > a_{11}(t) \frac{\overline{r}_2}{\overline{r}_1} \quad (t \in [0,T]). \tag{3.13}$$

From (3.12) and (3.13) it follows that

$$T\overline{r}_2 > \frac{\overline{r}_2}{\overline{r}_1}\int_0^T a_{11}(t)\exp[x_1^n(t-\tau_1(t,\exp x_1^n(t),\ldots,\exp x_3^n(t)))]dt + A_2^n \quad (n\in\mathbb{N})$$

which implies that

$$T\overline{r}_1 > \int_0^T a_{11}(t)\exp[x_1^n(t-\tau_1(t,\exp x_1^n(t),\ldots,\exp x_3^n(t)))]dt + \frac{\overline{r}_1}{\overline{r}_2}A_2^n \quad (n\in\mathbb{N})$$

from which we obtain that

$$\begin{aligned}-T\overline{r}_3\overline{r}_1 &< -\int_0^T \overline{r}_3 a_{11}(t)\exp[x_1^n(t-\tau_1(t,\exp x_1^n(t),\ldots,\exp x_3^n(t)))]dt \\ &- \frac{\overline{r}_3\overline{r}_1}{\overline{r}_2}A_2^n \quad (n\in\mathbb{N}). \qquad (3.14)\end{aligned}$$

On the other hand, from (3.12) we have

$$\begin{aligned}T\overline{r}_3\overline{r}_1 &= \int_0^T \overline{r}_1 a_{31}(t)\exp[x_1^n(t-\tau_1(t,\exp x_1^n(t),\ldots,\exp x_3^n(t)))]dt \\ &+ A_3^n\overline{r}_1 \quad (n\in\mathbb{N}). \qquad (3.15)\end{aligned}$$

By (3.14) and (3.15) we have

$$\begin{aligned}0 &< \int_0^T [\overline{r}_1 a_{31}(t) - \overline{r}_3 a_{11}(t)]\exp[x_1^n(t-\tau_1(t,\exp x_1^n(t),\ldots,\exp x_3^n(t)))]dt \\ &+ A_3^n\overline{r}_1 - \frac{\overline{r}_3\overline{r}_1}{\overline{r}_2}A_2^n \quad (n\in\mathbb{N}). \qquad (3.16)\end{aligned}$$

On the other hand, from (1.6) (for $(i,j)=(3,1)$) we obtain that

$$\overline{r}_1 a_{31}(t) - \overline{r}_3 a_{11}(t) < 0 \quad (t\in[0,T]). \qquad (3.17)$$

Using the fact that $A_i^n \underset{n}{\longrightarrow} 0 \quad (i=2,3)$ and (3.10), (3.16), (3.17) we deduce that $0<0$, a contradiction.

*We consider now the third situation.* Suppose that $I=\{3\}$, the other cases being treated in the same manner. Let $C_8$ be a constant such that

$$[x_i^n]_M \geq C_8 \quad (n\in\mathbb{N},\ i=1,2) \qquad (3.18)$$

and

$$[x_3^n]_M \underset{n}{\rightarrow} -\infty.$$

Using (3.4) and (3.18) we have

$$[x_i^n]_L \geq [x_i^n]_M - \|\dot{x}_i^n\|_{L'(0,T)} \geq C_8 - C_2 := C_9 \quad (i=1,2). \qquad (3.19)$$

Let $(t_1^n)_n$ be a sequence such that

$$x_1^n(t_1^n) = [x_1^n]_L \quad (n \in \mathbb{N}). \tag{3.20}$$

Using $Lx^n = \lambda_n N x^n \quad (n \in \mathbb{N})$ and (3.20) we deduce that

$$\begin{aligned} r_1(t_1^n) &= a_{11}(t_1^n) \exp[x_1^n]_L \\ &+ \sum_{\substack{j=1 \\ j \neq 1}}^{3} a_{1j}(t_1^n) \exp[x_j^n(t_1^n - \tau_j(t_1^n, \exp x_1^n(t_1^n), \ldots, \exp x_3^n(t_1^n)))] \\ & (n \in \mathbb{N}), \end{aligned}$$

which implies

$$[r_1]_L \leq [a_{11}]_M \exp[x_1^n]_L + \sum_{\substack{j=1 \\ j \neq 1}}^{3} [a_{1j}]_M \exp[x_j^n]_M \quad (n \in \mathbb{N}). \tag{3.21}$$

Let $(t_2^n)_n$ be a sequence such that

$$x_2^n(t_2^n) = [x_2^n]_M \quad (n \in \mathbb{N}). \tag{3.22}$$

Using again the relation $Lx^n = \lambda_n N x^n \quad (n \in \mathbb{N})$ and (3.22) it follows that

$$\begin{aligned} & r_2(t_2^n) = a_{22}(t_2^n) \exp[x_2^n]_M \\ + & \sum_{\substack{j=1 \\ j \neq 2}}^{3} a_{2j}(t_2^n) \exp[x_j^n(t_2^n - \tau_j(t_2^n, \exp x_1^n(t_2^n), \ldots, \exp x_3^n(t_2^n)))] \quad (n \in \mathbb{N}), \end{aligned}$$

which implies

$$[a_{22}]_L \exp[x_2^n]_M + \sum_{\substack{j=1 \\ j \neq 2}}^{3} [a_{2j}]_L \exp[x_j^n]_L \leq [r_2]_M \quad (n \in \mathbb{N}). \tag{3.23}$$

By (3.23) we have that

$$\begin{aligned} - \quad & [a_{12}]_M [r_2]_M \leq -[a_{12}]_M [a_{22}]_L \exp[x_2^n]_M \\ - \quad & \sum_{\substack{j=1 \\ j \neq 2}}^{3} [a_{12}]_M [a_{2j}]_L \exp[x_j^n]_L \quad (n \in \mathbb{N}). \end{aligned} \tag{3.24}$$

From (3.21) we obtain that

$$
\begin{aligned}
&[r_1]_L[a_{22}]_L \leq [a_{11}]_M[a_{22}]_L \exp[x_1^n]_L \\
+ &\sum_{\substack{j=1\\ j\neq 1}}^{3} [a_{1j}]_M[a_{22}]_L \exp[x_j^n]_M \quad (n \in \mathbb{N}).
\end{aligned} \tag{3.25}
$$

In view of (3.24) and (3.25) we have that

$$
\begin{aligned}
&[r_1]_L[a_{22}]_L - [a_{12}]_M[r_2]_M \leq \{[a_{11}]_M[a_{22}]_L - [a_{12}]_M[a_{21}]_L\} \exp[x_1^n]_L \\
+ \quad &[a_{13}]_M[a_{22}]_L \exp[x_3^n]_M - [a_{12}]_M[a_{23}]_L \exp[x_3^n]_L \quad (n \in \mathbb{N}).
\end{aligned} \tag{3.26}
$$

Using the fact that

$$[x_3^n]_M \underset{n}{\rightarrow} -\infty$$

and the relations (1.6) (for (i,j)=(1,2)), (3.6), (3.19), (3.26) it follows that

$$0 < [a_{11}]_M[a_{22}]_L - [a_{12}]_M[a_{21}]_L. \tag{3.27}$$

From (3.21) we have that

$$
\begin{aligned}
- \quad &[a_{21}]_L[r_1]_L \geq -[a_{21}]_L[a_{11}]_M \exp[x_1^n]_L \\
- \quad &\sum_{\substack{j=1\\ j\neq 1}}^{3} [a_{21}]_L[a_{1j}]_M \exp[x_j^n]_M \quad (n \in \mathbb{N}).
\end{aligned} \tag{3.28}
$$

By (3.23), we have that

$$
\begin{aligned}
[r_2]_M[a_{11}]_M \quad \geq \quad &[a_{11}]_M[a_{22}]_L \exp[x_2^n]_M \\
+ \quad &\sum_{\substack{j=1\\ j\neq 2}}^{3} [a_{2j}]_L[a_{11}]_M \exp[x_j^n]_L \quad (n \in \mathbb{N}).
\end{aligned} \tag{3.29}
$$

Using (3.28) and (3.29), we deduce that

$$
\begin{aligned}
&[r_2]_M[a_{11}]_M - [a_{21}]_L[r_1]_L \geq \{[a_{11}]_M[a_{22}]_L - [a_{21}]_L[a_{12}]_M\} \exp[x_2^n]_M \\
+ \quad &[a_{23}]_L[a_{11}]_M \exp[x_3^n]_L - [a_{21}]_L[a_{13}]_M \exp[x_3^n]_M \quad (n \in \mathbb{N}).
\end{aligned} \tag{3.30}
$$

From

$$[x_3^n]_M \underset{n}{\rightarrow} -\infty$$

and (1.6) (for (i,j)=(1,2)), (3.6), (3.18) and (3.30) we have that

$$0 > [a_{11}]_M[a_{22}]_L - [a_{21}]_L[a_{12}]_M. \tag{3.31}$$

In view of (3.27) and (3.31) we have obtained a contradiction. Consequently relation (3.7) is true.

From (3.4) and (3.7) we have that

$$[x_i]_L \geq [x_i]_M - \|\dot{x}_i\|_{L^1(0,T)} \geq C_5 - C_1 \quad (i = 1, 2, 3). \tag{3.32}$$

By (3.6) and (3.32) we deduce the existence of a constant $C > 0$, independent of $\lambda \in ]0, 1]$ such that the relation $Lx = \lambda Nx,\ x \in D(L)$ implies that $|x|_0 \leq C$.

The following lemma is proved in [2].

**Lemma 3.2** *Let $\varphi_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ \varphi_1(x) = y$ where*

$$y_i = \overline{r}_i - \sum_{j=1}^{3} \overline{a}_{ij} \exp(x_j) \quad (i = 1, 2, 3).$$

*Suppose that condition (1.6) holds. Then there exist an open, bounded set $\Omega_1 \subset \mathbb{R}^3$ such that*

$$\{x \in \mathbb{R}^3 :\ \varphi_1(x) = 0\} \subset \Omega_1 \text{ and } \deg(\varphi_1, \Omega_1, 0) \neq 0.$$

**Theorem 3.1** *Assume that relation (1.6) holds. Then system (1.3) has at least one T-periodic positive solution.*

*Proof.* See the proof of Theorem 2.1.

**Theorem 3.2** *If*

$$0 < \alpha_i(t) < 1 < \beta_i(t) \quad (t \in [0, T],\ i = 1, 2, 3), \tag{3.33}$$

*then system (1.5) has at least one T-periodic, positive solution.*

*Proof.* We apply Theorem 3.1 for

$$\begin{aligned} r_i &\equiv 1 \equiv a_{ii} \quad (i = 1, 2, 3) \\ a_{12} &= \alpha_1,\ a_{13} = \beta_1,\ a_{21} = \beta_2,\ a_{23} = \alpha_2, a_{31} = \alpha_3,\ a_{32} = \beta_3. \end{aligned}$$

## 4 Delay-prey-predator systems

Let

$$C_T := \{x : \mathbb{R} \to \mathbb{R}^2 | x \text{ is a continuous T-periodic function}\}.$$

It is know that $(C_T, ||\cdot||)$ is a Banach space with the norm $||x|| = \sup_{t\in\mathbb{R}} |x(t)|$ . We search for a positive function $x \in C_T$ such that $x$ is a solution of (1.7).

Consider the following system

$$\begin{aligned} \dot{u}(t) &= a(t) - b(t)\exp u(t) - c(t)\exp[v(t - \beta(t, \exp u(t), \exp v(t)))], \\ \dot{v}(t) &= d(t) + f(t)\exp[u(t - \alpha(t, \exp u(t), \exp v(t)))] - g(t)\exp v(t). \end{aligned} \quad (4.1)$$

It is obvious that if system (4.1) has a T-periodic solution, then system (1.7) has a positive T-periodic solution.

We reformulate problem (4.1) so we can use the continuation theorem. Let $(L, D(L))$ be the operator defined by

$$D(L) = C_T \cap C^1(\mathbb{R}, \mathbb{R}^2), Lx = \dot{x}$$

and $N : C_T \to C_T, Nx = y$ where

$$\begin{aligned} y_1(t) &= a(t) - b(t)\exp u(t) - c(t)\exp[v(t - \beta(t, \exp u(t), \exp v(t)))], \\ y_2(t) &= d(t) + f(t)\exp[u(t - \alpha(t, \exp u(t), \exp v(t)))] - g(t)\exp v(t). \end{aligned}$$

It is obvious that $x \in C_T$ is a solution of (4.1) iff $x \in D(L)$ and $Lx = Nx$. Define the continuous projectors $P, Q$

$$\begin{aligned} P &: C_T \to C_T,\ Px = x(0), \\ Q &: C_T \to C_T,\ Qx = \bar{x}. \end{aligned}$$

We know that

$$\begin{aligned} ImP &= kerL,\ kerQ = ImL, \\ C_T &= kerL \oplus kerP = ImL \oplus ImQ, \\ kerL &= ImQ \simeq \mathbb{R}^2. \end{aligned}$$

Consequently, $L$ is a Fredholm operator of index zero. It is easy to prove that $N$ is an $L$-compact operator.

**Lemma 4.1** *Suppose that condition (1.8) holds. Then there is a bounded, open set $\Omega \subset C_T$ such that*

$$\bigcup_{\lambda\in]0,1]} \{x \in D(L) :\ Lx = \lambda Nx\} \subset \Omega.$$

*Proof.* Let $\lambda \in ]0, 1]$ and $x \in D(L)$ such that

$$Lx = \lambda Nx. \quad (4.2)$$

In what follows $C_i$ denotes a fixed constant independent of $\lambda$ and $x$. Integrating (4.2), we obtain

$$\overline{a}T = \int_0^T b(t)\exp u(t)dt + \int_0^T c(t)\exp[v(t - \beta(t, \exp u(t), \exp v(t)))]dt, \quad (4.3)$$

$$\overline{d}T = -\int_0^T f(t)\exp[u(t-\alpha(t,\exp u(t),\exp v(t)))]dt + \int_0^T g(t)\exp v(t)dt. \quad (4.4)$$

On the other hand, from (4.2) we have that

$$|\dot{u}(t)| \leq a(t) + b(t)\exp u(t) + c(t)\exp[v(t-\beta(t,\exp u(t),\exp v(t)))]. \quad (4.5)$$

Using (4.3) and (4.5) we deduce that

$$\|\dot{u}\|_{L^1(0,T)} \leq 2T\overline{a} := C_2. \quad (4.6)$$

Using again (4.3) we obtain that

$$\overline{a} \geq \overline{b}\exp[u]_L + \overline{c}\exp[v]_L$$

which implies the existence of a constant $C_3 > 0$, such that

$$[u]_L, [v]_L \leq C_3. \quad (4.7)$$

By (4.6) and (4.7) we have that

$$[u]_M \leq [u]_L + \|\dot{u}\|_{L^1(0,T)} \leq C_3 + C_2 := C_4. \quad (4.8)$$

Using (4.4), we have that

$$\begin{aligned} 0 &< [g]_L \int_0^T \exp v(t)dt \leq |\overline{d}|T \quad (4.9) \\ &+ [f]_M \int_0^T \exp[u(t-\alpha(t,\exp u(t),\exp v(t)))]dt \leq |\overline{d}|T + [f]_M \exp[u]_M T. \end{aligned}$$

From (4.8) and (4.9) we deduce the existence of a constant $C_5 > 0$ such that

$$\int_0^T \exp v(t)dt \leq C_5. \quad (4.10)$$

Using (4.2), (4.8) and (4.10) we obtain, as for (4.6), the existence of a constant $C_6 > 0$ such that

$$\|\dot{v}\|_{L^1(0,T)} \leq C_6. \quad (4.11)$$

From (4.7) and (4.11) we have that

$$[v]_M \leq [v]_L + \|\dot{v}\|_{L^1(0,T)} \leq C_3 + C_6 := C_7. \quad (4.12)$$

Now we show the existence of a constant $\tilde{c}$ such that:

$$[u]_M, [v]_M \geq \tilde{c}. \quad (4.13)$$

Assume, by contradiction, that the relation (4.13) is not true. Then there exist $(\lambda_n)_n \subset ]0,1]$, $(x_n = (u_n, v_n))_n \subset D(L)$, $Lx_n = \lambda_n N x_n$ such that one of the three following situations holds:

I. $[u_n]_M \underset{n}{\rightarrow} -\infty, [v_n]_M \underset{n}{\rightarrow} -\infty$

II. $[u_n]_M \rightarrow -\infty, \exists C_8, [v_n]_M \geq C_8 \quad (n \in \mathbb{N})$

III. $\exists C_9, [u_n]_M \geq C_9, \ \ (n \in \mathbb{N}), [v_n]_M \underset{n}{\rightarrow} -\infty.$

*Let us first deal with situation I.* Using (4.3), we obtain that

$$\overline{a} \leq [b]_M \exp[u_n]_M + [c]_M \exp[v_n]_M \underset{n}{\longrightarrow} 0,$$

but $[a]_L > 0$, so we have obtained the desired contradiction.

*Consider now the situation II.* Using the relation (4.11) and II we have that

$$[v_n]_L \geq [v_n]_M - \|\dot{v}_n\|_{L^1(0,T)} \geq C_8 - C_5 := C_{10} \quad (n \in \mathbb{N}). \tag{4.14}$$

From (4.12) and (4.14) we obtain that the sequence $(v_n)_n$ is bounded in $C(0,T)$ and equicontinuous (clear from the relations (4.2), (4.8) and (4.12)), so, using Arzela-Ascoli's theorem, we can admit that there is a function $v \in C(0,T)$ such that $||v - v_n|| \underset{n}{\longrightarrow} 0$. We deal with two situations: II.**1** $[d]_M > 0$.

Consider the sequences $(t_n^1)_n, (t_n^2)_n \subset [0,T]$ such that

$$u_n(t_n^1) = [u_n]_M, v_n(t_n^2) = [v_n]_M \quad (n \in \mathbb{N}). \tag{4.15}$$

Because $[0,T]$ is compact we can assume that there are $t^1, t^2 \in [0,T]$ such that

$$t_n^i \underset{n}{\longrightarrow} t^i \quad (i = 1, 2).$$

Using (4.15) and the fact that $||v - v_n|| \underset{n}{\longrightarrow} 0$ we deduce that

$$v_n(t_n^2) \underset{n}{\longrightarrow} v(t^2) = [v]_M$$

and

$$v_n(t_n^1 - \beta(t_n^1, \exp u_n(t_n^1), \exp v_n(t_n^1))) = v_n(\tilde{t_n}) \underset{n}{\longrightarrow} v(\tilde{t}).$$

(where $\tilde{t} \in [0,T]$ such that $\tilde{t_n} \underset{n}{\longrightarrow} \tilde{t}$ ). So, for every $n \in \mathbb{N}$ we have

$$a(t_n^1) = b(t_n^1) \exp[u_n]_M + c(t_n^1) \exp v(\tilde{t_n}),$$

$$d(t_n^2) = -f(t_n^2) \exp u_n(t_n^2 - \alpha(t_n^2, \exp u_n(t_n^2), \exp v_n(t_n^2))) + g(t_n^2) \exp[v_n]_M.$$

Taking the limit we obtain that

$$a(t^1) = c(t^1) \exp v(\tilde{t}), \quad d(t^2) = g(t^2) \exp[v]_M,$$

so we have that

$$\frac{a}{c}(t^1) \leq \frac{d}{g}(t^2)$$

and from (1.8) and II.1 we have that

$$\left[\frac{a}{c}\right]_L > \left[\frac{d}{g}\right]_M,$$

a contradiction.

II.**2** $[d]_M \leq 0$ : Using the notation in II.1, we obtain that

$$d(t^2) = g(t^2)\exp[v]_M,$$

which is impossible.

*Consider now the last possible situation.* Using the same method (see II) we can show that this situation proves to be also impossible (for example we can use the fact that

$$-\frac{[f]_L}{[b]_M} < \min\left\{\frac{[d]_L}{[a]_M}, \frac{[d]_L}{[a]_L}\right\}.$$

Consequently (4.13) is true. Using (4.6), (4.11) and (4.13) we obtain a constant $C_{11}$ such that

$$[u]_L, [v]_L \geq C_{11}. \tag{4.16}$$

From (4.8), (4.12) and (4.16) we have that there is a constant $C_{12}$ such that

$$||u||, ||v|| \leq C_{12},$$

which completes the proof.

**Lemma 4.2** *Let* $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ \varphi(x) = y$ *where*

$$\begin{array}{rcl} y_1 &=& \overline{a} - \overline{b}\exp(x_1) - \overline{c}\exp(x_2), \\ y_2 &=& \overline{d} + \overline{f}\exp(x_1) - \overline{g}\exp(x_2). \end{array}$$

*If the relation (1.8) holds, then there is an open, bounded set* $\Omega_1 \subset \mathbb{R}^2$ *such that*

$$\{x \in \mathbb{R}^2 :\ \varphi(x) = 0\} \subset \Omega_1 \text{ and } \deg(\varphi, \Omega_1, 0) = 1.$$

*Proof.* It is obvious that from relation (1.8) we can deduce that

$$\overline{a} > 0,\ -\left[\frac{f}{b}\right]_L < \frac{\overline{d}}{\overline{a}} < \left[\frac{g}{c}\right]_L. \tag{4.17}$$

We have that

$$\left[\frac{f}{b}\right]_L \leq \frac{\overline{f}}{\overline{b}}, \quad \left[\frac{g}{c}\right]_L \leq \frac{\overline{g}}{\overline{c}}. \tag{4.18}$$

From (4.17) and (4.18) we have that

$$\overline{ag} - \overline{dc} > 0, \quad \overline{bd} + \overline{fa} > 0. \tag{4.19}$$

Because $b_L, c_L, f_L, g_L > 0$, it follows that

$$\overline{b}\overline{g} + \overline{c}\overline{f} > 0. \tag{4.20}$$

From (4.19) and (4.20) we deduce that there is only one point $(x_1^0, x_2^0) \in \mathbb{R}^2$ such that

$$\varphi(x_1^0, x_2^0) = 0.$$

Furthermore,

$$J_\varphi(x_1^0, x_2^0) = \begin{vmatrix} -\overline{b}\exp(x_1^0) & -\overline{c}\exp(x_2^0) \\ \overline{f}\exp(x_1^0) & -\overline{g}\exp(x_2^0) \end{vmatrix} = \exp(x_1^0)\exp(x_2^0)[\overline{b}\overline{g} + \overline{f}\overline{c}] > 0$$

and $\deg\,(\varphi, B(0,R), 0) = 1$. We can choose $\Omega_1 = B(0,R), R > 0$ such that $(x_1^0, x_1^0) \in B(0,R)$.

**Theorem 4.1** *Assume that relation (1.8) holds. Then system (1.7) has at least one T-periodic positive solution.*

*Proof.* See the proof of Theorem 2.1.

Next we state and prove our second result. As in the case of system (1.7), we consider the following system

$$\begin{aligned} \dot{u}(t) &= a(t) - b(t)\exp u(t) - c(t)\exp v(t) \\ \dot{v}(t) &= \tau(t)[\exp[u(t - \alpha(t, \exp u(t), \exp v(t)))] - \sigma(t)]. \end{aligned} \tag{4.21}$$

If system (4.21) has a T-periodic solution, then system (1.7) has a positive T-periodic solution. Let $N_1 : C_T \to C_T, N_1 x = y$ where

$$\begin{aligned} y_1(t) &= a(t) - b(t)\exp u(t) - c(t)\exp v(t), \\ y_2(t) &= \tau(t)[\exp[u(t - \alpha(t, \exp u(t), \exp v(t)))] - \sigma(t)]. \end{aligned}$$

It is obvious that $x \in C_T$ is a solution of (4.21) if and only if $x \in D(L)$ and $Lx = N_1 x$. We notice that $N_1$ is an L-compact operator.

**Lemma 4.3** *Suppose that condition (1.10) holds. Then there is a bounded, open set $\Omega \subset C_T$ such that*

$$\bigcup_{\lambda \in ]0,1]} \{x \in D(L) :\ Lx = \lambda N_1 x\} \subset \Omega.$$

*Proof.* The proof is similar to the proof of Lemma 4.2 and will be omitted.

**Lemma 4.4** *Let $\varphi_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ \varphi_1(x) = y$ where*

$$y_1 = \overline{a} - \overline{b}\exp(x_1) - \overline{c}\exp(x_2), \quad y_2 = \overline{\tau}[\exp(x_1) - \overline{\sigma}].$$

*If relation (1.10) is true, then there is an open, bounded set $\Omega_1 \subset \mathbb{R}^2$ such that*

$$\{x \in \mathbb{R}^2 :\ \varphi_1(x) = 0\} \subset \Omega_1 \textit{ and } \deg(\varphi, \Omega_1, 0) = 1.$$

*Proof.* From relation (1.10) we deduce that

$$\overline{a} - \overline{b}\overline{\sigma} > 0 \tag{4.22}$$

From the relation (4.22) we obtain that there is only one point $(x_1^0, x_1^0)$ such that $\varphi_1(x_1^0, x_1^0) = 0$. Furthermore

$$J_\varphi(x_1^0, x_2^0) = \begin{vmatrix} -\overline{b}\exp(x_1^0) & -\overline{c}\exp(x_2^0) \\ \overline{\tau}\exp(x_1^0) & 0 \end{vmatrix} = \overline{\tau}\overline{c}\exp(x_1^0)\exp(x_2^0) > 0.$$

We can choose $\Omega_1 = B(0, R), R > 0$, such that $(x_1^0, x_1^0) \in B(0, R)$.

**Theorem 4.2** *Assume that relation (1.10) holds. Then system (1.9) has at least one T-periodic positive solution.*

*Proof.* See the proof of Theorem 2.1.

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# References

[1] S. Ahmad, *On the nonautonomous Volterra-Lotka competition equations*, Proc. Amer. Math. Soc. **117** (1993), 199–204.

[2] A. Batauz, F. Zanolin, *Coexistence States for Periodic Competitive Kolmogorov Systems*, J. Math. Anal. Appl. **219** (1998), 179–199.

[3] J. Hofbauer, *On the accurence of limit cycle in the Volterra-Lotka equation*, Nonlinear Anal. TMA **5** (1981), 1003–1007.

[4] A. Leung, *Periodic solutions for a prey-predator differential delay equation*, J. Differential Equations **26** (1977), 391–403.

[5] Y. Li, *Periodic Solutions for Delay Lotka-Volterra Competition Systems*, J. Math. Anal. Appl. **246** (2000), 230–244.

[6] Y. Li, Y.Kuang, *Periodic Solution of Periodic Delay Lotka-Volterra Equations and Systems*, J. Math. Anal. Appl. **255** (2001), 260–280.

[7] R.M. May, W.J. Leonard, *Nonlinear aspects of competition between three species*, SIAM, J. Appl. Math. **29** (1975), 243–253.

[8] J. Mawhin, Topological Degree Methods in Nonlinear Boundary Value Problems, CBMS Series No. 40, AMS, Providence, 1979.

[9] F. Zanolin, *Permanence and positive periodic solutions for Kolmagorov Competing species systems*, Results Math. **21** (1992), 224–250.

[10] A. Zitan, R. Ortega, *A periodic prey-predator system*, J. Math. Anal. Appl. **185** (1994), 477–489.