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Existence and multiplicity results for some nonlinear problems with singular ϕ-Laplacian

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Abstract

Using Leray–Schauder degree theory we obtain various existence and multiplicity results for nonlinear boundary value problems

$$\big(\phi(u')\big)' = f(t,u,u'), \qquad l(u,u') = 0$$

where $l(u,u') = 0$ denotes the Dirichlet, periodic or Neumann boundary conditions on $[0,T]$, $\phi :]-a,a[\to \mathbb{R}$ is an increasing homeomorphism, $\phi(0) = 0$. The Dirichlet problem is always solvable. For Neumann or periodic boundary conditions, we obtain in particular existence conditions for nonlinearities which satisfy some sign conditions, upper and lower solutions theorems, Ambrosetti–Prodi type results. We prove Lazer–Solimini type results for singular nonlinearities and periodic boundary conditions.

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1. Introduction

The purpose of this article is to obtain some existence and multiplicity result for nonlinear problems

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$$\big(\phi(u')\big)' = f(t,u,u'), \qquad l(u,u') = 0 \tag{1}$$

where $l(u,u') = 0$ denotes the periodic, Neumann or Dirichlet boundary conditions on $[0,T]$, $\phi :]-a,a[\to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$ (we call it *singular*), and $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. For the Neumann problems, this type of equations has been introduced in [2]. Of course, a *solution* of (1) is a function $u \in C^1([0,T])$ such that $\max_{[0,T]} |u'| < a$ and $\phi \circ u' \in C^1([0,T])$.

The various boundary value problems are reduced to the search of fixed point for some operators defined on the whole space X of function $u \in C^1([0,T])$ such that $l(u,u') = 0$. Those operators are completely continuous, and a novel feature linked to the nature of the function ϕ lies in the fact that those operators map X into the cylinder of functions $v \in X$ such that $\max_{[0,T]} |v'| < a$. This property greatly simplifies the search of the *a priori* bounds for possible fixed points required by a Leray–Schauder approach, and allows existence results under milder conditions than in the corresponding problems with classical $\phi : \mathbb{R} \to \mathbb{R}$ or bounded $\phi : \mathbb{R} \to]-a,a[$ considered in earlier papers.

In Section 2, we study the ϕ-Laplacian operator with a time-dependent forcing term and the three boundary conditions, and we recall how to reduce, following [9], the nonlinear boundary value problems to some fixed points problems in X. Those results (and their proofs) are entirely analogous to the known ones for the case of a classical ϕ.

A first consequence of this reduction, considered in Section 3, is the somewhat surprising result that the Dirichlet problem

$$\big(\phi(u')\big)' = f(t,u,u'), \qquad u(0) = 0 = u(T)$$

is solvable for *any* right-hand member f (Theorem 1).

This is not the case for the other boundary conditions, for which we prove, in Section 4, existence results when the right-hand member f only satisfies some sign conditions (Theorem 2). The result is similar to the one proved in [3] when $\phi : \mathbb{R} \to]-a,a[$, except that it requires no one-sided boundedness or growth condition on f. Those sign conditions are specialized into Landesman–Lazer conditions for bounded or unbounded nonlinearities.

We extend in Section 5 the classical method of upper and lower solutions to this type of problems for periodic and Neumann conditions (Theorem 3). The methodology of the proof follows the one introduced in [12], together with an idea inspired by Mawhin's continuation theorem (see [11]). It extends to singular ϕ the results of [14] for the p-Laplacian situation. Notice that, in contrast to this case, no Nagumo-like growth condition for the dependence of $f(t,u,v)$ with respect to v is required. We also adapt the Amann, Ambrosetti and Mancini approach [1] to give an existence result when the lower and upper solutions are not ordered (Theorem 4).

Combining the method of upper and lower solutions and Leray–Schauder degree, we prove in Section 6 an Ambrosetti–Prodi type multiplicity result (Theorems 5 and 6). For the proof, we adapt the ideas of [6] to the present situation. A corresponding Ambrosetti–Prodi type result for nonlinear perturbations of the p-Laplacian can be found in [14].

In the last section, we study positive solutions of periodic problems for nonlinearities singular at 0, and prove Lazer–Solimini type results [8] (Theorems 7 and 8), following the methodology of [13], which has been applied to the p-Laplacian case in [7]. The case of $\phi : \mathbb{R} \to \mathbb{R}$ has been recently considered by Rachunková and Tvrdý [17].

2. Notations and equivalent fixed point problems

We denote the usual norm in $L^1(0,T)$ by $\|\cdot\|_1$. Let C denote the Banach space of continuous functions on $[0,T]$ endowed with the uniform norm $\|\cdot\|_\infty$, C^1 denote the Banach space of continuously differentiable functions on $[0,T]$, equipped with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$. We consider its closed subspaces

$$\begin{aligned} C_0^1 &:= \{u \in C^1\colon\ u(0) = 0 = u(T)\}, \\ C_\dagger^1 &= \{u \in C^1\colon\ u'(0) = 0 = u'(T)\}, \\ C_\#^1 &= \{u \in C^1\colon\ u(0) = u(T),\ u'(0) = u'(T)\}, \end{aligned}$$

and denote corresponding open balls of center 0 and radius r by B_r. We denote by $P, Q : C \to C$ the continuous projectors defined by

$$P, Q : C \to C, \qquad Pu(t) = u(0), \qquad Qu(t) = \frac{1}{T}\int_0^T u(s)\,ds \quad \big(t \in [0,T]\big),$$

and define the continuous linear operator $H : C \to C^1$ by

$$Hu(t) = \int_0^t u(s)\,ds \quad \big(t \in [0,T]\big).$$

If $u \in C$, we write

$$u^+ = \max\{u, 0\}, \qquad u^- = \max\{-u, 0\}, \qquad u_L = \min_{[0,T]} u, \qquad u_M = \max_{[0,T]} u.$$

The following assumption upon ϕ is made throughout the paper:

(H_ϕ) $\phi :]-a, a[\to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$.

A technical result is needed for the construction of the equivalent fixed point problems.

Lemma 1. *For each $h \in C$, there exists a unique $\alpha := Q_\phi(h) \in \operatorname{Range} h$ such that*

$$\int_0^T \phi^{-1}\big(h(t) - \alpha\big)\,dt = 0.$$

Moreover, the function $Q_\phi : C \to \mathbb{R}$ is continuous.

Proof. Let $h \in C$. We first prove uniqueness. Let $\alpha_i \in \mathbb{R}$ be such that $\int_0^T \phi^{-1}(h(t) - \alpha_i)\,dt = 0$ $(i = 1, 2)$. It follows that there exists $t_0 \in [0,T]$ such that $\phi^{-1}(h(t_0) - \alpha_1) = \phi^{-1}(h(t_0) - \alpha_2)$, and using the injectivity of ϕ^{-1} we deduce that $\alpha_1 = \alpha_2$. For existence, the function

$$\gamma : [h_L, h_M] \to \mathbb{R}, \qquad s \mapsto \int_0^T \phi^{-1}\big(h(t) - s\big)\,dt$$

is well defined and continuous. On the other hand, because ϕ^{-1} is strictly monotone and $\phi^{-1}(0) = 0$, we see that $\gamma(h_L)\gamma(h_M) \leqslant 0$, and the existence of $\alpha \in [h_L, h_M]$ such that $\gamma(\alpha) = 0$ follows. Finally, we show that Q_ϕ is continuous on C. Let $(h_n)_n \subset C$ such that $h_n \to h_0$ in C. Without loss of generality, passing if necessary to a subsequence, we may assume that $Q_\phi(h_n) \to \alpha_0$. Using the dominated convergence theorem we deduce that $\int_0^T \phi^{-1}(h_0(t) - \alpha_0)\,dt = 0$, so we have that $\alpha_0 = Q_\phi(h_0)$. $\square$

Remark 1. Lemma 1 shows that the function Q_ϕ verifies the identity

$$Q \circ \phi^{-1} \circ (I - Q_\phi) \circ u = 0 \quad \text{for all } u \in C. \tag{2}$$

Furthermore, from the homeomorphic character of ϕ and $\phi(0) = 0$, we have

$$Q_\phi(0) = 0. \tag{3}$$

To construct the associated fixed point operators, following the approach in [9], we first recall some easily proved criteria for the solvability, under various boundary conditions, of the forced equation

$$\big(\phi(u')\big)' = f(t) \tag{4}$$

with $f \in C$.

Proposition 1. *For each $f \in C$, problem* (4) *has a unique solution verifying the boundary conditions*

$$u(0) = 0 = u(T), \tag{5}$$

given by

$$u(t) = \int_0^t \phi^{-1} \circ (I - Q_\phi) H f(s)\,ds \quad \big(t \in [0, T]\big).$$

Problem (4) *has a solution satisfying the conditions*

$$u'(0) = 0 = u'(T)$$

if and only if

$$Qf = 0 \tag{6}$$

in which case the solutions are given by the family

$$u(t) = u(0) + \int_0^t \phi^{-1}\big(Hf(s)\big)\,ds.$$

Problem (4) *has a solution satisfying the conditions*

$$u(0) - u(T) = 0 = u'(0) - u'(T)$$

if and only if condition (6) *holds, in which case the solutions are given by the family*

$$u(t) = u(0) + \int_0^t \big[\phi^{-1}(I - Q_\phi)Hf(s)\big]\,ds \quad \big(t \in [0, T]\big).$$

Remark 2. The results above still hold for $f \in L^1(0, T)$ if a *solution* of $(\phi(u'))' = f(t)$ is defined as a function $u \in C^1$ such that $\max_{[0,T]} |u'| < a$ and $\phi(u')$ is absolutely continuous.

We now define fixed point operators which are similar to the ones introduced in [9] and [3] for $\phi : \mathbb{R} \to \mathbb{R}$ and $\phi : \mathbb{R} \to]-a, a[$, respectively. The proofs of equivalence of the boundary value and the fixed point problems are identical and are not repeated here. Assume that

(H_F) $F : C^1 \to C$ is continuous and takes bounded sets into bounded sets.

Proposition 2. *u is a solution of the abstract Dirichlet problem*

$$\big(\phi(u')\big)' = F(u), \qquad u(0) = 0 = u(T) \tag{7}$$

if and only if $u \in C_0^1$ is a fixed point of the operator M_0 defined on C_0^1 by

$$M_0(u) := H \circ \phi^{-1} \circ (I - Q_\phi) \circ [HF](u). \tag{8}$$

u is a solution of the abstract Neumann problem

$$\big(\phi(u')\big)' = F(u), \qquad u'(0) = 0 = u'(T)$$

if and only if $u \in C_\dagger^1$ is a fixed point of the operator $M_\dagger$ defined on $C_\dagger^1$ by

$$M_\dagger(u) := Pu + QF(u) + H \circ \phi^{-1} \circ \big[H(I - Q)F\big](u).$$

u is a solution of the abstract periodic problem

$$\big(\phi(u')\big)' = F(u), \qquad u(0) - u(T) = 0 = u'(0) - u'(T)$$

if and only if $u \in C_\#^1$ is a fixed point of the operator $M_\#$ defined on $C_\#^1$ by

$$M_\#(u) = Pu + QF(u) + H \circ \phi^{-1} \circ (I - Q_\phi) \circ \big[H(I - Q)F\big](u).$$

Furthermore, $\|(M_j(u))'\|_\infty < a$ *for all* $u \in C^1_j$ *and* M_j *is completely continuous on* C^1_j $(j = 0, \dagger, \#)$.

3. The case of Dirichlet boundary conditions

A somewhat surprising consequence of the above results is a 'universal' solvability result for the nonlinear Dirichlet problem (7).

Theorem 1. *For each continuous* $F : C^1_0 \to C$ *taking bounded sets into bounded sets, the abstract nonlinear Dirichlet problem* (7) *has at least one solution.*

Proof. It suffices to prove that the operator M_0 defined in (8) has a fixed point. We use Leray–Schauder degree [5] and consider the homotopy

$$\big(\phi(u')\big)' = \lambda F(u), \qquad u(0) = 0 = u(T) \quad \big(\lambda \in [0,1]\big),$$

which is equivalent to the family of fixed point problems in C^1_0

$$u = H \circ \phi^{-1} \circ (I - Q_\phi) \circ [\lambda H F](u) := \mathcal{M}(\lambda, u) \quad \big(\lambda \in [0,1]\big). \tag{9}$$

Notice that

$$\mathcal{M}(0, u) = H \circ \phi^{-1} \circ (I - Q_\phi)(0) = 0 \tag{10}$$

because of (3) and $\phi^{-1}(0) = 0$. If u is a possible solution of (9), then

$$\|u'\|_\infty = \big\|\phi^{-1} \circ (I - Q_\phi) \circ [\lambda H F](u)\big\|_\infty < a,$$

and, using the boundary condition at 0, we have, for all $t \in [0, T]$,

$$\big|u(t)\big| = \left|\int_0^t u'(s)\,ds\right| < aT,$$

which implies that $\|u\| < a(T + 1)$. Hence, the homotopy invariance of Leray–Schauder degree [5] and (10) give

$$\begin{aligned} d_{LS}\big[I - \mathcal{M}(1,\cdot), B_{a(T+1)}, 0\big] &= d_{LS}\big[I - \mathcal{M}(0,\cdot), B_{a(T+1)}, 0\big] \\ &= d_{LS}\big[I, B_{a(T+1)}, 0\big] = 1. \end{aligned}$$

The existence of a fixed point of $\mathcal{M}(1,\cdot) = M_0$ follows from the existence property of Leray–Schauder degree [5]. □

Remark 3. As ϕ is not homogeneous, it is not clear how to develop a spectral theory for ϕ with its various boundary conditions. But Theorem 1 seems to indicate that, for Dirichlet conditions, such a spectrum should be empty.

To each continuous function $f:[0,T]\times\mathbb{R}^2\to\mathbb{R}$, we associate its *Nemytskii operator* $N_f:C^1\to C$ defined by

$$N_f(u)(t)=f\big(t,u(t),u'(t)\big).$$

It is easy to show that N_f is continuous and takes bounded sets into bounded sets. Hence an immediate consequence of Theorem 1 is the following

Corollary 1. *For each continuous $f:[0,T]\times\mathbb{R}^2\to\mathbb{R}$, the Dirichlet problem*

$$\big(\phi(u')\big)'=f(t,u,u'),\qquad u(0)=0=u(T) \tag{11}$$

has at least one solution.

Remark 4. When ϕ is an increasing *diffeomorphism*, $u'=\phi^{-1}\circ\phi(u')$, each solution is of class C^2 and problem (11) can be written in the classical way

$$u''=\big[\phi'(u')\big]^{-1}f(t,u,u'),\qquad u(0)=0=u(T). \tag{12}$$

For example, for $\phi(s)=s/\sqrt{1-s^2}$, one gets

$$u''=\big(1-u'^2\big)^{3/2}f(t,u,u'),\qquad u(0)=0=u(T), \tag{13}$$

which shows that the right-hand member, defined for $|u'|\leqslant 1$, tends to zero when $u'\to\pm1$. So (13) 'looks like' the restriction to $|u'|<1$ of a problem of the type

$$u''=g(t,u,u'),\qquad u(0)=0=u(T)$$

with $g(t,u,v)=0$ for $|v|\geqslant 1$, which sheds some light on the 'universal' character of the existence theorem.

Our approach covers nondifferentiable homeomorphisms like ϕ defined on $]-1,1[$, by

$$\phi(s)=\begin{cases} s & \text{for } s\in[0,\frac{1}{2}[,\\ ns-\sum_{k=1}^{n-1}\frac{k}{k+1} & \text{for } s\in[\frac{n-1}{n},\frac{n}{n+1}[,\ n=2,3,\dots,\\ -\phi(-s) & \text{for } s\in\,]-1,0[. \end{cases}$$

Even in the differentiable case, the treatment using the fixed point reduction of (11) given here seems much more transparent than a classical fixed point reduction of problem (12).

4. Periodic or Neumann problems with nonlinearities satisfying a sign condition

The counterexamples

$$\big(\phi(u')\big)'=1,\quad u'(0)=0=u'(T)\quad\text{or}\quad u(0)-u(T)=0=u'(0)-u'(T),$$

which have no solution, show that no result corresponding to Theorem 1 exists for the Neumann or periodic problems. We show that some rather general sign condition upon f suffices to get existence.

Consider the periodic boundary value problems

$$\big(\phi(u')\big)' = f(t,u,u'), \qquad u(0)-u(T)=0=u'(0)-u'(T). \tag{14}$$

In order to apply Leray–Schauder degree to the equivalent fixed point operator, we introduce, for $\lambda \in [0,1]$, the family of abstract periodic boundary value problems

$$\big(\phi(u')\big)' = \lambda N_f(u) + (1-\lambda)QN_f(u), \qquad u(0)-u(T)=0=u'(0)-u'(T). \tag{15}$$

Notice that (15) coincide with (14) for $\lambda = 1$. For each $\lambda \in [0,1]$, the nonlinear operator $M_\#$ on $C^1_\#$ associated to (15) by Proposition 2 is the operator $\mathcal{M}(\lambda,\cdot)$, where $\mathcal{M}$ is defined on $[0,1]\times C^1_\#$ by

$$\mathcal{M}(\lambda,u) = Pu + QN_f(u) + H\circ\phi^{-1}\circ(I-Q_\phi)\circ\big[\lambda H(I-Q)N_f\big](u). \tag{16}$$

Using Lemma 1 and Arzelá–Ascoli's theorem it is not difficult to see that $\mathcal{M}:[0,1]\times C^1_\# \to C^1_\#$ is completely continuous.

The first lemma gives *a priori* bounds for the possible fixed points.

Lemma 2. *Assume that there exist $R>0$ and $\epsilon \in \{-1,1\}$ such that*

$$\begin{aligned} &\epsilon\int_0^T f\big(t,u(t),u'(t)\big)\,dt > 0 \quad \text{if } u_L \geqslant R, \qquad \|u'\|_\infty < a,\\ &\epsilon\int_0^T f\big(t,u(t),u'(t)\big)\,dt < 0 \quad \text{if } u_M \leqslant -R, \qquad \|u'\|_\infty < a. \end{aligned} \tag{17}$$

If $(\lambda,u)\in[0,1]\times C^1_\#$ is such that $u=\mathcal{M}(\lambda,u)$, then $\|u\| < R + a(T+1)$.

Proof. Let $(\lambda,u)\in[0,1]\times C^1_\#$ be such that $u=\mathcal{M}(\lambda,u)$. Then, taking $t=0$ we get

$$QN_f(u)=0 \tag{18}$$

and from

$$u' = \big(\mathcal{M}(\lambda,u)\big)' = \phi^{-1}\circ(I-Q_\phi)\circ\big[\lambda H(I-Q)N_f\big](u),$$

we obtain

$$\|u'\|_\infty < a. \tag{19}$$

If $u_M \leqslant -R$ (respectively $u_L \geqslant R$) then, from (19) and (17), it follows that

$$\epsilon \int_0^T f\big(t, u(t), u'(t)\big)\, dt < 0 \quad \left(\text{respectively} \quad \epsilon \int_0^T f\big(t, u(t), u'(t)\big)\, dt > 0\right).$$

Using (18) we deduce that

$$u_M > -R \quad \text{and} \quad u_L < R. \tag{20}$$

From

$$u_M \leqslant u_L + \int_0^T \big|u'(t)\big|\, dt$$

and relations (19) and (20), we obtain

$$-(R + aT) < u_L \leqslant u_M < R + aT. \tag{21}$$

Using (19) and (21) it follows that $\|u\| < R + a(T + 1)$. □

The following result shows that a solution exists and Leray–Schauder degree is not zero.

Lemma 3. *If f satisfies condition* (17) *of Lemma* 2*, then $d_{LS}[I - \mathcal{M}(1, \cdot), B_\rho, 0] = -\epsilon$, and problem* (14) *has at least one solution.*

Proof. Let $\mathcal{M}$ be the operator given by (16) and let $\rho > R + a(T + 1)$. Lemma 2 and the homotopy invariance of the Leray–Schauder degree imply that

$$d_{LS}\big[I - \mathcal{M}(0, \cdot), B_\rho, 0\big] = d_{LS}\big[I - \mathcal{M}(1, \cdot), B_\rho, 0\big]. \tag{22}$$

On the other hand, we have that

$$d_{LS}\big[I - \mathcal{M}(0, \cdot), B_\rho, 0\big] = d_{LS}\big[I - (P + QN_f), B_\rho, 0\big]. \tag{23}$$

But the range of $P + QN_f$ is contained in the subspace of constant functions, isomorphic to $\mathbb{R}$, so, using a reduction property of Leray–Schauder degree [11]

$$\begin{aligned} d_{LS}\big[I - (P + QN_f), B_\rho, 0\big] &= d_B\big[I - (P + QN_f)|_{\mathbb{R}},]{-\rho}, \rho[, 0\big] \\ &= d_B\big[-QN_f,]{-\rho}, \rho[, 0\big] \\ &= \frac{\operatorname{sign}(-QN_f(\rho)) - \operatorname{sign}(-QN_f(-\rho))}{2} = -\epsilon, \end{aligned}$$

where d_B denotes the Brouwer degree. But, using (17) and the fact that $\rho > R$, we see that $QN_f(\pm\rho) = \frac{1}{T}\int_0^T f(t, \pm\rho, 0)\, dt$ have opposite signs, which implies, using (22) and (23) that $d_{LS}[I - \mathcal{M}(1, \cdot), B_\rho, 0] = -\epsilon$. Then, from the existence property of the Leray–Schauder degree, there exists $u \in B_\rho$ such that $u = \mathcal{M}(1, u)$, which is a solution for (14). □

A limiting argument allows to weaken the sign condition, but this generalization can also be proved directly using another approach, used in Section 5, and based upon the following lemma. Let us decompose any $u \in C^1_\#$ in the form

$$u = \bar{u} + \tilde{u} \quad \big(\bar{u} = u(0),\ \tilde{u}(0) = 0\big),$$

and let

$$\widetilde{C}^1_\# = \big\{u \in C^1_\#:\ u(0) = 0\big\}.$$

Lemma 4. *The set $\mathcal{S}$ of the solutions $(\bar{u}, \tilde{u}) \in \mathbb{R} \times \widetilde{C}^1_\#$ of problem*

$$\begin{gathered}\big(\phi(\tilde{u}')\big)' = f(t, \bar{u} + \tilde{u}, \tilde{u}') - \frac{1}{T}\int_0^T f\big(s, \bar{u} + \tilde{u}(s), \tilde{u}'(s)\big)\,ds,\\ u(0) - u(T) = 0 = u'(0) - u'(T)\end{gathered} \tag{24}$$

contains a continuum $\mathcal{C}$ whose projection on $\mathbb{R}$ is $\mathbb{R}$ and projection on $\widetilde{C}^1_\#$ is contained in the ball $B_{a(T+1)}$.

Proof. Using an argument similar to the one introduced in Section 2, it is easy to see that, for each fixed $\bar{u} \in \mathbb{R}$, problem (24) is equivalent to the fixed point problem in $\widetilde{C}^1_\#$

$$\tilde{u} = H \circ \phi^{-1} \circ (I - Q_\phi) \circ \big[H(I - Q)N_f(\bar{u} + \tilde{u})\big] := \widetilde{M}(\bar{u}, \tilde{u}).$$

Again, $\widetilde{M}$ is completely continuous on $\mathbb{R} \times \widetilde{C}^1_\#$, and, for each $(\bar{u}, \tilde{u}) \in \mathbb{R} \times \widetilde{C}^1_\#$, we have

$$\big\|\widetilde{M}(\bar{u}, \tilde{u})\big\| = \big\|\widetilde{M}(\bar{u}, \tilde{u})\big\|_\infty + \big\|\big(\widetilde{M}(\bar{u}, \tilde{u})\big)'\big\|_\infty < a(T + 1). \tag{25}$$

It follows from (25) that, for each $\bar{u} \in \mathbb{R}$, any possible fixed point $\tilde{u}$ of $\widetilde{M}(\bar{u}, \cdot)$ is such that

$$\|\tilde{u}\| < a(T + 1). \tag{26}$$

Furthermore, for each $\lambda \in [0, 1]$, each possible fixed point $\tilde{u}$ of

$$\widetilde{\mathcal{M}}(\lambda, 0, \cdot) := H \circ \phi^{-1} \circ (I - Q_\phi) \circ \big[\lambda H(I - Q)N_f(\cdot)\big]$$

satisfies, for the same reasons, inequality (26), which implies that

$$\begin{aligned} &d_{LS}\big[I - \widetilde{M}(0, \cdot), B_{a(T+1)}, 0\big]\\ &\quad = d_{LS}\big[I - \widetilde{\mathcal{M}}(1, 0, \cdot), B_{a(T+1)}, 0\big]\\ &\quad = d_{LS}\big[I - \widetilde{\mathcal{M}}(0, 0, \cdot), B_{a(T+1)}, 0\big] = d_{LS}\big[I, B_{a(T+1)}, 0\big] = 1. \end{aligned} \tag{27}$$

Conditions (26), (27) and Theorem 1.2 in [10] or Lemma 2.3 in [15] then imply the existence of $\mathcal{C}$. $\square$

Theorem 2. *Assume that there exist* $R > 0$ *and* $\epsilon \in \{-1, 1\}$ *such that*

$$\epsilon \int_0^T f\big(t, u(t), u'(t)\big)\, dt \geqslant 0 \quad \text{if } u_L \geqslant R, \qquad \|u'\|_\infty < a,$$
$$\epsilon \int_0^T f\big(t, u(t), u'(t)\big)\, dt \leqslant 0 \quad \text{if } u_M \leqslant -R, \qquad \|u'\|_\infty < a. \tag{28}$$

Then (14) *has at least one solution.*

Proof. Consider the continuum $\mathcal{C}$ given by Lemma 4. If $(R + aT, \tilde{u}) \in \mathcal{C}$, then, for each $t \in [0, T]$,

$$R + aT + \tilde{u}(t) > R$$

and hence, using (28)

$$\epsilon \int_0^T f\big(t, R + aT + \tilde{u}(t), \tilde{u}'(t)\big)\, dt \geqslant 0.$$

Similarly, if $(-R - aT, \tilde{u}) \in \mathcal{C}$, then

$$\epsilon \int_0^T f\big(t, -R - aT + \tilde{u}(t), \tilde{u}'(t)\big)\, dt \leqslant 0.$$

The existence of $(\bar{u}, \tilde{u}) \in \mathcal{C}$ such that

$$\epsilon \int_0^T f\big(t, \bar{u} + \tilde{u}(t), \tilde{u}'(t)\big)\, dt = 0,$$

and hence such that $u = \bar{u} + \tilde{u}$ is a solution of (14) follows from the intermediate value theorem for a continuous function on a connected set. $\square$

Corollary 2. *Let* $h : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ *and* $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ *be continuous, with* h *is bounded on* $[0, T] \times \mathbb{R} \times {]-a, a[}$ *and* g *satisfies condition*

$$\lim_{u \to -\infty} g(t, u) = +\infty, \qquad \lim_{u \to +\infty} g(t, u) = -\infty$$
$$\Big(\textit{respectively} \quad \lim_{u \to -\infty} g(t, u) = -\infty, \qquad \lim_{u \to +\infty} g(t, u) = +\infty\Big)$$

uniformly in $t \in [0, T]$. *Then the problem*

$$\big(\phi(u')\big)' + g(t,u) = h(t,u,u'), \qquad u(0) - u(T) = 0 = u'(0) - u'(T)$$

has at least one solution.

Corollary 3. *Let the continuous function $h : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be bounded on $[0,T] \times \mathbb{R} \times]-a,a[$. Then, for each $\mu \neq 0$, the problem*

$$\big(\phi(u')\big)' + \mu u = h(t,u,u'), \qquad u(0) - u(T) = 0 = u'(0) - u'(T)$$

has at least one solution, with associated Leray–Schauder degree equal to sign μ.

Example 1. If $e \in C$, $c \in \mathbb{R} \setminus 0$, $d \in \mathbb{R}$, $q \geqslant 0$ and $p > 1$, the problem

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' + d|u'|^q + c|u|^{p-1}u = e(t),$$
$$u(0) - u(T) = 0 = u'(0) - u'(T),$$

has at least one solution.

Another easy consequence is a *Landesman–Lazer type existence condition*.

Corollary 4. *Let $g : \mathbb{R} \to \mathbb{R}$ be continuous. Then problem*

$$\big(\phi(u')\big)' + g(u) = e(t), \qquad u(0) - u(T) = 0 = u'(0) - u'(T)$$

has at least one solution for each $e \in C$ such that

$$\limsup_{u \to -\infty} g(u) < \frac{1}{T} \int_0^T e(s)\,ds < \liminf_{u \to +\infty} g(u)$$

or

$$\limsup_{u \to +\infty} g(u) < \frac{1}{T} \int_0^T e(s)\,ds < \liminf_{u \to -\infty} g(u).$$

Remark 5. Using the family of equations

$$\big(\phi(u')\big)' = \lambda N_f(u) + (1-\lambda) Q N_f(u), \qquad u'(0) = 0 = u'(T)$$

which gives the completely continuous homotopy $\mathcal{M} : [0,1] \times C^1_\dagger \to C^1_\dagger$ defined by

$$\mathcal{M}(\lambda, u) = Pu + QN_f(u) + H \circ \phi^{-1} \circ \big[\lambda H(I-Q)N_f\big](u),$$

and similar *a priori* estimates as in the periodic case, it is not difficult to see that the above theorems hold also for Neumann boundary conditions.

Remark 6. Any possible reasonable concept of spectrum for ϕ with Neumann or periodic boundary conditions should reduce to $\{0\}$.

5. Upper and lower solutions

In this section, we extend the method of upper and lower solutions (see e.g. [4]) to the periodic boundary value problem

$$\big(\phi(u')\big)' = f(t,u,u'), \qquad u(0)-u(T)=0=u'(0)-u'(T). \tag{29}$$

Definition 1. A *lower solution* α (respectively *upper solution* β) of (29) is a function $\alpha \in C^1$ such that $\|\alpha'\|_\infty < a$, $\phi(\alpha') \in C^1$, $\alpha(0)=\alpha(T)$, $\alpha'(0) \geqslant \alpha'(T)$ (respectively $\beta \in C^1$, $\|\beta'\|_\infty < a$, $\phi(\beta') \in C^1$, $\beta(0)=\beta(T)$, $\beta'(0) \leqslant \beta'(T)$) and

$$\big(\phi\big(\alpha'(t)\big)\big)' \geqslant f\big(t,\alpha(t),\alpha'(t)\big) \quad \big(\text{respectively} \quad \big(\phi\big(\beta'(t)\big)\big)' \leqslant f\big(t,\beta(t),\beta'(t)\big)\big) \tag{30}$$

for all $t \in [0,T]$. Such a lower or upper solution is called *strict* if the inequality (30) is strict for all $t \in [0,T]$.

Theorem 3. *If* (29) *has a lower solution α and an upper solution β such that $\alpha(t) \leqslant \beta(t)$ for all $t \in [0,T]$, then problem* (29) *has a solution u such that $\alpha(t) \leqslant u(t) \leqslant \beta(t)$ for all $t \in [0,T]$. Moreover, if α and β are strict, then $\alpha(t) < u(t) < \beta(t)$ for all $t \in [0,T]$, and $d_{LS}[I - M_\#, \Omega_{\alpha,\beta}, 0] = -1$, where*

$$\Omega_{\alpha,\beta} = \big\{u \in C^1_\#\colon\ \alpha(t) < u(t) < \beta(t) \text{ for all } t \in [0,T],\ \|u'\| < a\big\},$$

and $M_\#$ is the fixed point operator associated to (29).

Proof. Let $\gamma : [0,T] \times \mathbb{R} \to \mathbb{R}$ be the continuous function defined by

$$\gamma(t,u) = \begin{cases} \beta(t) & \text{if } u > \beta(t), \\ u & \text{if } \alpha(t) \leqslant u \leqslant \beta(t), \\ \alpha(t) & \text{if } u < \alpha(t), \end{cases}$$

and define $F : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ by $F(t,u,v) = f(t,\gamma(t,u),v)$. We consider the modified problem

$$\big(\phi(u')\big)' = F(t,u,u') + u - \gamma(t,u), \qquad u(0)-u(T)=0=u'(0)-u'(T) \tag{31}$$

and first show that if u is a solution of (31) then $\alpha(t) \leqslant u(t) \leqslant \beta(t)$ for all $t \in [0,T]$, so that u is a solution of (29). Suppose by contradiction that there is some $t_0 \in [0,T]$ such that $[\alpha - u]_M = \alpha(t_0) - u(t_0) > 0$. If $t_0 \in]0,T[$ then $\alpha'(t_0) = u'(t_0)$ and there are sequences (t_k) in $[t_0 - \varepsilon, t_0[$ and (t_k') in $]t_0, t_0+\varepsilon]$ converging to t_0 such that $\alpha'(t_k) - u'(t_k) \geqslant 0$ and $\alpha'(t_k') - u'(t_k') \leqslant 0$. As ϕ is an increasing homeomorphism, this implies $(\phi(\alpha'(t_0)))' \leqslant (\phi(u'(t_0)))'$. Hence, because α is a lower solution of (29) we obtain

$$\big(\phi\big(\alpha'(t_0)\big)\big)' \leqslant \big(\phi\big(u'(t_0)\big)\big)' = f\big(t_0, \alpha(t_0), \alpha'(t_0)\big) + u(t_0) - \alpha(t_0)$$
$$< f\big(t_0, \alpha(t_0), \alpha'(t_0)\big) \leqslant \big(\phi\big(\alpha'(t_0)\big)\big)',$$

a contradiction. If $[\alpha - u]_M = \alpha(0) - u(0) = \alpha(T) - u(T)$, then $\alpha'(0) - u'(0) \leqslant 0$, $\alpha'(T) - u'(T) \geqslant 0$. Using that $\alpha'(0) \geqslant \alpha'(T)$, we deduce that $\alpha'(0) - u'(0) = 0 = \alpha'(T) - u'(T)$. This implies that $\phi(\alpha'(0)) = \phi(u'(0))$. On the other hand, $[\alpha - u]_M = \alpha(0) - u(0)$ implies, reasoning in a similar way as for $t_0 \in]0, T[$, that

$$\big(\phi\big(\alpha'(0)\big)\big)' \leqslant \big(\phi\big(u'(0)\big)\big)'.$$

Using the inequality above and $\alpha'(0) = u'(0)$, we can proceed as in the case $t_0 \in]0, T[$ to obtain again a contradiction. In consequence we have that $\alpha(t) \leqslant u(t)$ for all $t \in [0, T]$. Analogously, using the fact that β is an upper solution of (29), we can show that $u(t) \leqslant \beta(t)$ for all $t \in [0, T]$. We remark that if α, β are strict, then $\alpha(t) < u(t) < \beta(t)$ for all $t \in [0, T]$.

We now apply Corollary 3 to the modified problem (31) to obtain the existence of a solution, and the relation

$$d_{LS}[I - \widetilde{M}, B_\rho, 0] = -1 \tag{32}$$

for the equivalent fixed point operator $\widetilde{M}$ and all sufficiently large $\rho > 0$.

Moreover, if α and β are strict, then $\alpha(t) < u(t) < \beta(t)$ for all $t \in [0, T]$. If ρ is large enough, then, using (32) and the additivity–excision property of the Leray–Schauder degree [5], we have

$$d_{LS}[I - \widetilde{M}, \Omega_{\alpha,\beta}, 0] = d_{LS}[I - \widetilde{M}, B_\rho, 0] = -1.$$

On the other hand, as the completely continuous operator $M_\#$ associated to (29) is equal to $\widetilde{M}$ on $\overline{\Omega_{\alpha,\beta}}$, we deduce that $d_{LS}[I - M_\#, \Omega_{\alpha,\beta}, 0] = -1$. □

Remark 7. In contrast to the classical p-Laplacian case, no Nagumo type condition is required upon f in Theorem 3.

Remark 8. A careful analysis of the above proof implies that Theorem 3 holds also if $f : [0, T] \times]0, +\infty[\times \mathbb{R} \to \mathbb{R}$ is continuous.

We now show, using an argument of Amann, Ambrosetti and Mancini [1], that the existence conclusion in Theorem 3 also holds when the lower and upper solutions are not ordered. See [16] for the case where $\phi : \mathbb{R} \to \mathbb{R}$.

Theorem 4. *If* (29) *has a lower solution α and an upper solution β, then problem* (29) *has at least one solution.*

Proof. Let $\mathcal{C}$ be given by Lemma 4. If there is some $(\bar{u}, \tilde{u}) \in \mathcal{C}$ such that

$$\int_0^T f\big(t, \bar{u} + \tilde{u}(t), \tilde{u}'(t)\big)\, dt = 0,$$

then $\bar{u} + \tilde{u}$ solves (29). If

$$\int_0^T f\big(t, \bar{u} + \tilde{u}(t), \tilde{u}'(t)\big)\, dt > 0$$

for all $(\bar{u}, \tilde{u}) \in \mathcal{C}$, then, using (24), $\bar{u} + \tilde{u}$ is an upper solution for (29) for each $(\bar{u}, \tilde{u}) \in \mathcal{C}$. Then, for $(\alpha_M + aT, \tilde{u}) \in \mathcal{C}$, $\alpha_M + aT + \tilde{u}(t) \geqslant \alpha(t)$ for all $t \in [0, T]$ is an upper solution and the existence of a solution to (29) follows from Theorem 3. Similarly, if

$$\int_0^T f\big(t, \bar{u} + \tilde{u}(t), \tilde{u}'(t)\big)\, dt < 0$$

for all $(\bar{u}, \tilde{u}) \in \mathcal{C}$, then $(\beta_L - aT, \tilde{u}) \in \mathcal{C}$ gives the lower solution $\beta_L - aT + \tilde{u}(t) \leqslant \beta(t)$ for all $t \in [0, T]$ and the existence of a solution. □

The choice of constant lower and upper solutions in Theorems 3 and 4 leads to the following simple existence condition.

Corollary 5. *Problem* (29) *has at least one solution if there exist constants a and b such that*

$$f(t, a, 0) \cdot f(t, b, 0) \leqslant 0$$

for all $t \in [0, T]$.

Remark 9. All the results of this section hold for Neumann boundary conditions.

6. An Ambrosetti–Prodi type result

In this section, we consider periodic problems of the type

$$\big(\phi(u')\big)' + f(t, u, u') = s, \qquad u(0) - u(T) = 0 = u'(0) - u'(T) \tag{33}$$

when $s \in \mathbb{R}$ and f satisfies the *coercivity condition*

$$(H_f) \qquad f(t, u, v) \to +\infty \quad \text{if } |u| \to \infty \text{ uniformly in } [0, T] \times \,]-a, a[. \tag{34}$$

We are interested in studying the existence and multiplicity of the solutions of (33) in terms of the value of the parameter s (*Ambrosetti–Prodi problem*).

We first obtain an *a priori* bound for the set of possible solutions.

Lemma 5. *For each* $b \in \mathbb{R}$, *there exists* $\rho = \rho(b) > 0$ *such that any possible solution* u *of* (33) *with* $s \leqslant b$ *belongs to the open ball* B_ρ.

Proof. Let $s \leqslant b$ and u be a solution of (33). This implies that u satisfies

$$\|u'\|_\infty < a \tag{35}$$

and

$$QN_f(u) = s. \tag{36}$$

Using (34) we can find $R > 0$ such that

$$f(t,u,v) > b \quad \text{if } |u| \geqslant R,\ (t,v) \in [0,T] \times]-a,a[. \tag{37}$$

If $u_L \geqslant R$, then using (35) and (37), we deduce that $QN_f(u) > b$, which, together with (36), gives $s > b$, a contradiction. So we have $u_L < R$. Analogously we can show that $u_M > -R$. Then using the inequality

$$u_M \leqslant u_L + \int_0^T \left|u'(\tau)\right| d\tau,$$

we obtain $\|u\|_\infty < R + Ta$. We can take any $\rho \geqslant R + (T+1)a$. $\square$

Theorem 5. *If f satisfies condition* (34)*, there exists $s_1 \in \mathbb{R}$ such that problem* (33) *has zero, at least one or at least two solutions according to $s < s_1$, $s = s_1$ or $s > s_1$.*

Proof. Let $S_j = \{s \in \mathbb{R}\colon$ (29) has at least j solutions$\}$ $(j \geqslant 1)$.

(a) $S_1 \neq \emptyset$.

Take $s^* > \max_{t\in[0,T]} f(t,0,0)$ and use (34) to find $R_-^* < 0$ such that

$$\min_{t\in[0,T]} f\big(t, R_-^*, 0\big) > s^*.$$

Then $\alpha \equiv R_-^* < 0$ is a strict lower solution and $\beta \equiv 0$ is a strict upper solution for (33) with $s = s^*$. Hence, using Theorem 3, $s^* \in S_1$.

(b) *If $\tilde{s} \in S_1$ and $s > \tilde{s}$ then $s \in S_1$.*

Let $\tilde{u}$ be a solution of (33) with $s = \tilde{s}$, and let $s > \tilde{s}$. Then $\tilde{u}$ is a strict upper solution for (33). Take now $R_- < \tilde{u}_L$ such that $\min_{t\in[0,T]} f(t, R_-, 0) > s$: $\alpha \equiv R_-$ is a strict lower solution for (33). From Theorem 3, $s \in S_1$.

(c) $s_1 = \inf S_1$ *is finite and* $S_1 \supset]s_1, \infty[$.

Let $s \in \mathbb{R}$ and suppose that (33) has a solution u. Then (35) and (36) hold, implying that $s \geqslant c$, with $c = \inf_{[0,T]\times\mathbb{R}\times]-a,a[} f$. To obtain the second part of claim (c), we apply (b).

(d) $S_2 \supset]s_1, \infty[$.

Let $s_3 < s_1 < s_2$. For each $s \in \mathbb{R}$, let $\mathcal{M}(s,\cdot)$ be the fixed point operator in $C^1_\#$ associated to problem (33). Using Lemma 5 we find ρ such that each possible zero of $I - \mathcal{M}(s,\cdot)$ with $s \in [s_3, s_2]$ is such that $u \in B_\rho$. Consequently, the Leray–Schauder degree $d_{LS}[I - \mathcal{M}(s,\cdot), B_\rho, 0]$ is well defined and does not depend upon $s \in [s_3, s_2]$. However, using (c), we see that $u - \mathcal{M}(s_3, u) \neq 0$ for all $u \in C^1_\#$. This implies that $d_{LS}[I - \mathcal{M}(s_3,\cdot), B_\rho, 0] = 0$, so that $d_{LS}[I - \mathcal{M}(s_2,\cdot), B_\rho, 0] = 0$ and, by excision property of Leray–Schauder degree [5], $d_{LS}[I -$

$\mathcal{M}(s_2, \cdot), B_{\rho'}, 0] = 0$ if $\rho' > \rho$. Let $s \in]s_1, s_2[$ and $\hat{u}$ be a solution of (33) (using (c)). Then $\hat{u}$ is a strict upper solution of (33) with $s = s_2$. Let $R < \hat{u}_L$ be such that $\min_{t \in [0,T]} f(t, R, 0) > s_2$. Then R is a strict lower solution of (33) with $s = s_2$. Consequently, using Theorem 3, problem (33) with $s = s_2$ has a solution in $\Omega_{R,\hat{u}}$ and $d_{LS}[I - \mathcal{M}(s_2, \cdot), \Omega_{R,\hat{u}}, 0] = -1$. Taking ρ' sufficiently large, we deduce from the additivity property of Leray–Schauder degree [5] that

$$\begin{aligned} d_{LS}\big[I - \mathcal{M}(s_2, \cdot), B_{\rho'} \setminus \overline{\Omega}_{R,\hat{u}}, 0\big] &= d_{LS}\big[I - \mathcal{M}(s_2, \cdot), B_{\rho'}, 0\big] \\ -d_{LS}\big[I - \mathcal{M}(s_2, \cdot), \Omega_{R,\hat{u}}, 0\big] &= -d_{LS}\big[I - \mathcal{M}(s_2, \cdot), \Omega_{R,\hat{u}}, 0\big] = 1, \end{aligned}$$

and (33) with $s = s_2$ has a second solution in $B_{\rho'} \setminus \overline{\Omega}_{R,\hat{u}}$.

(e) $s_1 \in S_1$.

Let (τ_k) be a sequence in $]s_1, +\infty[$ converging to s_1, and let u_k be a solution of (33) with $s = \tau_k$ given by (c). Using Proposition 2 we deduce that

$$u_k = \mathcal{M}(\tau_k, u_k). \tag{38}$$

From Lemma 5, there exists $\rho > 0$ such that $\|u_k\| < \rho$ for all $k \geqslant 1$. The complete continuity of $\mathcal{M}$ implies that, up to a subsequence, the right-hand member of (38) converges in $C^1_\#$, and hence (u_k) converges to some $u \in C^1_\#$ such that $u = \mathcal{M}(s_1, u)$, i.e. to a solution of (33) with $s = s_1$. □

A similar proof provides the following dual Ambrosetti–Prodi condition.

Theorem 6. *If f satisfies the anticoercivity condition*

$$f(t, u, v) \to -\infty \quad \text{if } |u| \to \infty \text{ uniformly in } [0, T] \times]-a, a[, \tag{39}$$

there exists $s_1 \in \mathbb{R}$ such that problem (33) *has zero, at least one or at least two solutions according to $s > s_1$, $s = s_1$ or $s < s_1$.*

Corollary 6. *Let $h : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ and $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ be continuous, such that h is bounded on $[0, T] \times \mathbb{R} \times]-a, a[$ and g satisfies the condition*

$$g(t, u) \to +\infty \quad (\text{respectively } -\infty) \quad \text{if } |u| \to \infty \text{ uniformly in } t \in [0, T].$$

Then, there exists $s_1 \in \mathbb{R}$ such that the problem

$$\big(\phi(u')\big)' + g(t, u) = s + h(t, u, u'), \qquad u(0) - u(T) = 0 = u'(0) - u'(T)$$

has no solution if $s < s_1$ (respectively $s > s_1$), at least one solution if $s = s_1$ and at least two solutions if $s > s_1$ (respectively $s < s_1$).

Example 2. For each $e \in C$, $p > 0$, $q \geqslant 0$, $d \in \mathbb{R}$ and $c > 0$ (respectively $c < 0$), there exists $s_1 \in \mathbb{R}$ such that the problem

$$\begin{gathered} \left(\frac{u'}{\sqrt{1 - u'^2}}\right)' + d|u'|^q + c|u|^p = s + e(t), \\ u(0) - u(T) = 0 = u'(0) - u'(T) \end{gathered}$$

has no solution if $s < s_1$ (respectively $s > s_1$), at least one solution if $s = s_1$ and at least two solutions if $s > s_1$ (respectively $s < s_1$).

Remark 10. It is not difficult to show that the results proved in this section hold also for Neumann boundary condition, but, from Theorem 1, it is clear that Ambrosetti–Prodi type results do not hold for Dirichlet boundary conditions.

7. Singular nonlinearities

In this section we prove the existence of *positive* solutions for the following periodic problems with *singular attractive* restoring force

$$\big(\phi(u')\big)' + g(u) = e(t), \qquad u(0) - u(T) = 0 = u'(0) - u'(T) \tag{40}$$

or with *singular repulsive* restoring force

$$\big(\phi(u')\big)' - g(u) = e(t), \qquad u(0) - u(T) = 0 = u'(0) - u'(T) \tag{41}$$

where $e \in C$ and $g :]0, +\infty[\to]0, +\infty[$ is continuous and such that

$$g(u) \to +\infty \quad \text{as } u \to 0+, \tag{42}$$

$$g(u) \to 0 \quad \text{as } u \to +\infty. \tag{43}$$

Theorem 7. *Suppose that g satisfies conditions* (42) *and* (43). *Then* (40) *has at least one solution if and only if $Qe > 0$.*

Proof. If u is a solution of (40), then $Qe = QN_g(u) > 0$ because g is positive. Conversely, suppose that $Qe > 0$. Using (42), there exists $\epsilon > 0$ such that $g(\epsilon) > e(t)$ for all $t \in [0, T]$. Hence, $\alpha \equiv \epsilon$ is a strict lower solution for (40). On the other hand, using Proposition 1, there exists $w \in C^1_\#$ such that $(\phi(w'))' = e(t) - Qe$. Using (43), there exists some $\delta > 0$ such that $\beta(t) = \delta + w(t) > \alpha(t)$ and $g(\beta(t)) < Qe$ for all $t \in [0, T]$. Then, β is a strict upper solution for (40) and Theorem 3 and Remark 8 with $f = -g + e$ imply the result. □

Example 3. If $\mu > 0$ and $e \in C$, the problem

$$\left(\frac{u'}{\sqrt{1 - u'^2}}\right)' + \frac{1}{u^\mu} = e(t), \qquad u(0) - u(T) = 0 = u'(0) - u'(T)$$

has at least one solution if and only if $Qe > 0$.

To solve (41) we need the following supplementary condition

$$\int_0^1 g(u)\, du = +\infty. \tag{44}$$

Lemma 6. *Suppose that g satisfies conditions* (42)–(44). *There exists $\epsilon > 0$ such that if $\lambda \in [0,1]$ and u is any positive solution of*

$$\begin{aligned}\big(\phi(u')\big)' = (1-\lambda)\big[QN_g(u)+Qe\big]+\lambda g(u)+\lambda e(t),\\ u(0)-u(T)=0=u'(0)-u'(T),\end{aligned} \tag{45}$$

then $u(t) > \epsilon$ for all $t \in [0,T]$.

Proof. Let $\lambda \in [0,1]$ and u be a possible positive solution of (45). Then

$$QN_g(u)+Qe=0 \tag{46}$$

and hence, if $\lambda \in]0,1]$, (45) is equivalent to

$$\big(\phi(u')\big)' = \lambda g(u)+\lambda e(t), \qquad u(0)-u(T)=0=u'(0)-u'(T). \tag{47}$$

Using the positivity of g, we deduce that

$$\big|g(u)+e(t)\big| \leqslant g(u)+\big|e(t)\big| = g(u)+e(t)+2e^-(t) \tag{48}$$

for all $(t,u) \in [0,T]\times\mathbb{R}$. From (47), (46) and (48) it follows that

$$\big\|\big(\phi(u')\big)'\big\|_1 = \lambda\big\|N_g(u)+e\big\|_1 \leqslant 2\lambda\|e^-\|_1. \tag{49}$$

Because $u \in C^1$ is such that $u(0)=u(T)$, there exists $\eta \in [0,T]$ such that $u'(\eta)=0$, which implies $\phi(u'(\eta))=0$ and

$$\phi\big(u'(t)\big) = \int_\eta^t \big(\phi\big(u'(s)\big)\big)'\,ds \quad \big(t \in [0,T]\big).$$

Using the equality above and (49) we have that

$$\big|\phi\big(u'(t)\big)\big| \leqslant 2\lambda\|e^-\|_1 \quad \big(t \in [0,T]\big). \tag{50}$$

Using (42), there exists $\xi > 0$ such that

$$g(u) > -Qe \quad \text{for all } 0 < u \leqslant \xi, \tag{51}$$

and therefore, by (51) and (46), there exists $t_1 \in [0,T]$ such that $u(t_1) > \xi$. Now, let

$$x(t) = \phi\big(u'(t)\big) \tag{52}$$

which implies

$$u'(t) = \phi^{-1}\big(x(t)\big) \tag{53}$$

for all $t \in [0, T]$. Introducing (52) in (47) we obtain

$$x'(t) - \lambda g\big(u(t)\big) = \lambda e(t) \tag{54}$$

for all $t \in [0, T]$. Multiplying (53) by $x'(t)$ and (54) by $u'(t)$ and subtracting we get

$$x'(t)\phi^{-1}\big(x(t)\big) - \lambda g\big(u(t)\big)u'(t) = \lambda e(t)u'(t),$$

i.e.

$$\left(\int_0^{x(t)} \phi^{-1}(s)\,ds\right)' - \lambda g\big(u(t)\big)u'(t) = \lambda e(t)u'(t)$$

for all $t \in [0, T]$. This implies that

$$\int_0^{x(t)} \phi^{-1}(s)\,ds - \int_0^{x(t_1)} \phi^{-1}(s)\,ds - \lambda \int_{u(t_1)}^{u(t)} g(s)\,ds = \lambda \int_{t_1}^{t} e(s)u'(s)\,ds$$

for all $t \in [0, T]$. Using the fact that $\int_0^v \phi^{-1}(s)\,ds \geqslant 0$ for all $v \in \mathbb{R}$, we deduce that

$$\lambda \int_{u(t)}^{u(t_1)} g(s)\,ds \leqslant \int_0^{x(t_1)} \phi^{-1}(s)\,ds + \lambda \int_{t_1}^{t} e(s)u'(s)\,ds \tag{55}$$

for all $t \in [0, T]$. Using (50), (52) and (55), we obtain

$$\begin{aligned}\int_{u(t)}^{u(t_1)} g(s)\,ds &\leqslant \frac{1}{\lambda}\left(\int_0^{2\lambda\|e^-\|_1} \phi^{-1}(s)\,ds + \int_0^{-2\lambda\|e^-\|_1} \phi^{-1}(s)\,ds\right) + a\|e\|_1 \\ &\leqslant \max_{[-2\|e^-\|_1, 2\|e^-\|_1]} 2\big|\phi^{-1}\big|\|e^-\|_1 + a\|e\|_1 := c\end{aligned} \tag{56}$$

for all $t \in [0, T]$. Using (44) we can find $0 < \epsilon < \xi$ such that

$$\int_\epsilon^\xi g(t)\,dt > c. \tag{57}$$

Since $u(t_1) > \xi$ and g is positive, from (56) and (57) one gets $u(t) > \epsilon$ for all $t \in [0, T]$. Now, for $\lambda = 0$, the solutions of (45) are the constant functions u solutions of

$$g(u) + Qe = 0$$

and they satisfy $u > \xi > \epsilon$. □

Theorem 8. *Suppose that $e \in C$ and g satisfies conditions* (42)–(44). *Then* (41) *has at least one positive solution if and only if $Qe < 0$.*

Proof. If u is a solution, then $Qe = -QN_g(u) < 0$. For sufficiency, we use the homotopy (45) and the corresponding homotopy for the associated family of fixed point operators $\mathcal{M}(\lambda, \cdot)$ defined in (16) with $f = g + e$. Let $\lambda \in [0,1]$ and u be a possible positive solution of (45). We already know from Lemma 6 that $u(t) > \epsilon$ for some $\epsilon > 0$ and all $t \in [0,T]$. From assumption (43) follows easily the existence of $R > 0$ such that $g(u) < -Qe$ if $u \geqslant R$. Hence, because of (46), there exists $t_2 \in [0,T]$ such that $u(t_2) < R$, which implies $u(t) < R + aT$ $(t \in [0,T])$. Hence, all the possible positive solutions of problem (45) are contained in the open bounded set

$$\Omega := \left\{u \in C^1_{\#}\colon\ \epsilon < u(t) < R + aT\ (0 \leqslant t \leqslant T),\ \|u'\|_\infty < a\right\}.$$

From the homotopy invariance of Leray–Schauder degree, we obtain

$$\begin{aligned} d_{LS}\big[I - \mathcal{M}(1,\cdot), \Omega, 0\big] &= d_{LS}\big[I - \mathcal{M}(0,\cdot), \Omega, 0\big] \\ &= d_B[g + Qh, \Omega \cap \mathbb{R}, 0] = d_B\big[g + Qh,]\epsilon, R[, 0\big] = -1, \end{aligned}$$

so that $\mathcal{M}(1,\cdot)$ has a fixed point. $\square$

Example 4. If $\mu \geqslant 1$ and $e \in C$, problem

$$\left(\frac{u'}{\sqrt{1 - u'^2}}\right)' - \frac{1}{u^\mu} = e(t), \qquad u(0) - u(T) = 0 = u'(0) - u'(T)$$

has at least one positive solution if and only if $Qe < 0$.

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