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BOUNDARY-VALUE PROBLEMS WITH NON-SURJECTIVE ϕ-LAPLACIAN AND ONE-SIDED BOUNDED NONLINEARITY

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Abstract. Using Leray-Schauder degree theory we obtain various existence results for nonlinear boundary-value problems

$$(\phi(u'))' = f(t,u,u'), \quad l(u,u') = 0$$

where $l(u,u') = 0$ denotes the periodic, Neumann or Dirichlet boundary conditions on $[0,T]$, $\phi : \mathbb{R} \to (-a,a)$ is a homeomorphism, $\phi(0) = 0$.

1. Introduction

The aim of this article is to obtain existence results for nonlinear boundary value problems of the form

$$(\phi(u'))' = f(t,u,u'), \quad l(u,u') = 0,$$

where $l(u,u') = 0$ denotes the periodic, Neumann or Dirichlet boundary conditions on $[0,T]$, $\phi : \mathbb{R} \to (-a,a)$ is a homeomorphism such that $\phi(0) = 0$. Such homeomorphisms ϕ are in particular motivated by the one-dimensional version of mean curvature problems or of capillary surfaces, for which $\phi(v) = \frac{v}{\sqrt{1+v^2}}$.

Several papers have been recently devoted to the Dirichlet problem for prescribed mean curvature problems

$$\nabla \cdot \Big(\frac{\nabla u}{1 + |\nabla u|^2} \Big) + f(x,u) = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega,$$

and to the corresponding one-dimensional version, with special attention on positive solutions (see e.g. [7, 18, 16, 4, 17, 3, 10, 1, 8]).

We show in Section 3 that the Dirichlet problem

$$(\phi(u'))' = f(t), \quad u(0) = 0 = u(T)$$

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is solvable if and only if there exists $\tau \in [0, T]$ such that

$$\|F_\tau\|_\infty < a \quad \text{and} \quad \int_0^T \phi^{-1}(F_\tau(s))\, ds = 0, \tag{1.1}$$

where

$$F_\tau(t) := \int_\tau^t f(s)\, ds.$$

This implies in particular that the simple problem

$$\Big(\frac{u'}{\sqrt{1+u'^2}}\Big)' = \alpha, \quad u(0) = 0 = u(T)$$

is solvable if and only if

$$|\alpha| < \frac{2}{T}.$$

This contrasts with situations where $\phi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism, like the p-Laplacian case ($p > 1$)

$$\big(|u'|^{p-2}u'\big)' = \alpha, \quad u(0) = 0 = u(T),$$

and the linear case $p = 2$

$$u'' = \alpha, \quad u(0) = 0 = u(T),$$

which are solvable for any $\alpha \in \mathbb{R}$.

For the more general problem

$$(\phi(u'))' = f(t, u, u'), \quad u(0) = 0 = u(T),$$

with $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ continuous and such that

$$|f(t, u, v)| \leq c \quad \text{for all} \quad (t, u, v) \in [0, T] \times \mathbb{R}^2, \tag{1.2}$$

we obtain in Section 7 the sufficient condition

$$c < \frac{a}{2T}$$

for solvability.

Mean curvature-like equations with Neumann or periodic boundary conditions seem to have been much less studied. For the Neumann problem

$$(\phi(u'))' = f(t, u, u'), \quad u'(0) = 0 = u'(T), \tag{1.3}$$

with $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ continuous verifying (1.2), it is proved in [2] that a solution exists when

$$c < \frac{\sqrt{3}a}{T} \tag{1.4}$$

and f satisfies a suitable sign condition. We show in Section 5 that (1.4) can be improved into

$$c < \frac{2a}{T}, \tag{1.5}$$

and we use a weaker sign condition, which may be seen as a nonlinear variant of the first necessary condition

$$\int_0^T f(s)\,ds = 0 \tag{1.6}$$

for the solvability of the simpler problem

$$(\phi(u'))' = f(t), \quad u'(0) = 0 = u'(T).$$

It is shown in Section 3 that a second necessary condition for the solvability of this problem is

$$\|F_0\|_\infty < a. \tag{1.7}$$

Inequalities (1.2) and (1.5) may be seen as extensions of (1.7) to the more general problem (1.3).

The periodic problem

$$(\phi(u'))' = f(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

which combines the technical difficulties of the Dirichlet and of the Neumann cases, is shown in Section 3 to be solvable if and only if (1.6) holds and if there exists $\tau \in [0, T]$ such that (1.1) is satisfied. Consequently, existence conditions for the more general situation

$$(\phi(u'))' = f(t, u, u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{1.8}$$

with $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ continuous and verifying (1.2) should require both a sign condition on f and a restriction on c. We prove in Section 6 that one can take

$$c < \frac{a}{T}$$

for the growth restriction.

The proofs in [2] and in Sections 6 and 7 consist in a reduction of the boundary-value problem to a fixed-point problem in a suitable function space, to which Leray-Schauder degree is applied using some homotopy. With respect to the case where $\phi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism, which was treated in this way in [12, 13], new difficulties occur when the range of ϕ is not $\mathbb{R}$, from the fact that the function ϕ^{-1}, occurring in the fixed-point operator, is no longer defined everywhere. When f verifies the boundedness conditions (1.2) and (1.4), it has been shown in [2] for the Neumann problem that the associated fixed-point operator remains defined everywhere. This is extended in Section 6, under the more general bound (1.5) and a more general sign condition, and to the case of periodic boundary conditions, which presents similarities with the Neumann problem, but is associated to a more complicated fixed-point operator.

But the principal aim of this paper is to study the more difficult case where the nonlinearity f is only bounded from below or from above, both for periodic and Neumann boundary conditions. The one-sided bound on f, even if suitably restricted in terms of a and T, does not insure that the associated fixed-point operator is defined everywhere. In this situation, the homotopy used in [2], inspired by the basic continuation theorem of coincidence degree theory [15] and by Theorem 3.1 in [12], only makes sense when considered on a well chosen open set depending upon the homotopy parameter. Hence, we need to rely upon the following extended homotopy invariance property of the Leray-Schauder degree, that we recall for the convenience of the reader (see [14] for references).

Proposition 1. *Let X be a real Banach space, $V \subset [0,1] \times X$ be an open, bounded set and $\mathcal{M}$ be a completely continuous operator on $\overline{V}$ such that $x \neq \mathcal{M}(\lambda, x)$ for each $(\lambda, x) \in \partial V$. Then the Leray-Schauder degree*

$$d_{LS}[I - \mathcal{M}(\lambda, \cdot), V_\lambda, 0]$$

is well defined and independent of λ in $[0,1]$, where V_λ is the open, bounded (possibly empty) set defined by $V_\lambda = \{x \in X : (\lambda, x) \in V\}$.

With this tool and the extension of a priori estimates inspired by a technique introduced by Ward [19] for semilinear periodic problems, and adapted to our quasi-linear situation, we are able to prove existence theorems for nonlinearities which are bounded below or above by a constant depending upon a and T, and which satisfy the sign condition mentioned above (see Theorems 1 and 2 in Sections 4 and 5 for periodic and Neumann boundary conditions respectively). Several concrete examples are given. The technique for getting a priori estimates does not work for the case of Dirichlet conditions, and the corresponding solvability problem when the nonlinearity is only bounded from above or from below remains open.

Because a number of technicalities and preliminary results reach their maximal difficulty in the case of perturbations bounded only from below or from above and periodic boundary conditions, we treat this situation first in Section 4. When considering similar perturbations with Neumann boundary conditions in Section 5, and the case of bounded perturbations in Sections 6 and 7, we can then avoid repeating the proofs of some lemmas, as they are similar, but simpler, to the corresponding ones proved in Section 4.

2. Notation and preliminaries

We first introduce some notation. We denote the usual norms in $L^1(0,T)$ and $L^\infty(0,T)$ respectively by $\|\cdot\|_1$ and $\|\cdot\|_\infty$. Let C denote the Banach space of continuous functions on $[0,T]$ endowed with the norm $\|\cdot\|_\infty$, C^1 denote the Banach space of continuously differentiable functions on $[0,T]$ equipped with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$, C^1_0 denotes the closed subspace of C^1 defined by $C^1_0 = \{u \in C^1 : u(0) = 0 = u(T)\}$, $C^1_\#$ denotes the closed subspace of C^1 defined by $C^1_\# = \{u \in C^1 : u'(0) = 0 = u'(T)\}$, C^1_{per} denotes the closed subspace of C^1 defined by $C^1_{per} = \{u \in C^1 : u(0) = u(T), u'(0) = u'(T)\}$. We denote by P, Q the projectors

$$P, Q : C \to C, \quad Pu(t) = u(0), \quad Qu(t) = \frac{1}{T}\int_0^T u(s)\,ds,$$

and we define $H : C \to C^1$ by

$$Hu(t) = \int_0^t u(s)\,ds.$$

If $u \in C$, we write

$$u^+ = \max\{u, 0\}, \quad u^- = \max\{-u, 0\}, \quad u_L = \min_{[0,T]} u, \quad u_M = \max_{[0,T]} u.$$

We need the following elementary lemmas. The first one is an inequality for functions in $L^\infty(0,T)$ which is used in Section 5 and improves Lemma 1 of [2] (when the uniform norm is used on both sides of the inequality), in replacing $T/\sqrt{3}$ by $T/2$ in the right-hand side of the inequality. An example shows that this new constant is best possible.

Lemma 1. *If $w \in L^\infty(0,T)$, then*

$$\|H(I-Q)w\|_\infty \le \frac{T}{2}\|w\|_\infty. \tag{2.1}$$

Proof. If $w \in L^\infty(0,T)$, we have, for $t \in [0,T]$,

$$\begin{aligned} H(I-Q)w(t) &= \int_0^t w(s)\,ds - \frac{t}{T}\int_0^T w(s)\,ds \qquad (2.2)\\ &= \big(1 - \frac{t}{T}\big)\int_0^t w(s)\,ds - \frac{t}{T}\int_t^T w(s)\,ds = \int_0^T G(t,s)w(s)\,ds, \end{aligned}$$

where

$$G(t,s) = \begin{cases} 1 - \frac{t}{T} & \text{if} \quad 0 \le s \le t \\ -\frac{t}{T} & \text{if} \quad t < s \le T. \end{cases} \tag{2.3}$$

Hence, for each $t \in [0,T]$, one has

$$\begin{aligned} |H(I-Q)w(t)| &\le \int_0^T |G(t,s)||w(s)|\,ds \\ &\le \|w\|_\infty \int_0^T |G(t,s)|\,ds = 2t\big(1 - \frac{t}{T}\big)\|w\|_\infty \le \frac{T}{2}\|w\|_\infty. \quad \square \end{aligned}$$

Remark 1. Inequality (2.1) is sharp as shown by the function $w \in L^\infty(0,T)$ defined by

$$w(t) = \begin{cases} 1 & \text{if} \quad 0 \le t \le \frac{T}{2} \\ -1 & \text{if} \quad \frac{T}{2} < t \le T, \end{cases} \tag{2.4}$$

which is such that $Qw = 0$, $\|w\|_\infty = 1$, and

$$\int_0^t w(s)\,ds = \begin{cases} t & \text{if} \quad 0 \le t \le \frac{T}{2} \\ T - t & \text{if} \quad \frac{T}{2} \le t \le T. \end{cases}$$

Consequently,

$$\|Hw\|_\infty = \frac{T}{2} = \frac{T}{2}\|w\|_\infty.$$

It is sharp also in the space C, as shown by the continuous functions w_ε $(0 < \varepsilon < \frac{T}{2})$ defined by

$$w_\varepsilon(t) = \begin{cases} 1 & \text{if} \quad 0 \le t \le \frac{T}{2} - \varepsilon \\ \frac{1}{\varepsilon}\left(\frac{T}{2} - t\right) & \text{if} \quad \frac{T}{2} - \varepsilon < t < \frac{T}{2} + \varepsilon \\ -1 & \text{if} \quad \frac{T}{2} + \varepsilon < t \le T, \end{cases} \tag{2.5}$$

which are such that $Qw_\varepsilon = 0$, $\|w_\varepsilon\|_\infty = 1$,

$$\int_0^t w_\varepsilon(s)\,ds = \begin{cases} t & \text{if} \quad 0 \le t \le \frac{T}{2} - \varepsilon \\ \frac{T-\varepsilon}{2} - \frac{1}{2\varepsilon}\left(\frac{T}{2} - t\right)^2 & \text{if} \quad \frac{T}{2} - \varepsilon < t < \frac{T}{2} + \varepsilon \\ T - t & \text{if} \quad \frac{T}{2} \le t \le T, \end{cases}$$

and hence,

$$\|Hw_\varepsilon\|_\infty = \frac{T}{2} - \frac{\varepsilon}{2} = \Big(\frac{T-\varepsilon}{2}\Big)\|w_\varepsilon\|_\infty.$$

The second lemma, used in Sections 3 and 4, gives an inequality similar to (2.1), except that the norm $\|\cdot\|_\infty$ is replaced by the norm $\|\cdot\|_1$ in the right-hand member. Again, the constant is best possible.

Lemma 2. *If $w \in L^1(0,T)$, then*

$$\|H(I-Q)w\|_\infty \le \|w\|_1. \tag{2.6}$$

Proof. We use again formula (2.2) with G defined in (2.3). Then, for each $t \in [0,T]$, we have

$$\begin{aligned} |H(I-Q)w(t)| &= \Big|\big(1-\frac{t}{T}\big)\int_0^t w(s)\,ds - \frac{t}{T}\int_t^T w(s)\,ds\Big| \\ &\le \big(1-\frac{t}{T}\big)\int_0^t |w(s)|\,ds + \frac{t}{T}\int_t^T |w(s)|\,ds \le \|w\|_1. \quad \square \end{aligned}$$

The third lemma is an adaptation of a result of [12] to the case of a homeomorphism which is not defined everywhere.

Lemma 3. *Let $B = \{h \in C : \|h\|_\infty < \frac{a}{2}\}$. For each $h \in B$, there exists a unique $\alpha \in \mathbb{R}$ such that*

$$\int_0^T \phi^{-1}(h(t)-\alpha)\,dt = 0.$$

Moreover, if $\|h\|_\infty \le \varepsilon$, then $\alpha \in [-\varepsilon,\varepsilon]$. The function $Q_\phi : B \to \mathbb{R}$ defined by $Q_\phi(h) := \alpha$ is continuous.

Proof. Let $h \in B$. We first prove uniqueness. Let $\alpha_i \in \mathbb{R}$ be such that $h(t)-\alpha_i \in (-a,a)$ for all $t \in [0,T]$ and $\int_0^T \phi^{-1}(h(t)-\alpha_i)\,dt = 0$ $(i=1,2)$. It follows that there exists $t_0 \in [0,T]$ such that $\phi^{-1}(h(t_0)-\alpha_1) = \phi^{-1}(h(t_0)-\alpha_2)$, and using the injectivity of ϕ^{-1} we deduce that $\alpha_1 = \alpha_2$. For existence, let $0 < \varepsilon < \frac{a}{2}$ be such that $\|h\|_\infty \le \varepsilon$. It is clear that the function

$$\gamma : [-\varepsilon,\varepsilon] \to \mathbb{R}, \quad s \mapsto \int_0^T \phi^{-1}(h(t)-s)\,dt$$

is well defined and continuous. On the other hand, because ϕ^{-1} is strictly monotone and $\phi^{-1}(0) = 0$, we see that $\gamma(-\varepsilon)\gamma(\varepsilon) \le 0$, and the existence of $\alpha \in [-\varepsilon,\varepsilon]$ such that $\gamma(\alpha) = 0$ follows. Finally, we show that Q_ϕ is continuous on B. Let $(h_n)_n \subset B$ such that $h_n \to h_0$ in C and $h_0 \in B$. Without loss of generality, we may assume that there is $0 < \varepsilon < \frac{a}{2}$ such that $(\|h_n\|_\infty)_n \subset [-\varepsilon,\varepsilon]$, and, passing if necessary to a subsequence, we may assume that $Q_\phi(h_n) \to \alpha_0 \in [-\varepsilon,\varepsilon]$. Using the dominated convergence theorem we deduce that $\int_0^T \phi^{-1}(h_0(t)-\alpha_0)\,dt = 0$, so we have that $\alpha_0 = Q_\phi(h_0)$. Hence, the function Q_ϕ is continuous. $\square$

Remark 2. The above result shows that *the function Q_ϕ verifies the identity*

$$Q \circ \phi^{-1} \circ (I - Q_\phi) \circ h = 0 \quad \textit{for all} \quad h \in C \quad \textit{with} \quad \|h\|_\infty < \frac{a}{2}. \qquad (2.7)$$

Finally, to each continuous function $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, we associate its Nemytskii operator $N_f : C^1 \to C$ defined by

$$N_f(u)(t) = f(t, u(t), u'(t)).$$

All the above defined operators P, Q, H, N_f are continuous.

Remark 3. All the results below remain true if the continuity of f is replaced by Carathéodory conditions. The details are left to the reader.

3. Forced ϕ-Laplacian with Dirichlet, Neumann or periodic boundary conditions

To motivate the assumptions of the theorems proved here and the construction of the associated fixed-point operators, we first study the solvability of the forced equation

$$(\phi(u'))' = f(t) \qquad (3.1)$$

with $f \in L^1(0,T)$, submitted to various boundary conditions. For each $\tau \in [0,T]$, we define $F_\tau : [0,T] \to \mathbb{R}$ by

$$F_\tau(t) := \int_\tau^t f(s)\, ds, \qquad (3.2)$$

so that

$$F_\tau(t) = F_0(t) - F_0(\tau).$$

We first consider the Neumann boundary conditions

$$u'(0) = 0 = u'(T). \qquad (3.3)$$

Proposition 2. *Problem* (3.1)-(3.3) *has a solution if and only if*

$$\int_0^T f(s)\, ds = 0 \qquad (3.4)$$

and

$$\|F_0\|_\infty < a, \qquad (3.5)$$

in which case problem (3.1)-(3.3) *has the family of solutions*

$$u(t) = u(0) + \int_0^t \phi^{-1}(F_0(s))\, ds. \qquad (3.6)$$

Proof. If u is a solution of problem (3.1)-(3.3), then (3.4) follows from integrating both members of (3.1) on $[0,T]$ and using the boundary condition (3.3). If (3.4) holds, we get, by integrating both members of (3.1) on $[0,t]$ and using the boundary condition (3.3)

$$\phi(u'(t)) = F_0(t) \quad (t \in [0,T]), \tag{3.7}$$

which implies (3.5). Now, if conditions (3.4) and (3.5) hold, problem (3.1)-(3.3) is equivalent to (3.7), hence to

$$u'(t) = \phi^{-1}(F_0(t)) \quad (t \in [0,T]),$$

which gives (3.6) by integration from 0 to t. □

Example 1. The problem

$$\Big(\frac{u'}{\sqrt{1+u'^2}}\Big)' = \alpha \cos t, \quad u'(0) = 0 = u'(\pi),$$

is solvable if and only if $|\alpha| < 1$.

Example 2. The problem

$$\Big(\frac{u'}{\sqrt{1+u'^2}}\Big)' = \alpha\Big(t - \frac{1}{2}\Big), \quad u'(0) = 0 = u'(1),$$

is solvable if and only if $|\alpha| < 8$.

Example 3. If w is defined in (2.4), the problem

$$\Big(\frac{u'}{\sqrt{1+u'^2}}\Big)' = \alpha w(t), \quad u'(0) = 0 = u'(T),$$

is solvable if and only if $|\alpha| < \frac{2}{T}$.

Remark 4. Using the inequality (2.1), we see that if $f \in L^\infty(0,T)$, the conditions

$$\int_0^T f(s)\,ds = 0, \quad \|f\|_\infty < \frac{2a}{T}, \tag{3.8}$$

are sufficient for the solvability of problem (3.1)-(3.3). The second inequality gives $|\alpha| < \frac{2}{\pi}$ in Example 1, $|\alpha| < 2$ in Example 2, and $|\alpha| < \frac{2}{T}$ in Example 3.

If we consider now the Dirichlet boundary conditions

$$u(0) = 0 = u(T), \tag{3.9}$$

we obtain the following necessary and sufficient conditions for the solvability of problem (3.1)-(3.9).

Proposition 3. *Problem* (3.1)-(3.9) *has a solution if and only if there exists* $\tau \in [0, T]$ *such that*

$$\|F_\tau\|_\infty < a \quad \textit{and} \quad \int_0^T \phi^{-1}(F_\tau(s))\, ds = 0, \tag{3.10}$$

in which case problem (3.1)-(3.9) *has the solution*

$$u(t) = \int_0^t \phi^{-1}(F_\tau(s))\, ds \quad (t \in [0, T]). \tag{3.11}$$

Proof. If u is a solution of problem (3.1)-(3.9), it follows from (3.9) and Rolle's theorem that there exists $\tau \in [0, T]$ such that $u'(\tau) = 0$. Then equation (3.1) gives

$$\phi(u'(t)) = F_\tau(t) \quad (t \in [0, T]),$$

which implies the first condition in (3.10) and the equivalent form

$$u'(t) = \phi^{-1}(F_\tau(t)) \quad (t \in [0, T]),$$

which, using the first boundary condition, gives (3.11). Then the second boundary condition implies the second condition in (3.10). Now, if condition (3.10) holds, it is immediate to check that (3.11) solves (3.1)-(3.9). □

Example 4. For the problem

$$\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = \alpha, \quad u(0) = 0 = u(1), \tag{3.12}$$

$F_\tau(t) = \alpha(t - \tau)$, and it is easily checked that

$$\int_0^1 \frac{\alpha(s-\tau)}{\sqrt{1-\alpha^2(s-\tau)^2}}\, ds = \frac{1}{\alpha}\left[\sqrt{1-\alpha^2\tau^2} - \sqrt{1-\alpha^2(1-\tau)^2}\right] = 0$$

if and only if $\tau = \frac{1}{2}$. Consequently, Problem (3.12) is solvable if and only if $|\alpha| < 2$. Elementary computations show that the (unique) solution is given by

$$u(t) = \frac{1}{\alpha}\left[\sqrt{1-\tfrac{\alpha^2}{4}} - \sqrt{1-\alpha^2(t-\tfrac{1}{2})^2}\right] \quad (t \in [0, 1]).$$

Remark 5. The computation of τ in conditions (3.10) may be difficult. As $F_\tau(\tau) = 0$, one has the inequalities

$$\frac{1}{2} Osc_{[0,T]} F_0 \le \|F_\tau\|_\infty \le Osc_{[0,T]} F_0 \le \|f\|_1, \tag{3.13}$$

which provide the less sharp but more explicit necessary condition for solvability of (3.1)-(3.9)

$$Osc_{[0,T]}F_0 < 2a,$$

and the less sharp versions of the first sufficient condition in (3.10)

$$Osc_{[0,T]}F_0 < a, \quad \text{or} \quad \|f\|_1 < a,$$

or, noticing that, for $f \in L^\infty(0,T)$, $\|f\|_1 \leq T\|f\|_\infty$,

$$\|f\|_\infty < \frac{a}{T}.$$

In Example 4, all those conditions reduce to $|\alpha| < 1$.

Remark 6. The existence of $\tau \in [0,T]$ such that

$$\int_0^T \phi^{-1}(F_\tau(s))\,ds = 0$$

i.e., such that

$$\int_0^T \phi^{-1}(F_0(s) - F_0(\tau))\,ds = 0$$

is equivalent to the existence of $c \in \text{Range } F_0$ such that

$$\int_0^T \phi^{-1}(F_0(s) - c)\,ds = 0,$$

and hence is guaranteed by Lemma 3 when $\|F_0\|_\infty < \frac{a}{2}$.

Finally, for the periodic boundary conditions

$$u(0) - u(T) = 0 = u'(0) - u'(T) \tag{3.14}$$

we have the following necessary and sufficient condition for the solvability of (3.1)-(3.14).

Proposition 4. *Problem* (3.1)-(3.14) *has a solution if and only if*

$$\int_0^T f(s)\,ds = 0 \tag{3.15}$$

and if there exists $\tau \in [0,T]$ *such that*

$$\|F_\tau\|_\infty < a \quad \textit{and} \quad \int_0^T \phi^{-1}(F_\tau(s))\,ds = 0, \tag{3.16}$$

in which case problem (3.1)-(3.14) *has the family of solutions*

$$u(t) = u(0) + \int_0^t \phi^{-1}(F_\tau(s))\,ds \quad (t \in [0,T]). \tag{3.17}$$

Proof. It is a combination of the ideas of the proofs of the Neumann and Dirichlet cases, and the details are left to the reader. □

Example 5. Consider the problem

$$\Big(\frac{u'}{\sqrt{1+u'^2}}\Big)' = \alpha w(t), \quad u(0)-u(T) = 0 = u'(0)-u'(T), \tag{3.18}$$

where $\alpha \in \mathbb{R}$ and w is defined in (2.4). Then for each $(t \in [0,T])$,

$$W_\tau(t) := \int_\tau^t w(s)\,ds = \Big|\tau - \frac{T}{2}\Big| - \Big|t - \frac{T}{2}\Big|.$$

Hence,

$$\begin{aligned}&\int_0^T \frac{\alpha W_\tau(s)}{\sqrt{1-\alpha^2 W_\tau^2(s)}}\,ds\\ = \;&\frac{2}{\alpha}\Big[\sqrt{1-\alpha^2\Big(\big|\tau - \frac{T}{2}\big| - \frac{T}{2}\Big)^2} - \sqrt{1-\alpha^2\big|\tau - \frac{T}{2}\big|^2}\Big] = 0,\end{aligned}$$

if and only if $\tau = \frac{T}{4}$ or $\tau = \frac{3T}{4}$. Now

$$\|W_{T/4}\|_\infty = \|W_{3T/4}\|_\infty = \frac{|\alpha|T}{4},$$

so that Problem (3.18) is solvable if and only $|\alpha| < \frac{4}{T}$.

4. Periodic problems with nonlinearities bounded from below or from above

In this section we are interested in periodic boundary-value problems of the type

$$(\phi(u'))' = f(t,u,u'), \quad u(0)-u(T) = 0 = u'(0)-u'(T), \tag{4.1}$$

where $\phi : \mathbb{R} \to (-a,a)$ is a homeomorphism, $\phi(0) = 0$ and $f : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ is continuous.

For $\lambda \in [0,1]$, consider the family of abstract periodic boundary-value problems

$$(\phi(u'))' = \lambda N_f(u) + (1-\lambda)QN_f(u), \;\; u(0)-u(T) = 0 = u'(0)-u'(T). \tag{4.2}$$

Notice that (4.2) coincide, for $\lambda = 1$, with (4.1).

Let

$$\Omega = \left\{(\lambda, u) \in [0,1] \times C^1_{per} : \|\lambda H(I-Q)N_f(u)\|_\infty < \frac{a}{2}\right\}. \tag{4.3}$$

It is clear that Ω is open in $[0,1]\times C^1_{per}$, and is nonempty because $\{0\}\times C^1_{per} \subset \Omega$. So, using Lemma 3, we can define on Ω the operator $\mathcal{M}$ by

$$\mathcal{M}(\lambda, u) = Pu + QN_f(u) + H \circ \phi^{-1} \circ (I - Q_\phi) \circ [\lambda H(I-Q)N_f](u). \tag{4.4}$$

Such an operator extends to the present situation of $\phi : \mathbb{R} \to (-a, a)$ the one introduced in [12] for $\phi : \mathbb{R} \to \mathbb{R}$.

Lemma 4. *The operator $\mathcal{M} : \Omega \to C^1_{per}$ is well defined and continuous and if $(\lambda, u) \in \Omega$ is such that $u = \mathcal{M}(\lambda, u)$, then u is a solution of* (4.2).

Proof. Let $(\lambda, u) \in \Omega$. It is clear that $\mathcal{M}(\lambda, u) \in C^1$. We show that in fact $\mathcal{M}(\lambda, u) \in C^1_{per}$. Using Remark 2, we deduce that

$$\begin{aligned}
\mathcal{M}(\lambda, u)(T) &= Pu + QN_f(u) + TQ \circ \phi^{-1} \circ (I - Q_\phi) \circ [\lambda H(I-Q)N_f](u) \\
&= Pu + QN_f(u) = \mathcal{M}(\lambda, u)(0).
\end{aligned}$$

On the other hand we have that

$$(\mathcal{M}(\lambda, u))' = \phi^{-1} \circ (I - Q_\phi) \circ [\lambda H(I-Q)N_f](u),$$

which implies that

$$(\mathcal{M}(\lambda, u))'(0) = \phi^{-1}(-Q_\phi(\lambda H(I-Q)N_f(u))) = (\mathcal{M}(\lambda, u))'(T).$$

Consequently $\mathcal{M}(\lambda, u) \in C^1_{per}$ and the operator $\mathcal{M} : \Omega \to C^1_{per}$ is well defined. Its continuity follows by the continuity of the operators which compose $\mathcal{M}$.

Now suppose that $(\lambda, u) \in \Omega$ is such that $u = \mathcal{M}(\lambda, u)$. It follows that

$$u - Pu - H \circ \phi^{-1} \circ (I - Q_\phi) \circ [\lambda H(I-Q)N_f](u) = QN_f(u),$$

which gives

$$u = Pu + H \circ \phi^{-1} \circ (I - Q_\phi) \circ [\lambda H(I-Q)N_f](u), \quad QN_f(u) = 0,$$

so that $u \in C^1_{per}$ and u is a solution for (4.2) by differentiating the first equation, applying ϕ to both of its members, differentiating again and using the second equation. $\square$

The following lemma, giving a priori bounds for the possible fixed points of $\mathcal{M}$, adapts a technique introduced by Ward [19].

Lemma 5. *Assume that f satisfies the following conditions.*

(1) *There exists* $c \in C$ *such that* $\|c^-\|_1 < \frac{a}{4}$ *and*

$$f(t,u,v) \geq c(t) \tag{4.5}$$

for all $(t,u,v) \in [0,T] \times \mathbb{R}^2$.

(2) *There exist* $R > 0$ *and* $\epsilon \in \{-1,1\}$ *such that*

$$\begin{aligned} \epsilon \int_0^T f(t,u(t),u'(t))\,dt \quad &> \quad 0 \quad \text{if} \quad u_L \geq R, \quad \|u'\|_\infty \leq M, \\ \epsilon \int_0^T f(t,u(t),u'(t))\,dt \quad &< \quad 0 \quad \text{if} \quad u_M \leq -R, \quad \|u'\|_\infty \leq M, \end{aligned} \tag{4.6}$$

where $M = \max\{|\phi^{-1}(2\|c^-\|_1)|, |\phi^{-1}(-2\|c^-\|_1)\}$.

If $(\lambda,u) \in \Omega$ *is such that* $u = \mathcal{M}(\lambda,u)$, *then* $\|\lambda H(I-Q)N_f(u)\|_\infty \leq 2\|c^-\|_1$ *and* $\|u\| < R + M(T+1)$.

Proof. Let $(\lambda,u) \in \Omega$ such that $u = \mathcal{M}(\lambda,u)$. Using Lemma 4 we have that u is a solution of (4.2), which implies that

$$QN_f(u) = 0, \tag{4.7}$$

and

$$(\phi(u'))' = \lambda N_f(u), \quad u(0) - u(T) = 0 = u'(0) - u'(T). \tag{4.8}$$

Using the fact that f is bounded from below by c, we deduce the elementary inequality

$$|f(t,u,v)| \leq f(t,u,v) + 2c^-(t) \quad \text{for all} \quad (t,u,v) \in [0,T] \times \mathbb{R}^2. \tag{4.9}$$

From (4.7), (4.8) and (4.9) it follows that

$$\|(\phi(u'))'\|_1 = \lambda \|N_f(u)\|_1 \leq \int_0^T N_f(u)(s)\,ds + 2\|c^-\|_1 = 2\|c^-\|_1. \tag{4.10}$$

Using (4.10) and (2.6), it follows that

$$\|\lambda H(I-Q)N_f(u)\|_\infty \leq 2\|c^-\|_1 < \frac{a}{2}.$$

Because $u \in C^1$ is such that $u(0) = u(T)$, there exists $\xi \in [0,T]$ such that $u'(\xi) = 0$, which implies $\phi(u'(\xi)) = 0$ and

$$\phi(u'(t)) = \int_\xi^t (\phi(u'(s)))'\,ds \quad (t \in [0,T]).$$

Using the equality above and (4.10) we have that

$$|\phi(u'(t))| \leq 2\|c^-\|_1 \quad (t \in [0,T]), \tag{4.11}$$

and hence

$$\|u'\|_\infty \leq M. \tag{4.12}$$

If $u_M \leq -R$ (respectively $u_L \geq R$) then, from (4.12) and (4.6), it follows that

$$\epsilon \int_0^T f(t, u(t), u'(t))\, dt < 0 \quad \text{(respectively} \quad \epsilon \int_0^T f(t, u(t), u'(t))\, dt > 0).$$

Using (4.7) we have that

$$u_M > -R \text{ and } u_L < R. \tag{4.13}$$

It is clear that

$$u_M \leq u_L + \int_0^T |u'(t)|\, dt. \tag{4.14}$$

From relations (4.12), (4.13) and (4.14), we obtain that

$$-(R + MT) < u_L \leq u_M < R + MT. \tag{4.15}$$

Using (4.12) and (4.15) it follows that $\|u\| < R + M(T+1)$. □

Remark 7. When $c^+ = 0$, inequality (4.9) with $c \leq 0$ is indeed equivalent to inequality (4.5), because (4.9) is equivalent to

$$f^-(t, u, v) \leq c^-(t) = -c(t) \quad ((t, u, v) \in [0, T] \times \mathbb{R}^2),$$

and hence to

$$f(t, u, v) \geq -f^-(t, u, v) \geq c(t) \quad ((t, u, v) \in [0, T] \times \mathbb{R}^2).$$

Let $K, \rho \in \mathbb{R}$ be such that $2\|c^-\|_1 < K < \frac{a}{2}$, $\rho > R + M(T+1)$ and consider the set

$$V = \{(\lambda, u) \in [0,1] \times C^1_{per} : \|\lambda H(I - Q)N_f(u)\|_\infty < K, \|u\| < \rho\}.$$

V is nonempty because $V \cap \{0\} \times C^1_{per} = \{0\} \times \{u \in C^1_{per} : \|u\| < \rho\}$. On the other hand it is clear that V is open and bounded in $[0,1] \times C^1_{per}$ and $\overline{V} \subset \Omega$. Using Lemma 4 and Lemma 5 we deduce that the operator $\mathcal{M} : \overline{V} \to C^1_{per}$ is well defined, continuous and

$$u \neq \mathcal{M}(\lambda, u) \quad \text{for all} \quad (\lambda, u) \in \partial V. \tag{4.16}$$

Lemma 6. *The operator $\mathcal{M} : \overline{V} \to C^1_{per}$ is completely continuous.*

Proof. Let $(\lambda_n, u_n)_n \subset \overline{V}$. We may assume that $\lambda_n \to \lambda_0$. Let $v_n = \mathcal{M}(\lambda_n, u_n)$ $(n \in \mathbb{N})$. Because $\|\lambda_n H(I-Q)N_f(u_n)\|_\infty \le K < \frac{a}{2}$ for all $n \in \mathbb{N}$, it follows, by Lemma 3, that

$$\|(I - Q_\phi) \circ [\lambda_n H(I-Q)N_f](u_n)\|_\infty \le 2K < a \quad (n \in \mathbb{N}). \tag{4.17}$$

Using (4.17) we deduce that, for all $n \in \mathbb{N}$,

$$\begin{aligned} &\|\phi^{-1} \circ (I - Q_\phi) \circ [\lambda_n H(I-Q)N_f](u_n)\|_\infty \\ &\qquad \le \max\{|\phi^{-1}(-2K)|, |\phi^{-1}(2K)|\} = M_1, \end{aligned} \tag{4.18}$$

which implies that $(v_n)_n$ is bounded in C. Let $t_1, t_2 \in [0,T]$. Then, using (4.18), we have, for all $n \in \mathbb{N}$

$$\begin{aligned} |v_n(t_1) - v_n(t_2)| &= \Big| \int_{t_1}^{t_2} \phi^{-1} \circ (I - Q_\phi) \circ [\lambda_n H(I-Q)N_f](u_n)(s)\, ds \Big| \\ &\le M_1 |t_1 - t_2|, \end{aligned}$$

which implies that $(v_n)_n$ is equicontinuous. Applying the Arzela-Ascoli theorem, passing if necessary to a subsequence, we may assume that $v_n \to v$ in C. On the other hand, for all $n \in \mathbb{N}$, we have that

$$v_n' = \phi^{-1} \circ (I - Q_\phi) \circ [\lambda_n H(I-Q)N_f](u_n) \tag{4.19}$$

Using (4.18) and (4.19) it follows that $\|v_n'\|_\infty \le M_1$ for all $n \in \mathbb{N}$. Furthermore, if $t_1, t_2 \in [0,T]$, then

$$|\phi(v_n'(t_1)) - \phi(v_n'(t_2))| = \Big| \int_{t_1}^{t_2} (I-Q)N_f(u_n)(s)\, ds \Big|,$$

so, using the relations (4.17), (4.19) and the uniform continuity of ϕ^{-1} on $[-2K, 2K]$, it follows that $(v_n')_n$ is equicontinuous. Applying the Arzela-Ascoli theorem, we may assume, passing to a subsequence, that $v_n' \to w$ in C. It follows that $v \in C^1_{per}$, $v' = w$, so that $v_n \to v$ in C^1. $\square$

Theorem 1. *Let f be continuous and satisfy condition* (1) *and* (2) *of Lemma* 5. *Then* (4.1) *has at least one solution.*

Proof. Let $\mathcal{M}$ be the operator given by (4.4). Using Lemma 6, relation (4.16) and Proposition 1, we deduce that

$$d_{LS}[I - \mathcal{M}(0,\cdot), V_0, 0] = d_{LS}[I - \mathcal{M}(1,\cdot), V_1, 0]. \tag{4.20}$$

On the other hand, we have that

$$d_{LS}[I - \mathcal{M}(0,\cdot), V_0, 0] = d_{LS}[I - (P + QN_f), B_\rho, 0], \tag{4.21}$$

where $B_\rho = \{u \in C^1_{per} : \|u\| < \rho\}$. But the range of $P + QN_f$ is contained in the subset of constant functions, isomorphic to $\mathbb{R}$, so, using a property of the Leray-Schauder degree we have that

$$d_{LS}[I - (P + QN_f), B_\rho, 0] = d_B[I - (P + QN_f)|_{\mathbb{R}}, (-\rho, \rho), 0]$$
$$= d_B[-QN_f, (-\rho, \rho), 0] = \frac{sign(-QN_f(\rho)) - sign(-QN_f(-\rho))}{2},$$

where d_B denotes the Brouwer degree. But, using (4.6) and the fact that $\rho > R$, we see that $QN_f(\pm\rho) = \frac{1}{T}\int_0^T f(t, \pm\rho, 0)\,dt$ have opposite signs, which implies, using the relations (4.20) and (4.21) that

$$|d_{LS}[I - \mathcal{M}(1, \cdot), V_1, 0]| = 1.$$

Then, from the existence property of the Leray-Schauder degree, the set V_1 is nonempty and there is $u \in V_1$ such that $u = \mathcal{M}(1, u)$, which is a solution for (4.1) by Lemma 4. □

Remark 8. The conclusion of Theorem 1 also holds if $f(t, u, v) \le c(t)$ for some $c \in C$ such that $\|c^+\|_1 < \frac{a}{4}$ and all $(t, u, v) \in [0, T] \times \mathbb{R}^2$, and f satisfies the sign condition (4.6). It suffices to replace ϕ by $-\phi$, f by $-f$ and to apply Theorem 1.

Remark 9. The conclusions of Theorem 1 and Remark 8 still hold if the sign condition (4.6) is weakened into

(2') There exist $R > 0$ and $\epsilon \in \{-1, 1\}$ such that

$$\begin{aligned} \epsilon\int_0^T f(t, u(t), u'(t))\,dt &\ge 0 \quad \text{if} \quad u_L \ge R, \quad \|u'\|_\infty \le M, \\ \epsilon\int_0^T f(t, u(t), u'(t))\,dt &\le 0 \quad \text{if} \quad u_M \le -R, \quad \|u'\|_\infty \le M, \end{aligned} \tag{4.22}$$

where $M > \max\{|\phi^{-1}(2\|c^-\|_1)|, |\phi^{-1}(-2\|c^-\|_1)|\}$.

Letting $f_n(t, u, v) = f(t, u, v) + \frac{\epsilon}{n}\frac{u}{\sqrt{1+u^2}}$, it is easy to see that, for $n \ge N$ with N sufficiently large, the problems

$$(\phi(u'))' = f_n(t, u, u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T) \tag{4.23}$$

satisfy the conditions of Theorem 1 for c replaced by a suitable c_N with $\|c_N^-\|_1 < \frac{a}{4}$ and M by a suitable M_N. Consequently, each problem (4.23) for $n \ge N$ admits at least one solution u_n satisfying suitable a priori bounds which allow us, as in the proof of Lemma 6, to extract a convergent subsequence whose limit is a solution of (4.1).

Notice that, for $f = f(t)$, condition (4.22) reduces to

$$\int_0^T f(t)\,dt = 0,$$

which is necessary for the solvability of

$$(\phi(u'))' = f(t), \quad u(0) - u(T) = 0 = u'(0) - u'(T).$$

Example 6. Using Theorem 1 and Remark 8, we see that the periodic boundary value problems

$$\Big(\frac{u'}{\sqrt{1+u'^2}}\Big)' - a(t)\exp u + h(t) = 0, \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

$$\Big(\frac{u'}{\sqrt{1+u'^2}}\Big)' + a(t)\exp u - h(t) = 0, \quad u(0) - u(T) = 0 = u'(0) - u'(T),$$

have at least one solution if $a \in C$ is positive and $h \in C$ is such that

$$\|h^-\|_1 < \|h^+\|_1 < \frac{1}{4}.$$

This is in particular the case if $0 < h_L \le h_M < \frac{1}{4T}$.

5. Neumann problems with nonlinearities bounded from below or from above

In this section we are interested in Neumann boundary-value problems of the type

$$(\phi(u'))' = f(t, u, u'), \quad u'(0) = 0 = u'(T), \tag{5.1}$$

where $\phi : \mathbb{R} \to (-a, a)$ is a homeomorphism, $\phi(0) = 0$ and $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. For $\lambda \in [0, 1]$ consider the family of abstract Neumann boundary-value problems

$$(\phi(u'))' = \lambda N_f(u) + (1 - \lambda)QN_f(u), \quad u'(0) = 0 = u'(T). \tag{5.2}$$

Notice that (5.2) coincides, for $\lambda = 1$, with (5.1). Let

$$\Omega = \{(\lambda, u) \in [0, 1] \times C^1_\# : \|\lambda H(I - Q)N_f(u)\|_\infty < a\}.$$

It is clear that Ω is open, and nonempty because $\{0\} \times C^1_\# \subset \Omega$. We consider the operator $\mathcal{M}$ given for $(\lambda, u) \in \Omega$ by

$$\mathcal{M}(\lambda, u) = Pu + QN_f(u) + H \circ \phi^{-1} \circ [\lambda H(I - Q)N_f](u). \tag{5.3}$$

When $\phi : \mathbb{R} \to \mathbb{R}$, such an operator has been considered by Garcia-Huidobro, Manásevich and Zanolin [5].

Lemma 7. *The operator $\mathcal{M} : \Omega \to C^1_{\#}$ is well defined, continuous and if $(\lambda, u) \in \Omega$ is such that $u = \mathcal{M}(\lambda, u)$, then u is a solution of* (5.2).

Proof. Let $(\lambda, u) \in \Omega$. It is clear that $\mathcal{M}(\lambda, u) \in C^1$ and

$$(\mathcal{M}(\lambda, u))' = \phi^{-1} \circ [\lambda H(I - Q)N_f](u).$$

Using the relations

$$H(I - Q)N_f(u)(0) = 0 = H(I - Q)N_f(u)(T), \quad \phi^{-1}(0) = 0,$$

we deduce that

$$(\mathcal{M}(\lambda, u))'(0) = 0 = (\mathcal{M}(\lambda, u))'(T).$$

So, we have $\mathcal{M}(\Omega) \subset C^1_{\#}$, which implies that the operator $\mathcal{M} : \Omega \to C^1_{\#}$ is well defined. The continuity of $\mathcal{M}$ follows from the continuity of the operators P, Q, H and N_f. Now, consider $(\lambda, u) \in \Omega$ such that $u = \mathcal{M}(\lambda, u)$. It follows that

$$u - Pu - H \circ \phi^{-1} \circ [\lambda H(I - Q)N_f](u) = QN_f(u),$$

which gives

$$u = Pu + H \circ \phi^{-1} \circ [\lambda H(I - Q)N_f](u), \quad QN_f(u) = 0,$$

so that $u \in C^1_{\#}$ and u is a solution for (5.2) by differentiating the first equation, applying ϕ to both members, differentiating again and using the second equation. □

In the next lemma, as in Lemma 5, we extend some techniques of Ward [19] to obtain the required a priori bounds.

Lemma 8. *Assume that f satisfies the conditions*

(1) *There exists $c \in C$ such that $\|c^-\|_1 < \frac{a}{2}$ and $f(t, u, v) \geq c(t)$ for all $(t, u, v) \in [0, T] \times \mathbb{R}^2$.*
(2) *There exist $R > 0$ and $\epsilon \in \{-1, 1\}$ such that*

$$\begin{aligned} \epsilon \int_0^T f(t, u(t), u'(t))\, dt &> 0 \quad if \quad u_L \geq R, \quad \|u'\|_\infty \leq M, \\ \epsilon \int_0^T f(t, u(t), u'(t))\, dt &< 0 \quad if \quad u_M \leq -R, \quad \|u'\|_\infty \leq M, \end{aligned} \tag{5.4}$$

where $M = \max\{|\phi^{-1}(2\|c^-\|_1)|, |\phi^{-1}(-2\|c^-\|_1)|\}$.

If $(\lambda, u) \in \Omega$ is such that $u = \mathcal{M}(\lambda, u)$, then $\|\lambda H(I - Q)N_f(u)\|_\infty \leq 2\|c^-\|_1$ and $\|u\| < R + M(T + 1)$.

Proof. Let $(\lambda, u) \in \Omega$ be such that $u = \mathcal{M}(\lambda, u)$. Using Lemma 7, u is a solution of (5.2), which implies that

$$QN_f(u) = 0. \tag{5.5}$$

Hence,

$$(\phi(u'))' = \lambda N_f(u), \quad u'(0) = 0 = u'(T),$$

which implies that

$$\phi(u') = \lambda H N_f(u) = \lambda H(I - Q)N_f(u). \tag{5.6}$$

On the other hand, as in the proof of Lemma 5, using the fact that the function f is bounded from below by the function c, we deduce that

$$|f(t, u, v)| \leq f(t, u, v) + 2c^-(t), \quad ((t, u, v) \in [0, T] \times \mathbb{R}^2). \tag{5.7}$$

Using (5.5), (5.6), (5.7) and (2.6), it follows that

$$\begin{aligned} \|\phi(u')\|_\infty &= \|\lambda H(I - Q)N_f(u)\|_\infty \leq \|N_f(u)\|_1 \\ &\leq \int_0^T N_f(u)(s)\, ds + 2\|c^-\|_1 = 2\|c^-\|_1 < a, \end{aligned}$$

which implies $\|u'\|_\infty \leq M$. The end of the proof is then entirely similar to that of Lemma 5. □

Let $K, \rho \in \mathbb{R}$ be such that $2\|c^-\|_1 < K < a$ and $\rho > R + M(T + 1)$. Consider the set

$$V = \{(\lambda, u) \in [0, 1] \times C^1_\# : \|\lambda H(I - Q)N_f(u)\|_\infty < K, \|u\| < \rho\}.$$

It is clear that V is open and bounded in $[0, 1] \times C^1_\#$, and is nonempty, because

$$\{0\} \times \{u \in C^1_\# : \|u\| < \rho\} \subset V.$$

On the other hand $\overline{V} \subset \Omega$, so, we can consider the operator $\mathcal{M}$ on $\overline{V}$. Using Lemma 7 and Lemma 8 we have

$$u \neq \mathcal{M}(\lambda, u) \quad \text{for all} \quad (\lambda, u) \in \partial V,$$

and if $(\lambda, u) \in V$ is such that $u = \mathcal{M}(\lambda, u)$, then u is a solution of (5.2). Using the same arguments as in the proof of Lemma 6 we show that the operator $\mathcal{M}$ is completely continuous on $\overline{V}$. So, with a proof similar to that of Theorem 1, we obtain the following existence result.

Theorem 2. *Let f be continuous and satisfy conditions* (1) *and* (2) *of Lemma* 8. *Then* (5.1) *has at least one solution.*

Remark 10. The conclusion of Theorem 2 also holds if $f(t,u,v) \le c(t)$ for some $c \in C$ such that $\|c^+\|_1 < \frac{a}{2}$ and all $(t,u,v) \in [0,T] \times \mathbb{R}^2$, and f satisfies the sign condition (5.4). It suffices to replace ϕ by $-\phi$, f by $-f$ and to apply Theorem 2.

Remark 11. One can weaken the sign condition (5.4) in a similar way as in Remark 9.

Example 7. Using Theorem 2 we obtain that the Neumann boundary-value problems

$$\Big(\frac{u'}{\sqrt{1+u'^2}}\Big)' - a(t)\exp u + h(t) = 0, \quad u'(0) = 0 = u'(T),$$
$$\Big(\frac{u'}{\sqrt{1+u'^2}}\Big)' + a(t)\exp u - h(t) = 0, \quad u'(0) = 0 = u'(T),$$

have at least one solution if $a \in C$ is positive and $h \in C$ is such that

$$\|h^-\|_1 < \|h^+\|_1 < \frac{1}{2},$$

which is in particular the case if $0 < [h]_L \le [h]_M < \frac{1}{2T}$.

6. Periodic or Neumann problems with bounded nonlinearities

Let $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy the condition

$$|f(t,u,v)| \le c < \frac{a}{T} \quad \text{for all} \quad (t,u,v) \in [0,T] \times \mathbb{R}^2. \tag{6.1}$$

Using Lemma 1 and (6.1), it follows that

$$\|\lambda H(I-Q)N_f(u)\|_\infty < \frac{a}{2} \quad \text{for all} \quad (\lambda,u) \in [0,1] \times C^1_{per}. \tag{6.2}$$

Using Lemma 3 and (6.2), we see that the operator $\mathcal{M}$ given by (4.4) is well defined and continuous on $[0,1] \times C^1_{per}$. As in Lemma 4 we show that if $(\lambda,u) \in [0,1] \times C^1_{per}$ is such that $u = \mathcal{M}(\lambda,u)$, then u is a solution of (4.2), and as in Lemma 6 we show that the operator $\mathcal{M}$ is completely continuous on $[0,1] \times C^1_{per}$. On the other hand, we obtain a priori estimates for the possible fixed points of $\mathcal{M}(\lambda,\cdot)$.

Lemma 9. *If condition* (6.1) *holds and if there exists* $R > 0$ *and* $\epsilon \in \{-1,1\}$ *such that, with*

$$M = \max\{|\phi^{-1}(-cT)|, |\phi^{-1}(cT)|\},$$

one has

$$\epsilon \int_0^T f(t,u(t),u'(t))\,dt > 0 \quad \text{if} \quad u_L \geq R, \quad |u'|_\infty \leq M,$$
$$\epsilon \int_0^T f(t,u(t),u'(t))\,dt < 0 \quad \text{if} \quad u_M \leq R, \quad |u'|_\infty \leq M, \tag{6.3}$$

then there is a constant $\rho > R$ such that for each $\lambda \in [0,1]$, each possible fixed point u of $\mathcal{M}(\lambda,\cdot)$ verifies the inequality $\|u\| < \rho$.

Proof. Let $(\lambda,u) \in [0,1] \times C^1_{per}$ such that $u = \mathcal{M}(\lambda,u)$. Then, u is a solution of (4.1), and, as in the proof of Lemma 5 we have (4.8). From (4.8) and (6.1), it follows that

$$\int_0^T |(\phi(u'(t)))'|\,dt \leq cT. \tag{6.4}$$

To finish the proof it suffices to use the same arguments as in the proof of Lemma 5. □

Using now the classical homotopy invariance property of the Leray-Schauder degree, we obtain, as in Theorem 1, the following existence result.

Theorem 3. *Let $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy conditions* (6.1) *and* (6.3). *Then* (4.1) *has at least one solution.*

For the Neumann problem, let $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy the condition

$$|f(t,u,v)| \leq c < \frac{2a}{T} \quad \text{for all} \quad (t,u,v) \in [0,T] \times \mathbb{R}^2. \tag{6.5}$$

Using Lemma 1 and (6.5), it follows that

$$\|\lambda H(I-Q)N_f(u)\|_\infty < a \quad \text{for all} \quad (\lambda,u) \in [0,1] \times C^1_{per}. \tag{6.6}$$

Hence, the operator $\mathcal{M}$ given by (5.3) is well defined and continuous on $[0,1] \times C^1_{per}$. As in Lemma 7 we show that if $(\lambda,u) \in [0,1] \times C^1_{per}$ is such that $u = \mathcal{M}(\lambda,u)$, then u is a solution of (5.2), and as in Lemma 6 we show that the operator $\mathcal{M}$ is completely continuous on $[0,1] \times C^1_{per}$. On the other hand, using arguments similar to those of Lemma 8 we obtain the analogue of Lemma 9 with condition (6.1) replaced by condition (6.5). Again the classical invariance property of Leray-Schauder degree implies the following result.

Theorem 4. *Let $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy conditions* (6.5) *and* (6.3). *Then* (5.1) *has at least one solution.*

Remark 12. Theorem 4 improves Theorem 1 in [2] by weakening both the bound and the sign conditions upon f.

Remark 13. For f bounded, condition (6.1) is better than the condition

$$|f(t,u,v)| \le c < \frac{a}{4T}$$

given by Theorem 1 or Remark 8 in the periodic case, and condition (6.5) is better than the condition

$$|f(t,u,v)| \le c < \frac{a}{2T}$$

given by Theorem 2 or Remark 10 in the Neumann case.

Remark 14. In Theorems 3 and 4, one can weaken the sign condition (6.3) in a similar way as in Remark 9.

Example 8. Using respectively Theorem 3 and Theorem 4, we obtain that the periodic boundary-value problem

$$\Big(\frac{u'}{\sqrt{1+u'^2}}\Big)' = \alpha(\arctan u + \sin t), \quad u(0)-u(1) = 0 = u'(0)-u'(1),$$

has at least one solution if $|\alpha| \le 0.4145$, and the Neumann boundary-value problem

$$\Big(\frac{u'}{\sqrt{1+u'^2}}\Big)' = \alpha(\arctan u + \sin t), \quad u'(0) = 0 = u'(1),$$

has at least one solution if $|\alpha| \le 0.8290$.

Example 9. Using respectively Theorem 3 and Theorem 4, we obtain that the periodic boundary-value problem

$$\begin{aligned} \Big(\frac{u'}{\sqrt{1+u'^2}}\Big)' &= \frac{1}{4}\arctan(u+t) + \frac{1}{3}\sin(u'+t^2), \\ u(0)-u(1) &= 0 = u'(0)-u'(1), \end{aligned}$$

and the Neumann boundary-value problem

$$\begin{aligned} \Big(\frac{u'}{\sqrt{1+u'^2}}\Big)' &= \frac{1}{2}\arctan(u+t) + \frac{2}{3}\sin(u'+t^2), \\ u'(0) &= 0 = u'(1), \end{aligned}$$

have at least one nonconstant solution.

7. Dirichlet problems with bounded nonlinearities

We finally consider Dirichlet problems of the form

$$(\phi(u'))' = f(t, u, u'), \quad u(0) = u(T) = 0 \tag{7.1}$$

where $\phi : \mathbb{R} \to (-a, a)$ is a homeomorphism such that $\phi(0) = 0$, $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and such that

$$|f(t, u, v)| \leq c < \frac{a}{2T} \quad \text{for all} \quad (t, u, v) \in [0, T] \times \mathbb{R}^2. \tag{7.2}$$

For $\lambda \in [0, 1]$, consider the family of Dirichlet problems

$$(\phi(u'))' = \lambda f(t, u, u'), \quad u(0) = u(T) = 0, \tag{7.3}$$

which reduces to (7.1) for $\lambda = 1$ and to

$$(\phi(u'))' = 0, \quad u(0) = u(T) = 0 \tag{7.4}$$

for $\lambda = 0$. This last problem has only the trivial solution.

Notice that if $u \in C_0^1$, it follows from (7.2) that for all $t \in [0, T]$, one has

$$|H \circ N_f(u)(t)| = \left| \int_0^t f(s, u(s), u'(s))\, ds \right| \leq cT < \frac{a}{2}, \tag{7.5}$$

and hence, using Lemma 3, $Q_\phi \circ H \circ N_f$ is well defined on C_0^1. Consequently, the nonlinear operator $\mathcal{M}$ given by

$$\mathcal{M}(\lambda, u) = H \circ \phi^{-1} \circ (I - Q_\phi) \circ H \circ (\lambda N_f)(u) \tag{7.6}$$

is well defined on $[0, 1] \times C_0^1$. When $\phi : \mathbb{R} \to \mathbb{R}$, such an operator has been considered by Garcia-Huidobro, Manásevich and Zanolin [6] and by Huang and Metzen [9].

Lemma 10. *The operator $\mathcal{M} : [0, 1] \times C_0^1 \to C_0^1$ is completely continuous and if $(\lambda, u) \in [0, 1] \times C_0^1$ is such that $u = \mathcal{M}(\lambda, u)$, then u is a solution of* (7.3).

Proof. The continuity follows from the continuity of the composing operators, and the proof of the complete continuity is entirely similar to that of Lemma 6. Let $(\lambda, u) \in \Omega$. It is clear that $\mathcal{M}(\lambda, u) \in C^1$. We show that in fact $\mathcal{M}(\lambda, u) \in C_0^1$. Using Remark 2, we deduce that

$$\begin{aligned} \mathcal{M}(\lambda, u)(T) &= TQ \circ \phi^{-1} \circ (I - Q_\phi) \circ H \circ [\lambda N_f(u)](s) \\ &= 0 = \mathcal{M}(\lambda, u)(0). \end{aligned}$$

Consequently $\mathcal{M}(\lambda, u) \in C_0^1$. Now suppose that $(\lambda, u) \in [0, 1] \times C_0^1$ is such that $u = \mathcal{M}(\lambda, u)$. It follows that

$$u' = \phi^{-1} \circ (I - Q_\phi) \circ H \circ (\lambda N_f)(u)$$

$$\phi(u') = (I - Q_\phi) \circ H \circ (\lambda N_f)(u)$$
$$(\phi(u'))' = \lambda f(t, u, u'). \qquad \square$$

Theorem 5. *Let $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy conditions* (7.2). *Then* (7.1) *has at least one solution.*

Proof. Let $\lambda \in [0,1]$ and u be a possible fixed point of $\mathcal{M}(\lambda, \cdot)$. Then, using Lemma 3,

$$\|\phi(u')\|_\infty = \|(I - Q_\phi) \circ H \circ (\lambda N_f)(u)\|_\infty \leq cT + cT < a.$$

Consequently,

$$\|u'\|_\infty < \max\{|\phi^{-1}(-2cT)|, |\phi^{-1}(2cT)|\} := M$$

and

$$\|u\|_\infty = \left\| \int_0^\cdot u'(s)\, ds \right\|_\infty < TM.$$

Therefore, if $\Omega = \{u \in C_0^1 : \|u\|_\infty < TM, \quad \|u'\|_\infty < M\}$, it follows from the homotopy invariance of Leray-Schauder degree that $d_{LS}[I - \mathcal{M}(\cdot, \lambda), \Omega, 0]$ is independent of $\lambda \in [0,1]$ so that, if we notice that $\mathcal{M}(0, \cdot) = 0$,

$$d_{LS}[I - \mathcal{M}(\cdot, 1), \Omega, 0] = d_{LS}[I - \mathcal{M}(\cdot, 0), \Omega, 0] = 1.$$

Hence, $\mathcal{M}(1, \cdot)$ has a fixed point u, which is a solution of (7.1) by Lemma 10. $\square$

Example 10. If follows from Theorem 5 that the Dirichlet problem

$$\left(\frac{u'}{\sqrt{1 + u'^2}}\right)' = \alpha(\sin u + \cos t), \quad u(0) = 0 = u(\pi/2)$$

has at least one solution if $|\alpha| < \frac{1}{2\pi}$.

Remark 15. In contrast to the periodic and Neumann cases, the solvability of the Dirichlet problem with bounded right-hand side f does not require any sign condition upon f. This is related to the absence of a necessary condition like (1.6) for the solvability of the simple Dirichlet problem

$$(\phi(u'))' = f(t), \quad u(0) = 0 = u(T).$$

On the other hand, the approach used in Sections 4 and 5 to study periodic or Neumann problems with one-sided nonlinearities does not work in the Dirichlet case.

References

[1] P. Amster and M.C. Mariani, *The prescribed mean curvature equation for nonparametric surfaces*, Nonlinear Anal., 52 (2003), 1069–1077.

[2] C. Bereanu and J. Mawhin, *Nonlinear Neumann boundary value problems with $\phi -$ Laplacian operators*, Analele Universitaii Ovidius Constanta, Seria Matematica, special issue dedicated to D. Pascali, 2005, to appear.

[3] Ph. Clément, R. Manásevich, and E. Mitidieri, *On a modified capillary equation*, J. Differential Equations, 124 (1996), 343–358.

[4] C.V. Coffman and W.K. Ziemer, *A prescribed mean curvature problem on domains without radial symmetry*, SIAM J. Math. Anal., 22 (1991), 982–990.

[5] M. García-Huidobro, R. Manásevich, and F. Zanolin, *Strongly nonlinear second-order ODE's with unilateral conditions*, Differential Integral Equations, 6 (1993), 1057–1078.

[6] M. García-Huidobro, R. Manásevich, and F. Zanolin, *A Fredholm-like result for strongly nonlinear second order ODE's*, J. Differential Equations, 114 (1994), 132–167.

[7] E. Giusti, *Boundary value problems for nonparametric surfaces of prescribed mean curvature*, Ann. Scuola Norm. Sup. Pisa, 3 (1976), 501–548.

[8] P. Habets and P. Omari, *Positive solutions of an indefinite prescribed mean curvature problem on a general domain*, Advanced Nonlinear Studies, 4 (2004), 1–14.

[9] Y.X. Huang and G. Metzen, *The existence of solutions to a class of semilinear equations*, Differential Integral Equations, 8 (1995), 429–452.

[10] T. Kusahara and H. Usami, *A barrier method for quasilinear differential equations of the curvature type,* Czechoslovak Math. J., 50 (125) (2000), 185–196.

[11] J. Leray and J. Schauder, *Topologie et équations fonctionnelles*, Ann. Ec. Norm. Sup., 51 (1934), 45–78.

[12] R. Manásevich and J. Mawhin, *Periodic solutions for nonlinear systems with p-Laplacian-like operators*, J. Differential Equations, 145 (1998), 367–393.

[13] R. Manásevich and J. Mawhin, *Boundary value problems for nonlinear perturbations of vector p-Laplacian-like operators*, J. Korean Math. Soc., 37 (2000), 665–685.

[14] J. Mawhin, *Leray-Schauder degree: A half century of extensions and applications*, Topological Methods Nonlinear Anal., 14 (1999), 195–228.

[15] J. Mawhin, "Topological Degree Methods in Nonlinear Boundary Value Problems," CBMS Series No.40, AMS, Providence RI, 1979.

[16] M. Nakao, *A bifurcation problem for a quasi-linear elliptic boundary value problem*, Nonlinear Anal., 8 (1990), 251–262.

[17] E.S. Noussair, Ch.A. Swanson, and Jianfu Yang, *A barrier method for mean curvature problems*, Nonlinear Anal., 21 (1993), 631–641.

[18] J. Serrin, *Positive solutions of a prescribed mean curvature problem*, in Calculus of Variations and Partial Differential Equations, S. Hildebrandt, D. Kinderlehrer and M. Miranda eds., Lecture Notes in Math. No. 1340, Springer, Berlin, 1988, 248–255.

[19] J.R. Ward Jr, *Asymptotic conditions for periodic solutions of ordinary differential equations*, Proc. Amer. Math. Soc., 81 (1981), 415–420.