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FUNCTIONAL COMMUTANT LIFTING AND INTERPOLATION ON GENERALIZED ANALYTIC POLYHEDRA

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Abstract. We show that for every generalized analytic polyhedron $D \subset \mathbb{C}^n$ there exists a suitable functional Hilbert space over D, allowing to apply the commutant lifting technique to various interpolation problems of Carathéodory-Féjér type. The existence of those solutions that belong to the Schur class of fractional transforms ($\subset$ in the closed unit ball of $H^\infty(D)$) is thus characterized in terms of positivity conditions. Also, we show on a concrete example how to obtain such solutions.

1. Preliminaries

Introduction This work is concerned with applications and extensions of a functional commutant lifting result obtained in [4]. We remind a few topics with this respect. As it is known, the commutant lifting theorem for contractions can be used, by a Sarason's idea [32], to solve various interpolation problems for bounded analytic functions on the unit disc $\mathbb{D} \subset \mathbb{C}$. Namely, consider a contraction T of class $C_{\cdot 0}$ on a Hilbert space. Then T is unitarily equivalent to the compression $P_M T_z|_M$ of the operator $T_z \in B(H^2(\mathbb{D}, E))$ of multiplication by the variable z to a $*$-invariant closed linear subspace M of $H^2(\mathbb{D}, E)$ [34]. Here $H^2(\mathbb{D}, E) \equiv H^2(\mathbb{D}) \otimes E$ is an E-valued Hardy space over the unit disk $\mathbb{D}$, where E is a Hilbert space. Then a functional version of the commutant lifting theorem [23], suitable for applications to interpolation problems, says that each operator $X : M \to M$ in the commutant of the compression $T = P_M T_z|M$ dilates to an operator from the commutant of T_z (that is, to a multiplication operator T_f

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given by an operator-valued bounded analytic function $f : \mathbb{D} \to B(E))$ such that $\|T_f\| = \|X\|$. Since $\|T_f\| = \|f\|_\infty$ $(:= \sup_{z\in D} \|f(z)\|)$, we have $\|f\|_\infty = \|X\|$. Moreover, $T_f^*|_M = X^*$ or, equivalently, $XP_M = P_M T_f$.

When seeking for bounded analytic functions $f : D \to B(E)$ of controlled sup-norm $\|f\|_\infty \le 1$ and satisfying certain linear interpolation conditions, a commutant lifting setting usually can be associated to the interpolation data, so that the symbol f of the operator T_f from above is a solution of the interpolation problem iff $\|X\| \le 1$. The simplest example in this sense is the Nevanlinna-Pick problem on the unit disc: given $z_j \in \mathbb{D}$ and $w_j \in \mathbb{C}$ for $j = \overline{1,m}$, one asks to study the existence of an analytic function f on $\mathbb{D}$ with $\|f\|_\infty \le 1$ such that $f(z_j) = w_j$ for all j. Consider then the Hardy space $H^2(\mathbb{D}) \subset L^2(\partial\mathbb{D})$, endowed with the reproducing kernel $C(z, \overline{w}) = (1 - z\overline{w})^{-1}$ for z, $w \in \mathbb{D}$. The functions $C_w \in H^2(\mathbb{D})$ defined by $C_w(z) = C(z, \overline{w})$ satisfy the *reproducing kernel property*: $h(w) = \langle h, C_w\rangle$ for every $h \in H^2(\mathbb{D})$ and $w \in \mathbb{D}$, namely we have $h(w) = \frac{1}{2\pi}\int_0^{2\pi} h(\mathrm{e}^{it})\frac{1}{1-w\mathrm{e}^{-it}}dt$ via Cauchy's formula. Then we let $M = sp_{j=1}^m C_{z_j}$ be the linear span of the reproducing kernels $C_{z_j}(\,\cdot\,)$ and define $X : M \to M$ by mean of its adjoint, by $X^*C_{z_j} = \overline{w}_j C_{z_j}$ for $j = \overline{1,m}$. Whenever f is a solution of the problem, by the reproducing kernel property we have $T_f^* C_{z_j} = \overline{f(z_j)}C_{z_j} = \overline{w}_j C_{z_j} = X^* C_{z_j}$, namely $T_f^*|_M = X^*$. Using the intertwining lifting theorem, one checks that a solution f exists iff $\|X\| \le 1$. Explicitely writing that $I - XX^* \overset{\text{not}}{=} \Gamma$ is nonnegative on M gives the well known Pick's condition that the matrix $[\Gamma_{ij}]_{i,j} \in M_m(\mathbb{C})$ defined by $1 - w_i\overline{w}_j = \Gamma_{ij}(1 - z_i\overline{z}_j)$ for $i, j = \overline{1,m}$ be nonnegative definite. Moreover, as it is known, all solutions f can be described as transfer functions $f(z) = d + c(1 - za)^{-1}zb$ of a linear system described by a unitary operator $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ on a space of the form $K \oplus \mathbb{C}$ with $K =$Hilbert space. Similarly we can treat more general interpolation problems, of Carathéodory-Féjér type for instance, involving given derivatives $\frac{d^k f}{dz^k}(z_j) = w_{kj}$ of f in certain points z_j, in which case the space M is spanned by corresponding partial derivatives $(\frac{\partial^k}{\partial\overline{w}^k})(C_w(\,\cdot\,))$, while X is suitably defined in terms of w_{kj} etc.

The n-dimensional case Subsequent developments of this technique were obtained by substituting for $\mathbb{D}$ various domains D in n variables $z_1, \dots, z_n$ and correspondingly replacing T by appropriate classes of multi–contractions $T = (T_1, \dots, T_n)$, like: for $D =$ the Euclidian unit ball (by [9], [30] and independently, by [18]); for the unit polydisc [2, 3, 14, 22]; for the noncommutative unit ball [28, 29]; for domains D : $\|d(z)\| < 1$ with matrix-valued analytic defining functions

$d = d(z) \in M_{p,q}(\mathbb{C})$ [4, 6, 13], see also [10, 15, 21, 31]. We can thus characterize the existence of those solutions f that belong to the *Schur class* $\mathcal{S}_d(E, E.)$, defined, for E, $E.$ Hilbert spaces, as the unit ball consisting of all $f : D \to B(E, E.)$ analytic such that $\|f\|_{\mathcal{S}} \leq 1$ with respect to the norm $\|f\|_{\mathcal{S}} = \sup\{\|f(\mathcal{Z})\| : \|d(\mathcal{Z})\| < 1\}$ where $\mathcal{Z}$ runs the set of all commuting n-tuples of operators $\mathcal{Z}_j \in B(\ell^2)$ $(j = \overline{1, n})$ with Taylor joint spectra $\sigma(\mathcal{Z})$ in the domain of d. Condition $\|d(\mathcal{Z})\| < 1$ implies that $\sigma(\mathcal{Z}) \subset D$ [6, 7], and so $f(\mathcal{Z})$ makes sense by analytic functional calculus [36]. Note that $\|f\|_{\mathcal{S}}$ is not necessarily finite and generally we have $\|f\|_\infty \leq \|f\|_{\mathcal{S}}$. The equality $\|f\|_{\mathcal{S}} = \|f\|_\infty$ holds for the well known exceptions $D = \mathbb{D}$ and $D = \mathbb{D}^2$, by the (generalized) von Neumann inequality [8, 34]. The norm $\| \cdot \|_{\mathcal{S}}$ has been introduced by Agler in the pioneering works [2, 3] concerned with the unit polydisc $D = \mathbb{D}^n$. For general domains D in $\mathbb{C}^n$, the contractivity type condition concerning X so that $X^* = T_f^*|_M$ for an f with $\|f\|_{\mathcal{S}} \leq 1$ means that $1_M - XX^*$ belongs to a positive subcone, defined in terms of d, of the cone of all nonnegative operators. In various interpolation problems one seeks for solutions $f : D \to B(E, E.)$ that belong to the *multiplier space* $M_{\mathcal{H}}(E, E.)$ (consisting of all $f : D \to B(E, E.)$ analytic such that $f\,(\mathcal{H} \otimes E) \subset \mathcal{H} \otimes E.$) with respect to a suitable reproducing kernel Hilbert space $\mathcal{H}$ of analytic functions on D. By the closed graph theorem, for each such f the induced multiplication map $T_f : h \mapsto fh$ is necessarily bounded. One seeks then for solutions f with multiplier norm ≤ 1, that is, $\|T_f\| \leq 1$. As it is known, for $\mathcal{H} = H^2(\mathbb{D})$ an analytic function f on $\mathbb{C}$ is a multiplier iff $f \in H^\infty(\mathbb{D})$ in which case $\|T_f\| = \|f\|$. Generally, for suitable $\mathcal{H}$ we have $\|T_f\| \leq \|f\|_{\mathcal{S}}$ [4]. We considered in [4] a generalized analytic polyhedron $D : \|d(z)\| < 1$ in $\mathbb{C}^n$ (see Definition 1), supposed also that there exists a d-space $\mathcal{H}$ of analytic functions on D (see Definition 2), and let then $X \in B(M)$ be an operator commuting with the compressions $P_M T_{z_j}|_M$ of the multiplications $T_{z_1}, \ldots, T_{z_n}$ to a given $*$-invariant subsubpace $M \subset \mathcal{H} \otimes E$ where E is a a Hilbert space. For given such D, $\mathcal{H}$ and X (satisfying also the technical hypotheses $\overline{M^\sim} = M$, see Notation 1.7), the main result [4, Theorem 3.7] (here Theorem 2.1) characterizes the existence of the liftings T_f of X with $\|f\|_{\mathcal{S}} \leq 1$. These solutions f are described as fractional transforms, too. As observed in [4], Theorem 2.1 applies to interpolation problems over Cartan domains of type I – III, in which cases suitable examples of d-spaces are known to exist. This covers the known cases of the unit polydisc and Euclidian unit ball. A natural question is whether for every domain $D : \|d(z)\| < 1$ one can find a d-space $\mathcal{H}$ – in which case Theorem 2.1 could be applied to interpolation problems over various such domains. Moreover, the reproducing kernel should have a concrete form to be used in applications. We give a positive answer to this question by

Theorem 2.3, making use of the reproducing kernels of the weighted Bergman spaces of a Cartan domain of type I [26, 35]. Then we can apply the functional intertwining lifting technique to obtain various interpolation results of Agler-Pick and Carathéodory-Féjér type, see statements 2.4 – 2.7 and the example in Section 3 that we completely work out to show how the solutions f can be obtained once the reproducing kernel of $\mathcal{H}$ is known.

Notation For every complex Hilbert spaces H and K, we write $B(H,K)$ for the space of all bounded linear operators from H to K endowed with the uniform norm, and $H \otimes K$ for the hilbertian tensor product of H and K.

For p, q positive integers, endow the space $M_{p,q}(\mathbb{C})$ of the $p \times q$ matrices with the operator norm induced by the identification $M_{p,q}(\mathbb{C}) \equiv B(\mathbb{C}^q, \mathbb{C}^p)$.

For any complex Hilbert space H, we identify $B(H^q, H^p)$ with the space $M_{p,q}(B(H))$ of all $p \times q$ – operator matrices with entries in $B(H) = B(H,H)$ where $H^p = \oplus_1^p H$.

For every open set U in $\mathbb{C}^n$ ($n =$ a fixed integer) and complex Banach space X, we denote by $O(U,X)$ the Fréchet space of all analytic X-valued functions on U; set $O(U) = O(U,\mathbb{C})$. For any compact subset $K \subset \mathbb{C}^n$, let $O(K)$ denote the algebra of germs of analytic functions on open neighborhoods of K.

For any n-tuple $T = (T_1, \ldots, T_n)$ with $T_j \in B(X)$ commuting on a Banach space X, $\sigma(T)$ denotes the Taylor joint spectrum of T on X. We write $\Phi : O(\sigma(T)) \to B(X)$, $f \mapsto f(T)$ for Taylor's analytic functional calculus of T, see [20, 36]. As it is known [20], if X is a Hilbert space then for any Hilbert spaces E, $E.$ and open neighborhood U of $\sigma(T)$ there is a unique continuous linear map $\Phi_{E,E.} : O(U, B(E,E.)) \cong O(U)\tilde{\otimes} B(E,E.) \to B(X \otimes E, X \otimes E.)$ such that $\Phi_{E,E.}(f \otimes A) = f(T) \otimes A$ for $f \in O(U)$ and $A \in B(E,E.)$. For simplicity, set again $f(T) := \Phi_{E,E.}(f)$. For any commuting tuple T on a Hilbert space H, we denote by M_T the commuting $2n$-tuple consisting of the left multiplications $L_{T_j} : B \mapsto T_j B$ and of the right multiplications $R_{T_j^*} : B \mapsto B T_j^*$ acting on $B(H)$. It is known [19] that $\sigma(M_T) = \sigma(T) \times \sigma(T^*)$. Then by analytic functional calculus $(f(M_T))(B)$ makes sens for every function $f \in O(\sigma(T) \times \sigma(T^*))$.

Definition 1. A *generalized analytic polyhedron* is an bounded open set $D \subset \mathbb{C}^n$ with polynomially convex closure $\overline{D}$, of the form $D = \{z \in W \,;\, \|d(z)\| < 1\}$ where $W \subset \mathbb{C}^n$ is open with $\overline{D} \subset W$ and $d : W \to B(\mathbb{C}^q, \mathbb{C}^p)$ is analytic with $0 \in W$ and $d(0) = 0$.

Definition 2. Let $D : \|d(z)\| < 1$ be a generalized analytic polyhedron. A *d–space* is a Hilbert space $\mathcal{H}$ of functions analytic on D such that:

(i) all the point evaluations $h \mapsto h(w)$, $w \in D$ are continuous on $\mathcal{H}$ (and so we have vectors $C_w \in \mathcal{H}$ such that $h(w) = \langle h, C_w \rangle$ for every $h \in \mathcal{H}$);

(ii) the function $C(z,u) = C_{\overline{u}}(z)$ does not vanish on $\Delta := \{(z,u) : z, \overline{u} \in D\}$ and $1/C$ extends analytically on a neighborhood of $\overline{\Delta}$; assume also $C_0(\,\cdot\,) \equiv 1$;

(iii) $\mathcal{H}$ is $O(\overline{D})$–invariant under multiplications $h \mapsto fh$ and $O(\overline{D})$ is dense in $\mathcal{H}$;

(iv) the Toeplitz operator $T_d : \mathcal{H}^q \to \mathcal{H}^p$ acting on column vectors from $\mathcal{H}^q$ as left multiplication by the $p \times q$ matrix–valued function $d|_D$ satisfies $\|T_d\| \leq 1$.

It is known that given a space $\mathcal{H}$ satisfying (i), the *reproducing kernel* function $C = C(z,u)$ defined by (ii) is necessarily analytic. If $f \in O(D, B(E, E.))$ is a multiplier then it induces a bounded linear multiplication operator $T_f : \mathcal{H} \otimes E \to \mathcal{H} \otimes E.$ by $g \mapsto fg$, see [4]. Condition (iii) implies then that all T_f with $f \in O(\overline{D})$ are continuous.

Lemma 1.1. *Let $\mathcal{H}$ be a Hilbert space of analytic functions on D, satisfying the hypotheses* (i) *and* (ii) *of Definition* 1*. Set $(Z_j h)(z) = z_j h(z)$ for any $h \in \mathcal{H}$, $z \in D$ and let $Z = (Z_1, \ldots, Z_n)$. Condition* (iii) *is then equivalent to the following condition:*

(iii)' *the space $\mathcal{H}$ is invariant under Z, $\sigma(Z) = \overline{D}$ and $\mathbb{C}[z]$ is dense in $\mathcal{H}$.*

If (i)-(iii) *hold, then for every $f \in O(\overline{D})$ we have $f(Z) = T_f$. Moreover, for every Hilbert spaces E, $E.$ and function $f : U \to B(E, E.)$ analytic on an open set $U \supset \overline{D}$, the multiplication operator $T_f \otimes 1_E : \mathcal{H} \otimes E \to \mathcal{H} \otimes E.$ given by $h \mapsto fh$ is well defined, continuous and we have $f(Z \otimes 1_E) = T_f \otimes 1_E$, that is, $((f(Z) \otimes 1_E)h)(z) = f(z)h(z)$ for $h \in \mathcal{H} \otimes E$ and $z \in D$.*

Proof. (iii) $\Rightarrow$ (iii)' . By the reproducing kernel property (i), for every $z \in D$ we have the equalities $Z_j^* C_z = \overline{z}_j C_z$ $(j = \overline{1,n})$. This implies $\overline{z} \in \sigma(Z^*)$, and so $z \in \sigma(Z)$. Thus $\overline{D} \subset \sigma(Z)$. To prove the opposite inclusion, let $z_0 \in \mathbb{C}^n \setminus \overline{D}$. Since $\overline{D}$ is polynomially convex, there is a polynomial p such that $|p(z_0)| > \max_{\overline{D}} |p|$. Define f on D by $f(z) = (p(z_0) - p(z))^{-1}$. Thus $f \in O(\overline{D})$. Now $T_f : \mathcal{H} \to \mathcal{H}$ is the inverse of the operator $p(z_0)1_{\mathcal{H}} - T_p$. Then $p(z_0) \notin \sigma(T_p)$. We can easily check the equality $T_p = p(Z)$. Hence $p(z_0) \notin \sigma(p(Z))$ $(= p(\sigma(Z))$ by the spectral mapping theorem). Then $z_0 \notin \sigma(Z)$. Therefore we have the inclusion $\sigma(Z) \subset \overline{D}$, too. Thus $\sigma(Z) = \overline{D}$. Let f be analytic on an open subset U of $\mathbb{C}^n$ with $U \supset \overline{D}$. By the closed graph theorem, the map $O(U) \hookrightarrow B(\mathcal{H})$ given by $f \mapsto T_f$ is continuous. The uniqueness property of the analytic functional calculus then gives $T_f = f(Z)$.

The generalization to operator-valued functions is straightforward. The density of $\mathbb{C}[z]$ in $\mathcal{H}$ follows since $\overline{D}$ is polynomially convex. Namely, $\overline{D}$ has a basis of neighborhoods consisting of open polynomial polyhedra P (use [25, lemma 2.7.4]). Fix such a P with $P \subset U$. Now P is a Runge domain, and so, by the approximation theorem, the polynomials are dense in $O(P)$ with respect to the uniform convergence on compact sets. Then use the continuity of the map $f \mapsto f(Z)\,1$ to get $\overline{\mathbb{C}[z]} = \mathcal{H}$. Also, (iii)' $\Rightarrow$ (iii) holds by the remarks preceding [Lemma 1.1, 4]. $\square$

By Lemma 1.1, conditions (i) – (iii) imply that $\mathcal{H}$ is densely spanned by the space $\mathbb{C}[z]$ of the analytic polynomial functions on D. Then by a Gram–Schmidt orthonormalization of all monomials z^α ($\alpha \in \mathbb{Z}_+^n$) in some arbitrary order we can find an orthonormal basis of $\mathcal{H}$ consisting of polynomials $e_k \in \mathbb{C}[z]$ ($k \geq 0$) the linear span of which is $\mathbb{C}[z]$.

Notation Let $\mathcal{H}$ be a d-space. Let $M \subset \mathcal{H} \otimes E$ be a $*$-*invariant* closed linear subspace, that is, we have $(Z_j \otimes 1_E)^* M \subset M$ for all j. Set $T_j = P_M(Z_j \otimes 1_E)|_M$ where P_M denotes the orthogonal projection from $\mathcal{H}$ onto M. Then $T_iT_j = T_jT_i$ for all i, j since $T_j^* = (Z_j \otimes 1_E)^*|_M$. Set $T = (T_1, \ldots, T_n)$. Let $M^\sim = \{m \in M : \sum_k \|e_k(T)^* m\|^2 < \infty\}$, where $(e_k)_{k\geq 0}$ is any orthonormal basis of $\mathcal{H}$ consisting of polynomials. The definition of $M^\sim$ proves to be independent of the choice of $(e_k)_{k\geq 0}$.

Lemma 1.2. [4] *Let $\mathcal{H}$ be a d-space over the domain $D : \|d(z)\| < 1$. Then:*

(a) *T is a commuting n–tuple with $\sigma(T) \subset \overline{D}$ and for any function $f \in O(\overline{D})$ we have $f(Z \otimes 1_E)^* M \subset M$ and $f(T) = P_M f(Z \otimes 1_E)|M$;*

(b) *For any $h \in \mathcal{H} \otimes E$, $z \in D$ and $x \in E$ we have the identity $\langle h, C_z \otimes x\rangle = \langle h(z), x\rangle$;*

(c) *For any $f \in M_{\mathcal{H}}(E, E.)$, $z \in D$ and $x \in E.$ we have $T_f^*(C_z \otimes x) = C_z \otimes f(z)^* x$.*

2. Main results

Theorem 2.1. [4] *Let $D : \|d(z)\| < 1$ be a generalized analytic polyhedron. Let $\mathcal{H}$ be a d-space, with reproducing kernel C. Let E, $E.$ be Hilbert spaces and $M \subset \mathcal{H} \otimes E$, $M. \subset \mathcal{H} \otimes E.$ be $*$-invariant subspaces. Set $T_j = P_M(Z_j \otimes I)|M$ and $T._j = P_{M.}(Z_j \otimes I)|M.$ for $j = \overline{1, n}$. Let $X \in B(M, M.)$ such that $XT_j = T._jX$ for all j. Assume that $M.^\sim$ is dense in $M.$ and set $T = (T_1, \ldots, T_n)$, $T. = (T._1, \ldots, T._n)$. The following are equivalent:*

(i) *there exists* $F : D \to B(E, E.)$ *analytic with* $\|f\|_{\mathcal{S}} \leq 1$ *such that* $XP_M = P_{M.}T_F$*;*

(ii) *there exists a nonnegative operator* $\Gamma = [\Gamma_{ij}]_{i,j=1}^p \in B(M.^p)$ *such that*

$$(1)\qquad \big(\frac{1}{C}(M_{T.})\big)(1_{M.} - XX^*) = \sum_{j=1}^p \Gamma_{jj} - \sum_{i,j=1}^p \sum_{k=1}^q d_{ik}(T.)\Gamma_{ij} d_{jk}(T.)^*;$$

(iii) *there are a Hilbert space* K *and a unitary operator*

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} : \begin{array}{c} K^p \\ \oplus \\ E \end{array} \to \begin{array}{c} K^q \\ \oplus \\ E. \end{array}$$

such that if the function $F : D \to B(E, E.)$ *is given by*

$$(2)\qquad F(z) = u_{22} + u_{21}(1_{K^p} - d(z) \cdot u_{11})^{-1} d(z) \cdot u_{12}$$

then $XP_M = P_{M.}T_F$*, where* $d(z) : K^q \to K^p$ *acts on column vectors* $(k_j)_{j=1}^q \in K^q$ *by matrix multiplication to the left,*

$$(k_j)_{j=1}^q \mapsto \big(\sum_{j=1}^q d_{ij}(z) k_j\big)_{i=1}^p \in K^p\,.$$

For every unitary U as in (iii), the function F defined by (2) is a Schur class multiplier. Hence $T_F : \mathcal{H} \otimes E \to \mathcal{H} \otimes E.$ is bounded, with $\|T_F\| \leq \|F\|_{\mathcal{S}} \leq 1$, see [4, 7]. We call F as usual a *fractional transform.* The right hand side of the (1) can be written as well

$$(3)\quad \sum_j \Gamma_{jj} - \sum_{i,j,k} d_{ik}(T.)\Gamma_{ij} d_{jk}(T.)^* = (\mathrm{tr}_p \otimes 1)\big(\Gamma\big) - (\mathrm{tr}_q \otimes 1)\big(d^t(T.)\,\Gamma\, d^t(T.)^*\big)$$

where $d^t(z) \in M_{q,p}(\mathbb{C})$ denotes the transposed of the matrix $d(z)$ for $z \in D$, so that $d^t(T.) \in M_{q,p}(B(M.))$, while $(\mathrm{tr}_p \otimes 1)(A) = \sum_{j=1}^p A_{jj}$ for $A \in M_p(B(M.))$ is the extension of the trace $\mathrm{tr}_p : M_p(\mathbb{C}) \to \mathbb{C}$ to the algebra $M_p(B(M.)) \equiv M_p(\mathbb{C}) \otimes B(M.)$.

Proposition 2.2. [4] *The hypotheses* $\overline{M.^\sim} = M.$ *in Theorem* 2.1 *is automatically fulfilled in any of the following cases: if* $\dim M. < \infty$*; if* $\sigma(T.) \subset D$ *(in particular, if* $M. \subset \overline{sp}_{z\in K} C_z \otimes E.$ *for* $K \subset D$ *compact); if* $M. = \overline{sp}_i M_i$ *for an arbitrary family of* $*$*-invariant subspaces* M_i *with* $\overline{M_i^\sim} = M_i$ *(in particular, if* $M. = \overline{sp}_{z\in D, \alpha\in\mathbb{Z}_+^n} \ker(T.^* - \overline{z})^\alpha$*); also, it is not required if* $p = 1$.

Remind that a function $K : \Lambda \times \Lambda \to B(H)$ (with Λ an arbitrary set and H a Hilbert space) is called *positive definite* if $\sum_{i,j=1}^{k} \langle K(\lambda_i, \lambda_j)c_i, c_j\rangle \geq 0$ whenever k is a positive integer, $\lambda_1, \ldots, \lambda_k \in \Lambda$ and $c_1, \ldots, c_k \in H$.

Remark. It is known [16, 17] that $f \in O(D, B(E, E.))$ belongs to the closed unit ball of $M_{\mathcal{H}}(E, E.)$ with respect to the multiplier norm $f \mapsto \|T_f\|$ iff the $B(E.)$–valued map defined on $D \times D$ by $(z, w) \mapsto C(w, \overline{z})(1_{E.} - f(w)f(z)^*)$ is positive definite.

Example Let $\mathbb{D}_{p,q} = \{z = [z_{ij}]_{i,j} \in M_{p,q}(\mathbb{C}) : \|z\| < 1\}$ where $p \leq q$. Then $\mathbb{D}_{p,q}$ is a generalized analytic polyhedron with defining function $d : \mathbb{C}^{pq} \to M_{p,q}(\mathbb{C})$, $d(z) = z$. For any integer $\lambda \geq q$, the λ-Bergman space $H^2_\lambda(\mathbb{D}_{p,q})$ is a d-space with reproducing kernel $C(z, \overline{w}) = \det(1_p - zw^*)^{-\lambda}$. Moreover, there exists a probability measure $\nu = \nu_\lambda$ on $\overline{\mathbb{D}}_{p,q}$ such that $H^2_\lambda(\mathbb{D}_{p,q}) \subset L^2(\nu)$ isometrically, and the pq-tuple $Z = (Z_{ij})_{i,j}$ is subnormal (we refer to [12, 26, 35]).

Remark. Any Cartan domain D of type I–III is a generalized analytic polyhedron and for each integer $\lambda > 0$ in the continuous part $W_c(D)$ of the Wallach set [35] there corresponds a generalized λ–Bergman space $\mathcal{H} := H^2_\lambda(D)$. Then $\mathcal{H}$ satisfies conditions (i)–(iii) of Definition 2 and $1/C(z, \overline{w})$ is a concrete polynomial of $2n$ variables $z, \overline{w}$. Whenever λ is sufficiently large, for example if $\lambda > g - 1$ where g is the genus of D [35], condition (iv) also is fulfilled (it is an interesting question if this holds for all $\lambda \in W_c(D)$). The Cartan domains of type I can be realized as operator unit balls $\mathbb{D}_{p,q}$ of spaces $B(\mathbb{C}^q, \mathbb{C}^p)$ of $p \times q$ matrices $z = [z_{ij}]_{i,j}$ (Example 2.4). In this case $W_c(D) = (p - 1, \infty)$ and $g = p + q$. The Hardy and Bergman space of $\mathbb{D}_{p,q}$ are obtained for $\lambda = q$ and $p + q$, respectively. The Cartan domains of type II correspond to the operator unit balls of the spaces of the symmetric matrices, namely we take $p = q$, $n = p(p + 1)/2$ and $d(z) = [d_{ij}(z)]_{i,j} \in B(\mathbb{C}^p)$ where $d_{ij}(z) = z_{ij}$ if $i \leq j$ and $d_{ij}(z) = z_{ji}$ if $i > j$. The Cartan domains of type III are the unit balls $\|z\| < 1$ of the skew–symmetric matrices $z = -z^t \in B(\mathbb{C}^p)$.

Theorem 2.3. *For every generalized analytic polyhedron $D : \|d(z)\| < 1$ there exists a d-space $\mathcal{H}$. Moreover, if there exists a map $r \in O(\overline{d(D)}, \mathbb{C}^n)$ such that $r(d(z)) \equiv z$ on a neighborhood of $\overline{D}$ and $p \leq q$, then we can let $\mathcal{H} := \overline{sp}_{z \in D}\, C_z(\cdot)$ be the functional Hilbert space with reproducing kernel $C(z, \overline{w}) :=$ $\det(1_p - d(z)d(w)^*)^{-q}$.*

Proof. We can aways assume the existence of a map r as above. To this aim, fix an $\epsilon > 0$ sufficiently small so that $\epsilon|z_j| < 1$ for every $z = (z_1, \ldots, z_n) \in D$ and $j = \overline{1, n}$. Let $\tilde{z}$ be the diagonal $n \times n$ matrix with entries $z_1, \ldots, z_n$. We replace, if necessary, the defining function $d : W \to M_{p,q}(\mathbb{C})$ by the map $\tilde{d} : W \to M_{p+n,q+n}(\mathbb{C})$ given by $\tilde{d}(z) = (\varepsilon\tilde{z}) \oplus d(z)$. Then let r take any $(n+p)\times(n+q)$ matrix $[x_{ij}]_{i,j}$ into $\epsilon^{-1}(x_{11},\ldots,x_{nn})$. Suppose therefore that d has a retraction r as stated in the enunciation.

For every $z, w \in D$, set $C(z,\overline{w}) = \det(1_p - d(z)d(w)^*)^{-q}$. For each $w \in D$, define the function $C_w : D \to \mathbb{C}$ by $C_w(z) = C(z,\overline{w})$ for $z \in D$. Let $\mathcal{H}_0$ be the linear span of all functions $C_w(\,\cdot\,)$ with $w \in D$. Thus $C = C_{p,q}\circ(d,d)$, where $(d,d)(z,w) = (d(z),d(w))$ and $C_{p,q}(x,\overline{y}) = \det(1_p - xy^*)^{-q}$ for $x, y \in \mathbb{D}_{p,q}$ is the reprodcing kernel of the q–Bergman space $H^2_q(\mathbb{D}_{p,q})$ over the Cartan domain $\mathbb{D}_{p,q}$. Since $C_{p,q}$ is positive–definite, the kernel $C = C_{p,q}\circ(d,d)$ also is positive definite. Hence there exists a well defined inner product on $\mathcal{H}_0$ given on the generators $(C_w)_{w\in D}$ by the formula $\langle C_w, C_z\rangle := C_w(z)$. Let $\mathcal{H}$ be the completion of $\mathcal{H}_0$ with respect to the norm defined by this inner product.

We prove that $\mathcal{H}$ is a d-space. The reproducing kernel property $h_0(z) = \langle h_0, C_z\rangle$ $(z \in D)$ obviously holds for all $h \in \mathcal{H}_0$. Hence for any $h \in \mathcal{H}$ and $z \in D$ there is a uniquely determined complex number, denoted by $h(z)$, defined as the limit of $h_{0k}(z)$ over a sequence $(h_{0k})_k$ of vectors $h_{0k} \in \mathcal{H}_0$ such that $h_{0k} \to h$ in $\mathcal{H}$ as $k \to \infty$. The limit $h(z)$ is independent of the choice of the sequence $(h_{0k})_k$. Also, we obtain $h(z) = \langle h, C_z\rangle$ for any $h \in \mathcal{H}$ and $z \in D$. Then we can associate a function $(h(z))_{z\in D}$ with any $h \in \mathcal{H}$. Moreover this representation of $\mathcal{H}$ as a functional space is injective, due to the density of $\mathcal{H}_0$ in $\mathcal{H}$. Therefore there is a continuous inclusion $\mathcal{H} \subset O(D)$ and the reproducing kernel of $\mathcal{H}$ is C. Condition (i) of Definition 2 is then fulfilled. Obviously condition (ii) is satisfied, too. Also, $\mathcal{H}$ is densely spanned by $C_z \in O(\overline{D})$ $(z \in D)$.

We prove that $\mathcal{H}$ is $O(\overline{D})$–invariant. Let then $f \in O(\overline{D})$. Thus $f \in O(U)$ where $U \subset \mathbb{C}^n$ is open with $\overline{D} \subset U$. Since $\overline{D}$ is compact, $d(\overline{D})$ is compact, too. Then from $d(D) \subset d(\overline{D})$ we can derive the inclusion $\overline{d(D)} \subset d(\overline{D})$. Hence $r(\,\overline{d(D)}\,) \subset r(d(\overline{D}))$. Now $r(d(\overline{D})) \subset \overline{D}$ due to the identity $r(d(z)) \equiv z$ for $z \in \overline{D}$. Thus $r(\,\overline{d(D)}\,) \subset \overline{D}$. Hence $\overline{d(D)} \subset r^{-1}(\overline{D}) \subset r^{-1}(U)$, that is, $\overline{d(D)}$ is a (compact) subset of the open set $r^{-1}(U)$. Then we may set $g := f\circ r \in O(\,\overline{d(D)}\,)$. Since $r\circ d = id$ on a neighbourhood $\widetilde{D}$ of $\overline{D}$ with $\widetilde{D} \subset W$, it follows that $d|_{\widetilde{D}}$ is an analytic embedding. Then we can find an $\varepsilon > 0$ such that the set $d(\widetilde{D})\cap(1+\varepsilon)\mathbb{D}_{p,q}$ be a closed analytic submanifold of the open set $(1+\varepsilon)\mathbb{D}_{p,q}$. Now $(1+\varepsilon)\mathbb{D}_{p,q}$ is a Stein domain. Hence by known cohomological arguments (Cartan's theorem B, see for instance [Corollary 4.1.8, 24]), there exists $G \in O((1+\varepsilon)\mathbb{D}_{p,q})$ such that $G|_{d(\tilde{D})\cap(1+\varepsilon)\mathbb{D}_{p,q}} = g$. Compose the equality $f\circ r|_{d(D)} = g|_{d(D)} = G|_{d(D)}$ with the map d to the right, use the identity $r\circ d = id$ and derive that $f = G\circ d$ on D. Now

$G \in O(\overline{D}_{p,q})$ and so G is a multiplier on the Hardy space $H^2_q(D_{p,q})$. Then there is a finite constant $c > 0$ such that the map $(y,x) \mapsto C_{p,q}(x,\overline{y})(c - G(x)\overline{G(y)})$ is positive definite on $\mathbb{D}_{p,q} \times \mathbb{D}_{p,q}$, see the Remark on the unit ball with respect to the multiplier norm. Compose this map with (d,d) and use $G \circ d = f$. Hence the map $(w,z) \mapsto C(z,\overline{w})(c - f(z)\overline{f(w)})$ also is positive definite, on $D \times D$. Then f is a multiplier of $\mathcal{H}$. Thus condition (iii) is fulfilled.

To prove now that $\|d(Z)\| \leq 1$, we follow the same idea as above. Set $d'(x) = x$ on $\mathbb{D}_{p,q}$. Let Z' be the pq–tuple of the multiplications by the variables x_{ij} on $H^2_q(\mathbb{D}_{p,q})$. Then we have the the estimate $\|d'(Z')\| \leq \|d'\|_{\infty,\mathbb{D}_{p,q}} = \sup_{x\in\mathbb{D}_{p,q}} \|x\| = 1$. The map $(y,x) \mapsto C_{p,q}(x,\overline{y})(1_p - xy^*)$ is then positive definite. Compose it with (d,d). Hence $(w,z) \mapsto C(z,\overline{w})(1_p - d(z)d(w)^*)$ also is positive definite. Then d is a $B(\mathbb{C}^q\mathbb{C}^p)$–valued contractive multiplier on $\mathcal{H}$, that is, $\|d(Z)\| \leq 1$. Condition (iv) also is thus fulfilled. $\square$

Remark. The class of generalized analytic polyhedra is closed under intersections $D = \cap_{i=1}^m D_i$ for $D_i \subset \mathbb{C}^n$, cartesian products $D = \prod_{i=1}^m D_i$ for $D_i \subset \mathbb{C}^{n_i}$, intersections $D = D' \cap L$ with certain analytic submanifolds $L \subset \mathbb{C}^n$ (in particular, with linear subspaces L), and biholomorphic transforms. Whenever there are d-spaces $\mathcal{H}_i$ (resp. $\mathcal{H}'$) over the domains D_i (resp. D'), we can define a suitable d-space $\mathcal{H}$ over D, the reproducing kernel of which can be easily expressed in terms of the kernels of $\mathcal{H}_i$ (resp. $\mathcal{H}'$), see [5, 7].

The following Theorem 2.4, concerning the *Nevanlinna-Pick problem*, has been directly obtained in [13] (see also [4, 7]). One can also derive it as a corollary of Theorem 2.1 as shown below.

Theorem 2.4. [13] *Let $D : \|d(z)\| < 1$ be a generalized analytic polyhedron. Let $S \subset D$ and $f : S \to B(E, E.)$ be arbitrary. Then the following are equivalent:*

(i) *f extends to a $B(E, E.)$-valued analytic map from $\mathcal{S}_d(E, E.)$;*

(ii) *there exists a positive definite map $\Gamma : S \times S \to B(E^p_.)$ such that*

$$1_{E.} - f(t)f(s)^* = \mathrm{tr}_p\big((1_p - d^t(s)^* d^t(t))\Gamma(s,t)\big) \tag{4}$$

$$= \sum_{j=1}^{p} \Gamma_{jj}(s,t) - \sum_{i,j=1}^{p}\sum_{k=1}^{q} \overline{d_{jk}(s)} d_{ik}(t)\Gamma_{ij}(s,t) \qquad (s,t \in S)$$

where tr_p is the trace and $d^t(s) \in M_{q,p}(\mathbb{C})$ is the transposed of $d(s)$;

(iii) *there are a Hilbert space K and a unitary operator*

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} : \begin{array}{c} K^p \\ \oplus \\ E \end{array} \to \begin{array}{c} K^q \\ \oplus \\ E. \end{array}$$

such that for every $s \in S$, $f(s) = u_{22} + u_{21}(1_{K^p} - d(s) \cdot u_{11})^{-1} d(s) \cdot u_{12}$.

Proof. Let $\mathcal{H}$ be any d-space, the existence of which has been established by Theorem 2.3. Let $C = C(z, \overline{w})$ $(z, w \in D)$ denote the reproducing kernel of $\mathcal{H}$. Define the subspaces $M := \overline{sp}_{s \in S} C_s \otimes E$ and $M_. := \overline{sp}_{s \in S} C_s \otimes E_.$ of $\mathcal{H} \otimes E$ and $\mathcal{H} \otimes E_.$, respectively. Then both M and $M_.$ are $*$-invariant. Let the operator $X : M \to M_.$ be defined on generators by $X^*(C_s \otimes e_.) = C_s \otimes f(s)^* e_.$ for every $e_. \in E_.$ and $s \in S$. To obtain the equality (4), apply (1) to $C_s \otimes e_.$ and take then the inner product with $C_t \otimes e'_.$ for arbitrary vectors $e_., e'_. \in E_.$ and points s, $t \in S$. We omit the details. □

Remark. Remind that taking $S = D$ in Theorem 2.4 provides, via (i) ⇔ (iii), the characterization of the elements of $\mathcal{S}_d(E, E_.)$ as fractional transforms, see (2).

Given points $s_k \in D$ and vectors $\xi_k \in E$ and $\xi_{\cdot k} \in E_.$ for $k = \overline{1, m}$ where E, $E_.$ are Hilbert spaces, the *tangential Nevanlinna–Pick problem* asks for the existence of a bounded analytic function $F : D \to B(E, E_.)$ with $\|F\|_\infty \le 1$ such that $F(s_k)^* \xi_{\cdot k} = \xi_k$ for every k. In this context we have the following proposition.

Proposition 2.5. *Let* $D : \|d(z)\| < 1$ *be a generalized analytic polyhedron. Let* $E, E_.$ *be Hilbert spaces. Suppose we are given points* $s_k \in D$ *and vectors* $\xi_k \in E$, $\xi_{\cdot k} \in E_.$ *for* $k = \overline{1, m}$. *The following statements are equivalent:*

(a) *there is* $F : D \to B(E, E_.)$ *analytic with* $\|F\|_{\mathcal{S}} \le 1$ *such that* $F(s_k)^* \xi_{\cdot k} = \xi_k$ $\forall k$;

(b) *there is a nonnegative* $pm \times pm$ *matrix* $[G_{\rho\sigma}]_{\rho,\sigma}$ *of complex numbers, the indices* ρ, σ *of which run the set* $\{1, \ldots, p\} \times \{1, \ldots, m\}$, *such that for all* $k, l = \overline{1, m}$

$$\langle \xi_{\cdot k}, \xi_{\cdot l} \rangle - \langle \xi_k, \xi_l \rangle = \sum_{j=1}^{p} G_{(j,l)(j,k)} - \sum_{i,j=1}^{p} \sum_{r=1}^{m} d_{ir}(s_l) \overline{d_{jr}(s_k)} G_{(i,l)(j,k)} \,.$$

Proof. By Theorem 2.3, there exists a d-space $\mathcal{H}$. Let the subspaces $M \subset \mathcal{H} \otimes E$ and $M_. \subset \mathcal{H} \otimes E_.$ be defined as the linear spans $M = sp_{k=1}^{m}\, C_{s_k} \otimes \xi_k$ and $M_. = sp_{k=1}^{m}\, C_{s_k} \otimes \xi_{\cdot k}$, respectively. By the formula $T_F^* (C_s \otimes \xi_.) = C_s \otimes (F(s)^* \xi_.)$ where $s \in D$ and $\xi_. \in E_.$ (see Lemma 1.2, (c)), a function $F \in H^\infty(D, B(E, E_.))$ is a solution of the equation (a) iff $T_F^* (C_{s_k} \otimes \xi_{\cdot k}) = C_{s_k} \otimes \xi_k$ for all $k = \overline{1, m}$. Also, both M and $M_.$ are $*$-invariant. Define $X : M \to M_.$ by means of its adjoint, according to the formula $X^* (C_{s_k} \otimes \xi_{\cdot k}) = C_{s_k} \otimes \xi_k$ for every k. Hence $X^* T_i^* = T_{\cdot i}^* X^*$ for all $i = \overline{1, n}$. The equations $F(s_k)^* \xi_{\cdot k} = \xi_k$ are now equivalent to $X P_M = P_{M_.} T_F$. Moreover $M_.^{\sim}$ is dense in $M_.$, see Proposition 2.2. We can apply then Theorem 2.1. Whenever condition (b) holds, we can obtain an operator

$\Gamma = [\Gamma_{ij}]_{i,j=1}^p \in B(M_\cdot^p)$ with $\Gamma_{ij} : M_\cdot \to M_\cdot$ defined on generators by the formula $\langle \Gamma_{ij}(C_{s_k} \otimes \xi_{\cdot k}), C_{s_l} \otimes \xi_{\cdot l}\rangle = G_{(i,l)(j,k)}$ for $i, j = \overline{1,p}$ and $k, l = \overline{1,m}$. Since $[G_{\rho\sigma}]_{\rho,\sigma}$ is positive definite, $\Gamma \geq 0$. Moreover, a straightforward calculation leads to (1). Then by the implication (ii) $\Rightarrow$ (i) we get (a). Conversely, if (a) holds, implication (i) $\Rightarrow$ (ii) gives the existence of a nonnegative Γ satisfying (1), which provides a nonnegative matrix $[G_{\rho\sigma}]_{\rho,\delta}$ by the formula from above. Moreover, condition (b) is fulfilled. We omit the details, that follow a known line as in the case of the polydisc [14, 15]. □

Fix a subset $S \subset D$. Suppose that for every $s \in S$, a set $A_s \subset \mathbb{Z}_+^n$ of multiindices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ is given such that whenever $\alpha \in A_s$, the whole segment $[0, \alpha] \subset A_s$, too, where $[0, \alpha] := \{\gamma \in \mathbb{Z}_+^n; \gamma \leq \alpha\}$ and the order $\gamma \leq \alpha$ is defined componentwise: $\gamma_i \leq \alpha_i$ for all $i = \overline{1,n}$. For each $s \in S$, let $(c_{s,\alpha})_{\alpha\in A_s}$ be a fixed family of operators $c_{s,\alpha} \in B(E, E_\cdot)$ where E, $E_\cdot$ are Hilbert spaces. The *Carathéodory-Féjér problem* asks then for the existence of a bounded analytic function $f : D \to B(E, E_\cdot)$ with $\|f\|_\infty \leq 1$ whose Taylor series in each point $s \in S$ has the form

$$f(z) = \sum_{\alpha\in A_s} c_{s,\alpha}(z-s)^\alpha + \sum_{\alpha\in\mathbb{Z}_+^n\setminus A_s} \frac{(\partial^\alpha f)(s)}{\alpha!}(z-a)^\alpha \qquad (|z-s| < \varepsilon,\, \varepsilon = \varepsilon_s > 0),$$

namely of a function f with $(\partial^\alpha f)(s)/\alpha! = c_{s,\alpha}$ for each $s \in S$ and $\alpha \in A_s$. As usual $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and $\alpha! = \alpha_1! \cdots \alpha_n!$.

The following result was proved in [4] for E, $E_\cdot = \mathbb{C}$ and finite sets S, A_s under the additional assumption that there exists a d-space $\mathcal{H}$ over D. We know, by Theorem 2.3, that this hypotheses is redundant.

Theorem 2.6. (see [4]) *Let $D : \|d(z)\| < 1$ be a generalized analytic polyhedron. Let $S \subset D$ be a subset. Let $A_s \subset \mathbb{Z}_+^n$ ($s \in S$) be sets of multiindices, with the property that whenever $\alpha \in A_s$ the whole segment $[0, \alpha] \subset A_s$, too. Let E, $E_\cdot$ be Hilbert spaces. Let $c_{s,\alpha} \in B(E, E_\cdot)$ be given for each $s \in S$ and $\alpha \in A_s$. Then the following are equivalent:*

(a) *there exists a bounded analytic function $f : D \to B(E, E_\cdot)$ with $\|f\|_{\mathcal{S}} \leq 1$ such that $(\partial^\alpha f)(s)/\alpha! = c_{s,\alpha}$ for every point $s \in S$ and multiindex $\alpha \in A_s$;*
(b) *there exists a positive definite map $G : \Lambda \times \Lambda \to B(E_\cdot)$, $G = G_{\rho\sigma}$ for $\rho, \sigma \in \Lambda$, where $\Lambda := \{\rho = (j, s, \alpha) : j = 1, \ldots, p;\, s \in S;\, \alpha \in A_s\}$, such*

that for every $s,t \in S$ and $\alpha \in A_s$, $\beta \in A_t$ we have the equality

$$\begin{aligned} \delta_{(\alpha,\beta),(0,0)} 1_{E.} - c_{s,\alpha} c^*_{t,\beta} \; &= \; \sum_{j=1}^{p} G_{(j,s,\alpha)(j,t,\beta)} \\ &- \sum_{\substack{0\le\delta\le\alpha \\ 0\le\lambda\le\beta}} \sum_{i,j=1}^{p} \left(\sum_{k=1}^{q} \frac{(\partial^{\alpha-\delta} d_{ik})(s)}{(\alpha-\delta)!} \frac{\overline{(\partial^{\beta-\lambda} d_{jk})(t)}}{(\beta-\lambda)!} \right) G_{(i,s,\delta)(j,t,\lambda)} \end{aligned} \tag{5}$$

where δ is Kronecker's symbol, $\delta_{(\alpha,\beta),(0,0)} = 1$ if $(\alpha,\beta) = (0,0)$ and 0 otherwise.

Proof. By Theorem 2.3, there exists a d-space $\mathcal{H}$ over D. Then we can apply Theorem 2.1, following the lines in [4]. To this aim, one shows firstly that for each $w \in D$ and $\alpha \in \mathbb{Z}_+^n$ there exists a unique function $C_w^\alpha \in \mathcal{H}$ such that

$$(\partial^\alpha f)(w)/\alpha! = \langle f, C_w^\alpha \rangle \qquad (f \in \mathcal{H}), \tag{6}$$

namely $C_w^\alpha(z) = \overline{(\partial^\alpha C_z)(w)} \,/\, \alpha!$ $\;(z \in D)$. Then one proves that for any multiplier $\varphi \in O(D)$ of $\mathcal{H}$, the identity

$$T_\varphi^* C_w^\alpha = \sum_{0\le\gamma\le\alpha} \frac{\overline{(\partial^{\alpha-\gamma} f)(w)}}{(\alpha-\gamma)!} C_w^\gamma \tag{7}$$

holds for any $w \in D$ and $\alpha \in \mathbb{Z}_+^n$ [4, lemma 4.1]. Hence the linear subspaces $M := \mathrm{sp}\{C_s^\alpha : s \in S\,;\, \alpha \in A_s\} \otimes E$ of $\mathcal{H} \otimes E$ and $M. := \mathrm{sp}\{C_s^\alpha : s \in S\,;\, \alpha \in A_s\} \otimes E.$ of $\mathcal{H} \otimes E.$ are $*$-invariant, see Lemma 1.2. Denote by $T \in B(M)^n$ and $T. \in B(M.)$ the compressions of $Z \otimes 1_E$ and $Z \otimes 1_{E.}$ to M and $M.$, respectively. Define then $X : M \to M.$ by

$$X^*\,(C_s^\alpha \otimes e.) = \sum_{0\le\gamma\le\alpha} C_s^\gamma \otimes (c^*_{s,\alpha-\gamma} e.) \tag{8}$$

for arbitrary $e. \in E.$ and $s \in S$, $\alpha \in A_s$. Then $T_j^* X^* = X^* T_{.j}^*$ for any $j = \overline{1,n}$. We have $\overline{M.^\sim} = M.^\sim$ by Proposition 2.2. These data fulfill then the hypotheses of Theorem 2.1. The existence of a Schur class solution f of (a) is equivalent to the existence of an $f \in \mathcal{S}_d(E,E.)$ such that $T_f^* M. \subset M.$ and $T_f^*|_{M.} = X^*$, or, equivalently, $XP_M = P_M T_f$. We have (a) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (b) and (b) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (a), where the equivalence (ii) $\Leftrightarrow$ (b) holds as follows. Assume (ii), namely we have a nonnegative $\Gamma = [\Gamma_{ij}]_{i,j=1}^p \in B(M.^p)$ such that (1) holds, where $\Gamma_{ij} \in B(M.)$. Note

$$\langle \Gamma_{ij} C_t^\beta, C_s^\alpha \rangle = G_{(i,s,\alpha)(j,t\beta)} \tag{9}$$

for arbitrary (i,s,α) and (j,t,β) in Λ. A map $G = G_{\rho\sigma}$ for $\rho,\sigma \in \Lambda$ is thus defined. Then (b) holds by applying the equality (1) to $C_t^\beta \otimes e.$ and taking the inner product with $C_s^\alpha \otimes e$ for arbitrary $e \in E$, $e. \in E.$, $s,t \in S$ and $\alpha \in A_s$, $\beta \in A_t$; use also the equality $\langle C_t^\nu, C_s^\alpha\rangle = \frac{(\partial_1^\alpha \overline{\partial}_2^\nu C)(s,\overline{t})}{\alpha!\nu!}$, see [4]. Since $\Gamma \geq 0$, the map $G = G_{\rho\sigma}$ is positive definite. Conversely, whenever a positive definite map G is given as in (b), an operator Γ satisfying (ii) can be defined by (9). We omit the details. $\square$

In what follows we explicitely write condition (1) for a simple Cartan domain D endowed with a Bergman-type functional Hilbert space $\mathcal{H}$ [26, 35]. Namely, we consider the case $p = q = 2$ of Example 2.4. Let D be the domain

$$\mathbb{D}_{2,2} = \{z = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix} \in M_{2,2}(\mathbb{C}) \,:\, \|z\| < 1\}.$$

The Shilov boundary $\partial_0 D$ of D consists then of all unitary 2×2 matrices. Let ν be the unique probability measure on $\partial_0 D$ ($\equiv$ the unitary group $U(2)$) that is invariant under the group $GL(D)$ of all linear automorphisms of D. Namely, ν is the Haar measure on $U(2)$. Then the Hardy space $H_2^2(D)$ of D is isometrically imbedded into $L^2(\partial_0 D, \nu)$, and the reproducing kernel of $H_2^2(D)$ is $C(z,\overline{w}) = \det(1_2 - zw^*)^{-2}$ [35].

Proposition 2.7. *Let $D = \mathbb{D}_{2,2}$ be the operator unit ball in $M_{2,2}(\mathbb{C})$. Let $\mathcal{H} = H_2^2(D)$ be the Hardy space of D. Let E be a Hilbert space and $M \subset \mathcal{H} \otimes E$ be a $*$-invariant subspace. Let $X \in B(M)$ such that $XT_j = T_jX$ for $j = \overline{1,4}$ where $T_j = P_M(Z_j \otimes I)|_M$. Suppose $\overline{M^\sim} = M$, too. Then the following are equivalent:*

(i) *there exists $F : D \to B(E)$ analytic with $\|F\|_{\mathcal{S}} \leq 1$ such that $XP_M = P_M T_F$;*

(ii) *there exists a nonnegative operator $\Gamma = [\Gamma_{ij}]_{i,j=1}^2 \in B(M \oplus M)$ such that*

$$1_M - XX^* - 2\sum_{j=1}^4 T_j(I - XX^*)T_j^* + 2(T_1T_4 - T_2T_3)(I - XX^*)(T_1^*T_4^* - T_2^*T_3^*)$$

$$+ \sum_{j,k=1}^4 T_jT_k(I - XX^*)T_j^*T_k^* - 2(T_1T_4 - T_2T_3)\sum_{j=1}^4 T_j(I - XX^*)T_j^*(T_1^*T_4^* - T_2^*T_3^*)$$

$$+(T_1T_4 - T_2T_3)^2(I - XX^*)(T_1^*T_4^* - T_2^*T_3^*)^2 = \Gamma_{11} + \Gamma_{22} - \sum_{i,j,k=1}^2 T_{2i+k-2}\Gamma_{ij}T_{2j+k-2}^*\,;$$

(iii) *there exist a Hilbert space K and a unitary operator $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B(K^2 \oplus E)$ such that if $F(z) = d + c(1_{K^2} - z\cdot a)^{-1}\, z\cdot b$ then $XP_M = P_M T_F$, where each $z \in D$ $(\subset M_{2,2}(\mathbb{C}))$ acts on column vectors from K^2 as multiplication to the left by z.*

PROOF. One verifies that $\mathcal{H} := H_2^2(D)$ is a d-space. Since $H_2^2(D) \subset L^2(\partial_0 D, \nu)$, we have $\|d(Z)\| \leq \|d\|_\infty = \sup_{z\in\mathbb{D}_{2,2}} \|z\| = 1$. The other conditions also are easily checked. Since the reproducing kernel C of $\mathcal{H}$ is given by the formula $C(z,\overline{w}) = \det(1_2 - zw^*)^{-2}$ for $z, w \in D$, we have

$$1/C(z,\overline{w}) = (\det(1_2 - zw^*))^2 = (1 - \mathrm{tr}\,(\mathrm{zw}^*) + \det\,(\mathrm{zw}^*))^2$$
$$= (1 - \sum_{j=1}^{4} z_j\overline{w}_j + (z_1z_4 - z_3z_2)(\overline{w}_1\overline{w}_4 - \overline{w}_3\overline{w}_2))^2 =$$
$$1 - 2\sum_{j=1}^{4} z_j\overline{w}_j + 2(z_1z_4 - z_2z_3)(\overline{w}_1\overline{w}_4 - \overline{w}_2\overline{w}_3) + \sum_{j,k=1}^{4} z_jz_k\overline{w}_j\overline{w}_k -$$
$$2(z_1z_4 - z_2z_3)\sum_{j=1}^{4} z_j\overline{w}_j(\overline{w}_1\overline{w}_4 - \overline{w}_2\overline{w}_3) + (z_1z_4 - z_2z_3)^2(\overline{w}_1\overline{w}_4 - \overline{w}_2\overline{w}_3)^2.$$

Note also that $d_{ij}(z) = z_{2i+j-2}$ for $i, j = 1, 2$. The conclusion follows by Theorem 2.1, using that for any polynomial $f(z,\overline{w}) := q(z)q(\overline{w})$, $f(M_T)(B) = q(T)Bq(T^*)$. □

Remark. Due to condition (iv): $\|d(Z)\| \leq 1$ of Definition 1, the function $(w,z) \mapsto C(z,\overline{w})(1_p - d(z)d(w)^*)$ is positive definite on $D \times D$. Hence by Kolmogorov's theorem it can be factored as

$$C(z,\overline{w})(1_p - d(z)d(w)^*) = a(z)a(w)^* \tag{10}$$

where $a = [a_{il}]_{i,l} : D \to B(\ell^2, \mathbb{C}^p)$. Moreover, we can take $a(\,\cdot\,)$ analytic on D. To this aim, use for instance the idea of [lemma 1.12, 1] providing analytic factorizations on compact subsets of D, together with Montel's theorem to get one globally. This factorization is not unique, in general. Suppose that there is a factorization (10) with all entries $a_{il} \in O(\overline{D})$ (it is an interesting question if this holds in general - to this aim, it would suffice to have it for $p \times q$ matrix balls). Then a slight generalization of Theorem 2.1 could be proved following the lines of [4, 14], so that the analogous of (1) be $I - XX^* = (\mathrm{tr}_p \otimes 1_{B(M.)})\left(a^t(T.)\Gamma a^t(T.)^*\right)$, where the right hand side term is defined as $so - \lim_{k\to\infty} \sum_{l=0}^{k} \sum_{i,j=1}^{p} a_{il}(T.)\Gamma_{ij} a_{jl}(T.)^*$; in particular, it follows that $I - XX^* \geq 0$. In this case the condition $\overline{M.^{\sim}} = M.$ is not necessary anymore. Whenever this condition holds, the previous equality would be an equivalent version of (1). To show for instance that it implies (1), we apply above $(1/C)(M_{T.})$, then – briefly speaking – apply to (10) the functional calculus of $M_{T.}$ and use $\frac{1}{C}(M_{T.}) \circ C(M_{T.}) = (\frac{1}{C} \cdot C)(M_{T.}) = I$ and the representation (3) of the right hand side of (1).

Remark. Let the domain D, the space $\mathcal{H}$ and the operator $X : M \to M.$ satisfy the hypotheses of Theorem 2.1. Let $\Gamma : M.^p \to M.^p$ be a nonnegative operator satisfying (1). Then any $F = F_U$ as in (iii) can be obtained by the known "lurking isometry" trick. In our present case, we proceed as follows [4]. Take any $L \in B(M.^p)$ such that $\Gamma = LL^*$. Set $L = [L_{ij}]_{ij,=1}^p$ with $L_{ij} : M. \to M.$, let $K_0 = M.^p$ and write $L = \begin{bmatrix} L_1 \\ \vdots \\ L_p \end{bmatrix} : K_0 \to M.^p$ where $L_j = [L_{j1} \ldots L_{jp}]$ for $j = \overline{1,p}$. The mapping

$$(11)\quad V : \ \Big(\sum_{j=1}^{p} L_j^* d_{jk}(T.)^* h\Big)_{k=1}^{q} \oplus (P_{E.} h) \ \mapsto \ (L_j^* h)_{j=1}^{p} \oplus (P_E X^* h) \qquad (h \in M.)$$

is a well defined isometry from the linear subspace of $K_0^q \oplus E.$ consisting of the vectors in the left hand side of (11) into $K_0^p \oplus E.$ Choose any Hilbert space $K \supset K_0$ and unitary $U : K^p \oplus E \to K^q \oplus E.$ such that U^* extends V. Then set $U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ with $u_{11} : K^p \to K^q$, $u_{12} : E \to K^q$, $u_{21} : K^p \to E.$, $u_{22} : E \to E.$ and write (2).

3. Obtaining concrete solutions

In what follows we show by an elementary example how Theorems 2.1, 2.6 and the last Remark from above could be applied to Carathéodory-Féjér interpolation problems.

Example Let $D = \{z \in \mathbb{C}^2 : \| \begin{bmatrix} z_1 & z_2 \\ 0 & z_1^2 \end{bmatrix} \| < 1\}$ and $a, b, c \in \mathbb{C}$ be given with $|a| < 1$. We show that the necessary and sufficient condition for the existence of a function f analytic on D with Schur norm $\|f\|_{\mathcal{S}} \le 1$ ($\Rightarrow \|f\|_\infty \le 1$) such that $f(0) = a$, $(\partial_1 f)(0) = b$ and $(\partial_2 f)(0) = c$ is that $|a|^2 + \sqrt{|b|^2 + |c|^2} \le 1$. Also, we find below a particular solution f. Define $l_1, \ldots, l_6$ as follows: if $|a|^2 + |b| < 1$, set $l_1 = \sqrt{1 - |a|^2}$, $l_2 = \frac{-\overline{a}b}{\sqrt{1-|a|^2}}$, $l_3 = \frac{\sqrt{(1-|a|^2)^2 - |b|^2}}{\sqrt{1-|a|^2}}$, $l_4 = \frac{-\overline{a}c}{\sqrt{1-|a|^2}}$, $l_5 = \frac{-\overline{b}c}{\sqrt{1-|a|^2}\sqrt{(1-|a|^2)^2-|b|^2}}$ and $l_6 = \frac{\sqrt{1-|a|^2}\sqrt{(1-|a|^2)^2-|b|^2-|c|^2}}{\sqrt{(1-|a|^2)^2-|b|^2}}$. If $|a|^2 + |b| = 1$

($\Rightarrow c = 0$), take l_1, l_2 as above and let $l_3, l_4, l_5 = 0$ and $l_6 := l_1$. Define also $\lambda_1, \dots, \lambda_4$ as follows: if $|a|^2 + \sqrt{|b|^2 + |c|^2} < 1$, set $\lambda_1 = \frac{\overline{a}\sqrt{(1-|a|^2)^2-|b|^2}}{(1-|a|^2)\sqrt{1+|c|^2}}$, $\lambda_2 = \frac{\overline{b}}{(1-|a|^2)\sqrt{1+|c|^2}}$, $\lambda_3 = \frac{\overline{c}}{\sqrt{(1-|a|^2)^2-|b|^2-|c|^2}\sqrt{1+|c|^2}}$ and $\lambda_4 = -\frac{\sqrt{(1-|a|^2)^2-|b|^2}}{\sqrt{1+|c|^2}}$. If $|a|^2 + \sqrt{|b|^2 + |c|^2} = 1$, let $\lambda_1, \lambda_2, \lambda_4 = 0$, $\lambda_3 = 1$ for $c \neq 0$, and $\lambda_1, \lambda_3, \lambda_4 = 0$, $\lambda_2 = 1$ for $c = 0$. Then for every a, b, c with $|a|^2 + \sqrt{|b|^2 + |c|^2} \leq 1$, set

$$\begin{aligned}\rho(z) = \quad & 1 - (\lambda_2 + \tfrac{l_2}{l_1})\, z_1 - \tfrac{l_6}{l_1}\, z_1^6 + (\tfrac{l_6}{l_1}\lambda_2 - \tfrac{l_5}{l_1}\lambda_3 + \tfrac{l_2 l_6}{l_1^2})\, z_1^7 + (\tfrac{l_2}{l_1^2}\lambda_2 - \tfrac{l_3}{l_1}\lambda_1)\, z_1^2 \\ & + (\tfrac{l_3 l_6}{l_1^2}\lambda_1 - \tfrac{l_2 l_6}{l_1^2}\lambda_2 + \tfrac{l_2 l_5 - l_3 l_4}{l_1^2}\lambda_3)\, z_1^8 - \tfrac{l_4}{l_1}\, z_2\end{aligned}$$

and define f on D by the equality

$$\begin{aligned}f(z) = a + \tfrac{1}{\rho(z)}\ \big(& b\, z_1 + c\, z_2 + (\lambda_4 l_3 - b\lambda_2)\, z_1^2 + (\lambda_4 l_5 - c\lambda_2)\, z_1 z_2 \\ & - b\tfrac{l_6}{l_1}\, z_1^7 + c\lambda_3 \tfrac{l_5}{l_1}\, z_1^7 z_2 + (c\lambda_3 \tfrac{l_3}{l_1} + b\lambda_2 \tfrac{l_6}{l_1} - b\lambda_3 \tfrac{l_5}{l_1} - \lambda_4 \tfrac{l_3 l_6}{l_1})\, z_1^8 \big).\end{aligned} \tag{12}$$

The solution f from above can be represented also as a fractional transform

$$f(z) = a + [\,v\ \ 0\,] \left(1_{K^2} - \begin{bmatrix} z_1 & z_2 \\ 0 & z_1^2 \end{bmatrix} \cdot \begin{bmatrix} r & s \\ t & u \end{bmatrix} \right)^{-1} \begin{bmatrix} z_1 & z_2 \\ 0 & z_1^2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \tag{13}$$

associated with the unitary U from below

$$U = \begin{bmatrix} \begin{bmatrix} r & s \\ t & u \end{bmatrix} & \begin{bmatrix} x \\ y \end{bmatrix} \\ [\ v \ \ 0\ ] & a \end{bmatrix} : \begin{matrix} K^2 \\ \oplus \\ \mathbb{C} \end{matrix} \rightarrow \begin{matrix} K^2 \\ \oplus \\ \mathbb{C} \end{matrix} \ ; \quad \begin{matrix} \begin{bmatrix} r & s \\ t & u \end{bmatrix} : K^2 \to K^2 \\[1ex] \begin{bmatrix} x \\ y \end{bmatrix} : \mathbb{C} \to K^2 \\[1ex] [\,v\ 0\,] : K^2 \to \mathbb{C} \end{matrix}$$

where $K = \mathbb{C}^6$ and the mappings $r, s, t, u : K \to K$, $x, y : \mathbb{C} \to K$ and $v : K \to \mathbb{C}$ are given by

$$r = \begin{bmatrix} l_2/l_1 & l_3/l_1 & 0 & 0 & 0 & 0 \\ \lambda_1 & \lambda_2 & \lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad s = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad x = \begin{bmatrix} b/l_1 \\ \lambda_4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$t = \begin{bmatrix} l_4/l_1 & l_5/l_1 & l_6/l_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad u = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad y = \begin{bmatrix} c/l_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$v = \begin{bmatrix} l_1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let us see how the function f from above has been obtained. By the equality $\begin{bmatrix} z_1 & z_2 \\ 0 & z_1^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i \cdot arg(z_1^2)} \end{bmatrix} \begin{bmatrix} z_1 & z_2 \\ 0 & |z_1^2| \end{bmatrix}$ we see that $z \in D$ if and only if $\| \begin{bmatrix} z_1 & z_2 \\ 0 & |z_1|^2 \end{bmatrix} \| < 1$, which easily shows that D is convex. The conditions of Definition 1.2 are then fulfilled.

Existence of the solutions We use Theorem 2.6. Let $S = \{0\}$ and $A_0 = \{(0,0),(1,0),(0,1)\}$. Set E, $E_\cdot = \mathbb{C}$ and $c_{(0,(0,0))} = a$, $c_{(0,(1,0))} = b$, $c_{(0,(0,1))} = c$. Order $\Lambda \equiv \{1,2\} \times A_0$ as follows: $\{ (1,(0,0)), (1,(1,0)), (1,(0,1)), (2,(0,0)), (2,(1,0)), (2,(0,1)) \} \overset{\text{not}}{=} \{1,2,3,4,5,6\}$. Then $G = [G_{\rho\sigma}]_{\rho,\sigma\in\Lambda} = [G_{\text{ij}}]_{\text{i,j}=1}^{6}$ is a 6×6 matrix of numbers. Write the equations $(5) = (5)_{\alpha\beta st}$ for α, $\beta \in A_0$ and s, $t = 0$. To this aim, note that $d_{21}(z) \equiv 0$ and $(\partial^\gamma d_{22})(0) = (\partial^\gamma(z_1^2))(0) = 0$ for all $\gamma \in A_0$. Hence if the indices i, $j \in \{1,2\}$ and $(i,j) \neq (1,1)$, then: either $i = 2$ whence $(\partial^\gamma d_{i1})(0) = (\partial^\gamma d_{21})(0) = 0$ and $(\partial^\gamma d_{i2})(0) = (\partial^\gamma d_{22})(0) = 0$ for $\gamma \in A_0$, namely $(\partial^\gamma d_{ik})(0) = 0$; or $j = 2$ whence $(\partial^\gamma d_{j1})(0) = (\partial^\gamma d_{21})(0) = 0$ and $(\partial^\gamma d_{j2})(0) = (\partial^\gamma d_{22})(0) = 0$, namely $(\partial^\gamma d_{jk})(0) = 0$. Hence the 2nd term, say

$\sigma_{\alpha\beta}$, in the right hand side of (5) is

$$\sigma_{\alpha\beta} = \sum_{\substack{\delta\le\alpha\\ \lambda\le\beta}} \sum_{k=1}^{2} \frac{(\partial^{\alpha-\delta} d_{1k})(0)}{(\alpha-\delta)!} \frac{\overline{(\partial^{\beta-\lambda} d_{1k})(0)}}{(\beta-\lambda)!} G_{(1,\delta)(1,\lambda)} =$$

$$\sum_{\substack{\delta\le\alpha\\ \lambda\le\beta}} (\partial^{\alpha-\delta} d_{11})(0)\overline{(\partial^{\beta-\lambda} d_{11})(0)} G_{(1,\delta)(1,\lambda)} + \sum_{\substack{\delta\le\alpha\\ \lambda\le\beta}} (\partial^{\alpha-\delta} d_{12})(0)\overline{(\partial^{\beta-\lambda} d_{12})(0)} G_{(1,\delta)(1,\lambda)}.$$

Now $(\partial^\gamma d_{11})(0) = (\partial^\gamma z_1)(0) = 1$ if $\gamma = (1,0)$, and it is 0 otherwise, while $(\partial^\gamma d_{12})(0) = (\partial^\gamma z_2)(0) = 1$ if $\gamma = (0,1)$, and it is 0 otherwise. Hence $\sigma_{\alpha\beta} = G_{(1,(0,0))(1,(0,0))}$ if either $\alpha = \beta = (1,0)$ or $\alpha = \beta = (0,1)$, and it is null otherwise. Since the matrix $[G_{\rho\delta}]_{\rho,\delta}$ is selfadjoint, equality $(5)_{\alpha\beta st}$ is identical to $(5)_{\beta\alpha ts}$. We write $(5)_{\alpha\beta 00}$ as follows:

$$\begin{array}{lll}
\alpha & \beta & \\
(0,0) & (0,0) & 1-|a|^2 = \sum_{j=1}^2 G_{(j,(0,0))(j,(0,0))} = G_{11} + G_{44} \\
(0,0) & (1,0) & -a\overline{b} = \sum_{j=1}^2 G_{(j,(0,0))(j,(1,0))} = G_{12} + G_{45} \\
(0,0) & (0,1) & -a\overline{c} = \sum_{j=1}^2 G_{(j,(0,0))(j,(0,1))} = G_{13} + G_{46} \\
(1,0) & (1,0) & -|b|^2 = \sum_{j=1}^2 G_{(j,(1,0))(j,(1,0))} - G_{(1,(0,0))(1,(0,0))} = G_{22} + G_{55} - G_{11} \\
(1,0) & (0,1) & -b\overline{c} = \sum_{j=1}^2 G_{(j,(1,0))(j,(0,1))} = G_{23} + G_{56} \\
(0,1) & (0,1) & -|c|^2 = \sum_{j=1}^2 G_{(j,(0,1))(j,(0,1))} - G_{(1,(0,0))(1,(0,0))} = G_{33} + G_{66} - G_{11}\,.
\end{array}$$

Note $G = \begin{bmatrix} g & * \\ * & g' \end{bmatrix}$ where $g = [G_{\mathrm{i\,j}}]_{\mathrm{i,j}=1}^3$ and $g' = [G_{3+\mathrm{i}\ 3+\mathrm{j}}]_{\mathrm{i,j}=1}^3$ while the symbol $*$ stands for compressions of G the entries $G_{\mathrm{i\,j}}$ of which are not involved in the equations (5). We can assume that $G = \begin{bmatrix} g & 0 \\ 0 & g' \end{bmatrix}$. Define the map π on $M_3(\mathbb{C})$ by $\pi\ [c_{ij}]_{i,j=1}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_{11} & 0 \\ 0 & 0 & c_{11} \end{bmatrix}$. Therefore (5) means the existence of two nonnegative 3×3 matrices g, g' such that $\mu_{abc} := \begin{bmatrix} 1-|a|^2 & -a\overline{b} & -a\overline{c} \\ -\overline{a}b & -|b|^2 & -b\overline{c} \\ -\overline{a}c & -\overline{b}c & -|c|^2 \end{bmatrix} = g - \pi g + g'$. One checks, using $\pi^2 = 0$, that this is equivalent to saying that $\mu_{abc} + \pi\mu_{abc}$ $(= g)$ is nonnegative, which leads to condition $|a|^2 + \sqrt{|b|^2+|c|^2} \le 1$. For general problems of this type, whenever numerical values of data like a, b, c are given we are lead to the question of finding a matrix $G \ge 0$ satisfying a set of linear restrictions.

The functional Hilbert space Let $\mathcal{H}$ be the d-space with reproducing kernel

$$C(z,\overline{w}) = \det(1_2 - d(z)d(w)^*)^{-2} = \det\left(1_2 - \begin{bmatrix} z_1 & z_2 \\ 0 & z_1^2 \end{bmatrix}\begin{bmatrix} \overline{w}_1 & 0 \\ \overline{w}_2 & \overline{w}_1^2 \end{bmatrix}\right)^{-2} =$$

$$\begin{vmatrix} 1 - z_1\overline{w}_1 - z_2\overline{w}_2 & -z_2\overline{w}_1^2 \\ -z_1^2\overline{w}_2 & 1 - z_1^2\overline{w}_1^2 \end{vmatrix}^{-2} = (1 - z_1\overline{w}_1 - z_2\overline{w}_2 - z_1^2\overline{w}_1^2 + z_1^3\overline{w}_1^3)^{-2},$$

see Theorem 2.3. For each $\alpha \in \mathbb{Z}_+^2$, let $C_0^\alpha \in \mathcal{H}$ be the unique function such that

$$(\partial^\alpha h)(0)/\alpha! = \langle h, C_0^\alpha\rangle \qquad (h \in \mathcal{H}), \tag{14}$$

namely $C_0^\alpha(z) = \overline{(\partial^\alpha C_z)(0)}/\alpha!$ $(z \in D)$, see (6). Letting $h := C_0^\nu$ with $\nu \in \mathbb{Z}_+^2$ in (14) and using the formula of C_0^α and the equality $C(x,\overline{z}) = \overline{C(z,\overline{x})}$ provides us with

$$\langle C_0^\nu, C_0^\alpha\rangle = \frac{(\partial_1^\alpha \partial_2^\nu C)(0,0)}{\alpha!\nu!} \qquad (\nu, \alpha \in \mathbb{Z}_+^2). \tag{15}$$

The factorization We find an operator L such that $\Gamma = LL^*$ where Γ is the operator in Theorem 2.1, (ii). To this aim, we factorize the matrix $g = \mu_{abc} + \pi\mu_{abc}$ as $g = ll^*$ with $l \in M_3(\mathbb{C})$. For instance, we can search for a triangular matrix $l = \begin{bmatrix} l_1 & 0 & 0 \\ l_2 & l_3 & 0 \\ l_4 & l_5 & l_6 \end{bmatrix}$, the entries $l_1, \ldots, l_6$ of which we can successively found by solving the equation $ll^* = \mu_{abc} + \pi\mu_{abc}$ (this is the case in our example). Then we follow the proof of Theorem 2.6. Let $M = M_\cdot = sp\,\{C_0^{(0,0)}, C_0^{(1,0)}, C_0^{(0,1)}\} \subset \mathcal{H}$. Let $\Gamma = [\Gamma_{ij}]_{i,j=1}^2 \in B(M^2)$, with $\Gamma_{ij} : M \to M$ for $i,j = 1,2$, be the nonnegative operator from Theorem 2.1. Remind that Γ provides the nonnegative matrix G in Theorem 2.6 via the equalities

$$\langle \Gamma_{ij} C_0^\beta, C_0^\alpha\rangle = G_{(i,\alpha)(j,\beta)} \qquad (\alpha,\, \beta \in A_0) \tag{16}$$

(see (9)). Using (15), we orthonormalize Gram–Schmidt the vectors $C_0^{(0,0)}, C_0^{(1,0)}$ and $C_0^{(0,1)}$. That is, we compute $\partial_2^{(1,0)} C$ and then $\langle C_0^{(1,0)}, C_0^{(0,1)}\rangle = \frac{(\partial_1^{(0,1)}\partial_2^{(1,0)} C)(0,0)}{(0,1)!\,(1,0)!}$ etc. It follows that the vectors $e_{(0,0)} := C_0$, $e_{(1,0)} := C_0^{(1,0)}/\sqrt{2}$ and $e_{(0,1)} := C_0^{(0,1)}/\sqrt{2}$ define an orthonormal basis of M. We identify $M \equiv \mathbb{C}^3$ so that $e_{(0,0)} \equiv (1,0,0)$, $e_{(1,0)} \equiv (0,1,0)$ and $e_{(0,1)} \equiv (0,0,1)$. Hence the vectors $e_{(0,0)} \oplus 0$, $e_{(1,0)} \oplus 0$, $e_{(0,1)} \oplus 0$ and $0 \oplus e_{(0,0)}$, $0 \oplus e_{(1,0)}$, $0 \oplus e_{(0,1)}$ define an orthonormal basis of $M \oplus M \equiv \mathbb{C}^6$. Let $G' = \left[G'_{(i,\alpha)(j,\beta)}\right]_{(i,\alpha),(j,\beta)\in\Lambda}$ be the matrix of the operator

$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \in B(M \oplus M) \equiv M_6(\mathbb{C})$ with respect to this basis. Then we easily check that $\langle \Gamma_{ij} e_\beta, e_\alpha \rangle = G'_{(i,\alpha)(j,\beta)}$ for $(i,\alpha),(j,\beta) \in \{1,2\} \times A_0$. Now, if a given basis $(C_0^\gamma)_\gamma$ of M provides an orthonormal basis $(e_\alpha)_\alpha$ as above and we know the coefficients $b_{\alpha\gamma}$ such that $e_\alpha = \sum_\gamma b_{\alpha\gamma} C_0^\gamma$, then plugging the e_α's in the previous equality and comparing with (16) gives $G'_{(i,\alpha)(j,\beta)} = \sum_{\lambda,\gamma} b_{\beta\lambda} \overline{b}_{\alpha\gamma} G_{(i,\gamma)(j,\lambda)}$. Using this equation we easily see that any factorization $G = \mathcal{L}\mathcal{L}^*$ of the matrix $G = [G_{\rho\delta}]_{\rho,\delta\in\Lambda}$ provides a factorization $G' = LL^*$ of the matrix G' of Γ, that is, a factorization $\Gamma = LL^*$ of the operator $\Gamma \equiv G'$ by means of the formulas $L_{(i,\alpha)(k,\nu)} = \sum_\gamma \overline{b}_{\alpha\gamma} \mathcal{L}_{(i,\gamma)(k,\nu)}$. In our present case, for $\mathcal{L} = \begin{bmatrix} l & 0 \\ 0 & 0 \end{bmatrix}$ the operator Γ $(\equiv G')$ has the form $\begin{bmatrix} \Gamma_{11} & 0 \\ 0 & 0 \end{bmatrix}$. Then $L = \begin{bmatrix} L_{11} & 0 \\ 0 & 0 \end{bmatrix}$, and we obtain $L_{11} = \begin{bmatrix} l_1 & 0 & 0 \\ l_2/\sqrt{2} & l_3/\sqrt{2} & 0 \\ l_4/\sqrt{2} & l_5/\sqrt{2} & l_6/\sqrt{2} \end{bmatrix}$. Hence

$$\begin{aligned} L_{11}^* e_{(0,0)} &= \overline{l}_1 e_{(0,0)} \\ L_{11}^* e_{(1,0)} &= \frac{\overline{l}_2}{\sqrt{2}} e_{(0,0)} + \frac{\overline{l}_3}{\sqrt{2}} e_{(1,0)} \\ L_{11}^* e_{(0,1)} &= \frac{\overline{l}_4}{\sqrt{2}} e_{(0,0)} + \frac{\overline{l}_5}{\sqrt{2}} e_{(1,0)} + \frac{\overline{l}_6}{\sqrt{2}} e_{(0,1)} \,. \end{aligned} \tag{17}$$

The induced isometry The operator L obtained in the previous subsection will provide now the isometry V, as described by the Remark at the end of Section 2. We compute the values, that we denote by v_1, v_2 resp. v_3, of the vector in left hand side of (11) for $h = e_{(0,0)}, e_{(1,0)}$ resp. $e_{(0,1)}$. Let $K_0 = K = M \oplus M$. Write $L = \begin{bmatrix} L_{11} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ where $L_1 = [\, L_{11} \;\; 0\,] : K \to M$ and $L_2 = [\,0 \;\; 0\,] : K \to M$. For any $h \in M$ we have $L_1^* h = (L_{11}^* h) \oplus 0$ and $L_2^* h = 0$. Hence

$$\begin{aligned} (\sum_{j=1}^{2} L_j^* d_{jk}(T)^* h)_{k=1}^2 &= (L_1^* d_{1k}(T)^* h)_{k=1}^2 = ((L_{11}^* d_{1k}(T)^* h) \oplus 0)_{k=1}^2 = \\ &((L_{11}^* d_{11}(T)^* h) \oplus (0,0,0)\,,\ (L_{11}^* d_{12}(T)^* h) \oplus (0,0,0))\,. \end{aligned}$$

We have $d_{1k}(T)^* = d_{1k}(Z)^*|_M = T_{d_{1k}}^*$ for $k = 1,2$. Then using (7) for the Toeplitz operator of symbol $\varphi(z) := d_{11}(z) = z_1$ we obtain $d_{11}(T)^* e_{(0,0)} = 0$, $d_{11}(T)^* e_{(1,0)} = \frac{1}{\sqrt{2}} e_{(0,0)}$ and $d_{11}(T)^* e_{(0,1)} = 0$. Using now (7) for $\varphi(z) := d_{12}(z) = z_2$ we obtain $d_{12}(T)^* e_{(0,0)} = 0$, $d_{12}(T)^* e_{(1,0)} = 0$ and $d_{12}(T)^* e_{(0,1)} =$

$\frac{1}{\sqrt{2}}e_{(0,0)}$. Also, we have $P_{\mathbb{C}}e_{(0,0)}=1$, $P_{\mathbb{C}}e_{(1,0)}=0$ and $P_{\mathbb{C}}e_{(0,1)}=0$. Using (17) and the computations of $d_{1k}(T)^*e_\alpha$ and $P_{\mathbb{C}}e_\alpha$ we get

$$v_1:=(\sum_{j=1}^{2}L_j^*d_{jk}(T)^*e_{(0,0)})_{k=1}^2\oplus P_{\mathbb{C}}e_{(0,0)}=((0,0,0)\oplus(0,0,0),\ (0,0,0)\oplus(0,0,0))\oplus 1,$$

as well as $v_2=((\frac{\bar{l}_1}{\sqrt{2}},0,0)\oplus 0_3\,,\ 0_3\oplus 0_3)\oplus 0$ and $v_3=(0_3\oplus 0_3\,,\ (\frac{\bar{l}_1}{\sqrt{2}},0,0)\oplus 0_3)\oplus 0$. We compute now the values w_1, w_2 and w_3 of the right hand side of (11) for $h=e_{(0,0)},e_{(1,0)}$ and $e_{(0,1)}$ respectively. Using (8), we obtain $X^*e_{(0,0)}=\overline{a}e_{(0,0)}$, $X^*e_{(1,0)}=\frac{1}{\sqrt{2}}\overline{b}e_{(0,0)}+\overline{a}e_{(1,0)}$ and $X^*e_{(0,1)}=\frac{1}{\sqrt{2}}\overline{c}e_{(0,0)}+\overline{a}e_{(0,1)}$. Hence $P_{\mathbb{C}}X^*e_{(0,0)}=\overline{a}$, $P_{\mathbb{C}}X^*e_{(1,0)}=\frac{1}{\sqrt{2}}\overline{b}$ and $P_{\mathbb{C}}X^*e_{(0,1)}=\frac{1}{\sqrt{2}}\overline{c}$. Also, for any $h\in M$ we have $(L_j^*h)_{j=1}^2=(L_1^*h,L_2^*h)=((L_{11}^*h)\oplus 0_3,0_3\oplus 0_3)$. Then using again (17) we get

$$w_1:=(L_j^*e_{(0,0)})_{j=1}^2\oplus P_{\mathbb{C}}X^*e_{(0,0)}=((\bar{l}_1,0,0)\oplus(0,0,0)\,,\ (0,0,0)\oplus(0,0,0))\oplus\overline{a}$$

and similarly $w_2=((\frac{\bar{l}_2}{\sqrt{2}},\frac{\bar{l}_3}{\sqrt{2}},0)\oplus 0_3,0_3\oplus 0_3)\oplus\frac{\overline{b}}{\sqrt{2}}$, $w_3=((\frac{\bar{l}_4}{\sqrt{2}},\frac{\bar{l}_5}{\sqrt{2}},\frac{\bar{l}_6}{\sqrt{2}})\oplus 0_3,0_3\oplus 0_3)\oplus\frac{\overline{c}}{\sqrt{2}}$. Therefore, the map (11) acts isometricaly between the linear subspaces $sp\,\{v_1,v_2,v_3\}\ \subset K^2\oplus\mathbb{C}$ and $sp\,\{w_1,w_2,w_3\}\ \subset K^2\oplus\mathbb{C}$ of the space $(K\times K)\oplus\mathbb{C}=((M\oplus M)\times(M\oplus M))\oplus\mathbb{C}\equiv(\mathbb{C}^6\times\mathbb{C}^6)\oplus\mathbb{C}\equiv\mathbb{C}^{13}$ by $v_j\ \mapsto\ w_j\ (j=1,2,3)$.

The solution as fractional transform We extend now the mapping V obtained above to a unitary matrix, that we shall write as U^*, on the whole space $(K\times K)\oplus\mathbb{C}$. To this aim, we use the canonical basis of $\mathbb{C}^{13}$ that we denote by $(f_j)_{j=1}^{13}$. Obviously we have $v_1=f_{13}$, $v_2=\frac{\bar{l}_1}{\sqrt{2}}f_1$ and $v_3=\frac{\bar{l}_1}{\sqrt{2}}f_7$, as well as $w_1=\bar{l}_1f_1+\overline{a}f_{13}$, $w_2=\frac{\bar{l}_2}{\sqrt{2}}f_1+\frac{\bar{l}_3}{\sqrt{2}}f_2+\frac{\overline{b}}{\sqrt{2}}f_{13}$ and $w_3=\frac{\bar{l}_4}{\sqrt{2}}f_1+\frac{\bar{l}_5}{\sqrt{2}}f_2+\frac{\bar{l}_6}{\sqrt{2}}f_3+\frac{\overline{c}}{\sqrt{2}}f_{13}$. Since $U^*v_j=w_j$ for $j=1,2,3$, the vectors U^*f_j for $j=1,7,13$ are known, that is, we set $U^*f_1=\frac{\sqrt{2}}{\bar{l}_1}w_2$, $U^*f_7=\frac{\sqrt{2}}{\bar{l}_1}w_3$ and $U^*f_{13}=w_1$. Note that $sp\,\{v_1,v_2,v_3\}=sp\,\{f_1,f_7,f_{13}\}$ is 3-dimensional and hence $sp\,\{w_1,w_2,w_3\}\subset sp\,\{f_1,f_2,f_3,f_{13}\}$ also is 3-dimensional. We shall find a unit vector $\omega\in sp\,\{f_1,f_2,f_3,f_{13}\}$, say $\omega=\overline{\lambda}_1f_1+\overline{\lambda}_2f_2+\overline{\lambda}_3f_3+\overline{\lambda}_4f_{13}$ with $\lambda_j\in\mathbb{C}$, such that $\omega\perp sp\,\{w_1,w_2,w_3\}$. Then set $U^*f_2=\omega$ and $U^*f_j=f_{j+1}$ $(j=3,\dots,6)$, $U^*f_j=f_j$ $(j=8,\dots,12)$. The numbers $\lambda_1,\dots,\lambda_4$ are obtained from $\langle\omega,f_j\rangle=0$ for $j=1,2,3,13$ and $\|\omega\|=1$. We can write now the matrix of U^* (and hence, of U) with respect to the basis $(f_j)_{j=1}^{13}$ of $K^2\oplus\mathbb{C}$ where $K=\mathbb{C}^6$. Then by (2) we obtain the solution $f=f_U$ given by (13). Obtaining the representation (12) of f is straightforward.

Remark. Doing the similar computation for the domain $\mathbb{D}^2$ ($\supset D$) instead of D shows that a solution f exists iff μ_{abc} can be represented as $g - pg + g' - p'g'$ for $g, g' \geq 0$ where p, p' are defined by $p\,[c_{ij}]_{i,j=1}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $p'\,[c_{ij}]_{i,j=1}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{11} \end{bmatrix}$ respectively. This condition is stronger than the one in section 3: $\mu_{abc} = g - \pi g + g'$. For example, the triple $(a, b, c) = (0, 1/\sqrt{2}, 1/\sqrt{2})$ satisfies $|a|^2 + \sqrt{|b|^2 + |c|^2} \leq 1$ but there are no matrices $g, g' \geq 0$ such that $\mu_{abc} = g - pg + g' - p'g'$, for in this case summing the equations $g_{11} + g'_{11} = 1 - |a|^2 = 1$, $g_{22} + g'_{22} - g_{11} = -|b|^2 = -1/2$ and $g_{33} + g'_{33} - g'_{11} = -|c|^2 = 1 - 1/2$ gives $g_{22} + g_{33} + g'_{22} + g'_{33} = 0$, and so g_{22}, g_{33}, g'_{22}, g'_{33} are null, whence g_{23}, $g'_{23} = 0$ too, and hence the equation $g_{23} + g'_{23} = -b\overline{c}$ gives $0 = -1/2$. Thus for these a, b, c there are no solutions f with $\|f\|_{\mathcal{S}} \leq 1$ over the bidisc $\mathbb{D}^2$. However, there exist solutions over D, for instance $f(z) = (z_1/\sqrt{2} + z_2/\sqrt{2} - z_1^7 z_2/2 + z_1^8)(1 + z_1^7/\sqrt{2})^{-1}$ obtained by replacing $a = 0$ and $b, c = 1/\sqrt{2}$ in (12).

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