Finding positive matrices subject to linear restrictions

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Abstract

We characterize the existence of a positive definite $l \times l$ matrix X the entries of which satisfy n nonhomogeneous linear conditions by the existence of a minimum for an associated function V, smooth and strictly convex on $\mathbb{R}^n$. If there exist solutions $X > 0$, then $\lim_{\|x\| \to \infty} V(x) = +\infty$ and the critical point x^0 of V can be approximated by the conjugate gradients method. Knowing x^0 provides, by a simple analytic formula, the unique solution X maximizing the entropy $-\mathrm{tr}\,(X \ln X) = -\sum_{j=1}^{l} \lambda_j \ln \lambda_j$ (where $\lambda_1, \ldots, \lambda_l$ are the eigenvalues of X) subject to the given restrictions. Related results are obtained in the semipositive definite case, too.

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Introduction

We consider the question of finding positive definite $l \times l$ matrices $X = [X_{jk}]_{j,k=1}^{l}$ of numbers X_{jk} satisfying a given set of n nonhomogeneous linear conditions: $\sum\limits_{j,k=1}^{l} \alpha_{ijk} X_{jk} = \beta_i$ for $i = 1, \ldots, n$. That is, we seek for $X > 0$ in the operator sense: $\langle Xc, c \rangle = \sum\limits_{j,k=1}^{l} X_{jk} c_k \overline{c}_j > 0$ for any tuple $c \neq 0$ of numbers $c_1, \ldots, c_l$, satisfying the equations

$$\mathrm{tr}\,(A_i X) = \beta_i \qquad (i = \overline{1, n}) \tag{1}$$

where $A_i := [\alpha_{ikj}]_{j,k=1}^{l}$ and tr denotes the trace, $\mathrm{tr}\, A = \sum\limits_{j=1}^{l} a_{jj}$ for any matrix $A = [a_{jk}]_{j,k=1}^{l}$. We can always suppose that A_i are selfadjoint and β_i real, since (in the complex case, for instance) conditions (1) are equivalent to $\mathrm{tr}\,((A_i + A_i^*)X) = \beta_i + \overline{\beta}_i$ and $\mathrm{tr}\,(\mathrm{i}(A_i^* - A_i)X) = \mathrm{i}(\overline{\beta}_i - \beta_i)$, using that $\overline{\mathrm{tr}\,(AX)} = \mathrm{tr}\,(AX)^* = \mathrm{tr}\,(A^*X)$. We shall simultaneously consider the real

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and complex cases. One can assume that $A_1, \dots, A_n$ are linearly independent over $\mathbb{R}$. Our main results are propositions 1 – 3, see also remarks 11 (i)-(iii).

The problem from above is an extensively studied one, its classical solutions and comprehensive references can be found in [3, 12, 16], see also [4, 7, 8]. Usually one considers also a dual problem, namely determining whether for given $l \times l$ matrices A_0, $A_1, \dots, A_n$, there exists a matrix $Y > 0$ of the form

$$Y = A_0 + \sum_{i=1}^{n} x_i A_i. \tag{2}$$

The latter problem is solved by maximizing the minimum eigenvalue of Y of the form (2) in the variables $x_1, \dots, x_n$, and for this a standard algorithm can be applied. When problem (2) admits solutions, and no positive definite matrix of the form $\sum_{i=1}^{n} x_i A_i$ exists, then there is also a distinguished solution of (1), which is the matrix maximizing $\ln \det Y$ over the set of all $Y > 0$ of the form (2). Actually, both problems (1) and (2) can be respectively associated with certain *semidefinite programs*, concerned with minimizing a linear functional subject to the constraint that an affine combination of symmetric matrices is positive semidefinite. Namely, set $\beta^t \cdot x = \sum_{i=1}^{n} \beta_i x_i$, let $A(x) = A_0 + \sum_{i=1}^{n} x_i A_i$ and define

$$p^* = \inf\{\, \beta^t \cdot x \,:\, A(x) \geq 0\}$$

and

$$q^* = \sup\{-\mathrm{tr}\,(A_0 X) \,:\, X \geq 0,\, \mathrm{tr}\,(A_i X) = \beta_i\}.$$

The key property of the dual program is that it yields bounds on the optimal value of the primal one and viceversa. That is, suppose that either (1) has solutions $X > 0$, or (2) has solutions x with $A(x) > 0$. Then $p^* = q^*$, see for instance [12, 16]. If both conditions hold, the optimal sets of the programs for p^* and q^* are nonempty. One defines in this case

$$\phi(x) = \ln \det A(x)^{-1} \qquad \text{if } A(x) > 0,$$

and $\phi(x) = +\infty$ otherwise. Then ϕ is a *barrier function* [12] for the set $\{x \,:\, A(x) > 0\}$, namely its values increase to infinity as x approaches to the boundary. Set $\overline{p} = \sup\{\beta^t \cdot x \,:\, A(x) > 0\}$. It can be shown [16] that for every $\gamma \in (p^*, \overline{p})$ there exist vectors x satisfying $A(x) \geq 0$, $\beta^t \cdot x = \gamma$, and the set of all such solutions x is bounded. Moreover, the *analytic center* of this linear matrix inequality, that is, the unique point x^* with $\beta^t \cdot x^* = 0$ such that

$$\min_{A(x)>0, \beta^t \cdot x = \gamma} \phi(x) = \phi(x^*),$$

satisfies the equations

$$\mathrm{tr}\,(A_i\, A(x^*)^{-1}) = \lambda \beta_i \qquad (i = \overline{1, n})$$

for a Lagrange multiplier λ, that proves to be positive (see for instance [16]). One obtains in this way a positive matrix $X := A(x^*)^{-1}/\lambda$ satisfying (1) and $-\mathrm{tr}\,(A_0 X) = \gamma - l/\lambda$. Thus briefly speaking, a necessary and sufficient condition for the existence of a solution $X \geq 0$ of (1) is that $\beta^t \cdot x \geq 0$ for all x with $\sum_{j=1}^{n} x_j A_j \geq 0$, in which case one can search for an analytic center x^* etc providing a solution X. The study of the optimality conditions and various properties for such semidefinte programs started in the sixties, and later on the related problem of minimizing the maximum eigenvalue for hermitian matrices was considered, too. Then the interior point method, introduced in [10] as an important tool for the linear programming, was generalized to a larger class of optimization problems by using barrier functions [12]. This method applies in particular to semidefinite programs as mentioned above.

The existence of a positive completion for a given partial matrix (which is a particular case of problem (1)) has been firstly characterized under certain hypotheses in [5, 6, 8]. The more general case of the linearly constrained completions has been considered subsequently in [11]. Namely, suppose that we seek for a matrix $X = [z_{ij}]_{i,j=1}^{l} > 0$ the entries z_{ij} of which are given for (i, j) in a prescribed subset $S \subset \{1, \ldots, l\} \times \{1, \ldots, l\}$. Moreover, assume that all the diagonal entries z_{ii} $(i = \overline{1, l})$ are specified. Since in particular X must be selfadjoint, the pattern S may be symmetric, with the condition $z_{ji} = \overline{z}_{ij}$ necessarily fulfilled. We call X a *completion* of the *partial matrix* $[z_{ij}]_{(i,j) \in S}$. Let S' be the set complementary to S, set $p = \mathit{card}\, S'$ and let $z = (z_{ij})_{(j,k) \in S'}$ denote a vector in $\mathbb{C}^p$, consisting of the unspecified entries of X. Let $X(z)$ denote our partial Hermitian matrix and $H(z) = X(z)^{-1}$ be the formal inverse of $X(z)$. Define also the vector $h(z) := (H_{ij}(z))_{(i,j) \in S'}$. Let c be a fixed vector and B a fixed matrix. It has been shown [11] that if the problem

$$\max \{\det X(z) \,:\, X(z) > 0,\, Bz = c\} \tag{3}$$

is feasible, then it has a unique optimal solution, which moreover is the unique feasible point z for which $h(z)$ is in the range of B^*. This provides in particular (for $B = 0$) a previously known result, that the det-maximizing completion of such a partial matrix is characterized as the unique completion having a zero in the inverse on every position in which X has an unspecified entry [5–8].

For certain positive matrix completion problems, one knows conditions that are sufficient for the existence of a solution. For example, suppose that all diagonal entries are given, and that $[z_{ij}]_{(i,j)\in S}$ is a *positive definite partial matrix*, that is, all diagonal minors that can be made of the given entries are positive. Then a positive completion exists whenever the undirected *graph associated to* S, the edges of which are $\{i, j\}$ for $(i, j) \in S$ with $i \neq j$, is *chordal* – that is, all its minimal cycles have length ≤ 3. For this topic we refer for instance to [1], where a problem similar to (1) also was studied, involving linear restrictions on both X, X^{-1} and solved by a fixed point procedure. Note that by our approach it is also allowed that the diagonal entries of the partial matrix be unspecified.

However, the question of recognizing whether a concrete given problem of the forms (1) – (3) admits solutions is generally open. Also, if the problem has solutions there is generally no closed formula solving it. Checking the feasibility and finding solutions requires then various algorithms of approximation. Our approach shows, in particular, that the barrier function $-\ln\det X$ usually involved in such algorithms can be substituted by $-tr\,(X \ln X)$.

Main results

Let $e^A = \sum\limits_{k\geq 0} \frac{1}{k!}A^k$ denote the exponential of a matrix A. Set $b_j = \beta_j e$.

1 Proposition *The function $V = V(x)$ defined for $x = (x_j)_{j=1}^n \in \mathbb{R}^n$ by*

$$V(x) = \mathrm{tr}\,(e^{\sum\limits_{j=1}^{n} x_j A_j}) - \sum_{j=1}^{n} x_j b_j$$

is strictly convex, $[\frac{\partial^2 V}{\partial x_i \partial x_j}(x)]_{i,j=1}^n > 0$ and $\frac{\partial V}{\partial x_j}(x) = tr\,(A_j e^{\sum\limits_{i=1}^{n} x_j A_j}) - b_j$.

2 Theorem *The system of equations* (1) *admits solutions $X > 0$ if and only if $\lim\limits_{\|x\|\to\infty} V(x) = +\infty$, that is, if and only if V has a critical point. In this* *case,* (1) *has also the particular solution*

$$X_0 = e^{\sum\limits_{j=1}^{n} x_j^0 A_j - I_l}$$

where $x^0 = (x_j^0)_{j=1}^n$ *is the unique (minimum) critical point of* V*. Also,* X_0 *is the unique matrix maximizing* $-tr\,(X \ln X) = -\sum\limits_{j=1}^{l} \lambda_j \ln \lambda_j$ *(where* $\lambda_1, \dots, \lambda_l$ *are the eigenvalues of* X*) amongst all solutions* $X \geq 0$ *of* (1).

3 Theorem *If the linear span of the* A_j*'s contains positive matrices, then:*

(a) *The system of equations* (1) *admits solutions* $X \geq 0$ *but all of them have* $\det X = 0$*, if and only if* V *is bounded from below and there exist vectors* $x \neq 0$ *such that* $\lim\limits_{t \to +\infty} V(tx)$ *is finite. The set* C *of all such vectors is then a convex cone,* V *has no critical points, and* $\lim\limits_{t \to +\infty} V(tx) = \dim \ker \sum\limits_{j=1}^{n} x_j A_j =$ *constant for all* $x \in C$*. Moreover,* C *consists of those* $x \neq 0$ *such that* $\sum\limits_{j=1}^{n} x_j A_j \leq 0$ *and* $\sum\limits_{j=1}^{n} \beta_j x_j = 0$*.*

In this case, for any integer $k \geq 1$ *there exists an* $X_k > 0$ *with* $tr\,(A_j X_k) = \beta_j + \frac{1}{k} tr\,(A_j)$ *for* $j = \overline{1, n}$*, any such sequence* $(X_k)_k$ *is bounded, and all its accumulation points* X *are solutions of* (1)*;*

(b) *The system* (1) *has no solutions* $X \geq 0$ *at all, if and only if* $\inf V = -\infty$*. In this case, there exist vectors* $x \neq 0$*, the set of which is a convex cone* C*, such that* $\lim\limits_{t \to +\infty} V(tx) = -\infty$*, and* V *has no critical points. Moreover,* C *consists of those* $x \neq 0$ *with* $\sum\limits_{j=1}^{n} x_j A_j \leq 0$ *and* $\sum\limits_{j=1}^{n} \beta_j x_j > 0$*.*

We state below some preliminaries, necessary later on in the proofs.

Let then $A_1, \dots, A_n$ be linearly independent selfadjoint $l \times l$ matrices and $b_1, \dots, b_n$ be real numbers. Let M_l denote the space of all $l \times l$ matrices, acting as usual on the l-dimensional Euclidian space and endowed with the operator norm $\|A\| = \sup\limits_{\|h\| \leq 1} \|Ah\|$ for $A \in M_l$. Let M_l^+ denote the cone of all nonnegative matrices $X \in M_l$, that is, $\langle Xh, h \rangle \geq 0$ for every vector h where $\langle \cdot, \cdot \rangle$ denotes the inner product. Since the function $h(t) = -t \ln t$ (with $0 \ln 0 = 0$) is continuous on $[0, \infty)$, it induces by continuous functional calculus a mapping $X \mapsto h(X)$ from M_l^+ into M_l. For every $X \geq 0$, set $H(X) = \operatorname{tr} h(X)$. Using that $h(U^* X U) = U^* h(X) U$ for every $X \geq 0$ and unitary $U \in M_l$, we obtain $H(X) = -\sum_{j=1}^{l} \lambda_j \ln \lambda_j$ where the λ_j's are the eigenvalues of X counted according to the multiplicity so that $0 \leq \lambda_1 \leq \cdots \leq \lambda_l$. By analogy with the corresponding Boltzman-Shannon's notion for probability densities, we call $H(X)$ the *entropy* of X. This differs from other notions used under the same name in [1, 2, 3, 8] as barrier functions (of the form $H(X) = \ln \det X +$ linear terms).

4 Lemma *The function H is continuous on M_l^+ and $\lim_{\|X\| \to \infty} H(X) = -\infty$. Also, H is of class C^1 on the interior of M_l^+ and the differential H' of H in a point $X > 0$ is given by $H'(X)Y = -tr((I + \ln X)Y)$ for all $Y \in M_l$.*

Proof Since the function h is concave, we have

$$\frac{1}{l} H(X) = -\frac{1}{l} \sum_{j=1}^{l} \lambda_j \ln \lambda_j = \frac{h(\lambda_1) + \cdots + h(\lambda_l)}{l} \leq h(\frac{\lambda_1 + \cdots + \lambda_l}{l})$$

$$= -\frac{\sum_{j=1}^{l} \lambda_j}{l} \ln \frac{\sum_{j=1}^{l} \lambda_j}{l} = -\frac{\operatorname{tr} X}{l} \ln \frac{\operatorname{tr} X}{l}.$$

The uniform norm $\| \cdot \|$ and the trace norm $\| \cdot \|_1$ are equivalent on M_l. Thus $\|X\| \to \infty$ implies that $\operatorname{tr} X$ $(= \|X\|_1$ for $X \geq 0) \to \infty$, too. Hence the right hand side of the previous estimate tends to $-\infty$. Then $H(X) \to -\infty$ as $\|X\| \to \infty$ with $X \geq 0$. The continuity of H holds using either the upper semicontinuity of the spectrum mapping $X \mapsto (\lambda_1, \ldots, \lambda_l)$, or the fact that all $X \geq 0$ are selfadjoint. For $X > 0$, $H(X) = -\operatorname{tr}(X \ln X)$ where the logarithm is defined on $\mathbb{C} \setminus (-\infty, 0]$ by $\ln(re^{\mathrm{i}\theta}) = \ln r + \mathrm{i}\theta$ for $r > 0$ and $-\pi < \theta < \pi$. Hence H is real analytic, in particular smooth, on the interior of M_l^+ (that consists of all $X > 0$). To check this, we use, for every $X_0 > 0$, a power series $\ln s = \sum_{k \geq 0} c_k (s - r/2)^k$ of the logarithm on a fixed interval $(0, r)$ with $r > \|X_0\|$ and the semicontinuity property of the spectrum, to show that there is an $\varepsilon = \varepsilon_{X_0} > 0$ such that $H(X) = -\operatorname{tr}(X \sum_{k \geq 0} c_k (X - r/2)^k)$, where the series is convergent for all X with $\|X - X_0\| < \varepsilon$. To find now the Fréchét differential $H'(X)$ we use the formula $H'(X)Y = \lim_{t \to 0} t^{-1}(H(X + tY) - H(X))$, which requires the computation of $\lim_{t \to 0} t^{-1} \operatorname{tr}(X(\ln(X + tY) - \ln X))$. To this aim: fix an $r > \|X\|$, apply the functional calculus of $X + tY$ (for small t) to the analytic function $\ln z = \sum_{k\geq 0} c_k(z-r/2)^k$ for $|z-r/2| < r/2$, substract $\ln X$ and multiply to the left by X. Neglect then the terms of order ≥ 2 in t and use for the others, that are linear in Y, the equalities $\mathrm{tr}\,(X^pYX^q) = \mathrm{tr}\,(X^{p+q}Y)$ (p, $q \geq 0$ integers) to get the coefficient of t.

5 Lemma *If the system* (1) *has any solutions* $X \geq 0$*, then it has in particular a solution* $X_* \geq 0$ *such that* $H(X_*) = \max\{H(X) : X \geq 0$ *satisfying* $(1)\}$.

Proof The set $S = \{X \in M_l^+ : \mathrm{tr}\,(A_jX) = \beta_j\ \forall j\}$ is nonempty and closed. By Lemma 4, $\sup_S H \leq \sup H < \infty$ and there exists a ball $B = \{X \in M_l : \|X\| \leq r\}$ of radius r sufficiently large such that $\sup_S H = \sup_{S\cap B} H$, and so $H|_S$ reaches its supremum (on the compact set $S \cap B$).

6 Proposition *If the system* (1) *has at least one positive solution* X*, then any maximum entropy solution* $X_* \geq 0$ *must be positive.*

Proof We fix a matrix $X > 0$ satisfying (1). Let $X_* \geq 0$ be any solution of (1) of maximum entropy (such solutions exist by Lemma 5). Let $X_t = tX + (1-t)X_*$ for $0 < t < 1$. In order to prove that $X_* > 0$, we can suppose $X_* \neq X$. Fix positive constants c and C such that $X \geq c$ and $\|X\| + \|X_*\| + \max H \leq C$, see Lemma 4. We have $\langle X_t h, h\rangle > 0$ for any vector $h \neq 0$, that is, $X_t > 0$. Hence $\sigma(X_t) \subset (0, C]$. Suppose that $0 \in \sigma(X_*)$. Now $X_t \to X_*$ as $t \to 0$. Then by using the upper semicontinuity property of the spectrum and $\ker X_* \neq \{0\}$, we easily derive that for each $t \in (0,1)$ there exists a partition $\sigma_t \cap \sigma_t'$ of $\sigma(X_t)$ with $\sigma_t \neq \emptyset$ such that $\lim_{t\to 0}(\max_{\lambda\in\sigma_t} \lambda) = 0$ while all points of σ_t' stay away from 0, that is, there are positive constants $b < 1$ and $B < C$ such that $\sigma_t' \subset [B, C]$ for all $t \in (0, b)$. We can also assume, for b sufficiently small, that all $\lambda \in \sigma_t$ are < 1. Let $e_1, \ldots, e_l$ be the canonical basis of $\mathbb{R}^l$. For every $t \in (0, b)$ we choose an arbitrary unitary $U_t \in M_l$ such that $U_t^* X_t U_t = \sum_{i=1}^{l} \lambda_{it}\langle\,\cdot\,, e_i\rangle e_i$, where $(\lambda_{1t}, \ldots, \lambda_{lt})$ is a list of (not necessarily distinct) eigenvalues of X_t counted according to the multiplicity. By a permutation, we can suppose that $\lambda_{1t}, \ldots, \lambda_{pt} \in \sigma_t$ and $\lambda_{p+1t}, \ldots, \lambda_{lt} \in \sigma_t'$ where $p = p(t) \geq 1$ is the dimension of the spectral space $\Sigma_t \subset \mathbb{R}^l$ of σ_t. Let $\Delta_t = \mathrm{diag}\,(\lambda_{1t}, \ldots, \lambda_{lt})$ be the matrix of $U_t^* X_t U_t$ with respect to the basis $e_1, \ldots, e_l$. Let $(e_{1t}, \ldots, e_{lt})$ be the orthonormal basis of $\mathbb{R}^l$ consisting of the eigenvectors $e_{it} = U_t e_i$ of X_t. The subspace Σ_t is spanned by the eigenvectors $e_{1t}, \ldots, e_{pt}$ of X_t corresponding to the eigenvalues $\lambda_{1t}, \ldots, \lambda_{pt}$, respectively. Let $P_t = \sum_{i=1}^{p} \langle \cdot \,, e_{it} \rangle e_{it}$ $(\neq 0)$ be the spectral projection on Σ_t. We have $\|X_t|_{\Sigma_t}\| = \max_{\lambda \in \sigma_t} \lambda \to 0$ as $t \to 0$. Hence $\|X_t P_t\| \to 0$ as $t \to 0$. Then by decreasing b if necessary, we can suppose that $P_t X_t P_t < c/4$ for all $t \in (0, b)$. Let $Q_t = U_t^* P_t U_t$ be the orthogonal projection onto the linear span of $e_1, \ldots, e_{p(t)}$. Then $U_t Q_t U_t^* X_t U_t Q_t U_t^* < c/4$, whence $Q_t U_t^* X_t U_t Q_t < c/4$. Also, if b is sufficienty small we have $\|Q_t U_t^* (X_* - X_t) U_t Q_t\| < c/4$ for all $t \in (0, b)$. Hence $Q_t U_t^* X_* U_t Q_t < c/2$. Since $X \geq c$, $U_t^* X U_t \geq c$ and so $Q_t U_t^* X U_t Q_t \geq c Q_t$. Hence

$$Q_t U_t^* (X - X_*) U_t Q_t = Q_t U_t^* X U_t Q_t - Q_t U_t^* X_* U_t Q_t \geq \frac{c}{2} Q_t.$$

Hence all the diagonal elements

$$[\, Q_t U_t^* (X - X_*) U_t Q_t \,]_{ii} := \langle Q_t U_t^* (X - X_*) U_t Q_t e_i, e_i \rangle \geq \frac{c}{2}, \tag{4}$$

with $i = \overline{1, p}$. For any $A \in M_l$ we have

$$\operatorname{tr}(Q_t A Q_t) = \sum_{i=1}^{p} \langle A e_i, e_i \rangle,$$

$$\operatorname{tr}((I - Q_t) A (I - Q_t)) = \sum_{i=p+1}^{l} \langle A e_i, e_i \rangle$$

and

$$\operatorname{tr} A = \operatorname{tr}(Q_t A Q_t) + \operatorname{tr}((I - Q_t) A (I - Q_t)).$$

Let $A = U_t^* (X - X_*) U_t \,(\ln \Delta_t)$. Then for every $t \in (0, b)$

$$-\operatorname{tr}((X - X_*) \ln X_t) = \operatorname{tr}(U_t^* (X - X_*) U_t \, U_t^* (\ln X_t) U_t)) =$$

$$-\operatorname{tr}(U_t^* (X - X_*) U_t \, \ln \Delta_t) = \tau_1(t) + \tau_2(t)$$

where

$$\tau_1(t) = -\operatorname{tr}(Q_t U_t^* (X - X_*) U_t \, Q_t \,(\ln \Delta_t) Q_t)$$

and

$$\tau_2(t) = -\operatorname{tr}((I - Q_t) U_t^* (X - X_*) U_t \,(I - Q_t)\,(\ln \Delta_t)(I - Q_t)),$$

using also that $(\ln\Delta_t)Q_t = Q_t(\ln\Delta_t)Q_t$ since the range of Q_t is invariant under Δ_t. By (4), we have

$$\tau_1(t) = \sum_{i=1}^{p}[\,Q_t U_t^*(X - X_*)U_t Q_t\,]_{ii}\,(-\ln\lambda_{it})$$

$$\geq \sum_{i=1}^{p}\frac{c}{2}(-\ln\lambda_{it}) \geq p\frac{c}{2}\ln\frac{1}{\max\limits_{1\leq i\leq p}\lambda_{it}} \geq \frac{c}{2}\frac{1}{\max\limits_{\lambda\in\sigma_t}\lambda} \to +\infty \quad \text{as } t\to 0.$$

Using $|\mathrm{tr}\,(AB)| \leq \|A\|\,\|B\|_1$ and $\|(I-Q_t)\Delta_t(I-Q_t)\|_1 = \sum\limits_{i=p+1}^{l}|\lambda_{it}|$, we obtain

$$|\tau_2(t)| \leq \|(I-Q_t)U_t^*(X-X_*)U_t\,(I-Q_t)\| \cdot \sum_{i=p+1}^{l}|\lambda_{it}| \leq$$

$$(\|X\| + \|X_*\|)\cdot l \max_{p+1\leq i\leq l}|\ln\lambda_{it}| \leq Cl\;\max\{|\ln B|,\, |\ln C|\}.$$

Therefore, using Lemma 4,

$$\frac{d}{dt}(H(X_t)) = H'(X_t)(X - X_*) = -tr\,((X-X_*)(I + \ln X_t)) =$$

$$-\mathrm{tr}\,(X - X_*) - \mathrm{tr}\,((X-X_*)(\ln X_t)) = -\mathrm{tr}\,(X-X_*) + \tau_1(t) + \tau_2(t) \to +\infty$$

as $t\to 0$. Hence $(H(X_t) - H(X_*))/t > 0$ if $t > 0$ is sufficiently small. Then we have matrices $X_t \neq X_*$ with $H(X_t) > H(X_*)$, which is impossible. It follows that X_* must be positive.

7 Lemma *Let A, $B \in M_l$ be selfadjoint with $A \neq 0$ and set $f(x) = tr\,e^{xA+B}$ for x real. Then $f'(x) = tr\,(Ae^{xA+B})$ and $f''(x) > 0$ for all x.*

Proof Fix x, set $M = xA + B$ and let t be a real variable. Then

$$\mathrm{tr}\,(e^{M+tA} - e^M) = \sum_{k\geq 0}\frac{1}{k!}\mathrm{tr}\,((M+tA)^k - M^k) =$$

$$t\sum_{k\geq 1}\frac{1}{k!}\mathrm{tr}\,(M^{k-1}A + M^{k-2}AM + \cdots + MAM^{k-2} + AM^{k-1}) + O(t^2)$$

$$= t\sum_{k\geq 1}\frac{1}{k!}\cdot k\,\mathrm{tr}\,(AM^{k-1}) + O(t^2) = t\,\mathrm{tr}\,(Ae^M) + O(t^2).$$

Hence $f'(x) = \lim_{t\to 0} t^{-1}\mathrm{tr}\,(e^{M+tA} - e^M) = \mathrm{tr}\,(Ae^{xA+B})$. We arbitrarily fix x_0 and show that $f''(x) > 0$ for all x in a small neighbourhood of x_0. Set again $M = xA + B$. Fix a constant $c = c_{x_0,A,B} > 0$ sufficiently large so that $x_0A + B + c > 0$. Writing $f(x) = e^{-c}\,\mathrm{tr}\,e^{xA+(B+c)}$ and replacing B by $B + c$ shows that we can assume, for $x \approx x_0$, that $M > 0$. Then we compute $f''(x) = \lim_{t\to 0} t^{-1}\mathrm{tr}\,(A(e^{M+tA} - e^M))$ as follows. We have

$$A(e^{M+tA} - e^M) = A\sum_{k\geq 0}\frac{1}{k!}((M + tA)^k - M^k) =$$

$$tA\sum_{k\geq 1}\frac{1}{k!}(M^{k-1}A + M^{k-2}AM + \cdots + AM^{k-1}) + O(t^2) =$$

$$tA(\frac{1}{1!}A + \frac{1}{2!}(MA + AM) + \frac{1}{3!}(M^2A + MAM + AM^2) + \cdots) + O(t^2).$$

Apply the trace to the previous equalities. The coefficient of t has the form $\sum_{p,q\geq 0} c_{pq}\mathrm{tr}\,(AM^pAM^q)$ with all $c_{pq} \geq 0$. Since $M > 0$ and $A = A^*$, then $M^q > 0$ and $AM^pA \geq 0$ for all p, q. Hence $\mathrm{tr}\,(AM^pA \cdot M^q) \geq 0$ using that $\mathrm{tr}\,(CD) = \mathrm{tr}\,(D^{1/2}CD^{1/2}) \geq 0$ for any $C, D \in M_l^+$. Then $f''(x) \geq 0$. If $f''(x) = 0$ for some x, then all terms $c_{pq}\mathrm{tr}\,(AM^pAM^q) = 0$, in particular $\mathrm{tr}\,(A^2) = 0$ whence $A = 0$ that is false.

Proof of Proposition 1 Let $x, y \in \mathbb{R}^n$ be arbitrary with $x \neq y$. For any real t, set $g(t) = V(ty+(1-t)x)$. Then g has the form $g(t) = \mathrm{tr}\,e^{tA+B} + ta + b$, where $A = \sum_{j=1}^{n}(y_j - x_j)A_j$ and $B = \sum_{j=1}^{n} x_jA_j$. Note that $A \neq 0$ since the A_j's are linearly independent. By Lemma 7, $g''(t) > 0$ for all t. Then g is strictly convex and so $V(\frac{1}{2}(x + y)) = g(\frac{0+1}{2}) < \frac{g(0)+g(1)}{2} = \frac{V(x)+V(y)}{2}$. Thus V is strictly convex, too. Moreover, for every $x \in \mathbb{R}^n$ and $c = (c_1, \ldots, c_n) \neq 0$, letting $y = x + c$ in condition $g''(0) > 0$ gives $\sum_{i,j=1}^{n}(\partial_i\partial_jV)(x)c_jc_i > 0$. The partial derivatives of V are computed also by Lemma 7.

8 Theorem *Suppose that* (1) *has positive solutions. Then there exists a unique vector* $x^0 = (x_j^0)_{j=1}^n$ *in* $\mathbb{R}^n$ *such that the matrix* $X_0 := e^{\sum_{j=1}^{n} x_j^0A_j - I}$

satisfies (1). *Moreover,* $V'(x_0) = 0$ *and there exists a unique maximum entropy solution* X_* *of* (1), *namely* $X_* = X_0$.

Proof We shall maximize the entropy functional H over the set of all nonnegative solutions of (1), by using Lagrange's method of the multipliers. Firstly, let $X_* \geq 0$ be a matrix provided by Lemma 5. By Proposition 6, we necessarily have $X_* > 0$. Let f be defined on the set $D := \{X \in M_l : X > 0\}$ by $f(X) = H(X)$. The function f is of class C^1 by Lemma 4. Define f_i on D by $f_i(X) = \mathrm{tr}\,(A_i X) - b_i$ for $i = \overline{1,n}$. Set $S = \{X \in D : f_i(X) = 0\,\forall\, i\}$. Thus $X_* \in S$ and $\max\limits_S f = f(X_*)$. Since D is open, we can apply the method of the multipliers. Thus there exist some real numbers $x_1^0, \ldots, x_n^0$ such that X_* is a critical point of the function $f + \sum\limits_{i=1}^n x_i^0 f_i$. By Lemma 4, $-\mathrm{tr}\,((I + \ln X_*)Y) + \sum\limits_{i=1}^n x_i^0 \mathrm{tr}\,(A_i Y) = 0$ for all $Y \in M_l$. Hence $I + \ln X_* = \sum\limits_{i=1}^n x_i^0 A_i$. Then $X_* = e^{\sum\limits_{i=1}^n x_i^0 A_i - I} = X_0$. Writing that X_0 fulfills the equations (1) shows that x^0 is a critical point of V, see Proposition 1. Since the function V is strictly convex, its critical point is unique. The vector x^0 is then uniquely determined.

9 Lemma (i) *If a strictly convex function* f *of class* C^2 *on* $\mathbb{R}^n$ *has a critical point, then* $\lim\limits_{\|x\| \to \infty} f(x) = +\infty$.

(ii) *If a convex function* f *on* $\mathbb{R}^2 \equiv \mathbb{C}$ *has finite radial limits* $l(s) := \lim\limits_{t \to +\infty} f(te^{is})$ *for* $-\frac{\pi}{2} < s < \frac{\pi}{2}$, *then* $l(\,\cdot\,)$ *is continuous on* $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Proof (i) By a translation, we can assume $f'(0) = 0$. The restriction $v = v(t) := f(tx)$ $(t \in \mathbb{R})$ of f to any line $\mathbb{R}x$ $(x \neq 0)$ is strictly convex with $v'(0) = 0$. For any points $P = (\alpha, v(\alpha))$ and $Q = (\beta, v(\beta))$ with $\alpha < \beta$ the graph of $v = v(t)$ is below the line PQ for $\alpha < t < \beta$ and above the line PQ for $t \notin [\alpha, \beta]$, in particular it is above any tangent. Hence there exist $a,\ b > 0$ such that $v(t) \geq v(0) - b + a|t|$ for all t, for example $a := \min\{v(1), v(-1)\}$ and $b := \max\{v(1), v(-1)\}$. Then $\lim\limits_{t \to \infty} f(tx) = +\infty$ for all x with $\|x\| = 1$. The function $g(t) = f'(tx)x$ satisfies $g(0) = 0$ and $g'(t) = f''(tx)(x,x) \geq 0$ for all $t \geq 0$. Then $g(t) \geq 0$ for all $t \geq 0$, and so

$$f(t_2 x) - f(t_1 x) = \int_{t_1}^{t_2} \frac{d}{dt}(f(tx))\, dt = \int_{t_1}^{t_2} g(t) dt \geq 0 \qquad (5)$$

for any positive t_1, t_2 with $t_2 > t_1$. Since all radial limits $\lim_{t\to\infty} f(tx)$ are $+\infty$ and the map $t \mapsto f(tx)$ is monotonically increasing, it follows using the compactness of the unit sphere that $\lim_{\|x\|\to\infty} f(x) = +\infty$.

(ii) Let s, $s' \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $c := \cos\frac{s-s'}{2} > 0$. We have $ce^{i\frac{s+s'}{2}} = \frac{e^{is}+e^{is'}}{2}$. Since f is convex, it follows that $f(cte^{i\frac{s+s'}{2}}) \leq \frac{f(te^{is})+f(te^{is'})}{2}$ for every $t > 0$. Letting $t \to \infty$ gives $l(\frac{s+s'}{2}) \leq \frac{l(s)+l(s')}{2}$. Hence l is convex, and so continuous.

Proof of Theorem 2 If the system (1) has positive solutions, then by Theorem 8 it has also a particular solution of the form $X_0 = e^{\sum_{j=1}^{n} x_j^0 A_j - I}$. Moreover $x^0 = (x_j^0)_{j=1}^n$ is a critical point of V. Then $\lim_{\|x\|\to\infty} V(x) = +\infty$, see Lemma 9, (i). The converse implication is obvious, if $\lim_{\|x\|\to\infty} V(x) = +\infty$ then the smooth function V reaches a minimum in a point x^0 and so $V'(x^0) = 0$, which provides the solution X_0 using again Proposition 1.

10 Lemma *For every $x \neq 0$ in $\mathbb{R}^n$, set $A_x = \sum_{j=1}^{n} x_j A_j$. Then $A_x \neq 0$ and:*

(i) *If the spectrum of A_x has both positive and negative eigenvalues, then $\lim_{t\to\pm\infty} V(tx) = +\infty$.*

(ii) *If $A_x \geq 0$ and $\sum_{j=1}^{n} b_j x_j > 0$, then $\lim_{t\to\pm\infty} V(tx) = +\infty$.*

(iii) *If $A_x \geq 0$ and $\sum_{j=1}^{n} b_j x_j = 0$, then we have $\lim_{t\to+\infty} V(tx) = +\infty$ and $\lim_{t\to-\infty} V(tx) = \dim\ker A_x$.*

(iv) *If $A_x \geq 0$ and $\sum_{j=1}^{n} b_j x_j < 0$, then we have $\lim_{t\to-\infty} V(tx) = -\infty$ and $\lim_{t\to\infty} V(tx) = +\infty$.*

Proof Since the A_j's are linearly independent, $A_x \neq 0$. Let $\sigma_1 \leq \cdots \leq \sigma_l$ be the eigenvalues of A_x counted according to their multiplicities. Then A_x is unitarily equivalent to a diagonal matrix $\operatorname{diag}(\sigma_1, \ldots, \sigma_l)$ with diagonal elements $\sigma_1, \ldots, \sigma_l$. Hence $\operatorname{tr} e^{tA_x} = \sum_{i=1}^{l} e^{t\sigma_i}$ for any real t.

(i) In this case, $\sigma_1 < 0$ and $\sigma_l > 0$. Then obviously $\lim_{t\to\pm\infty} V(tx) =$

$\lim_{t\to\pm\infty} \sum_{i=1}^{l} e^{t\sigma_i} = +\infty$.

(ii) Since all $\sigma_i \geq 0$ and $V(tx) = \sum_{i=1}^{l} e^{t\sigma_i} - t\sum_{j=1}^{n} b_j x_j$, then $V(tx) \to +\infty$ as $t \to -\infty$ due to the linear term. Also, $V(tx) \to +\infty$ as $t \to +\infty$ due to the exponential term, that is not constant since $A_x \neq 0$ and so $\sigma_l \neq 0$.

(iii) We have $\lim_{t\to-\infty} V(tx) = \lim_{t\to-\infty} \sum_{j=1}^{n} e^{t\sigma_j} =$ the number of null eigenvalues $(= \dim\ker A_x)$. Also, $\lim_{t\to+\infty} V(tx) = +\infty$ since $\sigma_l > 0$

(iv) In this case $\lim_{t\to-\infty} V(tx) = \lim_{t\to-\infty} (-t\sum_{j=1}^{n} x_j b_j) = -\infty$ and, obviously, $\lim_{t\to+\infty} V(tx) = +\infty$.

Proof of Theorem 3 Let us note firstly that there is a constant $C > 0$ such that whenever the problem (1) has nonnegative solutions X, all these solutions satisfy $\|X\| \leq C$. Indeed, the linear span of the A_j's contains positive matrices, and so there exists a linear combination of the equations (1) leading to an equality of the form $\mathrm{tr}\,(PX) = b$ with P a positive matrix. Also, we have $b \geq 0$ since there exists at least one solution $X \geq 0$. We fix a constant $c > 0$ such that $P \geq c$. Then for every solution $X \geq 0$ of (1) we have $c\,X \leq X^{1/2}PX^{1/2}$, whence $c\,\mathrm{tr}\,X \leq \mathrm{tr}\,(X^{1/2}PX^{1/2}) = \mathrm{tr}\,(PX) = b$, and so $\|X\| \leq c^{-1}b$.

(a) Suppose that problem (1) admits solutions $X \geq 0$ but none of them is positive. Firstly we show that V is bounded from below. Fix an $l \times l$ matrix $M \geq 0$ satisfying (1). For any vector x in $\mathbb{R}^n$, let $\sigma_1(x), \ldots, \sigma_l(x)$ be the eigenvalues of the matrix $A_x := \sum_{j=1}^{n} x_j A_j$ counted according to the multiplicity. Then there exists a unitary U_x in M_l such that $U_x^* A_x U_x = \mathrm{diag}\,(\sigma_1(x), \ldots \sigma_l(x))$. Hence $\sum_{j=1}^{n} b_j x_j = \sum_{j=1}^{n} x_j \mathrm{tr}\,(A_j M) = \mathrm{tr}\,(A_x M)$ and so $V(x) = \mathrm{tr}\, e^{A_x} - \mathrm{tr}\,(A_x M) = \sum_{i=1}^{l} (e^{\sigma_i(x)} - \sigma_i(x)\mu_i(x))$ where $\mu_1(x), \ldots, \mu_l(x)$ are the diagonal entries of the matrix $U_x^* M U_x$. Note that for every vector x in $\mathbb{R}^n$ and index $i = \overline{1, l}$ we have $0 \leq \mu_i(x) \leq \|U_x^* M U_x\| = \|M\|$. Now for any nonnegative μ with $\mu \leq \|M\|$ and real σ we have the estimate $e^\sigma - \sigma\mu \geq \mu - \mu\ln\mu \geq -\|M\|\ln\|M\|$ (minimize to this aim the function $\sigma \mapsto e^\sigma - \sigma\mu$ on the real line). This leads to the estimate $V(x) \geq -l\,\|M\|\ln\|M\|$ for all

x. Thus V is bounded from below.

We prove now the existence of a finite radial limit of V. For an arbitrary vector $x \neq 0$, there are several possibilities as described by Lemma 10 (using also that $A_{-x} = -A_x$). Case (iv) is excluded since $\inf V > -\infty$. Then we easily see that at least one vector x should be in case (iii). Indeed, if only the cases (i), (ii) occur then all radial limits of V are $+\infty$. This implies, as in the proof of Lemma 9, (i) (see (5)), that $\lim_{\|x\|\to\infty} V(x) = +\infty$. Then Theorem 2 provides positive solutions of (1), which is excluded by hypotheses. Let then C denote the set of those vectors $x \neq 0$ such that $A_x \leq 0$ and $\sum_{j=1}^{n} b_j x_j = 0$. Obviously, C $(\neq \emptyset)$ is a positive convex cone. For every $x \in C$, Lemma 10, (iii) gives $\lim_{t\to+\infty} V(tx) = \dim\ker A_x$. Also, all other radial limits of V are $+\infty$. Using Lemma 9, (ii), one easily proves that all radial limits of V along half-lines from C are equal. Namely, for any points x, y with $x \neq y$ in C, the restriction of the map $z \mapsto \lim_{t\to+\infty} V(tz)$ to the segment $[x, y] := \{tx + (1-t)y : 0 \leq t \leq 1\}$ is continuous, and hence locally constant (since it's integer-valued), and so it is constant. Then $\dim\ker A_x = \lim_{t\to+\infty} V(tx) =$ constant for all $x \in C$.

Conversely, suppose that $\inf V > -\infty$ and V has at least one finite radial limit. Then by Lemma 9, (i) the function V does not have critical points. Hence by Theorem 2 there are no positive solutions of (1). Let us show that problem (1) admits however nonnegative solutions. By Lemma 10, for every vector $x \neq 0$ either x or $-x$ must be in one of the cases (i)-(iii). In each of the these cases we have the implication $A_x \geq 0 \Rightarrow \sum_{j=1}^{n} b_j x_j \geq 0$. For every integer $k \geq 1$, set $\beta_{kj} = \beta_j + \frac{1}{k}\mathrm{tr}\, A_j$ and $b_{kj} = \beta_{kj} e$. Then for every k we have the implication $A_x \geq 0 \Rightarrow \sum_{j=1}^{n} x_j b_{kj} > 0$ $(x \neq 0)$. Indeed, if $A_x \geq 0$ then $\mathrm{tr}\, A_x \geq 0$. Also, $\mathrm{tr}\, A_x \neq 0$ since $A_x \neq 0$. Then $\sum_{j=1}^{n} x_j b_{kj} = \sum_{j=1}^{n} x_j (b_j + \frac{e}{k}\mathrm{tr}\, A_i) = \sum_{j=1}^{n} x_j b_j + \frac{e}{k}\mathrm{tr}\, A_j > 0$ since $\sum_{j=1}^{n} b_j x_j \geq 0$ and $\mathrm{tr}\, A_x > 0$. Using Lemma 10 (i), (ii) for the function $V_k(x) = \mathrm{tr}\, e^{A_x} - \sum_{j=1}^{n} b_{kj} x_j$, it follows that the radial limits of V_k are $+\infty$. Using again (5) as in the proof of Lemma 9, now for the function $f := V_k$ (whence $v_k(t) = V_k(tx)$ and $g(t) = V_k'(tx)x$), we obtain $\lim_{\|x\|\to\infty} V_k(x) = +\infty$. Then by Theorem 2, each problem $(1)_k$: $\mathrm{tr}\,(A_j X) = \beta_{kj}$ $(j = \overline{1,n})$ has a positive solution X_k. Now by the remark at the beginning of the proof, the sequence $(X_k)_k$ is bounded. Then it has some accumulation points $X \geq 0$, and all these X's are solutions of (1).

(b) It follows from (a) and Theorem 2 that problem (1) admits nonnegative solutions if and only if $\inf V > -\infty$. If $\inf V = -\infty$, then at least one radial limit should be $-\infty$, for otherwise only cases (i)-(iii) could appear in Lemma 10. This would lead again to the existence of some nonnegative solutions X of (1), either by Theorem 2 if only cases (i), (ii) occur, or by considering the problems $(1)_k$ if the case (iii) also appears.

11 Remarks (i) If problem (1) admits solutions $X > 0$, then Proposition 1 and Theorem 2 show that V fulfills the conditions for applying the conjugate gradients minimization method [9, 14]. This yields, in principle, a sequence of vectors $x^1, x^2, \ldots$ in $\mathbb{R}^n$ such that $V(x^k) \searrow \min V$ and $x^k \to x^0$ as $k \to \infty$, where x^0 is the critical point of V. Thus we can approximate our particular solution $X_0 = e^{\sum_{j=1}^{n} x_j^0 A_j - I}$. Calculating the gradients of V in the points $x = x^k$ $(k \geq 1)$ requires to compute at each step k a matrix exponential $e^{\sum_{j=1}^{n} x_j A_j}$. This can be performed for instance either by diagonalizing $A_x = \sum_{j=1}^{n} x_j A_j$ by Jacobi's method (see for example [7, section 8.4] and [13]), or by tridiagonalizing A_x by means of Householder reflections and using then an iterative eigenvalue algorithm (which in the gross requires $O(l^3)$ flops/step).

(ii) Let $(x^k)_k$ be a minimizing sequence as above, and $\mathcal{H}$ be the linear manifold of all hermitian solutions of (1). Let $P_{\mathcal{H}}(X)$ denote the projection of a hermitian $X \in M_l$ onto $\mathcal{H}$ with respect to the inner product $\langle X, Y \rangle = \mathrm{tr}\,(XY^*)$. Then $P_{\mathcal{H}}(X) = X + \sum_{j=1}^{n} c_j A_j$ with c_j given by the system of equations $\sum_{j=1}^{n} \mathrm{tr}\,(A_i A_j) c_j = \beta_i - \mathrm{tr}\,(A_i X)$ $(i = \overline{1,n})$. For large k so that $X^k := e^{\sum_{j=1}^{n} x_j^k A_j}$ is sufficiently close to X_0, the matrix $P_{\mathcal{H}}(X^k)$ $(\approx X_0)$ is also positive, and hence, an exact positive solution of (1).

(iii) In the case (a), resp. (b) of Theorem 3, for any minimizing sequence $(x^k)_k$ of V we have $\|x^k\| \to \infty$, with $\lim_{k\to\infty} V(x^k) \in \{0, 1, \ldots, l\}$, resp.

$\lim_{k\to\infty} V(x^k) = -\infty.$

12 Example (i) Consider the positive matrix completion problem for 4×4 matrices $X = [X_{jk}]_{j,k=1}^4 > 0$ the given entries of which are $X_{11} = 3$, $X_{12} = 1.45$, $X_{23} = 1.25$, $X_{34} = 2.35$ and $X_{41} = 2.75$. Thus $l = 4$, $n = 5$ and we take

$$A_1 = [a_{ij}^1]_{i,j=1}^4 \text{ with } a_{11}^1 = 1 \text{ and } a_{ij}^1 = 0 \text{ for } (i,j) \neq (1,1),$$

$$A_2 = [a_{ij}^2]_{i,j=1}^4 \text{ with } a_{12}^2,\ a_{21}^2 = 1 \text{ and } a_{ij}^2 = 0 \text{ for } (i,j) \neq (1,2),\ (2,1)$$

etc, as well as $\beta_1 = 3$, $\beta_2 = 2\cdot 1.45 = 2.9$ etc so that the requirements $X_{11} = 3$, $X_{12} = 1.45$ etc be written in the form (1).

We minimize V by the conjugate gradients method [9, 13–15] that we briefly remind below. Start with an initial point x^1 $(:= (0.01,\ldots,0.01) \in \mathbb{R}^5$ in our case). For every $k \geq 1$, let g_k be the gradient of V in x^k, and c_k be the conjugate gradient in x^k. Namely, $c_1 = -g_1$ and $c_k = -g_k + \alpha_{k-1}\cdot g_{k-1}$ for $k \geq 2$, where $\alpha_k = \frac{\langle g_{k+1}-g_k, g_{k+1}\rangle}{\langle g_{k+1}-g_k, c_k\rangle}$ is Polak-Ribière's coefficient. Once we have an x^k, we take $x^{k+1} := x^k + tc_k$ for that value $t = t_k$ minimizing V on the half-line $x^k + tc_k$ $(t \geq 0)$ etc. We stop when we get a point x^k with $\|g_k\|_{max}$ less than a prescribed tolerance (:= 0.001 in our case), where $\|(y_j)_{j=1}^n\|_\infty := \max_{1\leq j\leq n} |y_j|$ for $y \in \mathbb{R}^n$. For the line minimization, a value $t = \tilde{t}_k$ is accepted instead of t_k if $|\langle V'(x^k + tc_k), c_k\rangle| \leq \delta\|g_k\|_{max}$ where $\delta < 1$ is a fixed constat (:= 0.5 in our case). After 10 iterations providing the vectors $x^2,\ldots,x^{11} \in \mathbb{R}^5$ we have obtained, for $x := x^{11}$ with

$$x^{11} = (0.8571823445, 0.8822629161, 1.022695324, 1.904910646, 1.688386182),$$

the positive matrix $X_0 = e^{\sum_{j=1}^5 x_j A_j - I}$, namely

$$X_0 = \begin{bmatrix} 2.999990931 & 1.449938928 & 1.925776815 & 2.749864277 \\ 1.449938928 & 1.117124222 & 1.250052935 & 1.414252494 \\ 1.925776815 & 1.250052935 & 2.090027040 & 2.349861843 \\ 2.749864277 & 1.414252494 & 2.349861843 & 3.037633457 \end{bmatrix}$$

satisfying the equations (1) with error < 0.001. Then $P_{\mathcal{H}}(X_0)$ is simply X_0 corrected at the prescribed entries $X_{11} = 3$ etc.

Finding a solution X of (1) by using the barrier function $\ln\det X$, see [12], requires to consider a dual problem (2). We have $\beta^t \cdot x = 3x_1 + 2.9x_2 +$

$2.5x_3 + 4.7x_4 + 5.5x_5$. Note firstly that condition $\sum_{j=1}^{5} x_j A_j \geq 0$ (which means $x_1 \geq 0$, $x_{2,3,4,5} = 0$) implies that $\beta^t \cdot x \geq 0$. This guarantees that (1) has nonnegative solutions. Now we can take a positive $A_0 := I_4$ for example, so that $\{x : A(x) > 0\} \neq \emptyset$. Hence

$$A(x) = \begin{bmatrix} 1+x_1 & x_2 & 0 & x_5 \\ x_2 & 1 & x_3 & 0 \\ 0 & x_3 & 1 & x_4 \\ x_5 & 0 & x_4 & 1 \end{bmatrix}$$

and $p^* < 0 < \overline{p}$. Thus we may let $\gamma = 0$. Then we have to numerically solve the optimization problem

$$\sup\{\ln\det A(x) : x \in \mathbb{R}^5,\ A(x) > 0,\ \beta^t \cdot x = 0\} = \ln\det A(x^*).$$

Once the analytic center x^* is known, a solution $X_{x^*} > 0$ of (1) will be provided, of the form $X_{x^*} = A(x^*)^{-1}/\lambda$ for a Lagrange multiplier $\lambda > 0$. We can derive this as usual, by means of the formula $\frac{\mathrm{d}}{\mathrm{d}x}[\det(xA + B)] = \det(xA+B)\,\mathrm{tr}\,((xA+B)^{-1}A)$ where A, B are selfadjoint matrices with $xA+B$ invertible, which gives $\lambda = \det A(x^*)$.

To represent the problem in the form (3), too (see [5, 6, 8, 11]), we introduce the variable $z \equiv (z_{13}, z_{22}, z_{24}, z_{33}, z_{44})$ consisting of the unspecified entries of

$$X = X(z) = \begin{bmatrix} 3 & 1.45 & z_{13} & 2.75 \\ 1.45 & z_{22} & 1.25 & z_{24} \\ \overline{z}_{13} & 1.25 & z_{33} & 2.35 \\ 2.75 & \overline{z}_{24} & 2.35 & z_{44} \end{bmatrix}$$

with z_{22}, z_{33}, z_{44} real. Then set $H(z) = X(z)^{-1} = [H_{ij}(z)]_{i,j=1}^4$. The associated graph has vertices 1, 2, 3, 4 and edges $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, $\{1,4\}$, in particular has a cycle of length 4 and so is not chordal; also, the diagonal entries z_{22}, z_{33}, z_{44} are unspecified. The det-maximizing solution X_{x^*} from above may also be described as the unique positive completion such that $h(z) = (0, \lambda, 0, \lambda, \lambda)$, where $h(z) \equiv (H_{13}(z), H_{22}(z), H_{24}(z), H_{33}(z), H_{44}(z))$ with

$$H_{13}(z) = \frac{1}{\det X(z)} \det \begin{bmatrix} 1.45 & z_{22} & \overline{z}_{24} \\ z_{13} & 1.25 & 2.35 \\ 2.75 & z_{24} & z_{44} \end{bmatrix} =$$

$$\frac{1}{\det X(z)}(1.8125z_{44} - 3.4075z_{24} - z_{22}z_{13}z_{44} + 6.4625z_{22} + z_{13}|z_{24}|^2 - 3.4375\overline{z}_{24})$$

etc.

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