A Riesz-Haviland type result for truncated moment problems with solutions in L^1

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Abstract

We give a version of the Riesz-Haviland theorem for truncated moments problems, characterizing the existence of the representing measures that are absolutely continuous with respect to the Lebesgue measure. The existence of such representing densities describes the dense interior of the convex cone of all data having nonnegative Borel representing measures. A natural regularity assumption on the support is required.

Keywords: positive functional, moments problem, measure

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1 Introduction

In this paper we consider problems of moments in n real variables $t = (t_1, \ldots t_n)$, with respect to a finite number of monomial functions $t^i = t_1^{i_1} \cdots t_n^{i_n}$ where $i = (i_1, \ldots, i_n) \in (\mathbb{Z}_+)^n$, for Lebesgue integrable densities $f = f(t) \geq 0$ a.e. (almost everywhere) on closed subsets $T \subset \mathbb{R}^n$. Given a set of numbers g_i with $|i| = i_1 + \cdots + i_n \leq 2k$ where $k \in \mathbb{N}$ is fixed, the truncated problem of moments, called also T-problem of moments [6] when T is prescribed, is concerned with the existence of the Borel measures $\nu \geq 0$ on $\mathbb{R}^n$ supported on T such that $\int_T t^i d\nu(t) = g_i$ for all i [1], [18]. One calls ν a representing measure for g, and g_i the moments of ν. We are interested in those representing measures $\nu = f \, dt$, called *representing densities* of g, that are absolutely continuous with respect to the Lebesgue measure $dt = dt_1 \cdots dt_n$. For any subset $I \subset \mathbb{Z}_+^n$, let P_I denote the linear span of the monomials X^i ($i \in I$) in $\mathbb{R}[X_1, \ldots, X_n]$. Given $g = (g_i)_{i \in I}$, the linear *Riesz functional* $\varphi_g : P_I \to \mathbb{R}$ associated to g is defined as it is known [7] by $\varphi_g X^i = g_i$ for $i \in I$. We say that φ_g is T-positive [13] if $\varphi_g p \geq 0$ for all polynomials p such that $p(t) \geq 0$ for all $t \in T$. This condition is necessary for g to have representing measures ν on T, since in this case $\varphi_g p = \int_T p \, d\nu$ for all polynomials p. We remind below the Riesz-Haviland theorem [13], a basic result concerned with the full problem of moments when $I = \mathbb{Z}_+^n$.

Theorem 1 [13] *Let $T \subset \mathbb{R}^n$ be closed, and $g = (g_i)_{i \in \mathbb{Z}_+^n}$ be a set of reals. Then g has representing measures on T if and only if φ_g is T-positive.*

An analogue of the Riesz-Haviland theorem for the truncated case was established by R.E. Curto and L.A. Fialkow [6]. For $I = I_{2k} = \{i : |i| \leq 2k\}$,

when P_I is the space of all polynomials of degree $\leq 2k$, they proved that a set $g = (g_i)_{|i|\leq 2k}$ has representing measures on T if and only if the Riesz functional $\varphi_g : P_{I_{2k}} \to \mathbb{R}$ admits T-positive extensions $\tilde{\varphi}_g : P_{I_{2k+2}} \to \mathbb{R}$ to the space $P_{I_{2k+2}}$ of all polynomials of degree $\leq 2k + 2$.

By Theorem 7, we characterize the existence of the representing densities of g on regular supports T by the condition: $\varphi_g p > 0$ for all $p \in P_I \setminus \{0\}$ such that $p(t) \geq 0$ for all $t \in T$. A similar characterization [14] holds under more specific hypotheses, in particular if a distinguished moment τ_{i_0} $(= t^{i_0}$ or linear combination of the t^i's) exists such that $\lim_{\|t\| \to \infty} \frac{|t^i|}{1+\tau_{i_0}(t)} = 0$ for $i \neq i_0$. We mention also the existence of similar results [11] for quadratic $(k = 1)$ and some quartic $(k = 2, n = 2)$ T-problems of moments.

By Theorem 6, the set of all g having representing densities is the dense interior of the set of all g having representing measures. Our proofs are independent of the results from [6], [14] and rely mainly on Theorem 1 [13].

Started by works of Stieltjes, Hausdorff, Hamburger and Riesz, the area of the truncated problems of moments saw interesting development in various other directions, that we do not attempt to cover. A few recent works [5] - [9] should be mentioned in this sense, see also [3], [12], [15], [16].

2 Main results

Definitions Let $T \neq \emptyset$ be a closed subset of $\mathbb{R}^n$, such that for any $t \in T$ and $\varepsilon > 0$ the Lebesgue measure of the set $\{x \in T : \|x - t\| < \varepsilon\}$ is > 0. This always holds a.e. (in the density points of T [7.6, [17]]), but we require it in every point t. We call such a T *regular*. Here $\| \ \|$ is the usual Euclidian norm. For any multiindex $i \in \mathbb{Z}_+^n$, set $\sigma_i = \{j \in \mathbb{Z}_+^n : j_k = \text{either } 0 \text{ or } i_k, 1 \leq k \leq n\}$. Let $I \subset \mathbb{Z}_+^n$ be finite, $I \neq \emptyset$ such that $\sigma_i \subset I$ for all $i \in I$. We call such an I *regular*, too. Let $g = (g_i)_{i \in I}$ be a set of real numbers with $g_0 = 1$. We call a convex cone C in a real linear space F *acute* if $C \cap (-C) = \{0\}$. Given also a linear functional $f : F \to \mathbb{R}$, we write $f > 0$ if $fc > 0$ for all $c \in C \setminus \{0\}$.

Lemma 2 and its Corollary 3 follow from various well known arguments (Hahn-Banach, Krein, Mazur, Choquet [4], [10], [17]) on the extension of positive functionals. We found it easier to gather them in a short proof.

Lemma 2 *Let $C \subset F$ be an acute closed convex cone in a finite dimensional linear space F. Let $L \subset F$ be a linear subspace of codimension 1, and ϕ a linear functional on L such that $\phi l > 0$ for all $l \in L \cap C$ with $l \neq 0$. Then there is an extension Φ of ϕ to F such that $\Phi x > 0$ for all $x \in C$, $x \neq 0$.*

Proof. The sum $Y + \mathbb{R}v$ of a closed convex cone $Y \subset F$ and a 1-dimensional subspace $\mathbb{R}v$ is closed. Indeed, if both $\pm v \in Y$, $\mathbb{R}v \subset Y$ and so $Y + \mathbb{R}v = Y$; if not, we may suppose $v \notin Y$ by replacing v by $-v$ if necessary. Let

$x = \lim_{k \to \infty}(y_k + \lambda_k v)$ where $y_k \in Y$. If $(\lambda_k)_k$ is bounded, by compactness we obtain a number λ and vector $y \in Y$ such that $x = y + \lambda v$. If it is not, we can assume either $\lim_k \lambda_k = \infty$, or $\lim_k \lambda_k = -\infty$. From $y_k + \lambda_k v \to x$ we derive $\frac{1}{\lambda_k} y_k + v \to 0$. The case $\lambda_k \to -\infty$ is impossible since it leads to $v = \lim_k \frac{1}{-\lambda_k} y_k \in Y$. If $\lambda_k \to \infty$, we obtain $-v = \lim_k \frac{1}{\lambda_k} y_k \in Y$, and so the distance $d(x, Y)$ from x to Y satisfies $d(x, Y) = \lim_k d(y_k - \lambda_k(-v), Y) = \lim_k d(y_k, Y) = 0$ whence $x \in Y$. In particular we get (inductively) that the convex cone $K := C + \ker \phi$ is closed.

To find Φ, we may suppose $C \neq \{0\}$. If $\phi = 0$, $L \cap C = \{0\}$. Let then f be a linear functional on F with $\ker f = L$. Since C is acute, $C \setminus \{0\}$ is segmentwise connected. Hence the set $I := f(C \setminus \{0\})$ is connected, and so, an interval, that cannot contain 0 for: $fc = 0, c \in C \Rightarrow c \in L \cap C = \{0\}$. Then either $I \subset (0, \infty)$, in which case we let $\Phi = f$, or $I \subset (-\infty, 0)$ in which case $\Phi := -f$. Then $\Phi c \geq 0$ for all $c \in C$, with strict inequality if $c \neq 0$.

If $\phi \neq 0$, let $F' = F / \ker \phi$ and $\lambda : F \to F'$ be the factorization map. Then $C' := \lambda(C)$, $L' := \lambda(L)$ and the map ϕ' induced by ϕ on L' satisfy the hypotheses as well. Indeed, $K = C + \ker \lambda$ is closed, λ open and $\lambda(F \setminus K) = F' \setminus C'$ whence C' is closed. Also $\phi > 0, C$ acute $\neq \{0\} \Rightarrow \phi' > 0, C'$ acute $\neq \{0\}$. Since $\dim F' = 2$, we easily get the existence of an extension $\Phi' > 0$ of ϕ' to F', that will provide $\Phi := \Phi' \circ \lambda$: note that $\dim L' = 1$ and $\phi'|_{L'}$ is injective, and so increasing along a direction of L' given by a vector $\overline{e} \in \mathbb{R}^2$ (a drawing would help). Then $L' = \{r\overline{e}\}_{r \in \mathbb{R}}$ and $\phi'\overline{e} > \phi'0 = 0$. The (closed) convex cone $\mathcal{C} := \operatorname{co}(C', \overline{e})$ generated by C' and $\overline{e}$ is acute, for otherwise $\phi'|_{L'}$ would decrease and be < 0 along the half-line $\{r\overline{e}\}_{r<0}$ opposite to $\overline{e}$ and contained into C', which is impossible since $\phi' > 0$ on $L' \cap C' \setminus \{0\}$. Since $\mathcal{C}$ is acute, there is an extension $\Phi' > 0$ of ϕ', whose kernel is a supporting line for $\mathcal{C}$ in 0 only: fix $\overline{f} \in \mathbb{R}^2$ such that $(\mathbb{R}\overline{f}) \cap \mathcal{C} = \{0\}$, then for any $v = r\overline{e} + s\overline{f}$ with r, s real, set $\Phi'v = r\phi'\overline{e}$. If $v \in C' \setminus \{0\}$, $r > 0$ and so $\Phi'v = r\phi'\overline{e} > 0$. $\square$

Corollary 3 *Let F be a finite dimensional linear space and $K \subset F$ an acute closed convex cone. Let $f_0 : L \to \mathbb{R}$ be a linear functional on a linear subspace L of F such that $f_0 x > 0$ for every $x \in L \cap K$ with $x \neq 0$. Then there is a linear extension f of f_0 to F such that $fx > 0$ for every $x \in K$, $x \neq 0$.*

For any σ-finite measure $\mu \geq 0$ on T and $1 \leq p \leq \infty$ the symbols $L^p(T, \mu)$, $L^p_+(T, \mu)$ have the usual meaning. Lemma 4 is an extension of [Theorem 2.9, [2]]. We give its original proof adapted to our slightly different context.

Lemma 4 (see [2]) *Let $T \subset \mathbb{R}^n$ be closed with positive Lebesgue measure, finite or not. Let $\rho : T \to (0, \infty)$ be locally integrable, and $\mu = \rho dt$ be the measure on T with density ρ. Let $I \subset \mathbb{Z}^n_+$ be finite. Let $f \geq 0$ a.e. on T be measurable, $f \not\equiv 0$ a.e., such that $\int_T |t^i| f(t) d\mu(t) < \infty$ $(i \in I)$. There is an $r > 0$ such that for any $\beta = (\beta_i)_{i \in I}$ with $\|\beta\| < r$, there exists a $g = g_\beta \in L^\infty(T, \mu)$, $g > 0$ a.e. with the properties $\int_T |t^i| g(t) d\mu(t) < \infty$,*

$$\int_T t^i g(t) d\mu(t) = \int_T t^i f(t) d\mu(t) + \beta_i \quad (i \in I).$$

Proof. Set $T_l = \{t \in T : f(t) \geq 1/l, \|t\| \leq l\}$. Using $\{t \in T : f(t) > 0\} = \cup_{l \in \mathbb{N}} T_l$ we find $\delta > 0$ and $T_* \subset T$ bounded with $0 < \mu(T_*) < \infty$ such that $f(t) \geq \delta$ a.e. on T_*. The map $A : L^\infty(T_*) \to \mathbb{R}^N$ ($N = \text{card}\, I$) given by $Au = (\int_{T_*} u(t)t^i d\mu(t))_{i \in I}$ is surjective, for if there is a vector $\lambda = (\lambda_i)_i \neq 0$ orthogonal to its range, $\sum_i \lambda_i \int_{T_*} ut^i d\mu = 0 \; \forall i, u$ whence $\sum_i \lambda_i t^i = 0$ a.e. on T_*, that is impossible because the set of zeroes of a polynomial $\not\equiv 0$ has measure zero. Then A is open. Hence 0 is in the interior $\text{int}\, C$ of the set $C = \{Au : u \in L^\infty(T_*, dt), \|u\| < \delta/4\}$. Fix $r > 0$ such that the ball of center 0 and radius r is contained in C. Define f_k on T by $f_k(t) = \min(f(t), k) + \frac{1}{k} e^{-\|t\|}/\rho(t)$. Then $0 \leq |t^i| f_k \leq |t^i| f + |t^i| e^{-\|t\|}/\rho(t) \in L^1(T, \mu)$ and $f_k \to f$ a.e. as $k \to \infty$. The vector $(\int_T (f_k - f) t^i d\mu)_{i \in I} \to 0$ in $\mathbb{R}^N$ as $k \to \infty$, by Lebesgue's theorem of dominated convergence. Since $0 \in \text{int}\, C$, for large k we have $(\int_T (f_k - f) t^i d\mu)_i \in C$. Then $\int_T (f_k - f) t^i d\mu = \int_T ut^i d\mu$ for some $u \in L^\infty(T)$ with $\|u\| < \delta/4$ and $u = 0$ outside T_*. Also, if $\|\beta\| < r$, $\beta \in C$ and so there is a $v \in L^\infty(T)$, $v = 0$ outside T_*, with $\|v\| < \delta/4$ such that $\int_T vt^i d\mu = \beta_i$. Set $g = f_k - u + v$ for a sufficiently large fixed $k \, (\geq \delta)$. Hence $\int_T gt^i d\mu = \int_T (f_k - u + v) t^i d\mu = \int_T (f_k - u) t^i d\mu + \int_T vt^i d\mu = \int_T ft^i d\mu + \beta_i$. Since ρ is locally integrable and T_* contained in a ball, on which u, v and all t^i are bounded, the functions $t^i u, t^i v \in L^1(T, \mu)$. Hence $|t^i|g = |t^i|(f_k - u + v)$ is in $L^1(T, \mu)$. Moreover f_k, u, v (and hence, g) are in $L^\infty(T)$. On T_*, $f \geq \delta$ and $|u|, |v| \leq \delta/2$, whence $g = f_k - u + v \geq \min(f, k) - u + v \geq \delta - u + v \geq \delta/2$. Outside T_*, $g = f_k \geq \frac{1}{k} e^{-\|t\|}/\rho(t) > 0$. Then $g > 0$ a.e. $\square$

Notation Given any closed subset $T \subset \mathbb{R}^n$ and finite subset $I \subset \mathbb{Z}_+^n$, set
$\Gamma_{TI} = \{\gamma = (\gamma_i)_{i \in I} : \exists$ Borel measures $\nu \geq 0$ on T with $\int_T t^i d\nu(t) = \gamma_i, \, i \in I\}$
and
$G_{TI} = \{g = (g_i)_{i \in I} \neq 0 : \exists \; f \in L_+^1(T, dt)$ such that $\int_T t^i f(t) dt = g_i, \, i \in I\}$,
where $|t^i|$ are implicitly supposed to be integrable.

Lemma 5 *Let $T \subset \mathbb{R}^n$ be a closed regular set and $I \subset \mathbb{Z}_+^n$ a finite regular set. Then G_{TI} is dense in Γ_{TI}.*

Proof. Let $\gamma \in \Gamma_{TI}$. There is a measure $\nu \geq 0$ on T such that

$$\int_T y^i d\nu(y) = \gamma_i \quad (i \in I), \tag{1}$$

in particular $\nu(T) < \infty$ since $0 \in I$. For any $\varepsilon \in (0, 1)$ let h_ε be the characteristic function of the ball b_ε of center 0 and radius ε in $\mathbb{R}^n$, $h_\varepsilon = 1$ on b_ε and $h_\varepsilon = 0$ outside b_ε. For any $y \in T$, let $v_\varepsilon(y)$ be the n-dimensional volume of the set $\{x \in T : \|x - y\| < \varepsilon\}$. Then $v_\varepsilon = (h_\varepsilon * h_T)|_T$ is the convolution of h_ε with the characteristic function h_T of T. Hence the map $y \mapsto v_\varepsilon(y)$ is measurable. All $v_\varepsilon(y) > 0$ since T is regular. For $t \in T$, set $\nu_\varepsilon(t) = \int_T \frac{1}{v_\varepsilon(y)} h_\varepsilon(t - y) d\nu(y)$. By the Tonelli and Fubini theorems, $\nu_\varepsilon \in L_+^1(T, dt)$ has finite moments of orders $i \in I$ on T, that we compute by

$$\int_T t^i \nu_\varepsilon(t) dt = \int_T \frac{1}{v_\varepsilon(y)} \int_T t^i h_\varepsilon(t - y) dt \, d\nu(y) = \int_T \psi_{\varepsilon i}(y) d\nu(y) \tag{2}$$

where $\psi_{\varepsilon i}(y) = \frac{1}{v_\varepsilon(y)} \int_T t^i h_\varepsilon(t-y)dt$. By the change of variables $t - y = w$,

$$\psi_{\varepsilon i}(y) = \frac{1}{v_\varepsilon(y)} \int_{\|w\|<\varepsilon, w \in T-y} (y+w)^i dw = y^i \frac{1}{v_\varepsilon(y)} \int_{\|w\|<\varepsilon, w \in T-y} dw + E_i(\varepsilon, y)$$

where $E_i(\varepsilon, y)$ is a linear combination with binomial coefficients c_{ij},

$$E_i(\varepsilon, y) = \sum_{0 \le j \le i, j \ne i} c_{ij} y^j \frac{1}{v_\varepsilon(y)} \int_{\|w\|<\varepsilon, w \in T-y} w^{i-j} dw \tag{3}$$

and the order on $\mathbb{Z}_+^n$ is given as usual by $j \le i \Leftrightarrow j_k \le i_k$ for $1 \le k \le n$. The set $\{\|w\| < \varepsilon, w \in T - y\}$ is taken by the translation $w \mapsto w + y$ into the set $\{x \in T : \|x - y\| < \varepsilon\}$ the Lebesgue measure of which is $v_\varepsilon(y)$. Hence

$$\frac{1}{v_\varepsilon(y)} \int_{\|w\|<\varepsilon, w \in T-y} dw = 1. \tag{4}$$

Then

$$\psi_{\varepsilon i}(y) = y^i + E_i(\varepsilon, y). \tag{5}$$

Since $\|w\| < \varepsilon < 1$ and $|i - j| \ge 1$ for all j in (2), $|w^{i-j}| \le \|w\|^{|i-j|} \le \|w\| < \varepsilon$. Hence by (2) and (4), we obtain the estimate

$$|E_i(\varepsilon, y)| \le \varepsilon \sum_{0 \le j \le i} c_{ij} |y^j|. \tag{6}$$

Since I is regular, $j \le i \Rightarrow |y_k^{j_k}| \le |y_k^{i_k}| + 1 \Rightarrow |y^j| \le \prod_{k=1}^n (|y_k^{i_k}| + 1) = \sum_{\iota \in \sigma_i} |y^\iota| \le \sum_{\iota \in I} |y^\iota|$ and so we can integrate in (5), (6) with respect to ν on T. By (2) and (1), this gives $\lim_{\varepsilon \to 0} \int_T t^i \nu_\varepsilon(t)dt = \gamma_i$ for all $i \in I$. Set $\tilde\gamma_\varepsilon = (\int_T t^i(\nu_\varepsilon(t) + \varepsilon e^{-\|t\|})dt)_{i \in I}$. Then $\tilde\gamma_\varepsilon \in G_{TI}$ and $\lim_{\varepsilon \to 0} \tilde\gamma_\varepsilon = \gamma$. $\square$

Theorem 6 *Let $T \subset \mathbb{R}^n$ be a closed regular set, and $I \subset \mathbb{Z}_+^n$ a finite regular set. Then G_{TI} is the dense interior of Γ_{TI}.*

Proof. By Lemma 4 for $\rho \equiv 1$, G_{TI} is open, and so contained in the interior of Γ_{TI} (the regularity of T, I is not required here). Let γ be in the interior of Γ_{TI}. There is an $r > 0$ such that the ball B of center γ and radius r is contained in Γ_{TI} (a drawing will be helpful). By Lemma 5, there is a $\tilde\gamma \in G_{TI} \cap B$. By Lemma 4 applied to a representing density f of $\tilde\gamma$, there is an $r_0' > 0$ such that all balls $B(\tilde\gamma, r')$ of center $\tilde\gamma$ and radii $r' \in (0, r_0']$ satisfy $B(\tilde\gamma, r') \subset G_{TI}$. We can fix an r' sufficiently small so that $B(\tilde\gamma, r') \subset B$. Let γ' be the unique point such that $\gamma = \frac{1}{2}(\tilde\gamma + \gamma')$. Since $B(\tilde\gamma, r') \subset B$, then $B(\gamma', r') \subset B$; a quick argument to this aim is that B is symmetric with respect to its center γ and $B(\gamma', r')$, $B(\tilde\gamma, r')$ are symmetric to each other over γ. In particular $\gamma' \in B \subset \Gamma_{TI}$. By Lemma 5, there is a $\tilde\gamma' \in G_{TI} \cap B(\gamma', r')$. Since $\tilde\gamma' \in B(\gamma', r')$, the point v such that $\gamma = \frac{1}{2}(\tilde\gamma' + v)$ must be in $B(\tilde\gamma, r')$ ($\subset G_{TI}$). Hence $v \in G_{TI}$. Since G_{TI} is convex, $\gamma = \frac{1}{2}(\tilde\gamma' + v)$ and both $\tilde\gamma'$, $v \in G_{TI}$, then $\gamma \in G_{TI}$. $\square$

By the previous results, the following completion (Theorem 7) can be made to the Riesz-Haviland theorem [13] and its truncated version [6].

Condition (b) from below is equivalent to the existence of a constant $c > 0$ such that $\varphi_g p \geq c\|p\|$ for every $p \in P_I$ with $p(t) \geq 0$ for all $t \in T$, where $\|\,\|$ is any norm on P_I: use the compactness of $\{p \in P_I : p \geq_T 0, \|p\| = 1\}$ and write $\varphi_g(p/\|p\|) > 0$ for such $p \in P_I \setminus \{0\}$.

Theorem 7 *Let $T \subset \mathbb{R}^n$ be a closed regular set. Let $I \subset \mathbb{Z}_+^n$ be a finite regular set. Let $g = (g_i)_{i \in I}$ be a set of numbers with $g_0 = 1$. The following statements are equivalent:*

(a) There exist functions $f \in L_+^1(T, dt)$ such that $\int_T |t^i| f(t) dt < \infty$ and

$$\int_T t^i f(t) dt = g_i \ \ \text{for all } i \in I;$$

(b) The Riesz functional φ_g defined on the linear span $P_I \subset \mathbb{R}[X_1, \ldots, X_n]$ of the monomials X^i, $i \in I$ by

$$\varphi_g \sum_{i \in I} c_i X^i = \sum_{i \in I} g_i c_i$$

satisfies $\varphi_g \, p > 0$ for every $p \in P_I \setminus \{0\}$ such that $p(t) \geq 0$ for all $t \in T$.

Proof. (a) $\Rightarrow$ (b) From $\varphi_g p = \int_T pf \, dt$ we obtain as usual that $\varphi_g p \geq 0$ for every polynomial p such that $p(t) \geq 0$ for all $t \in T$. If moreover $p \neq 0$ in $\mathbb{R}[X_1, \ldots, X_n]$, the set $Z = \{t \in \mathbb{R}^n : p(t) = 0\}$ of the zeroes of p is an algebraic variety, (empty or) of dimension $\leq n - 1$, and so has null Lebesgue measure. Then $\varphi_g p > 0$, for the equality $\int_T pf \, dt = \int_{T \setminus Z} pf \, dt = 0$ with $p > 0$ on $T \setminus Z$ would compel $f = 0_{L^1(T, dt)}$ that is impossible since $\int_T f dt = g_0 = 1$.

(b) $\Rightarrow$ (a) Endow P_I with a norm and its dual P_I^* with the dual norm. Let C denote the convex cone of all $p \in \mathbb{R}[X_1, \ldots, X_n]$ such that $p(t) \geq 0$ for all $t \in T$. Since the Lebesgue measure of T is > 0, C is acute. There is a constant $c = c_g > 0$ such that $\varphi_g p \geq c\|p\|$ for every $p \in P_I \cap C$, see the comment just before the Theorem. Since the map $\gamma \mapsto \varphi_\gamma \in P_I^*$ is linear, there is a constant $c' > 0$ such that $\|\varphi_\gamma\| \leq c'\|\gamma\|$ for all γ. Then for every $\tilde{g}$ in the ball B of center g and radius $r = c/2c'$, we have $\varphi_{\tilde{g}} p \geq (c/2)\|p\|$ for all $p \in P_I \cap C$. Indeed, $\varphi_{\tilde{g}} p = \varphi_g p + \varphi_{\tilde{g}-g} p \geq c\|p\| - |\varphi_{\tilde{g}-g} p| \geq c\|p\| - c'\|\tilde{g} - g\| \|p\| \geq (c/2)\|p\|$. Hence condition (b), briefly $\varphi_{\tilde{g}} > 0$, holds as well for all $\tilde{g}$ in the neighborhood B of g. Write $\mathbb{Z}_+^n = \cup_{l \in \mathbb{N}} I_l$ as an increasing union of finite subsets I_l such that $I_1 = I$. Let P_l be the linear span of the monomials X^i with $i \in I_l$. Thus $P_1 = P_I$. Let $\tilde{g} = (\tilde{g}_i)_{i \in I}$ be an arbitrary point in B, that for the moment we fix. Since $\varphi_{\tilde{g}} > 0$, by applying successively Corollary 3 for $F = P_{l+1}$, $K = C \cap P_{l+1}$ and $L = P_l$ with $l \geq 1$ we obtain, starting from $\psi_{1\tilde{g}} := \varphi_{\tilde{g}} : P_1 \to \mathbb{R}$, a sequence of linear functionals $\psi_{l\tilde{g}} : P_l \to \mathbb{R}$ such that $\psi_{l+1\,\tilde{g}}|_{P_l} = \psi_{l\tilde{g}}$ for all $l \geq 1$ and $\psi_{l\tilde{g}} p > 0$ for all $p \in P_l \cap C \setminus \{0\}$. Then we have a linear functional $\psi_{\tilde{g}} : \mathbb{R}[X_1, \ldots, X_n] \to \mathbb{R}$, extending $\varphi_{\tilde{g}}$, determined by $\psi_{\tilde{g}}|_{P_l} = \psi_{l\tilde{g}}$ for all $l \geq 1$. For any polynomial $p \in C$, $p \not\equiv 0$ there is an $l \geq 1$ such that $p \in P_l$, and so $\psi_{\tilde{g}} p = \psi_{l\tilde{g}} p > 0$. Hence $\psi_{\tilde{g}} p \geq 0$

for every polynomial p such that $p(t) \geq 0$ for all $t \in T$. By the Riesz-Haviland's Theorem 1 there is a measure $\nu_{\tilde{g}} \geq 0$ on T, with finite moments of any order, such that $\psi_{\tilde{g}} p = \int_T p d\nu_{\tilde{g}}$ for every polynomial p. In particular for $p := X^i$ with $i \in I$, we obtain $\tilde{g}_i = \varphi_{\tilde{g}} X^i = \psi_{\tilde{g}} X^i = \int_T t^i d\nu_{\tilde{g}}(t)$. Thus $\tilde{g}_i$ ($i \in I$) are the moments of a measure $\nu_{\tilde{g}}$ on T, that is, $\tilde{g} \in \Gamma_{TI}$. Since $\tilde{g}$ was arbitrary in a neighborhood of g, it follows that g is in the interior of Γ_{TI}. Then by Theorem 6, $g \in G_{TI}$. $\square$

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