Central values of Rankin L-series over real quadratic fields

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Abstract

We study the Rankin L-series of a cuspidal automorphic representation of GL(2) of even weight over the rational numbers, twisted by a character of a real quadratic field. When the sign of the functional equation is +1, we give an explicit formula for the central value of the L-series, analogous to the formulae obtained by B. Gross, S.W. Zhang, and H. Xue in the imaginary case. The proof uses a version of the Rankin-Selberg method in which the theta correspondence plays an important role. We also discuss an arithmetic application to computing the order of the Tate-Shafarevich group of the base change to real quadratic fields of an elliptic curve over the rationals, and an analytic application to proving the equidistribution of individual closed geodesics on modular curves.

Contents

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1. Introduction

Let f be a newform of even weight $2k \geqslant 0$, level N, and trivial central character. Throughout the paper we adopt the language of automorphic representations, and denote by π_f the cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ associated to f, where $\mathbb{A}$ denotes the adeles of the rational numbers $\mathbb{Q}$. Let K be a real quadratic field of discriminant d_K, and let χ be a (unitary) Hecke character of K, trivial on $\mathbb{A}^\times$. Denote by π_χ the automorphic representation of $\mathrm{GL}(2, \mathbb{A})$, attached to χ via the Jacquet-Langlands correspondence.

When the sign of the functional equation for the (completed) Rankin L-series $L(s, \pi_f \times \pi_\chi)$ is $+1$, we give an explicit formula for its central value. Throughout the paper, we assume that N, d_K, and the conductor $c(\chi)$ of χ are pairwise coprime, and that N is square free. If f is a weight 0 Maass form, we also impose a mild restriction on the archimedean component of π_f (see §4).

Before describing our result in more generality, we state it in the simplest form, when f has weight 2, the character χ is unramified and all the primes dividing N split in the quadratic field K. Then χ can be viewed as a character of the narrow class group of K, and the formula can be written in entirely classical terms (see Theorem 6.3.1 for arbitrary weight):

$$L_{\mathrm{fin}}(1/2, \pi_f \times \pi_\chi) = \frac{1}{\sqrt{d_K}} \left| \sum_Q \chi^{-1}(Q) \int_{\gamma_Q} \omega_f \right|^2,$$

where $\omega_f = 2\pi i f(z) dz$ is a holomorphic differential on the compactified Riemann surface $X = \overline{\mathcal{H}/\Gamma_0(N)}$ (with $\mathcal{H}$ the upper half plane, and $\Gamma_0(N)$ the standard congruence subgroup of level N), and L_{fin} is the L-function with the archimedean component removed. The sum is over $\Gamma_0(N)$-equivalence classes of Heegner quadratic forms Q of discriminant d_K and level N, and the integral is over the closed geodesic γ_Q on X obtained by projecting the geodesic on the upper half plane connecting the two real roots of the quadratic polynomial $Q(z, 1)$. See §6.2 for the definition of Heegner forms, and for the correspondence between such forms and ideal classes in the narrow class group of K.

The formula can be seen as a generalization of a classical formula for the integral of a weight zero Eisenstein series over geodesic cycles attached to ideal classes in real quadratic fields [Si61, Ch. II, §3]. It extends to the real quadratic case results obtained in the imaginary case by B.H. Gross [Gr87] (for weight 2 forms of prime level over $\mathbb{Q}$), by S.W. Zhang [Zh01] (for weight two forms over a totally real number field), and by H. Xue [Xu02] (for even weight forms). The place of the Heegner points appearing in the imaginary case is taken in this paper by the geodesic cycles γ_Q, and our result opens the way for applying techniques that have been developed for studying Heegner points to the study of the geodesic cycles. For example, recent subconvexity bounds for the central value $L(1/2, \pi_f \times \pi_\chi)$, when the discriminant of the imaginary field K goes to infinity, have been shown in [HM04] to imply equidistribution results for Heegner points in Galois orbits, via S.W. Zhang's formula, thus generalizing a result of W. Duke [Du88]. In the real quadratic case, we show in the same way that *individual* "long" closed geodesics become equidistributed on $X_0(N)$ (Theorem 6.5.1).

Another application is a formula for the order of Tate-Shafarevich groups of the base change to real quadratic fields of an elliptic curve E defined over $\mathbb{Q}$, assuming the Birch and Swinnerton-Dyer conjecture (§6.4). Since it relates the order of this mysterious group with a geometric invariant attached to K (the homology class of the sum of geodesic cycles), this formula may be used in the future to shed light on this group.

For another application of our formula to a construction of Heegner point analogues over real quadratic fields using p-adic interpolation of special values of L-functions, see the upcoming paper

[BD05].

We proceed to describe our result in more detail, while also sketching the method of proof. Our approach can be applied, in principle, to the imaginary case as well, after suitable modifications at the infinite place. However, in that case the analogous formula is known, thanks to the work of S.W. Zhang and H. Xue, hence we concentrate on the real quadratic case in this paper.

Let S be the set of primes dividing N which are inert in K. The assumption about the sign of the functional equation implies that S has even cardinality, hence there exists a quaternion algebra B defined over $\mathbb{Q}$ and ramified at the primes in S. Fix an embedding of K into B.

By the Jacquet-Langlands correspondence, there is an automorphic representation of $B^\times(\mathbb{A})$, denoted by π_f^{JL}, having the same local L-factors as π_f. In [Wa85], J.-L. Waldspurger has connected the nonvanishing of the central value of the Rankin L-series to the nonvanishing of a toric linear form l defined on the space of adelic automorphic forms ϕ on $B^\times(\mathbb{A})$ on which the representation π_f^{JL} acts:

$$l(\phi) = \int_{\mathbb{A}^\times K^\times \backslash \mathbb{A}_K^\times} \phi(x)\chi^{-1}(x)dx.$$

In the present paper, we make this connection more precise by proving that the central value $L(1/2, \pi_f \times \pi_\chi)$ equals the absolute value of the linear form above evaluated on a specific automorphic form in the space of π_f^{JL}, up to a nonzero constant (for the precise statement see Theorem 5.3.9). In the case that χ is unramified, we also determine the constant explicitly (Theorem 5.4.1). Finally, assuming further that all the primes dividing N split in K, so that B is the matrix algebra, we rewrite the result in terms of the classical newform f, thus obtaining the formula stated in the beginning of the introduction (Theorem 6.3.1).

The proof is inspired by J.-L. Waldspurger's approach in [Wa85], and by the work of S.-W. Zhang in the imaginary case [Zh01]. In a first stage, we deduce an integral representation for the L-function using an adelic version of the Rankin-Selberg method, similar to that developed by S.-W. Zhang [Zh01]. The novelty here is that the Rankin-Selberg integral is taken over the subgroup $\mathrm{GL}_2(\mathbb{A})^+$ of $\mathrm{GL}_2(\mathbb{A})$ consisting of matrices with determinant belonging to $\mathrm{N}_{K/\mathbb{Q}}(\mathbb{A}_K^\times)$, and that two of the forms entering into the integral are constructed using the Weil representation:

$$L(s, \pi_f \times \pi_\chi) = M(s) \int_{Z(\mathbb{A})\mathrm{GL}_2(\mathbb{Q})^+ \backslash \mathrm{GL}_2(\mathbb{A})^+} \phi_f(g)\theta_\chi(g;\varphi_1)E(s,g;\varphi_2)dg.$$

Here ϕ_f is a newform in the space of π_f, while the theta series θ_χ, and the Eisenstein series E are constructed using the Weil representation attached to the field K, viewed as a quadratic space over $\mathbb{Q}$ with form given by the norm, and by a multiple of the norm respectively. The multiple is chosen such that the two quadratic spaces above provide an orthogonal decomposition of the four dimensional space B, with form given by the reduced norm. The automorphic forms θ_χ and E depend on two Schwartz functions φ_1, φ_2 on $\mathbb{A}_K$, which are carefully chosen at each place so that the local zeta integrals equal the local Rankin L-functions, up to simple factors. Essential in this step is the notion of Whittaker newform, which we review in Section 3.

The next step fits in the general philosophy of "seesaw dual pairs" of S. Kudla [Ku83], which in this setting has been considered by B. Roberts [Ro98]. Using the Siegel-Weil formula for SL(2) [KR88], extended to similitudes in §2.4, we realize the special value $E(1/2, g; \varphi_2)$ as the theta lift of the trivial character of $\mathbb{A}_K^\times$, and interchange the order of integration in the Rankin-Selberg formula to obtain:

$$L(1/2, \pi_f \times \pi_\chi) = M \int_{(\mathbb{A}^\times K^\times \backslash \mathbb{A}_K^\times)^2} \theta_f(x, y; \varphi)\chi(xy^{-1})dxdy,$$

with an explicit constant M, where $\theta_f(x, y)$ is the Shimizu theta lift of ϕ_f to the special similitude group $GSO(B_{\mathbb{A}}) \simeq B_{\mathbb{A}}^\times \times B_{\mathbb{A}}^\times/\mathbb{A}^\times$, the isomorphism being given by $(x,y)v = xvy^{-1}$. It depends on the Schwartz function $\varphi = \varphi_1 \otimes \varphi_2$ on $B_{\mathbb{A}}$.

To identify the form θ_f, we need to replace φ by a Schwartz function φ' which differs from φ at the primes dividing d_K and at infinity. By computing the level and weight of the theta lift, and using a result of Shimizu [Sh72], we show that if χ is unramified it decomposes as follows (for ramified χ see Proposition 5.3.6):

$$\theta_f(x,y;\varphi') = C\phi_f^{JL}(x)\phi_f^{JL}(y),$$

where C is a nonzero constant and ϕ_f^{JL} is an explicit automorphic form on $B^\times(\mathbb{A})$ belonging to the space of π_f^{JL}. The form ϕ_f^{JL} is determined up to a constant by its weight and level structure (Proposition 5.3.6), and it is the analogue of the "toric newform" defined by S.-W. Zhang in the imaginary quadratic setting. The effect of replacing φ by φ' is then computed locally, yielding Theorem 5.3.9.

When χ is unramified, the constant C is determined by using a result of T. Watson [Wa02]. We obtain the explicit formula:

$$L(1/2, \pi_f \times \pi_\chi) = \frac{R}{\sqrt{d_K}} \frac{\|\phi_f\|^2}{\|\phi_f^{JL}\|^2} |l(\phi_f^{JL})|^2$$

with R a rational number given in Theorem 5.4.1. The norms are with respect to the adelic Petersson inner products on $\mathrm{GL}_2(\mathbb{A})$ and $B^\times(\mathbb{A})$, with respect to Tamagawa measures on the two groups. Note that the right hand side is well-defined, even though ϕ_f^{JL} is only determined up to a constant.

In Section 6, we review the theory of optimal embeddings and Heegner forms, and use it to rewrite the previous formula in the classical language, when χ is unramified and the quaternion algebra B is the matrix algebra. We also discuss two applications of this classical formula.

We point out that the method developed here could equally apply over an arbitrary totally real base field F in place of $\mathbb{Q}$, and in fact the local computations are carried over an arbitrary local field (except for the even residue characteristic case). However, to keep the exposition clear and to avoid complications in constructing explicitly the quaternion algebra B, we have restricted ourselves to working over the rational field.

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1.1 Notation

Local fields. When F is a finite extension of a p-adic field $\mathbb{Q}_p$, we denote by $\varpi_F, \mathcal{O}_F, U_F$ a fixed uniformizer, the ring of integers, and the units of F respectively. For each integer $r \geqslant 0$, we let U_F^r denote the subgroup of U_F consisting of units congruent to 1 modulo $\varpi_F^r\mathcal{O}_F$. We normalize the absolute value on F, denoted $|\ |_F$ or simply $|\ |$ when there is no danger of confusion, by requiring that $|\varpi_F| = q^{-1}$, where q is the cardinality of the residue field of F. The valuation on F is denoted by ν_F, and it is always normalized by $\nu_F(\varpi_F) = 1$. If $F = \mathbb{R}$ then the absolute value is the usual one.

Norms. If K/F is a separable quadratic algebra extension of perfect field F, we denote by $\mathrm{N}_{K/F}$ the norm $\mathrm{N}_{K/F}(x) = x\bar{x}$, where the bar denotes the unique nontrivial involution of K fixing F. If

$K = F \oplus F$, we have $\overline{(x_1, x_2)} = (x_2, x_1)$. If F is a local field, we also denote by $\omega_{K/F}$ the character of $F^\times$ attached to the quadratic extension K, which is trivial if K is split and is the nontrivial quadratic character of $F^\times / \mathrm{N}_{K/F} K^\times$ otherwise.

Subgroups of GL(2). Throughout the paper we denote by G the algebraic group $\mathrm{GL}(2)$. We denote by Z the center of G, by B the Borel subgroup of upper triangular matrices, by N the unipotent subgroup of B, and by T_1 the subgroup of diagonal matrices with lower right entry equal to 1. If a is a scalar, we denote by $i(a) \in T_1$, $n(a) \in N$, the matrices having upper left, respectively upper right entry equal to a. If $g \in G(F)$ for some field F, we denote by $g_1 \in \mathrm{SL}_2(F)$ the matrix $\begin{pmatrix} 1 & 0 \\ 0 & \det g \end{pmatrix}^{-1} g$.

Adeles and ideles. If F is a number field, we denote by $\mathbb{A}_F$ the adele ring of F and by $\mathbb{A}_F^\times$ the group of ideles. We also write $\mathbb{A} = \mathbb{A}_\mathbb{Q}$, and occasionally write $F_\mathbb{A}$ or $F_\mathbb{A}^\times$ instead of $\mathbb{A}_F$ and $\mathbb{A}_F^\times$ respectively (especially when F is viewed as a vector space over $\mathbb{Q}$).

Congruence subgroups. When F is a local nonarchimedean field and $\alpha \in \mathcal{O}_F$, we consider the congruence subgroups of $\mathrm{GL}_2(F)$:

$$K_1(\alpha) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F) : \quad c \in \alpha\mathcal{O}_F, d \in 1 + \alpha\mathcal{O}_F \right\};$$

$$K_0(\alpha) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_F) : \quad c \in \alpha\mathcal{O}_F \right\}.$$

If χ is a character of conductor C, we often view χ as a character of the congruence subgroup $K_0(\pi_F^C)$ by acting on the lower right entry of a matrix.

If F is a global field and N is an integral ideal in the ring of integers of F, we denote by $K_0(N)$ the subgroup $\prod_v K_0(N_v)$ of $\mathrm{GL}_2(\mathbb{A}_F)$, where the product is over all finite places v, and N_v denotes the image of N under fixed embeddings $F \to F_v$.

Induced representations. For F a local field and μ_1, μ_2 two characters of $F^\times$, we denote by $B(\mu_1, \mu_2)$ the induced representation space of functions on $G(F)$ satisfying:

$$f\left(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g \right) = \mu_1(a)\mu_2(b)|a/b|^{1/2} f(g), \quad \text{for all } \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in G(F),$$

which are K-finite and locally constant (in the nonarchimedean case) or smooth (in the archimedean case).

2. Review of the Weil Representation

In this section we collect the facts about the Weil representation which are used throughout the paper. For our purposes, we only need to consider the Weil representation for the dual pair $(\mathrm{SL}(2), \mathrm{O}(V))$, where V is an even dimensional space with a nondegenerate quadratic form (over a local or a global field). The assumption that the quadratic space V is even dimensional implies that the Weil representation can be viewed as a representation of $\mathrm{SL}(2)$, and not of its metaplectic cover. The only possibly new contribution appears in §2.5, where we compute the level and weight of various Schwartz functions under the two dimensional Weil representation.

2.1 The local Weil representation for $\mathrm{SL}(2)$

Let F be a local field and let V be a $2n$-dimensional vector space over F endowed with a nondegenerate quadratic form q. We will also denote by q the bilinear form on V:

$$q(x, y) = q(x + y) - q(x) - q(y).$$

Let ψ be a fixed nondegenerate character of F, which we assume unramified if F is nonarchimedean.

The Weil representation r_ψ is a representation of $\mathrm{SL}_2(F)$ depending on ψ, attached to the quadratic space (V,q). In a model suitable for our needs, the Weil representation acts on the space $S(V)$ of Schwartz functions on V as follows:

$$r_\psi \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} f(x) = \psi(aq(x))f(x)$$
$$r_\psi \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} f(x) = |a|^n \omega(a) f(ax)$$
$$r_\psi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f(x) = \gamma \widehat{f}(x)$$

Here γ is an eighth root of unity whose precise value is given in Lemma 2.1.3, ω is the discriminant character of $F^\times$ associated with the quadratic space V, and $\widehat{f}(x) = \int_V f(y)\psi(q(x,y))dy$ denotes the Fourier transform with respect to a self-dual Haar measure dy.

The three equations above can be combined into one for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F)$, with $c \neq 0$:

$$r_\psi(g)\varphi(x) = |c^{-1}|\omega(-c^{-1})\gamma \int_K \varphi(y)\psi[c^{-1}(aq(x)+dq(y)-q(x,y))]dy. \tag{2.1.1}$$

The quadratic spaces we shall consider in this paper are either two or four-dimensional, and they fall in one of the following two cases:

i) V is a separable quadratic algebra K over F (a field or the split algebra);

ii) V is a quaternion algebra B over F.

These spaces are endowed with a natural quadratic form q, that is the norm in the algebra extension K/F in case (i), and the reduced norm in case (ii). The character ω appearing in the definition of r_ψ is trivial if $K = F \oplus F$ is a split algebra or a quaternion algebra, while it is the nontrivial character of $F^\times/\mathrm{N}_{K/F}(K^\times)$ if K is a quadratic field extension of F.

For later use, in the next lemma we record the normalization factor for the self-dual measure used in the Fourier transform, in case $V = K$ is a quadratic algebra extension of F.

Lemma 2.1.1. *If the different of the quadratic extension K/F is $\varpi_K^s \mathcal{O}_K$ $(s \geqslant 0)$, then the self-dual measure on K is the one which gives $\mathcal{O}_K$ measure $|\varpi_K|_K^{s/2}$.*

The constant γ appearing in the definition of the Weil representation is the p-adic version of a classical Gauss sum. Before giving its value, we recall the following connection between Gauss sums and local epsilon factors [Zh01, Section 2.1]:

Lemma 2.1.2. *Let ψ be an unramified nontrivial additive character of a nonarchimedean field F, and let η be a unitary character of $F^\times$, of conductor $s > 0$. Then:*

$$|a|^{1/2} \int_{U_F} \eta(ax)^{-1}\psi(ax)dx = \begin{cases} \epsilon(\eta,\psi) & \text{if } \nu_F(a) = -s\ , \\ 0 & \text{otherwise,} \end{cases}$$

where the measure is normalized such that $\mathcal{O}_F$ has unit measure, and $\epsilon(\eta,\psi)$ is the epsilon factor appearing in the functional equation for the zeta function of the character η, as in Tate's thesis.

An easy modification of the proof of Lemma 1.2 in [JL70] yields:

Lemma 2.1.3. *(i) If $V = B$ is a quaternion algebra over F, then $\gamma = 1$ if B is split, and $\gamma = -1$ if B is a division algebra;*

(ii) If $V = K$ is a quadratic extension of F then

$$\gamma = \epsilon(\omega, \psi) = \int_{\varpi_K^{-l}\mathcal{O}_K} \psi(q(x))dx \text{ for } l \geqslant C,$$

where C is the exponent of the different of the extension K/F. The measure used in the integral is normalized as in Lemma 2.1.1.

Finally, a scalar change in the quadratic form modifies the Weil representation as follows. For $\lambda \in F^\times$, let r'_ψ be the Weil representation associated to the quadratic form $q'(x) = \lambda q(x)$. Let ω', γ', dx', be the corresponding character, Gauss sum and self-dual measure for the representation r'_ψ. They are related to the original quantities as follows:

$$\omega' = \omega, \quad \gamma' = \omega(\lambda)\gamma, \quad dx' = |\lambda|_F dx.$$

2.2 Local theta correspondence for $\mathrm{GL}(2)$

H. Jacquet and R. Langlands used the Weil representation in [JL70] to construct representations of $\mathrm{GL}(2)$ over a local field F, attached to representations of $K^\times$ or $B^\times$, where K is a quadratic extension, and B is a quaternion algebra over F. This correspondence preserves $L-$ and $\epsilon-$ factors.

In this paper we are interested mostly in the global construction of the Jacquet-Langlands correspondence, which will be reviewed in the next section. Therefore we only state the local results, without getting into the details of the local construction.

First, let K be a quadratic extension of F, either a field or the split algebra, and let χ be a quadratic character of $K^\times$. Denote by π_χ the associated representation of $\mathrm{GL}_2(F)$ attached to χ by Jacquet-Langlands.

Theorem 2.2.1 [JL70], Theorem 4.6. *The representation π_χ of is admissible and irreducible, of central character $\omega_{K/F}\chi|_{F^\times}$. More precisely:*

(a) If $K = F \oplus F$ is split, then $\chi = (\chi_1, \chi_2)$ for two characters of $F^\times$, and π_χ is the principal series representation $\pi(\chi_1, \chi_2)$.

(b) If χ does not factor through the norm $N_{K/F}$ and F is nonarchimedean, then π_χ is supercuspidal;

(c) If $\chi = \delta \circ N_{K/F}$ for a character δ of $F^\times$, then π_χ is the principal series representation $\pi(\delta, \delta\omega_{K/F})$.

Let now B be a nonsplit quaternion algebra over F. Let χ be an irreducible (finite dimensional) representation of $B^\times$. Denote as before by π_χ the representation of $\mathrm{GL}_2(F)$ attached to χ.

Theorem 2.2.2 [JL70], Theorem 4.2. *The representation π_χ is admissible and irreducible, of central character $\chi|_{F^\times}$. More precisely*

(a) If the dimension of χ is greater then 1 and F is nonarchimedean, then π_χ is supercuspidal.

(b) If $\chi = \eta \circ \mathrm{N}_{B/F}$ for some character η of $F^\times$, then π_χ is the discrete series representation $\sigma(\eta|\ |_F^{1/2}, \eta|\ |_F^{-1/2})$.

Moreover, all special, and supercuspidal in the nonarchimedean case, representations of $\mathrm{GL}_2(F)$ can be obtained in this way.

As a matter of notation, if π is a special or supercuspidal representation of $\mathrm{GL}_2(F)$, we denote by π^{JL} the irreducible representation of $B^\times$ whose Jacquet-Langlands lift it is. For other admissible, irreducible representations of $\mathrm{GL}_2(F)$, we set $\pi^{JL} = 0$.

2.3 The global theta correspondence for GL(2)

At the global level, the Jacquet-Langlands correspondence of the previous section is a particular case of a more general construction that relates automorphic forms on GL(2) over a global field F to automorphic forms on the similitude group of a quadratic space over F. We first review this more general construction, following [HK92], and then specialize to the cases of interest. Throughout this section, we write $\mathbb{A} = \mathbb{A}_F$, and fix a nontrivial character ψ of $\mathbb{A}/F$.

Let (V, q) be a $2n$ dimensional quadratic vector space over F, and assume that V is anisotropic over F. By taking the restricted tensor product of the local Weil representations, we obtain a global representation:

$$r_\psi : O(V_\mathbb{A}) \times \mathrm{SL}_2(\mathbb{A}) \to \mathrm{Aut}\ S(V_\mathbb{A}).$$

Since we are interested in automorphic forms on $\mathrm{GL}_2(\mathbb{A})$, we would ideally like to extend this representation to the group $GO(V_\mathbb{A}) \times \mathrm{GL}_2(\mathbb{A})$. This is not possible without enlarging the representation space of r_ψ, as in [Wa85], and instead we proceed as follows [HK92, Sh72].

First extend the action of $O(V_\mathbb{A})$ to $GO(V_\mathbb{A})$ by:

$$L(h)f(x) = |\nu(h)|^{-n/2} f(h^{-1}x) \text{ for } h \in GO(V_\mathbb{A}), f \in S(V_A)$$

where $\nu : GO(V_\mathbb{A}) \to \mathbb{A}^\times$ denotes the similitude factor. This action does not commute with the action of $\mathrm{SL}_2(\mathbb{A})$; instead, it satisfies the following global version of Lemma 1.4 in [JL70]:

Lemma 2.3.1. *Let $h \in GO(V_\mathbb{A})$ and let $a = \nu(h) \in \mathbb{A}^\times$. Then:*

$$L(h) r_\psi(g) L(h)^{-1} = r_\psi\left(\begin{pmatrix}1 & 0\\ 0 & a\end{pmatrix} g \begin{pmatrix}1 & 0\\ 0 & a^{-1}\end{pmatrix}\right),$$

for all $g \in \mathrm{SL}_2(\mathbb{A})$.

The lemma allows us to extend the representation r_ψ to the adelic points of the algebraic group:

$$R := \{(h, g) \in GO(V) \times \mathrm{GL}(2) : \nu(h) = \det g\}$$

by defining:

$$r(h, g)f(x) = L(h) r_\psi(g_1) f(x) \text{ for } (h, g) \in R(\mathbb{A}),$$

where $g_1 = \begin{pmatrix}1 & 0\\ & \det g^{-1}\end{pmatrix} g \in \mathrm{SL}_2(\mathbb{A})$.

Remark 2.3.2. There is another possible extension of r_ψ to $R(\mathbb{A})$, given by

$$r'(h, g)f = r_\psi\left(g \begin{pmatrix}1 & 0\\ & \det g^{-1},\end{pmatrix}\right) L(h) f$$

as in[HK92]. The previous lemma shows that r' and r are isomorphic, but it turns out that the Siegel-Weil formula of the next section can be more easily generalized to similitudes if one uses r and not r'. The advantage of working with r rather than with r' was pointed out by Harris and Kudla in a later paper [HK01].

Using the extended representation, one can define a theta kernel for $(h, g) \in R(\mathbb{A})$ generalizing the usual theta kernel on $O(V_\mathbb{A}) \times \mathrm{SL}_2(\mathbb{A})$:

$$\theta(h, g; \varphi) = \sum_{x \in V_F} r(h, g)\varphi(x), \quad \text{for } \varphi \in S(V_\mathbb{A}).$$

The correspondence $\varphi \to \theta(\cdot\,; \varphi)$ defines a map from $S(V_\mathbb{A})$ into the space of functions on $R(F)\backslash R(\mathbb{A})$, which is intertwining for the action of $R(\mathbb{A})$ (Lemma 5.1.7 in [HK92]).

Let $G(\mathbb{A})^+$ be the subgroup of $G(\mathbb{A})$ consisting of elements whose determinant belongs to $\nu(GO(V_{\mathbb{A}}))$. Integrating against the theta kernel gives a correspondence between automorphic forms on $GO(V_F)\backslash GO(V_{\mathbb{A}})$ and automorphic forms on $G(F)\backslash G(\mathbb{A})$, as follows:

If χ is an automorphic form on $GO(V_F)\backslash GO(V_{\mathbb{A}})$, then for $g \in G(\mathbb{A})^+$ define:

$$\theta_\chi(g;\varphi) = \int_{O(V_F)\backslash O(V_{\mathbb{A}})} \theta(\sigma h, g; \varphi)\chi(\sigma h) d\sigma,$$

where $h \in GO(V_{\mathbb{A}})$ with $\nu(h) = \det g$. The assumption that V is an anisotropic space over F guarantees that the domain of integration is compact, so that the integral converges. From the properties of the theta kernel, it is easy to see that θ_χ is left invariant under $G(F)^+$, hence it can be extended to $G(F)G(\mathbb{A})^+$ by left invariance under $G(F)$, and to the whole of $G(\mathbb{A})$ by setting it equal to 0 off $G(F)G(\mathbb{A})^+$, a subgroup of index two in $G(\mathbb{A})$. The resulting form, still denoted by θ_χ, is an automorphic form on $G(F)\backslash G(\mathbb{A})$, with central character $\chi|_{F^\times}\omega_V$, where ω_V is the discriminant character of V.

Conversely, if f is an automorphic form on $G(F)\backslash G(\mathbb{A})$, define for $h \in GO(V_{\mathbb{A}})$:

$$\theta_f(h;\varphi) = \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \theta(h, \sigma g; \varphi) f(\sigma g) d\sigma,$$

where $g \in G(\mathbb{A})$ such that $\det g = \nu(h)$. The form θ_f is an automorphic form on $GO(V_F)\backslash GO(V_{\mathbb{A}})$, of central character $\omega_f \omega_V^n$, where ω_f is the central character of f.

Now we specialize the quadratic space (V, q) to the cases of interest in this paper. If $(V, q) = (K, N_{K/F})$ for a quadratic field extension K of F, then the similitude group $GO(V) \simeq K^\times \rtimes \mu_2$, with $K^\times$ acting by multiplication, and the nontrivial element in the order two group μ_2 acting by the nontrivial Galois automorphism of K/F. A Hecke character χ of $\mathbb{A}_K^\times/K^\times$ determines a unique irreducible automorphic representation $\Pi(\chi)$ of $GO(V_F)\backslash GO(V_{\mathbb{A}})$. If we denote by $\Theta(\chi)$ the automorphic representation of $G(\mathbb{A})$ generated by $\theta_{\widetilde{\chi}}(g;\varphi)$ when φ varies in $S(V_{\mathbb{A}})$ and $\widetilde{\chi}$ in the space of $\Pi(\chi)$, we have the following theorem ([HK91], Section 13):

Theorem 2.3.3 Local-global compatibility. *Let $V = K$ be a quadratic field extension of F, with quadratic form $\mathrm{N}_{K/F}$. If χ is a Hecke character of $\mathbb{A}_K^\times/K^\times \equiv GSO(V_A)/GSO(V_F)$ with local components χ_v, let π_χ be the restricted tensor product of the local representations π_{χ_v} defined in Theorem 2.2.1. Then π_χ is isomorphic to the automorphic representation $\Theta(\chi)$ of $G(\mathbb{A})$ attached to χ via the global correspondence.*

Let now $(V, q) = (B, \mathrm{N}_{B/F})$ for a quaternion algebra B over F. The similitude group $GO(V) \simeq GSO(V) \rtimes \mu_2$, with μ_2 generated by the principal involution of B/F. We identify the special similitude group $GSO(V)$ with $B^\times \times B^\times/F^\times$ via the action $(x, y)v = xvy^{-1}$, for $(x, y) \in B^\times \times B^\times$ and $v \in V$.

Let π be a cuspidal automorphic representation of $G(F)\backslash G(\mathbb{A})$, and let π^{JL} be the automorphic representation of $B_{\mathbb{A}}^\times$ attached to π by Jacquet-Langlands (Theorem 2.2.2). Let $\Theta(\pi)$ be the set of functions on $B_{\mathbb{A}}^\times \times B_{\mathbb{A}}^\times$ of the form $\theta_f(\cdot\,;\varphi)$, for f in the space of π and for $\varphi \in S(V_{\mathbb{A}})$. The following theorem is proved in [Sh72]; see also [Ha93] where it is stated in the form given here.

Theorem 2.3.4 Shimizu's correspondence. *With the notations above, assume that the central character of π is unitary. Then the set $\Theta(\pi)$ is spanned by $E \otimes E^c$, where E is the space of automorphic forms on $B_{\mathbb{A}}^\times$ on which π^{JL} acts, and E^c is the representation space of the contragredient representation.*

2.4 The Siegel-Weil formula

As in the previous section, let V be a $2n$ dimensional quadratic space over the totally real number field F. To ensure convergence of the theta integrals, assume that V is anisotropic over F. The Weil representation $r = r_\psi$, attached to V and to an additive character ψ, gives rise to Eisenstein series on $\mathrm{GL}_2(\mathbb{A})$ in the following way. For $g \in \mathrm{GL}_2(\mathbb{A}), s \in \mathbb{C}, \varphi \in S(V_\mathbb{A})$ define the function:

$$f(s,g;\varphi) = r(g_1)\varphi(0)|a(g)|^{s-s_0}|\det g|^{-n/2}\omega_V^{-1}(\det g)$$

where $s_0 = n-1$ and $|a(g)| = |a/b|^{1/2}$, if $g = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k$, for k in the standard maximal compact subgroup K of $G(\mathbb{A})$. An easy computation shows that $f(s,g;\varphi)$ belongs to the induced representation space $B(|\ |^{s/2}, \omega^{-1}|\ |^{-s/2})$, and that $f(s,g;\varphi)$ is a *flat section* in this representation space, that is, its restriction to the standard maximal compact subgroup of $G(\mathbb{A})$ is independent of s. The corresponding Eisenstein series is given by:

$$E(s,g;\varphi) = \sum_{\gamma \in B(F)\backslash G(F)} f(s,\gamma g;\varphi).$$

For $g \in \mathrm{SL}_2(\mathbb{A})$, the Eisenstein series coincides with the one for SL(2) defined in [KR88]. For a fixed $g \in GL_2(\mathbb{A})$ the series converges absolutely for $Re(s) > 1$, and has a meromorphic analytic continuation and functional equation, provided the function φ is K-finite [KR88], [Bu97, Ch. 3.7].

Recall the theta kernel $\theta(h,g;\varphi)$ for $(h,g) \in R(\mathbb{A})$, used to define the theta correspondence. For $g \in G^+(\mathbb{A})$ (the subgroup of $G(\mathbb{A})$ of matrices whose determinant belongs to $\nu(GO(V_\mathbb{A}))$), consider the integral:

$$I(g;\varphi) = \int_{O(V_F)\backslash O(V_\mathbb{A})} \theta(\sigma h, g;\varphi) d\sigma,$$

where $h \in GO(V_\mathbb{A})$ has $\nu(h) = \det g$ (ν is the similitude factor), and the measure on the compact group $O(V_\mathbb{A})/O(V_F)$ is normalized to give this group unit volume. Note that in the language of the previous section, $I(g;\varphi)$ is simply the theta lift of the constant function on $O(V_\mathbb{A})$. As before, it can be shown that $I(g;\varphi)$ is left-invariant under $(G(F) \cap G(\mathbb{A})^+)$, and therefore it extends to an automorphic form on $G(\mathbb{A})$.

The Siegel-Weil formula relates this theta lift with a special value of an Eisenstein series. It has been proved for $g \in \mathrm{SL}_2(\mathbb{A})$ in [KR88], and can be extended to similitudes by following the proof of Theorem 4.2 in [HK01].

Theorem 2.4.1 Siegel-Weil for similitude groups. *Let κ be 1 or 2 as $n > 1$ or $n = 1$ respectively. Then:*

$$E(s_0, g;\varphi) = \kappa I(g;\varphi) \quad \text{for all } g \in G(\mathbb{A})^+.$$

Recall that $s_0 = n-1$.

2.5 Special vectors in the Weil representation

Let F be a local field and let K be a quadratic extension of F (either a field or the split algebra). For $\Lambda \in F^\times$, let $r_\Lambda = r_{\Lambda,\psi}$ be the Weil representation of attached to the vector space K with quadratic form $\Lambda \mathrm{N}_{K/F}(x)$ and to a nontrivial character ψ of F. Let ω be the quadratic character of $F^\times$ attached to K.

The purpose of this section is twofold. First we identify Schwartz functions in $S(K)$ that are invariant under certain congruence subgroups of GL(2) in the nonarchimedean case, or that have prescribed weight in the archimedean case. These Schwartz functions are later used in Proposition 4.1 to determine the level of a global automorphic form which is a theta lift of a character of a real quadratic field.

Second, in Lemma 2.5.2 we compute the "Gaussian transforms" of various nonarchimedean Schwartz functions, which play a central role in solving the local Rankin-Selberg integrals in §4.2. We also use Lemma 2.5.2 to determine the image of Schwartz functions under the map $\varphi \to f(s,g;\varphi) \in B(|\ |^{s-1/2}, \omega^{-1}|\ |^{1/2-s})$, used to construct Eisenstein series the previous section:

$$f(s,g;\varphi) = r_\Lambda(g_1)\varphi(0)|a(g)|^{2s-1}|\det g|^{-1/2}\omega_{K/F}^{-1}(\det g). \tag{2.5.1}$$

For simplicity, denote by $I(s,\omega)$ the space $B(|\ |^{s-1/2}, \omega^{-1}|\ |^{1/2-s})$.

2.5.1 *Nonarchimedean case* Assume F is nonarchimedean. As before, let $\omega = \omega_{K/F}$ be the quadratic character of $F^\times$ whose kernel is $\mathrm{N}_{K/F}(K^\times)$ and let $\delta = \varpi_K^C\mathcal{O}_K$ be the different of the extension K/F. It is well-known that the conductor of ω is C. If K is a field we fix uniformizers ϖ_K, ϖ_F of K and F such that $\varpi_K = \varpi_F$ if K/F unramified and $\mathrm{N}_{K/F}\varpi_K = \varpi_F$ if K/F is ramified. Let γ, and dx be the gamma-factor and the self-dual measure for the Weil representation attached to the norm $\mathrm{N}_{K/F}(x)$; as pointed out in Section 2.1, the corresponding quantities for the representation attached to the norm $\Lambda\mathrm{N}_{K/F}(x)$ are $\omega(\Lambda)\gamma$, and $|\Lambda|dx$ respectively. We assume that the character ψ appearing in the definition of the Weil representation is unramified.

For $\Lambda = 1$, the level of the global theta series θ_χ in Proposition 4.1 will be determined using the following local computation.

Proposition 2.5.1. *(i) Let $\delta = \varpi^C\mathcal{O}_K$ be the different of the extension K/F ($C \geqslant 0$). If $\varphi = 1_{\mathcal{O}_K}$, then for all $k \in K_0(\varpi_F^C) \cap \mathrm{SL}_2(F)$ we have*

$$r_1(k)\varphi = \omega_{K/F}(k)\varphi$$

(ii) Assume K/F unramified, and let χ be a (unitary) character of $K^\times/F^\times$ of conductor $s > 0$. If $\varphi(x) = \chi(x)1_{U_K}(x)$, then for all $k \in K_0(\varpi^{2s}) \cap \mathrm{SL}_2(F)$ we have:

$$r_1(k)\varphi = \varphi$$

Proof. (i) It is enough to check the statement for a set of generators of $K_0(\varpi_F^C)$. Note that the function $\varphi = 1_{\mathcal{O}_K}$ has Fourier transform $\widehat{\varphi} = \mu(\mathcal{O}_K)1_{\delta^{-1}}$. We have two cases:
Case 1: K/F unramified. The group $K_0(1) \cap \mathrm{SL}_2(F)$ is generated by matrices of the type:

$$t(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},\ n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},\ w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

with $a \in U_F$, $b \in \mathcal{O}_F$. The first two matrices clearly fix $\varphi = 1_{\mathcal{O}_K}$, while for the third we have ($\gamma = 1$ in the unramified case) :

$$r_1(w)\varphi = \gamma\widehat{\varphi} = \varphi,$$

since $\gamma = 1$ and $\mu(\mathcal{O}_K) = 1$ in the unramified case.
Case 2: K/F ramified. The group $K_0(\varpi^C) \cap \mathrm{SL}_2(F)$ is generated by the matrices $t(a)$, $n(b)$, together with

$$v(\varpi_F^C u) = \begin{pmatrix} 1 & 0 \\ \varpi_F^C u & 1 \end{pmatrix} = -wn(-\varpi_F^C u)w,$$

for $a, u \in U_F$, $b \in \mathcal{O}_F$. The action of the first two matrices is easily seen to satisfy the claim, while $r_1[n(\varpi_F^C u)]$ fixes $\widehat{\varphi} = \mu(\mathcal{O}_K)1_{\delta^{-1}}$, therefore:

$$r_1[wn(-\varpi_F^C u)w]\varphi(x) = r_1[wn(\varpi_F^C u)]\gamma\widehat{\varphi}(x) = \gamma^2\varphi(-x),$$

by the Fourier inversion formula. Since $\gamma^2 = \omega_{K/F}(-1)$, the claim follows.

(ii) As in part *(i)*, we need to check that $\varphi = \chi(x)1_{U_K}(x)$ is fixed by the generators $t(a), n(b)$, and $v(\varpi^{2s}u)$ of $K_0(\varpi^{2s}) \cap \mathrm{SL}_2(F)$, for any $a, u \in U_F$, $b \in \mathcal{O}_F$. For the first two, the claim follows

from the definition, while for the last we need to consider two cases:
Case 1: K is a field. The character $\psi_K := \psi \circ \mathrm{Tr}$ of K is unramified, hence the Fourier transform of $\varphi = \chi 1_{U_K}$ can be computed using Lemma 2.1.2 (here $\varpi = \varpi_F = \varpi_K$):

$$\widehat{\varphi}(x) = \int_{U_K} \chi(y)\psi_K(x\overline{y})dy = |\varpi|^{-1/2}\epsilon(\chi,\psi)\chi(x)1_{\varpi^{-s}U_K}(x).$$

Since $r_1[n(\varpi^{2s}u)]$ fixes $\widehat{\varphi}$, it follows that:

$$r_1[wn(-\varpi^{2s}u)w]\varphi = r_1[wn(\varpi_F^{2s}u)]\widehat{\varphi} = \varphi(-x).$$

But $\chi(-1) = 1$ since χ is trivial on $F^\times$, hence the conclusion follows.
Case 2: $K = F \oplus F$. Since χ is trivial on $F^\times$, there is a character η of $F^\times$ of conductor s such that $\chi = (\eta, \eta^{-1})$. Using again Lemma 2.1.2, we have:

$$\begin{aligned}\widehat{\varphi}(x_1,x_2) &= \int_{U_F\times U_F} \eta(y_1)\eta(y_2)^{-1}\psi(x_1y_2 + x_2y_1)dy_1dy_2\\ &= |\varpi|^{-1}\eta(x_1)1_{\varpi^{-s}U_F}(x_1)\eta(x_2)^{-1}1_{\varpi^{-s}U_F}(x_2)\epsilon(\eta,\psi)\epsilon(\eta^{-1},\psi).\end{aligned}$$

Since $r_1[n(\varpi^{2s}u)]$ fixes $\widehat{\varphi}$, the conclusion follows as in the previous case. □

The following lemma, which determines the Gaussian transform of various Schwartz functions, plays an important role in computing local Rankin-Selberg integrals in Section 4.2.

Lemma 2.5.2 Nonarchimedean Gaussian. *Let ψ be an unramified nontrivial character of F, and let K/F be a quadratic algebra extension of different $\varpi_K^C\mathcal{O}_K$.*

(i) If $K = F \oplus F$ is split, then:

$$\int_{\varpi_F^s\mathcal{O}_F\times\varpi_F^t\mathcal{O}_F} \psi(\alpha xy)dxdy = \begin{cases} |\varpi_F|^{s+t} & \text{if } \nu(\alpha) \geqslant -s-t,\\ |\alpha|^{-1} & \text{if } \nu(\alpha) \leqslant -s-t.\end{cases}$$

(ii) If K/F is a field extension, t is an integer, and $\alpha \in F^\times$, then:

$$\int_{\varpi_K^t\mathcal{O}_K} \psi[\alpha\mathrm{N}(y)]dy = \begin{cases} |\varpi_F|^{ft}\mu(\mathcal{O}_K) & \text{if } \nu(\alpha) \geqslant -ft,\\ 0 & \text{if } -ft > \nu(\alpha) > -C-ft\\ \gamma|\alpha|^{-1}\omega(\alpha)^{-1} & \text{if } \nu(\alpha) \leqslant -C-ft,\end{cases}$$

where f is the residue class degree of the extension K/F (that is, $f = 1$ or 2 corresponding as K is ramified or not respectively).

(iii) If K/F is ramified, $t \in K$ with $\nu_K(t) = -r$, $0 < r \leqslant C$, and $\alpha \in F$, then:

$$\int_{t+\mathcal{O}_K} \psi[\alpha\mathrm{N}(y)]dy = \begin{cases} \psi[\alpha\mathrm{N}(t)]\mu(\mathcal{O}_K) & \text{if } \alpha \in \mathcal{O}_F,\\ 0 & \text{if } \alpha \notin \mathcal{O}_F, \text{ and } \nu_F(\alpha) \neq r-C.\end{cases}$$

The remaining case, $\alpha \notin \mathcal{O}_F$ and $\nu_F(\alpha) = r - C$, can occur only when the residue characteristic of F is 2. Assuming $F = \mathbb{Q}_2$ in this case, the integral has complex norm $2^{(r+1-2C)}$.

Proof. Part (i) follows easily by direct integration. To prove part (ii), first change coordinates $y = \varpi_K^t\mathcal{O}_K$ to obtain (recall that $\varpi_K = \varpi_F$ in the unramified case and $\mathrm{N}(\varpi_K) = \varpi_F$ in the ramified case):

$$\int_{\varpi_K^t\mathcal{O}_K} \psi(\alpha y\bar{y})dy = |\varpi_F|^{ft}\int_{\mathcal{O}_K} \psi(\alpha\varpi_F^{ft}x\bar{x})dx.$$

The statement of the lemma follows from Lemma 2.1.3 in the case $\nu(\alpha) + ft \leqslant -C$, while if $\nu(\alpha) + ft \geqslant 0$ it is obvious. We are left with the middle case (which can only occur if the residue characteristic of F is 2), for which it is enough to prove that:

$$\int_{\mathcal{O}_K} \psi(\varpi_F^{-r} x\bar{x})dx = 0 \text{ if } 0 < r < C.$$

A change of variables $z = x\bar{x}$ reduces the integral to:

$$\int_{\mathcal{O}_F} \psi(\varpi_F^{-r} z)(1+\omega(z))dz = \int_{\mathcal{O}_F} \psi(\varpi_F^{-r} z)\omega(z)dz.$$

To show that the last integral vanishes, it is enough to prove that the same integral over U_F vanishes. Writing $z = a + \varpi_F^r b + \varpi_F^C t$, with $a \in U_F/U_F^r, b \in \mathcal{O}_F/\varpi_F^{C-r}\mathcal{O}_F$, and $t \in O_F$, we have:

$$\int_{U_F} \psi(\varpi_F^{-r} z)\omega(z)dz = \sum_{a,b} \psi(a\varpi_F^{-r})\omega(a+\varpi_F^r b)$$

Since ω has conductor C, the sum over $b \in \mathcal{O}_F/\varpi_F^{C-r}$ vanishes for any $a \in U_F$, thus finishing the proof of this case.

For part (iii), we change variables $y = tx$, with $x \in U_K^r$. Denoting the integral by I, we have:

$$I = |t| \int_{U_K^r} \psi[\alpha \mathrm{N}(t)\mathrm{N}(x)]dx.$$

If $\alpha \in \mathcal{O}_F$, we further change variables $x = 1 + \varpi_K^r x'$ to conclude $I = \psi(\alpha\mathrm{N}(t))\mu(\mathcal{O}_K)$.

Assume therefore $\alpha \notin \mathcal{O}_F$. We change variables $z = N(x)$. If $r = C$, the norm maps U_K^C *onto* U_F^C, and the integral is easily seen to vanish. Assume further $0 < r < C$, which can only occur if the residue characteristic of F is 2. The norm maps U_K^r into U_F^r, and we have:

$$I = |t| \int_{U_F^r} \psi(\alpha\mathrm{N}(t)z)[1+w(z)]dz,$$

where the measure on F is normalized by $\mu(\mathcal{O}_F) = \mu(\mathcal{O}_K)$. It is easy to see that $\int_{U_F^r} \psi(\alpha N(t)z)dz = 0$, and to compute the remaining integral we change variables $z = 1 + \varpi_F^r a + \varpi_F^{C-r} y$ with $a \in \mathcal{O}_F/\varpi_F^{C-r}\mathcal{O}_F$, $y \in \mathcal{O}_F$:

$$I = |\varpi_F|^{C-r}\psi(\alpha\mathrm{N}(t)) \sum_a \int_{\mathcal{O}_F} \psi[\alpha\mathrm{N}(t)(\varpi_F^r a + \varpi_F^C y)]\omega(1+\varpi_F^r a)dy$$

The integral over y vanishes, unless $\nu_F(\alpha) \geqslant r - C$, which we assume. We are left to compute:

$$I = |\varpi_F|^{C-r}\psi(\alpha\mathrm{N}(t))\mu(\mathcal{O}_F) \sum_a \psi[\alpha\mathrm{N}(t)\varpi_F^r a]\omega(1+\varpi_F^r a),$$

where the sum is over $a \in \mathcal{O}_F/\varpi_F^{C-r}\mathcal{O}_F$.

It is here that we assume $F = \mathbb{Q}_2$. The sum over a has two or four terms, as $C - r = 1$ or $C - r = 2$ respectively, and it is easy to check, by examining the four possible cases for which $3 \geqslant C > r > 0$ and $0 > \nu_F(\alpha) \geqslant r - C$, that the sum vanishes unless $\nu_(\alpha) = r - C$, when its absolute value is $2^{(C-r+1)/2}$. Using the fact that $\mu(\mathcal{O}_F) = |\varpi_F|^{C/2}$, and $|\varpi_F| = 1/2$, we obtain that the (complex) absolute value of I is $2^{(r+1-2C)/2}$. Its exact value depends on the character ψ, but it will not be needed in the sequel. $\square$

The previous lemma allows us to determine the image of the map $\varphi \in S(K) \to f(s,g;\varphi) \in I(s,\omega)$ given by equation (2.5.1), for suitable functions φ. The function $f(s,g;\varphi)$ is determined by its restrictions to $K_0(1)$, which is independent of s, hence it is enough to compute the values $f(k;\varphi) := f(s,k;\varphi) = r_\Lambda(k_1)\varphi(0)\omega^{-1}(\det k)$, for $k \in K_0(1)$.

Proposition 2.5.3. *Let* $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(1)$.

(i) Assume that K/F is unramified (split or not), and that $\nu_F(\Lambda) = M \geqslant 0$. If $\varphi = 1_{\mathcal{O}_K}$, then:

$$f(k;\varphi) = \begin{cases} \omega(d)^{-1} = 1 & \text{if } k \in K_0(\varpi_F^M)\ , \\ |\Lambda c^{-1}|_F \omega(\Lambda c^{-1}) & \text{otherwise.} \end{cases}$$

(ii) Assume that K/F is ramified with different $\delta = \varpi_K^C \mathcal{O}_K$, and that $\nu_f(\Lambda) = M \geqslant 0$. If $\varphi = 1_{\varpi_K^{-M}\mathcal{O}_K}$, then:

$$f(k;\varphi) = \begin{cases} \omega(d)^{-1} & \text{if } k \in K_0(\varpi_F^C)\ , \\ 0 & \text{if } C > \nu_F(c) > 0 \\ |\varpi_F|_F^{-M} \mu(\mathcal{O}_K)\gamma |\Lambda| \omega(-\Lambda c^{-1}) & \text{if } c \in U_F. \end{cases}$$

The middle case occurs when $C \geqslant 2$, which can only happen if the residue characteristic of F is 2.

(iii) Assume that K/F is split, and $\nu(\Lambda) = M \geqslant 0$. If $\varphi = 1_{\mathcal{O}_F \times \varpi_F^{-M}\mathcal{O}_F}$, then $f(k;\varphi) = 1$ for all $k \in K_0(1)$.

Proof. Since $f(g;\varphi) \in B(|\ |^{s-1/2}, \omega^{-1}|\ |^{1/2-s})$, it is enough to check the claim for $k \in K_0(1) \cap \mathrm{SL}_2(F)$.

If $c = 0$, by the definition of the Weil representation we have:

$$r_\Lambda \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \varphi(x) = \omega(a)|a|\psi(ab\Lambda q(x))\varphi(ax)$$

and it follows that $f(k;\varphi) = \omega(a)\varphi(0) = \omega(a)$ as desired.

If $c \neq 0$, the conclusion follows from formula (2.1.1) together with the previous lemma. $\square$

2.5.2 *Archimedean case* Assume now that $F = \mathbb{R}$, and fix the character $\psi(x) = e^{2\pi i x}$. For future applications, we need to consider only the case when the extension K/F is split, that is $K = \mathbb{R}^2$. We are interested in functions $F(s,g;\varphi) \in I(s,\omega)$ given by Eq. (2.5.1) of arbitrary even weight under the action of r_Λ. Equivalently, it is enough to find Schwartz functions φ_k of weight $-2k$ under the action of r_Λ, for each integer k. We assume here that $\Lambda > 0$.

The basic computational tool for the action of the orthogonal group is the following:

Lemma 2.5.4. *Let $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$. Then:*

$$\int_{\mathbb{R}} y^{2n} e^{-\pi\lambda y^2} dy = \frac{(2n)!}{(4\pi)^n n!} \cdot \frac{1}{\lambda^n \sqrt{\lambda}}$$

Proof. The case $n = 0$ is the well-known Gaussian integral, and the general case follows by induction, using integration by parts. $\square$

Let now $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}_2(\mathbb{R})$.

Proposition 2.5.5. *For each integer $k \geqslant 0$, let $\varphi_k \in S(\mathbb{R}^2)$ be given by $\varphi_k(x_1, x_2) = P_k(x_1 - x_2)e^{-\pi\Lambda(x_1^2+x_2^2)}$, where P_k is the polynomial:*

$$P_k(X) = \sum_{j=0}^{k} \frac{(-4\pi\Lambda)^j j!}{(2j)!} \binom{k}{j} X^{2j}.$$

Then

$$r_\Lambda(k_\theta)\varphi_k(x) = e^{-2\pi i k\theta}\varphi_k(x).$$

Proof. Note that the map $\phi(x_1,x_2) \to \phi(\sqrt{\Lambda}x_1, \sqrt{\Lambda}x_2)$ provides an isomorphism between the Weil representations r_1 and r_Λ. Therefore we can assume without loss of generality that $\Lambda = 1$.

We are looking for a polynomial $P(x_1, x_2)$ of degree $2k$ such that:

$$r_1(k_\theta)\varphi(0) = e^{-2\pi ik\theta}, \text{ where } \varphi(x_1, x_2) = e^{-\pi(x_1^2+x_2^2)}P(x_1, x_2).$$

Let $a = \cos\theta, c = -\sin\theta$. Then formula (2.1.1) gives:

$$r_1(k_\theta)\varphi(0) = |c^{-1}| \int_{\mathbb{R}^2} e^{-\pi(x_1^2+x_2^2)}P(x_1, x_2)e^{2\pi ac^{-1}x_1x_2}dx_1dx_2.$$

We change variables $x_1 = y_1 + y_2, x_2 = y_1 - y_2$. Denoting by $Q(y_1, y_2)$ the polynomial $P(y_1 + y_2, y_1 - y_2)$, we have:

$$r_1(k_\theta)\varphi(0) = 2|c^{-1}| \int_{\mathbb{R}^2} e^{-2\pi y_1^2(1-iac^{-1})}e^{-2\pi y_2^2(1+iac^{-1})}Q(y_1, y_2)dy_1dy_2$$

We shall look for a polynomial of the shape:

$$Q(y_1, y_2) = \sum_{n=0}^{k} a_n y_2^{2n},$$

where a_n are to be determined. The integral is then easy to compute using Lemma 2.5.4:

$$r_1(k_\theta)\varphi(0) = \sum_{n=0}^{k} a_n \frac{(2n)!}{(8\pi)^n n!}(1 + ia/c)^{-n}.$$

Note that $(1 + ia/c)^{-1} = i\sin\theta e^{-i\theta}$, and that the following identity holds:

$$e^{-2ki\theta} = \sum_{n=0}^{k}(-2i\sin\theta e^{-i\theta})^n \binom{k}{n}. \tag{2.5.2}$$

It follows that $r_1(k_\theta)\varphi(0) = e^{-2ki\theta}$, if a_n satisfies:

$$a_n \frac{(2n)!}{(8\pi)^n n!} = (-2)^n \binom{k}{n}.$$

With these values, the polynomial $P(x_1, x_2) = Q[(y_1 + y_2)/2, (y_1 - y_2)/2]$ is seen to satisfy

$$P(x_1, x_2) = P_k(x_1 - x_2)$$

where P_k is the polynomial defined in the statement of the proposition.

It remains to check that $r_1(k)\varphi(x) = e^{-2ki\theta}\varphi(x)$ for all values of $x \in \mathbb{R}^2$, which can be done by direct computation using Lemma 2.5.4 and identity (2.5.2). □

Remark 2.5.6. A more conceptual proof would consist of defining P_k as the result of applying k times the lowering operator L to P_0, where L is a certain matrix in the complexification of the Lie algebra of $\mathrm{SL}_2(\mathbb{R})$. However the action of the lowering operator seems to be harder to compute than the direct method presented here.

The polynomials P_k have a very simple Gaussian transform, which we state here for later use:

Proposition 2.5.7. *If $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$, then:*

$$\int_{\mathbb{R}} P_k(x)e^{-\pi\lambda x^2}dx = \frac{1}{\sqrt{\lambda}}\left(1 - \frac{1}{\lambda}\right)^k$$

Proof. The identity follows from the binomial formula, using Lemma 2.5.4. □

3. Local L-function theory and Whittaker newforms

Let F be a local field, and let ψ be a fixed nontrivial, additive character of F. Let π be an admissible, irreducible, infinite dimensional representation of $G(F) = \mathrm{GL}_2(F)$, and let K be the standard maximal compact subgroup of G. We recall that any such π admits a Whittaker model, in which $G(F)$ acts by right translation on a space $W(\pi, \psi)$ of smooth[1] and K-finite functions W on $G(F)$ satisfying:

$$W\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) = \psi(x)W(g) \quad \text{for all } x \in F.$$

For the construction of the space $W(\pi, \psi)$, we refer the reader to [Go70], or to [Bu97]. For any $W \in W(\pi, \psi)$, define the "Mellin transform:"

$$\Psi_W(s, g) = \int_{F^\times} W\left(\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix} g\right) |x|^{s-1/2} d^\times x \tag{3.0.1}$$

where $d^\times x$ is the invariant measure on $F^\times$ such that the set of units in F has measure 1 in the nonarchimedean case, and the multiplicative Lebesgue measure on $\mathbb{R}^\times$ or $\mathbb{C}^\times$ if $F = \mathbb{R}$ or $F = \mathbb{C}$ respectively. The L-function $L(s, \pi)$ is defined as the greatest common denominator of all $\Psi_W(s, g)$, appropriately normalized. Our choice of exponent $s - 1/2$ in formula (3.0.1) guarantees that the Mellin transforms above have a functional equation for $s \to 1 - s$.

In this section, we review the theory of *Whittaker newforms*, which are elements $W_\pi \in W(\pi, \psi)$ such that:

$$\Psi_{W_\pi}(s, e) = L(s, \pi), \tag{3.0.2}$$

where e is the unit matrix in $G(F)$. Such elements are not unique, but there are natural choices that we review below.

3.1 The nonarchimedean case

Assume now that F is a nonarchimedean field, and that the nontrivial character ψ used to define the Whittaker model is unramified.

Following Casselman [Ca73], we define *the conductor of* π as the smallest integer $C \geqslant 0$ such that there is a nonzero function $W \in W(\pi, \psi)$ which is invariant under $K_1(\varpi_F^C)$. Casselman has shown that the space of $K_1(\varpi_F^C)$-invariant functions in $W(\pi, \psi)$ is one-dimensional, and we define the Whittaker newform as the function in this space that takes value 1 at the identity (that this is possible follows from [Ca73]). It is an easy check that W_π satisfies the identity (3.0.2).

The following proposition can be used to determine the values of W_π. Part (i) is well-known, while for part (ii) see [Zh01].

Proposition 3.1.1. *Let $C \geqslant 0$ be the conductor of π, and W_π the Whittaker newform.*

(i) Let $\alpha_1, \alpha_2 \in \mathbb{C}$ be such that $L(s, \pi) = \prod_{1,2}(1 - \alpha_i |\varpi_F|^s)^{-1}$. Then we have:

$$W_\pi \begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} = \begin{cases} 0 & \text{if } |a| > 1,\\ |a|^{1/2} \sum_{k+l=\nu(a)} \alpha_1^k \alpha_2^l & \text{otherwise,}\end{cases}$$

with the convention that $0^0 = 1$, in case α_1 or α_2 is 0.

(ii) Let $\epsilon(\pi, \psi)$ be the epsilon factor attached to π, and let $W_{\tilde{\pi}}$ be the Whittaker newform for the contragredient representation $\tilde{\pi}$. Then:

$$W_\pi(gh) = W_{\tilde{\pi}}(g)\omega(\det g)\epsilon(\pi, \psi),$$

[1] If F is nonarchimedean, smooth means locally constant.

where $h = \begin{pmatrix} 0 & 1 \\ -\varpi_F^C & 0 \end{pmatrix}$ *is the Atkin-Lehner operator of level* ϖ_F^C.

3.2 The archimedean case

Consider now the case $F = \mathbb{R}$ (the case $F = \mathbb{C}$ is not needed in this paper and it is treated in [Po04]). We fix the character $\psi(x) = e^{2\pi i x}$, and denote by $K = O_2(\mathbb{R})$ the maximal compact subgroup of $\mathrm{GL}_2(\mathbb{R})$. Let π be an admissible, irreducible, infinite dimensional representation of $\mathrm{GL}_2(\mathbb{R})$, that is a $(\mathfrak{g}, K)$-module, where $\mathfrak{g}$ is the complexification of the Lie algebra of $\mathrm{GL}_2(\mathbb{R})$. See [Bu97, p.200] for a definition of $(\mathfrak{g}, K)$-modules.

Since it will be often used, we recall the formula for the L-function $L(s, \pi)$. It is defined as follows in terms of the gamma factors:

$$G_1(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right), \quad G_2(s) = 2(2\pi)^{-s}\Gamma(s) = G_1(s)G_1(s+1).$$

If π is a principal series representation $\pi(\mu_1, \mu_2)$ of $G(\mathbb{R})$ with $\mu_i = |\ |^{r_i}\mathrm{sgn}^{m_i}$, $r_i \in \mathbb{C}, m_i \in \{0, 1\}$, then:

$$L(s, \pi) = \prod_{i=1,2} G_1(s + r_i + m_i).$$

On the other hand, if π is a discrete series representation $\sigma(\mu_1, \mu_2)$, we can assume without loss of generality that $\mu_1 = |\ |^{s_1}, \mu_2 = |\ |^{s_2}\mathrm{sgn}^{m_2}$, with $s_1 - s_2 = S$ a positive integer, $m_2 \in \{0, 1\}$, and $S - m_2$ odd. Then:

$$L(s, \pi) = G_2\left(s + s_1\right).$$

The notion of level in the nonarchimedean case is replaced by that of *weight* in the archimedean case. A Whittaker element $W \in \mathcal{W}(\pi, \psi)$ is said to have weight m if:

$$W(gk_\theta) = e^{im\theta}W(g) \quad \text{for all } k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}_2(\mathbb{R}).$$

The weight of π is the smallest nonnegative integer n such that $\mathcal{W}(\pi, \psi)$ contains a nontrivial vector of weight n. If π is a principal series representation with central character $\mu(t) = |t|^r\mathrm{sgn}(t)^m$ with $m \in \{0, 1\}$, then the weight of π is m; if π is a discrete series representation $\sigma(\mu_1, \mu_2)$ with $\mu_1\mu_2^{-1}(t) = t^p\mathrm{sgn}t$ for some integer $p > 0$, then the weight of π is $p + 1$.

We are now ready to identify Whittaker newforms in the real case. These statements are well-known, and for proofs we refer to [Po04].

Proposition 3.2.1. *(i) If π is an admissible, irreducible representation of $G(\mathbb{R})$ of weight k, which is not of the form $\pi(|\ |^{r_1}\mathrm{sgn}, |\ |^{r_2}\mathrm{sgn})$, then there is a Whittaker function $W_\pi \in W(\pi, \psi)$ of weight k such that $\Psi_{W_\pi}(s, e) = L(s, \pi)$.*
(ii) If π is the weight 0 representation $\pi(|\ |^{r_1}\mathrm{sgn}, |\ |^{r_2}\mathrm{sgn})$, then there is a Whittaker function $W \in W(\pi, \psi)$ of weight 2 such that $\Psi_W(s, e) = L(s, \pi)$.

We call the function W_π of Proposition 3.2.1 a Whittaker newform for the representation π, in case π is not of the form $\pi(|\ |^{r_1}\mathrm{sgn}, |\ |^{r_2}\mathrm{sgn})$. In the latter case, following [Zh01], we call Whittaker newform the weight 0 function W_π such that $W_\pi(g)\mathrm{sgn}(\det g)$ is a newform for the representation $\pi \otimes \mathrm{sgn} = \pi(|\ |^{r_1}, |\ |^{r_2})$. Even though $\Psi_{W_\pi}(s, e) = 0$ in this case, it will be important for the next section that the newform W_π have the same weight as the representation π.

We also need to know the values of W_π on the diagonal torus, for which we refer to [Po04], and [Zh01].

Proposition 3.2.2. *Let W_π be the Whittaker newform defined above. Then W_π is completely determined by the function $f_\pi(t) = W_\pi\begin{pmatrix} |t|^{1/2}\mathrm{sgn}t & 0 \\ 0 & |t|^{-1/2} \end{pmatrix}$, which is given by:*

- *If π is discrete of weight k, then:*

$$f_\pi(t) = \begin{cases} 2t^{\frac{k}{2}} e^{-2\pi t} & \textit{if } t > 0, \\ 0 & \textit{if } t < 0. \end{cases}$$

- *If π is the weight 0 principal series $\pi(|\ |^{r_1}\mathrm{sgn}^m, |\ |^{r_2}\mathrm{sgn}^m)$ with $m \in \{0,1\}$, and $r = r_1 - r_2$, then:*

$$(-1)^m f_\pi(-t) = f_\pi(t) = 2t^{1/2} J_{r/2}(2\pi t) \quad \textit{for } t > 0,$$

where $J_{r/2}$ is the Bessel function defined below.

For each complex number u, the Bessel function $J_u = J_{-u}$ is a solution of the following differential equation:

$$J_u''(y) + \frac{J_u'(y)}{y} - \left(1 + \frac{u^2}{y^2}\right) J_u(y) = 0 \ \text{ for } y > 0.$$

It can be shown that (up to a constant) this equation admits a unique solution of moderate growth at infinity. If normalized appropriately, this solution satisfies the following identities, which will be often used:

$$\int_0^\infty e^{-y(t+t^{-1})} t^u d^\times t = 2J_u(2y), \tag{3.2.1}$$

$$\int_0^\infty J_u(y) y^s d^\times y = 2^{s-2}\Gamma\left(\frac{s+u}{2}\right)\Gamma\left(\frac{s-u}{2}\right), \tag{3.2.2}$$

where $y > 0$ in the first equation, and $\mathrm{Re}\ s > |\mathrm{Re}\ u|$ in the second.

4. The Rankin-Selberg method

We start by outlining the classical Rankin-Selberg method, as in [Ja72] and [Zh01], and then describe a version that is suitable for our goals.

Recall the global setting considered in the Introduction. Let π_f be the cuspidal automorphic representation of $G(\mathbb{A})$, associated with a newform f of even weight $2k$, and trivial nebentypus for $\Gamma_0(N)$ over $\mathbb{Q}$. Let K be a real quadratic field of discriminant d_K, let χ be a character of $\mathbb{A}_K^\times / K^\times \mathbb{A}^\times$ of conductor $c(\chi) \in \mathbb{Z}$, and let π_χ be the associated representation of $G(\mathbb{A})$, whose local components are described in Theorem 2.2.1. The representation π_χ has conductor $D := d_K c(\chi)^2$, weight 0 at infinity, and central character $\omega = \omega_K$, the quadratic character of $\mathbb{A}^\times/\mathbb{Q}^\times$ attached to K by class field theory.

In this paper we assume that $N, d_K, c(\chi)$ are pairwise coprime, and that N is square free. The later assumption implies that the local components of $\pi_{f,p}$ at the primes dividing N are discrete series representations $\sigma(\eta_p|\ |^{1/2}, \eta_p|\ |^{-1/2})$ with η_p unramified. When π_f has weight 0, its archimedean component is a principal series $\pi(\mu, \mu^{-1})$, and we also assume that $\mu(-1) = \chi_{\infty,1}(-1)$ where $\chi_{\infty,1}$ is one of the archimedean components of χ (the other being $\chi_{\infty,1}^{-1}$).

The Rankin L-series $L(s, \pi_f \times \pi_\chi)$ satisfies a functional equation:

$$L(s, \pi_f \times \pi_\chi) = \epsilon(s, \pi_f \times \pi_\chi) L(1-s, \pi_f \times \pi_\chi).$$

Moreover, we have $\epsilon(1/2, \pi_f \times \pi_\chi) = \alpha_K(-N)$, independent of χ, where α_K is the Dirichlet character of $(\mathbb{Z}/d_K\mathbb{Z})^\times$ attached to the quadratic field K. In this paper we study the case when $\alpha_K(-N) = 1$, so that the sign of the functional equation is $+1$. Recalling that $\alpha_K(-1) = 1$ (since K is real quadratic) and that N is square free, this assumption means that the number of primes dividing N which are inert in K is even.

Let $W(\pi_f,\psi)$, $W(\pi_\chi,\psi)$ be the Whittaker models of π_f, π_χ with respect to the unramified character ψ of $\mathbb{A}/\mathbb{Q}$ having infinity component $\psi_\infty(x)=e^{2\pi ix}$. The Rankin-Selberg method for studying the L-function $L(s,\pi_f\times\pi_\chi)$, developed in general by H. Jacquet [Ja72], can be summarized as follows:

$$\begin{aligned} L(s,\pi_f\times\pi_\chi) &= \int_{Z(\mathbb{A})N(\mathbb{A})\backslash G(\mathbb{A})} W_f(g)W_\chi(\epsilon g)f(s,g)dg \\ &= \int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} \phi_f(g)\phi_\chi(g)E(s,g)dg. \end{aligned} \tag{4.0.1}$$

In the first integral, the Whittaker functions $W_f\in W(\pi_f,\psi), W_\chi\in W(\pi_\chi,\psi)$ are pure tensors whose local components are Whittaker newforms, and the section $f(s,g)$ is a suitably chosen pure tensor belonging to the induced representation space $I(s,\omega):=B(|\ |^{s-1/2},\omega_K^{-1}|\ |^{1/2-s})$ (see [Zh01] for the construction of $f(s,g)$ in the case K is imaginary quadratic). The first identity can be proved locally, using the definition of $L(s,\pi_f\times\pi_\chi)$ as an Euler product, and the properties of Whittaker newforms.

The second integral, the adelic version of the Petersson inner product, is obtained from the first by the folding process characteristic of the Rankin-Selberg method. Here ϕ_f, ϕ_χ are the automorphic forms in the space of π_f,π_χ whose Whittaker coefficients are W_f, W_χ respectively, and $E(s,g)$ is the Eisenstein series:

$$E(s,g)=\sum_{\gamma\in B(\mathbb{Q})\backslash G(\mathbb{Q})} f(s,\gamma g).$$

The functional equation and analytic continuation of $L(s,\pi_f\times\pi_\chi)$ then follow from the corresponding properties of the Eisenstein series, via the integral representation (4.0.1).

Our goal in this section is to prove a version of the Rankin-Selberg identity (4.0.1) in which the form $\phi_\chi(g)$ is replaced by a theta lift $\theta_\chi(g;\varphi_1)$ of the character χ via the Weil representation attached to the quadratic space $(K,\mathrm{N}_{K/\mathbb{Q}})$ as in §2.3, and the section $f(s,g)$ is replaced by the section $f(s,g;\varphi_2)$ constructed using the Weil representation attached to the quadratic space $(K,\Lambda\mathrm{N}_{K/\mathbb{Q}})$, as in §2.4. The Schwartz functions $\varphi_1,\varphi_2\in S(K_\mathbb{A})$ will be chosen later, and $\Lambda\in\mathbb{Z}$ is a constant chosen so that the quaternion algebra B ramified at the primes dividing N which are inert in K has global Hilbert symbol $(d_K,-\Lambda)$. Since θ_χ is defined as an integral only on $G(\mathbb{A})^+$, the subgroup of matrices with determinant belonging to $\mathrm{N}(\mathbb{A}_K^\times)$, it is not surprising that the following version of the Ranking-Selberg identity holds over $G(\mathbb{A})^+$:

$$\begin{aligned} L(s,\pi_f\times\pi_\chi) &= M(s)\int_{Z(\mathbb{A})N(\mathbb{A})\backslash G(\mathbb{A})^+} W_f(g)W_\chi(\epsilon g;\varphi_1)f(s,g;\varphi_2)dg \\ &= M(s)\int_{Z(\mathbb{A})G(\mathbb{Q})^+\backslash G(\mathbb{A})^+} \phi_f(g)\theta_\chi(g;\varphi_1)E(s,g;\varphi_2)dg, \end{aligned} \tag{4.0.2}$$

where $W_\chi(g;\varphi_1)$ is the Whittaker coefficient of $\theta_\chi(g;\varphi_1)$. The factor $M(s)$ is a product of local terms that will be computed while proving the first identity above.

This section is devoted to proving this identity, and it is organized as follows. In §4.1, we study the properties of the theta lift $\theta_\chi(g;\varphi_1)$, in particular we compute its level and its Whittaker coefficients for a suitable choice of φ_1. Using this information, the proof of the first identity in (4.0.2) is done in §4.2, which is entirely local. The second identity then follows just like the classical Rankin-Selberg identity, leading to Proposition 4.3.1, the main result of this section.

4.1 Theta series

In this section, we view K as a quadratic space over $\mathbb{Q}$ with quadratic form $q=\mathrm{N}_{K/\mathbb{Q}}$, and the character χ as an automorphic form on the adelic points of the special similitude group $GSO(K)=$

$K^\times$. We analyze the theta lift $\theta_\chi(g;\varphi)$ defined in section 2.3, and we show that for a suitable choice of Schwartz function φ, the Whittaker coefficient of $\theta_\chi(g;\varphi)$ decomposes as a product of Whittaker newforms for the local factors of the automorphic representation π_χ, for most $g\in G(\mathbb{A})^+$. For most of this section, we let χ be an arbitrary Hecke character of K, while for Proposition 4.1.3 we assume that χ is trivial on $\mathbb{A}^\times$.

We start by recalling the construction of θ_χ, since it is slightly different from that in §2.3. Denote by K^1 the group of elements of K of norm 1, identified with the special similitude group $SO(K)$. We denote by N the norm in the field extension $K/\mathbb{Q}$, as well as in the local and adelic extensions. We also denote by $G(\mathbb{Q})^+$, $G(\mathbb{Q}_v)^+$, $G(\mathbb{A})^+$ the subgroup of the corresponding linear groups consisting of matrices with determinants belonging to $\mathrm{N}(K^\times)$, $\mathrm{N}(K_v^\times)$, and $\mathrm{N}(\mathbb{A}_K^\times)$ respectively.

Let $r: R(\mathbb{A})\to \operatorname{Aut} S(K_{\mathbb{A}})$ be the Weil representation extended to the subgroup $R(\mathbb{A})$ of $GO(K_{\mathbb{A}})\times \mathrm{GL}_2(\mathbb{A})$ consisting of pairs (h,g) with $q(h)=\det g$ (see Section 2.3), and define the theta kernel:

$$\theta(h,g;\varphi)=\sum_{x\in K} r(h,g)\varphi(x) \quad \text{for } (h,g)\in R(\mathbb{A}),$$

depending on a choice of Schwartz function $\varphi\in S(K_{\mathbb{A}})$. Let $\theta_\chi(g;\varphi)$ be the theta lift of the "automorphic form" χ on $GSO(K_{\mathbb{A}})/GSO(K)=K_{\mathbb{A}}^\times/K^\times$:

$$\theta_\chi(g;\varphi):=\int_{SO(K)\backslash SO(K_{\mathbb{A}})}\theta(\sigma h,g;\varphi)\chi(\sigma h)d\sigma, \tag{4.1.1}$$

where h is any element in $GSO(K_{\mathbb{A}})$ with $q(h)=\det g$. Note that the integral is taken over the compact group $K^1\backslash K^1_{\mathbb{A}}$, hence it converges absolutely. The integral is independent of h and it defines a function on $G(\mathbb{A})^+$, left invariant under $G(\mathbb{Q})^+$. We extend it to $G(\mathbb{Q})G(\mathbb{A})^+$, an index two subgroup of $G(\mathbb{A})$, by left invariance under $G(\mathbb{Q})$.

We normalize the measure on $K^1_{\mathbb{A}}=\mathrm{SO}(K_{\mathbb{A}})$ by requiring that the compact sets $K^1_p\cap U_{K,p}$ have measure one if p is nonarchimedean, and on $K^1_\infty\simeq\mathbb{R}^\times$ we use the multiplicative Lebesgue measure dx/x. This measure normalization is compatible with the Hilbert exact sequence:

$$1\to\mathbb{A}^\times\to\mathbb{A}_K^\times\to K^1_{\mathbb{A}}\to 1,$$

where the measures on $\mathbb{A}^\times$ and $\mathbb{A}_K^\times$ are the restricted product measures for which the units have measure 1 at all primes p, and the multiplicative Lebesgue measure at infinity. We can then compute the total measure of $K^1\backslash K^1_{\mathbb{A}}\simeq \mathbb{A}^\times K^\times\backslash\mathbb{A}_K^\times$ using the decomposition:

$$\mathbb{A}^\times K^\times\backslash\mathbb{A}_K^\times/\widehat{\mathcal{O}}_K^\times\simeq\bigsqcup_{\mathfrak{a}\in H_K}\mathfrak{a}\cdot\epsilon_K^{\mathbb{Z}}\backslash K^1_\infty \tag{4.1.2}$$

where $\mathfrak{a}$ runs through a set of finite idele representatives of the narrow class group H_K, $\epsilon_K^{\mathbb{Z}}$ denotes the group generated by the smallest totally positive power ϵ_K of the fundamental unit (ϵ_K is either the fundamental unit or its square), and $K^1_\infty=\{(t,t^{-1})\in K_\infty : t>0\}$. It follows that the total measure of $\mathrm{SO}(K)\backslash\mathrm{SO}(K_{\mathbb{A}})$ equals $h_K\ln\epsilon_K$, with h_K the cardinality of H_K.

Remark 4.1.1. The definition of θ_χ given here differs from that in §2.3, where the domain of integration in Eq. (4.1.1) is the entire orthogonal group $O(K)\backslash O(K_{\mathbb{A}})$, and the character χ is replaced by an automorphic form $\widetilde{\chi}$ on $GO(K)\backslash GO(K_{\mathbb{A}})$ belonging to the representation space of $\Pi(\chi)$ (see Theorem 2.3.3 and the paragraph preceding it for the notation). This difference is responsible for not being able to identify the Whittaker coefficients of our θ_χ with Whittaker newforms in the representation space of π_χ, and for the slightly awkward statement of Proposition 4.1.3 (ii). We consider the less general theta series θ_χ defined above in order to avoid complications due to considering forms $\widetilde{\chi}$ and integrating over the orthogonal group, and since they are sufficient for our purposes. See also Remark 5.2.1 for a comparison between the two theta integrals for $\chi=1$.

Next we compute the Whittaker coefficient of $\theta_\chi(g;\varphi)$ under the assumption that $\varphi \in S(K_{\mathbb{A}})$ is a pure tensor.

Proposition 4.1.2. *Assume that the Schwartz function φ used to define the theta lift is a pure tensor $\varphi = \prod_v \varphi_v$, and let $W_\chi(g;\varphi)$ be the Whittaker coefficient of $\theta_\chi(g;\varphi)$.*

(i) For $g \in G(\mathbb{A})^+$, the Whittaker function $W_\chi(g;\varphi)$ decomposes into a product of local Whittaker functions on $G(\mathbb{Q}_v)^+$ given by:

$$W_{\chi,v}(g_v;\varphi_v) = \int_{K_v^1} L(h_v) r(g_{v,1})\varphi_v(\sigma^{-1})\chi_v(\sigma h_v) d\sigma, \tag{4.1.3}$$

where $h_v \in K_v$ is such that $\mathrm{N}(h_v) = \det g_v$.

(ii) For $\xi \in \mathbb{Q}^\times$, $g \in G(\mathbb{A})^+$, we have:

$$W_\chi\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix} g;\varphi\right) = 0 \quad \text{if } \xi \notin \mathrm{N}_{K/Q}(K^\times).$$

Proof. *(i)* We compute the Whittaker coefficient of θ_χ:

$$W_\chi(g;\varphi) = \int_{\mathbb{Q}\backslash\mathbb{A}} \theta_\chi(n(x)g;\varphi)\psi(-x)dx,$$

where $n(x) \in N(\mathbb{A})$ is the unipotent matrix with upper right entry equal to x. Assuming $g \in G(\mathbb{A})^+$ we have (after switching the order of integration):

$$W_\chi(g;\varphi) = \int_{K^1\backslash K^1_{\mathbb{A}}} \int_{\mathbb{Q}\backslash\mathbb{A}} \theta(\sigma h, n(x)g;\varphi)\psi(-x)\chi(\sigma h) dx d\sigma, \tag{4.1.4}$$

where $h \in \mathbb{A}_K$ with $\mathrm{N}_{K/\mathbb{Q}}(h) = \det g$.

The inner integral is seen to be:

$$\begin{aligned}\int\limits_{\mathbb{Q}\backslash\mathbb{A}} \theta(\sigma h, n(x)g;\varphi)\psi(-x)dx &= \sum_{t\in K}\int\limits_{\mathbb{Q}\backslash\mathbb{A}} L(h)r(g_1)\varphi(\sigma^{-1}t)\psi(xq(t)-x)dx \\ &= \sum_{t\in K^1} L(h)r(g_1)\varphi(\sigma^{-1}t).\end{aligned}$$

For the first equality we have used Lemma 2.3.1 together with the fact that the Weil representation for $\mathrm{SL}_2(\mathbb{A})$ commutes with the action of the orthogonal group $SO(K_{\mathbb{A}})$. Now the integral in (4.1.4) collapses with the summation, and we obtain:

$$W_\chi(g;\varphi) = \int_{K^1_{\mathbb{A}}} L(h)r(g_1)\varphi(\sigma^{-1})\chi(\sigma h)d\sigma.$$

This proves that $W_\chi(g;\varphi)$ is a product of the local Whittaker functions from Eq. (4.1.3).
(ii) Assume $\xi \in \mathbb{Q}^\times$, $g \in G(\mathbb{A})^+$. Then (see Notation for $i(x)$):

$$W_\chi[i(\xi)g;\varphi] = \int_{\mathbb{Q}\backslash\mathbb{A}} \theta_\chi(i(\xi)n(\xi^{-1}x)g;\varphi)\psi(-x)dx.$$

Using the fact that θ_χ is left $G(\mathbb{Q})$-invariant we obtain as before:

$$W_\chi[i(\xi)g;\varphi] = \int\limits_{K^1\backslash K^1_{\mathbb{A}}}\int\limits_{\mathbb{Q}\backslash\mathbb{A}} \theta(\sigma h, n(\xi^{-1}x)g;\varphi)\psi(-x)\chi(\sigma h)dxd\sigma,$$

where $h \in \mathbb{A}_K$ with $\mathrm{N}_{K/\mathbb{Q}}(h) = \det g$. The inner integral is now:

$$\sum_{t\in K}\int_{\mathbb{Q}\backslash\mathbb{A}} L(h)r(g_1)\varphi(\sigma^{-1}t)\psi(x\xi^{-1}q(t)-x)dx,$$

and we see that all the terms of this series vanish if $\xi \notin N_{K/\mathbb{Q}}(K^\times)$, which proves part *(ii)*. $\square$

Let W_χ be the Whittaker newform for the global representation π_χ, that is the pure tensor in the Whittaker model $W(\pi_\chi, \psi)$ whose local component at a place v is the Whittaker newform for $\pi_{\chi,v}$. The next proposition shows that, for an appropriate $\varphi \in S(K_\mathbb{A})$, the Whittaker coefficient $W_\chi(g,\varphi)$ of $\theta_\chi(g,\varphi)$ is almost equal to the Whittaker newform $W_\chi(g)$, for $g \in G(\mathbb{A})^+$. Recall that $D = d_K c(\chi)^2$ is the conductor of π_χ.

Proposition 4.1.3. *Assume that the unitary Hecke character χ is trivial on $\mathbb{A}^\times$, and that its conductor is coprime to d_K. Let $\varphi \in S(K_\mathbb{A})$ be the function whose local components are:*

$$\varphi_p(x) = \begin{cases} 1_{\mathcal{O}_{K_p}}(x) & \text{if } \chi_p \text{ is unramified,} \\ \chi_p(x) 1_{U_{K_p}}(x) & \text{if } \chi_p \text{ is ramified,} \\ e^{-\pi(x_1^2+x_2^2)} & \text{if } p = \infty. \end{cases}$$

where 1_A denotes the characteristic function of the set A.

(i) For $k \in K_0(D)^+ := K_0(D) \cap G(\mathbb{A})^+$, and $g \in G(\mathbb{A})^+$ we have:

$$\theta_\chi(gk;\varphi) = \omega_K(k)\theta_\chi(g;\varphi), \tag{4.1.5}$$

and moreover θ_χ has weight 0 at the archimedean place.

(ii) For all primes $p \nmid D$ (including $p = \infty$), and for $g \in G(\mathbb{Q}_p)^+$, the local component $W_{\chi,p}(g;\varphi_p)$ given by Proposition 4.1.2 equals the Whittaker newform $W_{\chi,p}(g)$. If $p|D$, then $W_{\chi,p}(t;\varphi_p) = W_{\chi,p}(t)$ for $t \in T_1(\mathbb{Q}_p)^+$.

Proof. (i) To show that the automorphic form $\theta_\chi(g;\varphi)$ has level $K_0(D)^+$, let first $k \in K_0(D) \cap \mathrm{SL}_2(\mathbb{A})$. We need to show that:

$$\theta_\chi(gk;\varphi) = \omega_K(k)\theta_\chi(g;\varphi),$$

hence it is enough to show that the theta kernel has the same invariance property. This boils down to showing that:

$$r(k)\varphi(x) = \omega_K(k)\varphi(x),$$

which follows from Lemma 2.5.1 (note that the places where χ ramifies are unramified in the extension K/F).

Take now $\delta(a) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \in K_0(D)^+$, with $a = \mathrm{N}(\alpha)$ for some $\alpha \in \mathbb{A}_K^\times$. By definition we have:

$$\theta_\chi(g\delta(a);\varphi) = \int_{SO(K)\backslash SO(K_\mathbb{A})} \theta(\alpha h\sigma, g\delta(a);\varphi)\chi(\alpha h\sigma)d\sigma,$$

for all $g \in G(\mathbb{A})^+$ and $h \in K_\mathbb{A}$ with $\nu(h) = \det g$. Using lemma 2.3.1, we can compute the theta kernel as follows:

$$\begin{aligned} \theta(\alpha h\sigma, g\delta(a);\varphi) &= \sum_{x\in K} L(\alpha h\sigma) r(\delta(a)^{-1} g_1 \delta(a))\varphi(x) \\ &= \sum_{x\in K} L(h\sigma) r(g_1)\varphi(\alpha^{-1}x). \end{aligned}$$

Since α is a unit at all finite local places, we have $\varphi(\alpha^{-1}x) = \chi(\alpha^{-1})\varphi(x)$, which shows that

$$\theta(\alpha h\sigma, g\delta(a);\varphi) = \theta(h\sigma, g;\varphi)\chi(\alpha^{-1}).$$

Coming back to the theta integral, it follows that $\theta_\chi(g\delta(a);\varphi) = \theta_\chi(g;\varphi)$, which proves Eq. (4.1.5).

At the archimedean place, it is easy to see that θ_χ has weight 0, that is $\theta_\chi(gk_\theta;\varphi) = \theta_\chi(g;\varphi)$ for $k_\theta \in \mathrm{SO}_2(\mathbb{R})$. Indeed, the theta kernel is invariant under the action of $\mathrm{SO}_2(\mathbb{R})$ by Proposition 2.5.5.

(ii) By Proposition 4.1.2, the Whittaker coefficient $W_\chi(g;\varphi)$ decomposes into a product of local Whittaker functions for $g \in G(\mathbb{A})^+$. For primes $p \nmid D$ and for $p=\infty$, part (i) implies that $W_{\chi,p}(g;\varphi)$ is invariant under the maximal compact subgroup $K_0(1)_p$, or under $\mathrm{SO}_2(\mathbb{R})$ for $p=\infty$, hence it is completely determined by its restriction to the diagonal torus $T_1(\mathbb{Q}_p)^+$. We are reduced to computing these diagonal values at all places.

Recall that the values of the Whittaker newforms $W_{\chi,p}$ on $T_1(\mathbb{Q}_p)$ have been computed in Proposition 3.1.1 for p a finite prime, and in Proposition 3.2.2 for $p=\infty$. On the other hand, for $a=\mathrm{N}(h)$ formula (4.1.3) gives:

$$W_{\chi,p}[i(a);\varphi_p] = |a|^{1/2}\int_{K_p^1} \varphi_p(a\sigma^{-1}h^{-1})\chi_p(\sigma h)d\sigma. \tag{4.1.6}$$

For p a finite prime, we only need to consider the case $a\in\mathbb{Z}_p$, since the right invariance of $W_{\chi,p}(\cdot;\varphi)$ under $K_0(D)_p^+$ shows that $W_{\chi,p}[i(a);\varphi]$ vanishes if $|a|_p>1$. For simplicity, we denote by $\mathcal{O}_p$ the ring of integers in K_p.

There are four cases to consider:

• $p=\mathfrak{p}\mathfrak{p}'$ splits in K. Then $K_p=\mathbb{Q}_p+\mathbb{Q}_p$, and $\chi_p=(\chi_\mathfrak{p},\chi_{\mathfrak{p}'})$ with $\chi_\mathfrak{p},\chi_{\mathfrak{p}'}$ characters of $\mathbb{Q}_p^\times$ such that $\chi_\mathfrak{p}\chi_{\mathfrak{p}'}=1$. Taking $h=(1,a)$ in the integral (4.1.6), we have:

$$W_{\chi,p}[i(a);\varphi_p] = |a|_p^{1/2}\int_{\mathbb{Q}_p^\times} \varphi_p(ax,x^{-1})\chi_\mathfrak{p}(x^{-1})\chi_{\mathfrak{p}'}(ax)d^\times x. \tag{4.1.7}$$

If $\chi_\mathfrak{p},\chi_{\mathfrak{p}'}$ are unramified and φ_p is the characteristic function of $\mathbb{Z}_p\times\mathbb{Z}_p$, then we have:

$$W_{\chi,p}[i(a);\varphi_p] = |a|^{1/2}\sum_{i=0}^{\nu_p(a)} \chi_\mathfrak{p}(p)^i\chi_{\mathfrak{p}'}(p)^{\nu_p(a)-i}.$$

On the other hand, if $\chi_\mathfrak{p},\chi_{\mathfrak{p}'}$ are ramified, and $\varphi_p=\chi_p 1_{U_{K,p}}$, the integral (4.1.7) vanishes unless $a\in\mathbb{Z}_p^\times$, when it equals 1. In both cases, it agrees with the values of the Whittaker newform for $\pi_{\chi,p}=\pi(\chi_\mathfrak{p},\chi_{\mathfrak{p}'})$.

• p is inert in K. Let $a=p^{2k}u$ with $u\in\mathbb{Z}_p^\times$, $k\geqslant 0$, and $h=p^k u'$ with $u'\in\mathcal{O}_p^\times$, $\mathrm{N}(u')=u$. If the character χ_p is unramified, then it must be trivial since it is trivial on $\mathbb{Q}_p^\times$, and for $\varphi_p=1_{\mathcal{O}_p}$ the integral in (4.1.6) equals 1. This agrees with the Whittaker newform for $\pi_{\chi,p}$, which is the principal series $\pi(1,\omega_{K,p})$.

On the other hand, if χ_p is ramified and $\varphi_p=\chi_p 1_{U_{K_p}}$, the integral (4.1.6) vanishes unless $a\in\mathbb{Z}_p^\times$, when it equals $\chi_p(a)=1$. It agrees therefore with the Whittaker newform for $\pi_{\chi,p}$, which is either supercuspidal, or of the type $\pi(\eta,\eta\omega_{K,p})$ if $\chi=\eta\circ\mathrm{N}$, for a ramified character η of $\mathbb{Q}_p^\times$.

• p is ramified in K. Let $a=p^k u$ with $u\in\mathbb{Z}_p^\times$, $k\geqslant 0$, and $h=\mathfrak{p}^k u'$ with $N(h)=a$. Since φ is the characteristic function of $\mathcal{O}_p$, we have:

$$W_{\chi,p}[i(a);\varphi_p] = |a|^{1/2}\int_{K_p^1} \chi_\mathfrak{p}(\mathfrak{p}^k)d\sigma.$$

The character χ is unramified, hence $\chi=\eta\circ\mathrm{N}$, for η an unramified character of $\mathbb{Q}_p^\times$, hence this formula agrees with the values of the Whittaker newform for $\pi(\eta,\eta\omega_{K,p})$.

• $p=\infty$. Let $\chi_1=|\ |^r\mathrm{sgn}^m$ be the first component of χ_∞, with $r\in\mathbb{C}$ and $m\in\{0,1\}$. Taking $h=(1,a)$ and $\sigma=(t,t^{-1})\in K_\infty^1$ in formula (4.1.6), we obtain:

$$W_{\chi,\infty}[i(a);\varphi_\infty] = |a|^{1/2}\int_{\mathbb{R}^\times} e^{-\pi(a^2t^{-2}+t^2)}|t|^{2r}|a|^{-r}\mathrm{sgn}(a)^m d^\times t.$$

The expression on the right is even or odd as a function of a as $m=0$ or $m=1$ respectively, in agreement with the formula for $W_{\chi,\infty}$ given in Proposition 3.2.2. Moreover, for $a>0$ we obtain, after a change of variables $t^2 = au$:

$$W_{\chi,\infty}[i(a);\varphi_\infty] = a^{1/2}\int_0^\infty e^{-\pi a(u+u^{-1})}u^r d^\times u = 2a^{1/2}J_r(2\pi a).$$

In the last equality we have used formula (3.2.1). Comparison with Proposition 3.2.2 shows that the local Whittaker functions agree in this case as well, which finishes the proof. □

4.2 Local Rankin-Selberg convolutions

Let F be a local field, and π_1, π_2 be two admissible, irreducible representations of $G(F) = \mathrm{GL}_2(F)$ of central characters ω_1, ω_2. Let $\omega = \omega_1\omega_2$, and fix as usually a nontrivial additive character ψ of F (which is unramified in the nonarchimedean case). The convolution L-function $L(s, \pi_1 \times \pi_2)$ is defined as the common denominator (appropriately normalized) of the following local Rankin–Selberg integrals:

$$\Psi(s, W_1, W_2, f) = \int_{Z(F)N(F)\backslash G(F)} W_1(g)W_2(\epsilon g)f(s,g)dg$$

where $W_i \in W(\pi_i, \psi)$, $\epsilon = i(-1) \in G(F)$, and $f(s, g)$ is a function in the induced representation space $I(s, \omega) := B(|\,|^{s-1/2}, \omega^{-1}|\,|^{1/2-s})$. Given our choice of exponents, the Rankin–Selberg integral satisfies a functional equation for $s \to 1 - s$. See [Zh01] for a concise account. When one of the representations π_1, π_2 is a principal series, the convolution L-function can be computed as follows:

$$L(s, \pi_1 \times \pi_2) = L(s, \mu_1 \otimes \pi_2)L(s, \mu_2 \otimes \pi_2) \ \text{ if } \pi_1 = \pi(\mu_1, \mu_2).$$

In this section we specialize π_1 and π_2 to the local factors of the global representations π_f and π_χ respectively. We consider a modified version of the Rankin-Selberg integral, in which W_1 is always taken to be a Whittaker newform for π_1, while W_2 and $f \in I(s, \omega)$ are constructed via the Weil representation from Schwartz functions $\varphi_1, \varphi_2 \in S(K)$ respectively, with K a quadratic separable extension of F. We show that $L(s, \pi_1 \times \pi_2)$ equals the modified Rankin-Selberg integral up to an explicit factor, for a suitable choice of φ_1, φ_2.

Let π be an admissible, irreducible representation of $G(F)$ with trivial central character, which is to be thought of as a local component of the global representation π_f. Let K/F be a quadratic separable extension with norm $\mathrm{N} = \mathrm{N}_{K/F}$, and let χ be a character of $K^\times$, trivial on $F^\times$. Let π_χ be the irreducible representation associated to χ as in Theorem 2.2.1, which has central character ω, the quadratic character of $F^\times$ determined by K. Fix also a constant $\Lambda \in F^\times$, and let r_1, r_Λ be the Weil representations associated with the quadratic spaces $(K, \mathrm{N}_{K/F})$, and $(K, \Lambda\mathrm{N}_{K/F})$ respectively. We denote by $G(F)^+$ the index two subgroup of $G(F)$ consisting of matrices with determinant in $\mathrm{N}(K^\times)$, and for any subgroup H of $G(F)$ we let $H^+ = H \cap G(F)^+$.

Fix a nontrivial character ψ of F, and let $W(\pi, \psi)$ be the corresponding Whittaker model of π. Let $W^+(\pi_\chi, \psi)$ be the set of functions on $G(F)^+$ of the type:

$$W_\chi(g;\varphi) = \int_{K^1} L(h)r_1(g_1)\varphi(\sigma^{-1})\chi(\sigma h)d\sigma,$$

for $\varphi \in S(K)$ and $h \in K$ with $\mathrm{N}(h) = \det g$. This is the local component of the Whittaker coefficient of the global form $\theta_\chi(g, \varphi)$ considered in §4.1, and the space $W^+(\pi_\chi, \psi)$ is closely related to the Whittaker model of π_χ. We will not be concerned however with the exact relationship between the two spaces.

We shall only consider sections $f(s, g; \varphi) \in I(s, \omega_K)$ constructed using the representation r_Λ as in §2.4.1:

$$f(s, g;\varphi) = r_\Lambda(g_1)\varphi(0)|a(g)|^{2s-1}|\det g|^{-1/2}\omega(\det g),$$

for $\varphi \in S(K)$.

The local Rankin-Selberg integral that we consider here depends on a choice of two Schwartz functions $\varphi_1, \varphi_2 \in S(K)$ as follows:

$$\Psi^+(s, \varphi_1, \varphi_2) = \int_{Z(F)N(F)\backslash G(F)^+} W_\pi(\epsilon g) W_\chi(g; \varphi_1) f(s, g; \varphi_2) dg \tag{4.2.1}$$

where W_π is the Whittaker newform for π. We shall show that under certain restrictions on the data π, K/F, χ, Λ, the local Rankin-Selberg integral exactly equals $L(s, \pi \times \pi_\chi)$ up to a simple factor, for a suitable choice of Schwartz functions. The restrictions are exactly the ones imposed at the local places in the global situation.

4.2.1 *Nonarchimedean case* Let F be a nonarchimedean local field, and assume that the character ψ is unramified. Let $q = |\varpi_F|^{-1}$ be the cardinality of the residue field of F.

If π_1, π_2 are two arbitrary representations of $G(F)$ such that $L(s, \pi_1) = \prod_{i=1}^2 (1 - \alpha_i q^{-s})^{-1}$ and $L(s, \pi_2) = \prod_{i=1}^2 (1 - \beta_i q^{-s})^{-1}$, the convolution L-function can be computed as follows:

$$L(s, \pi_1 \times \pi_2) = \prod_{i,j=1}^{2} (1 - \alpha_i \beta_j q^{-s})^{-1},$$

provided at least one of π_1, π_2 is a principal series representation. If that is the case, the following lemma is used in computing the local Rankin–Selberg integrals.

Lemma 4.2.1. *Assuming that least one of π_1, π_2 is a principal series representation, let W_i be the Whittaker newform for π_i, $i = 1, 2$. If the measure on $F^\times$ is such that U_F has measure 1, then:*

$$\int_{F^\times} W_1[i(a)] W_2[i(-a)] |a|^{s-1} d^\times a = \begin{cases} \dfrac{L(s, \pi_1 \times \pi_2)}{L(2s, \omega)} & \text{if } \pi_1, \pi_2 \text{ unram.}, \\ L(s, \pi_1 \times \pi_2) & \text{otherwise.} \end{cases}$$

Proof. This is a routine computation using the formulas for the diagonal values of the Whittaker newforms given in Proposition 3.1.1, together with a power series identity. □

We now specialize π_1, π_2 to the setting considered in §4.2. Our data consists of a representation π with trivial central character; a quadratic, separable extension K/F with discriminant δ_K and quadratic character $\omega = \omega_K$; a character χ of $K^\times$ trivial on $F^\times$ with Jacquet-Langlands lift π_χ; an element $\Lambda \in F^\times$. If $c(\chi) = S \geqslant 0$ denotes the conductor of χ, recall that π_χ has conductor $C + 2S$, where $C \geqslant 0$ is the exponent of δ_K (and also the conductor of ω).

The modified Rankin-Selberg integral can be written in this case:

$$\Psi^+(s, \varphi_1, \varphi_2) = \int_{\mathrm{N}(K^\times)} \int_{K_0(1)^+} W_\pi[i(-t)k] W_\chi[i(t)k; \varphi_1] f(k; \varphi_2) |t|^{s-1} dk d^\times t, \tag{4.2.2}$$

where $f(k; \varphi) := f(s, k; \varphi)$ is independent of s for $k \in K_0(1)$. We normalize the measure dg on $Z(F)N(F)\backslash G(F)^+$ in formula (4.2.1) as follows:

$$dg = |t|^{-1} d^\times t dk$$

for the decomposition $G(F)^+ = Z(F)N(F)T_1(F)^+ K_0(1)^+$, where $d^\times a$ is the measure on $F^\times$ such that $\mathrm{N}(U_K)$ has measure 1, and dk is a measure on $K_0(1)^+$ of total measure 1. Note that this measures give both $K_0(1)$ and U_F volume 1, except when K/F is ramified, when they both have volume 2.

In view of the global case, it is enough to consider the following restriction on our data:

Assumption 4.2.2. *At most one of the representation π, the extension K/F, and the character χ is ramified, and π is either unramified or special with unramified twist.*

If K/F is ramified, we further assume that $\Lambda \in U_F$, and $-\Lambda = \mathrm{N}(u)$ for some fixed $u \in U_K$; for each $\alpha \in \delta_K^{-1}/\mathcal{O}_K$, we define the functions:

$$\varphi_1^\alpha = 1_{\alpha+\mathcal{O}_K}, \quad \varphi_2^\alpha = 1_{\alpha u^{-1}+\mathcal{O}_K}.$$

Under the previous assumptions, we collect in a table the data that will be used to compute $\Psi^+(s, \varphi_1, \varphi_2)$ (for reasons of space, we abbreviate Unramified by Unr. and Ramified by Ram.). For future reference, in the last column we have included the primes p in the global case that correspond to a given local case. To state the next proposition in a more compact manner, we also define the integer M to be the largest of the conductors of π and π_χ. Thus, the integer M equals: 0 in the cases A1, A1'; 1 in case A2; C in case A3, and $2S$ in case $A4$.

Case	π	K/F	χ	$\nu_F(\Lambda)$	φ_1	φ_2	Global
A1.	Unr.	Unr.	Unr.	0	$1_{\mathcal{O}_K}$	$1_{\mathcal{O}_K}$	$p \nmid \Lambda d_K$
A1'.	Unr.	Split	Unr.	1	$1_{\mathcal{O}_K}$	$1_{\mathcal{O}_F} \times 1_{\varpi_F^{-1}\mathcal{O}_F}$	$p = \lambda$
A2.	Ram.	Unr.	Unr.	1	$1_{\mathcal{O}_K}$	$1_{\mathcal{O}_K}$	$p \mid N$
A3.	Unr.	Ram.	Unr.	0	φ_1^α	φ_2^α	$p \mid d_K$
A4.	Unr.	Unr.	Ram.	$2S$	$\chi 1_{U_K}$	$1_{\mathcal{O}_K}$	$p \mid c(\chi)$

Proposition 4.2.3. *In each of the cases above, we have:*

$$\Psi^+(s, \varphi_1, \varphi_2) = M(s)^{-1} L(s, \pi \times \pi_\chi)$$

where the factor $M(s)$ is given by:

- *A1., A1'.:*

$$M(s) = L(2s, \omega);$$

- *A2., A4.:*

$$M(s) = \frac{L(1, \omega)}{\mu[K_0(\pi_F^M)^+]};$$

- *A3.:*

$$M(s) = \frac{L(1, \omega)}{2\mu[K_0(\pi_F^M)^+]} \cdot \begin{cases} 2 & \text{if } \nu_F(\alpha) = -C, \\ 1 & \text{if } \alpha \in \mathcal{O}_K, \text{ or if } F = \mathbb{Q}_2 \text{ and } -C < \nu_F(\alpha) \leqslant -1. \end{cases}$$

Here $\mu[K_0(\pi_F^M)^+] = 1/(q^M + q^{M-1})$ is the measure of the compact subgroup $K_0(\pi_F^M)^+$, for $M \geqslant 1$.

Proof. In the cases when π_χ is unramified (that is A1., A1'., A2.), it has been shown in the proof of Proposition 4.1.5 that $W_\chi(g; \varphi_1) = W_\chi(g)$ for $g \in G(F)^+$, where W_χ is the Whittaker newform for π_χ. The proof in these cases is therefore simpler than in the remaining ones, which require a direct computation of $W_\chi(g; \varphi_1)$.

Cases A1. and A1'. We can apply proposition 2.5.3 (i) (in case A1.) or 2.5.3 (iii) (in case A1'.) to conclude that $f(k; \varphi_2) = 1$ for all $k \in K_0(1)$. Since W_π, W_χ are both invariant under $K_0(1)^+ = K_0(1)$, and $W_\chi[i(t)] = 0$ if $t \notin N(K^\times)$ (if K/F is a field), Lemma 4.2.1 applied to Eq.. (4.2.2) implies that:

$$\Psi^+(s, \varphi_1, \varphi_2) = \frac{L(s, \pi \times \pi_\chi)}{L(2s, \omega)}.$$

Case A2. The representation π is the special series $\sigma(\eta|\ |^{1/2}, \eta|\ |^{-1/2})$, with η unramified, hence the Whittaker newform is right invariant under $K_0(\varpi_F)$. Since $\nu(\Lambda) = 1$, Proposition 2.5.3 implies that the function $f(k;\varphi_2)$ is also right invariant under $K_0(\varpi_F)$, and formula (4.2.2) becomes:

$$\Psi^+(s,\varphi_1,\varphi_2)=\mu[K_0(\varpi_F)]\sum_{\xi\in K_0(1)/K_0(\varpi_F)}\int_{\mathrm{N}(K^\times)}W_\chi[i(-t)]W_\pi[i(t)\xi]f(\xi;\varphi_2)|t|^{s-1}d^\times a.$$

As system of representatives for the cosets $K_0(1)/K_0(\varpi_F)$ we take the set $\{I\}\cup\Sigma$, where:

$$\Sigma=\left\{\begin{pmatrix}\beta & 1\\ -1 & 0\end{pmatrix}\ :\ \beta\in\mathcal{O}_F/\varpi_F\mathcal{O}_F\right\}.$$

We are led to compute the following sum:

$$S(t)=\sum_{\xi\in\Sigma}W_\pi[i(t)\xi]f(\xi;\varphi_2),$$

for $t\in\mathcal{O}_F$, where the values $f(\xi;\varphi_2)$ are given by Proposition 2.5.3:

$$f\left[\begin{pmatrix}\beta & 1\\ -1 & 0\end{pmatrix};\varphi_2\right]=|\Lambda|\omega(\Lambda). \tag{4.2.3}$$

From the identity:

$$\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\beta & 1\\ -1 & 0\end{pmatrix}=\begin{pmatrix}1 & -\beta t\\ 0 & 1\end{pmatrix}\begin{pmatrix}t & 0\\ 0 & \varpi_F^{-1}\end{pmatrix}\begin{pmatrix}0 & 1\\ -\varpi_F^1 & 0\end{pmatrix},$$

together with the functional equation for the Whittaker newform (Proposition 3.1.1 (ii)), we deduce that for $\xi\in\Sigma$:

$$W_\pi[i(t)\xi]=\widetilde{W}_\pi[i(t\varpi_F)]\epsilon(\pi,\psi),$$

where $\widetilde{W}_\pi$ is the Whittaker newform for the contragredient representation $\tilde\pi$.

Since $\pi=\sigma(\eta|\ |^{1/2},\eta|\ |^{-1/2})$, with η unramified and $\eta^2=1$, we have that $\pi=\tilde\pi$, and the epsilon factor is $\epsilon(\pi,\psi)=-\eta(\varpi_F)$ (see [Zh01]). By Proposition 3.1.1, we have:

$$W_\pi[i(t\varpi_F)]=\eta(\varpi_F)|\varpi_F|W_\pi[i(t)],$$

hence we find:

$$W_\pi[i(t)\xi]=-|\varpi_F|W_\pi[i(t)]\ \text{ for all }\xi\in\Sigma.$$

Since there are $|\varpi_F|^{-1}$ terms in the sum S, we obtain (taking into account that $f(\xi;\varphi_2)=|\varpi_F|\omega(\varpi_F)$ for $\xi\in\Sigma$):

$$S(t)=-|\varpi_F|\omega(\varpi_F)W_\pi[i(t)].$$

It follows that:

$$\Psi^+(s,\varphi_1,\varphi_2)=\mu(K_0(\varpi_F))[1-|\varpi_F|\omega(\varpi_F)]\int_{\mathrm{N}(K^\times)}W_\chi[i(-a)]W_\pi[i(a)]|a|^{s-1}d^\times a.$$

The integral over $\mathrm{N}(K^\times)$ is the same as the integral over $F^\times$: when $K=F\oplus F$ this is clear, and when K is a field we have $W_\chi[i(-t)]=0$ if $t\notin N(K^\times)$. Hence the conclusion follows by applying lemma 4.2.1.

Case A3. In this case, we compute directly the Whittaker coefficient $W_\chi(i(t)k;\varphi_1^\alpha)$ and the function $f(k;\varphi_2^\alpha)$ appearing in the integral (4.2.2), using Lemma 2.5.2. The computation is more complicated when the residue characteristic of F is 2, when we take the liberty of assuming that $F=\mathbb{Q}_2$. Fix $\alpha\in\delta^{-1}/\mathcal{O}_K$, and let $r=-\nu_K(\alpha)$, so $0\leqslant r\leqslant C$.

Using formula (2.1.1), we have for $k \in K_0(1)^+$ with $k_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (see Notation):

$$\begin{aligned} f(k;\varphi_2^\alpha) &= \omega(-\Lambda c^{-1})|c|_F^{-1}\gamma \int_{\alpha u^{-1}+\mathcal{O}_K} \psi[\Lambda c^{-1} d\mathrm{N}(y)]dy \\ &= \omega(-\Lambda c^{-1})|c|_F^{-1}\gamma \int_{\alpha+\mathcal{O}_K} \psi[-c^{-1} d\mathrm{N}(y)]dy, \end{aligned}$$

where the second equality follows from a change of variables, using the fact that $\mathrm{N}(u) = -\Lambda$. The integral appearing above is computed in Lemma 2.5.2, parts (ii) and (iii), for $\alpha \in O_K$ and $\alpha \notin \mathcal{O}_K$ respectively.

The Whittaker function can be computed using Proposition 4.1.2:

$$W_\chi[i(t)k;\varphi_1^\alpha] = |t|_F^{1/2}\eta(t)\int_{K^1} r_1(k_1)\varphi_1^\alpha(th^{-1}\sigma^{-1})d\sigma,$$

where $h \in K$ with $\mathrm{N}(h) = t\det k$, and η is the unramified character of $F^\times$ such that $\eta(\varpi_F) = \chi(\varpi_K)$. Using the definition of the Weil representation (2.1.1), we have:

$$r_1(k_1)\varphi_1^\alpha(x) = |c^{-1}|\omega(-c^{-1})\gamma\int_{\alpha+\mathcal{O}_K} \psi[c^{-1}(a\mathrm{N}(x) + d\mathrm{N}(y) - \mathrm{Tr}(x\bar{y}))]dy.$$

Since $W_\pi[i(-t)] = 0$ if $t \notin \mathcal{O}_F$ in the integral (4.2.2), we can assume $t \in \mathcal{O}_F$ in the next formula, which implies that $x = th^{-1}\sigma \in \mathcal{O}_K$. Using the fact that $\psi[(c^{-1}a - c^{-1}d^{-1})N(x)] = 1$ for $x \in \mathcal{O}_K$, we have:

$$r_1(k_1)\varphi_1^\alpha(x) = |c^{-1}|\omega(-c^{-1})\gamma\int_{\alpha+\mathcal{O}_K} \psi[c^{-1}d\mathrm{N}(y)]dy.$$

Noting that the last integral is the complex conjugate of the integral appearing in the formula $f(k;\varphi_2^\alpha)$, it follows that:

$$W_\chi[i(t)k;\varphi_1^\alpha]f(k;\varphi_2^\alpha) = W_\chi[i(t)]|c|^{-2}\left|\int_{\alpha+\mathcal{O}_K} \psi[c^{-1}d\mathrm{N}(y)]dy\right|^2,$$

where we have used $W_\chi[i(t)] = |t|_F^{1/2}\eta(t)$ for the Whittaker newform W_χ. The integral appearing in the last formula is computed in Lemma 2.5.2 (ii) or (iii), depending on whether $\alpha \in \mathcal{O}_K$:

- $\alpha \in \mathcal{O}_K$. Then Lemma 2.5.2 (ii) applies and we obtain:

$$W_\chi[i(t)k;\varphi_1^\alpha]f(k;\varphi_2^\alpha) = \begin{cases} W_\chi[i(t)]|\pi_F|^C & \text{if } c \in U_F, \\ 0 & \text{if } 0 < \nu_F(c) < C, \\ W_\chi[i(t)] & \text{if } \nu_F(c) \geqslant C. \end{cases}$$

The Rankin-Selberg integral becomes:

$$\Psi^+(s,\varphi_1^\alpha,\varphi_2^\alpha) = \tau\int_{\mathrm{N}(K^\times)} W_f[i(-t)]W_\chi[i(t)]|t|^{s-1}dt, \tag{4.2.4}$$

where $\tau = |\pi_F|^C\mu[K_0(1)^+ - K_0(\varpi_F)^+] + \mu[K_0(\varpi_F^C)^+]$. Since the measure is normalized such that $\mu(K_0(1)^+) = 1$, we obtain

$$\tau = 2|\varpi_F|^{C-1}(1+|\varpi_F|^{-1})^{-1} = 2\mu[K_0(\varpi_F^C)^+].$$

Together with Lemma 4.2.1 and the fact that the measure of $N(U_K)$ is 1, this proves the desired identity.

• $\alpha \notin \mathcal{O}_K$. Then Lemma 2.5.2 (iii) applies and we obtain:

$$W_\chi[i(t)k;\varphi_1^\alpha]f(k;\varphi_2^\alpha) = \begin{cases} W_\chi[i(t)]|\varpi_F|^C & \text{if } c \in U_F, \\ 0 & \text{if } 0 < \nu_F(c) \neq C - r, \\ W_\chi[i(t)]|\varpi_F|^{2r-2C} & \text{if } 0 < \nu_F(c) = C - r, \end{cases}$$

where the last case can occur only when $C > 1$, that is $F = \mathbb{Q}_2$. The formula (4.2.4) still holds, but now τ is given by

$$\tau = \begin{cases} |\varpi_F|^{C-1}(1 + |\varpi_F|^{-1})^{-1} & \text{if } \nu_F(\alpha) = -C, \\ |\varpi_F|^{C-2}(1 + |\varpi_F|^{-1})^{-1} & \text{if } -C < \nu_F(\alpha) \leqslant -1. \end{cases}.$$

The conclusion follows by Lemma 4.2.1.

Case A4. Let the conductor of χ be $S > 0$. In this case, both $W_\chi(\cdot;\varphi_1)$ and $f(k;\varphi_2)$ are right invariant under $K_0(\varpi_F^{2S})$, by Propositions 4.1.5 and 2.5.3 respectively. Formula (4.2.2) becomes:

$$\Psi^+(s,\varphi_1,\varphi_2) = \mu[K_0(\varpi_F^{2S})] \sum_{\xi \in K_0(1)/K_0(\varpi_F^{2S})} \int_{\mathrm{N}(K^\times)} W_\pi[i(-a)]W_\chi[i(a)\xi;\varphi_1]f(\xi;\varphi_2)|a|^{s-1}d^\times a.$$

As system of representatives for the cosets $K_0(1)/K_0(\varpi_F^{2S})$ we take the set $\Sigma = \Sigma_1 \cup \Sigma_2$, where:

$$\Sigma_1 = \left\{ \begin{pmatrix} \beta & 1 \\ -1 & 0 \end{pmatrix} \ : \ \text{with } \beta \in \mathcal{O}_F/\varpi_F^{2S}\mathcal{O}_F \right\},$$
$$\Sigma_2 = \left\{ \begin{pmatrix} 1 & 0 \\ \varpi_F\alpha & 1 \end{pmatrix} \ : \ \text{with } \alpha \in \mathcal{O}_F/\varpi_F^{2S-1}\mathcal{O}_F \right\}.$$

We are lead to compute the sums:

$$S_i(a) = \sum_{\xi \in \Sigma_i} W_\chi[i(a)\xi;\varphi_1]f(\xi;\varphi_2), \quad i = 1, 2.$$

As in case A3., the Whittaker function can be computed as follows for $a \in \mathcal{O}_F$:

$$W_\chi[i(a)\xi;\varphi_1] = |a|_F^{1/2} \int_{K^1} r_1(\xi)\varphi_1(ah^{-1}\sigma^{-1})\chi(\sigma h)d\sigma,$$

where $\mathrm{N}h = a$. For $\xi \in \Sigma_1$, it is easy to see from formula (2.1.1) that $r_1(\xi)\varphi_1(th^{-1}\sigma^{-1}) = 0$ (since χ is ramified), hence $S_1(a) = 0$.

To compute S_2, let $\xi = \begin{pmatrix} 1 & 0 \\ \varpi_F\alpha & 1 \end{pmatrix} \in \Sigma_2$ with $\alpha \in \mathcal{O}_F/\varpi_F^{2S-1}\mathcal{O}_F$, $\alpha \notin \varpi_F^{2S-1}\mathcal{O}_F$. By Proposition 2.5.3 (i) and by formula (2.1.1) we have:

$$f(\xi,\varphi_2) = |\Lambda\varpi_F^{-1}\alpha^{-1}|\omega(\varpi_F\alpha)$$
$$r_1(\xi)\varphi_1(x) = \omega(\varpi_F\alpha)|\varpi_F^{-1}\alpha^{-1}| \int_{U_K} \chi(y)\psi[\varpi_F^{-1}\alpha^{-1}\mathrm{N}(y-x)]dy.$$

Writing $\alpha = \varpi_F^{r+1}u$, with $r = 1, \ldots, 2S-1$, $u \in U_F/U_F^{2S-r}$, the summation over α becomes:

$$S_2(a) = W_\chi[i(a)] + |a|^{1/2} \sum_{r=1}^{2S-1} \sum_{u \in U_F/U_F^{2S-r}} |\varpi_F|^{2S-2r}\cdot$$
$$\cdot \int_{K^1}\int_{U_K} \chi(\sigma h y)\psi[\varpi_F^{-r}u^{-1}\mathrm{N}(y - a\sigma^{-1}h^{-1})]dyd\sigma. \quad (4.2.5)$$

We compute the sum by keeping r fixed and summing over u, using the identity (valid for $n>0$):

$$\sum_{u\in U_F/U_F^n}\psi(ux)=\begin{cases}0 & \text{if } -n\leqslant \nu_F(x)\leqslant -2,\\ -q^{n-1} & \text{if } \nu_F(x)=-1,\\ (q-1)q^{n-1} & \text{if } x\in\mathcal{O}_F.\end{cases}\tag{4.2.6}$$

Note that the same formula holds if u is replaced by u^{-1} in the argument of ψ.

There are two cases to consider, depending on whether K is a field or the split algebra.

• K a field. When $a\in\varpi_F\mathcal{O}_F$ [implying that $\mathrm{N}(y-a\sigma^{-1}h^{-1})\in U_F$ independent of $y\in U_K$], we claim that the sum over u vanishes for each fixed r. Indeed, if $r\leqslant S$, identity (4.2.6) applies to conclude that the sum over u is independent of y, hence the integral over y vanishes. If $r>S$, then the integral over y is seen to vanish by a change of variables $y=v+\varpi_K^S z$, with $v\in U_K/U_K^S$, $z\in\mathcal{O}_K$.

Assume from now on that $a\in U_F$. After a change of variables $y=ah^{-1}\sigma^{-1}z$, the sum becomes (taking into account that the measure on K^1 is normalized such that K^1 has measure 1):

$$S_2(a)=W_\chi[i(a)]+\sum_{r=1}^{2S-1}\sum_{u\in U_F/U_F^{2S-r}}|\varpi_F|^{2S-2r}\int_{U_K}\chi(y)\psi[\varpi_F^{-r}u^{-1}a\mathrm{N}(y-1)]dy.\tag{4.2.7}$$

When $r\leqslant S$, we can sum over u first, using identity (4.2.6). When $r>S$, a change of variables $y=v+\varpi_K^S z$ as before reveals that the integral over z vanishes unless $v\in U_K^{r-S}$. In the later case the sum over u can again be computed using identity (4.2.6). The result is:

$$S_2(a)=W_\chi[i(a)]+\sum_{r=1}^{2S-1}\Big(\int\limits_{\nu[\mathrm{N}(y-1)]\geqslant r}q^r\chi(y)dy-\int\limits_{\nu[\mathrm{N}(y-1)]\geqslant r-1}q^{r-1}\chi(y)dy\Big).$$

We finally obtain:

$$S_2(a)=W_\chi[i(a)]+q^{2S-1}\int_{U_K^S}\chi(y)dy-\int_{U_K}\chi(y)dy=W_\chi[i(a)](1+q^{-1}),$$

where we have used the fact that the Whittaker newform $W_\chi[i(a)]$ is 1 if $a\in U_F$ and 0 otherwise (since $L(s,\pi_\chi)=1$). It follows that:

$$\Psi^+(s,\varphi_1,\varphi_2)=\mu[K_0(\varpi_F^{2S})]L(1,\omega)^{-1},$$

which is the desired identity since $L(s,\pi\times\pi_\chi)=1$.

• $K=F+F$ is split. Then $\chi(y_1,y_2)=\eta(y_1)\eta^{-1}(y_2)$, with η a character of $F^\times$ of conductor S. We let $\varpi=\varpi_F$ in the sequel. Formula (4.2.5) becomes in this case (recall that we assume $a\in\mathcal{O}_F$:

$$S_2(a)=W_\chi[i(a)]+|a|^{1/2}\sum_{r=1}^{2S-1}\sum_{u\in U_F/U_F^{2S-r}}|\varpi_F|^{2S-2r}\cdot$$
$$\cdot\int_{F^\times}\int_{U_F\times U_F}\eta(y_1\sigma y_2^{-1}\sigma a)\psi[\varpi_F^{-r}u^{-1}(y_1-\sigma^{-1})(y_2-a\sigma)]dy_1dy_2d\sigma$$

By lemma 2.1.2, the integral over $U_F\times U_F$ vanishes unless there exist $y_1,y_2\in U_F$ such that $\nu_F(y_1-\sigma^{-1})=\nu_F(y_2-a\sigma)=r-S$, which can only happen if both $\sigma^{-1},a\sigma\in\mathcal{O}_F$. Moreover, if $a\in\varpi\mathcal{O}_F$, the integral vanishes unless $r=S$. In the latter case, the identity (4.2.6) applies to conclude that the sum over $u\in U_F/U_F^S$ is independent of either y_1 (if $\sigma^{-1}\in\varpi\mathcal{O}_F$) or y_2 (if $a\sigma\in\varpi\mathcal{O}_F$). Therefore the integral over either y_1 or y_2 vanishes after performing the summation over u, hence the sum vanishes for $a\in\varpi\mathcal{O}_F$.

Assume therefore $a\in U_F$ from now on. After a change of variables, the sum becomes (taking into account that the measure on K^1 is normalized such that $K^1 \cap U_K$ has measure 1):

$$S_2(a) = W_\chi[i(a)] + \sum_{r=1}^{2S-1} \sum_{u \in U_F/U_F^{2S-r}} |\varpi_F|^{2S-2r}.$$

$$\cdot \int_{U_F \times U_F} \eta(y_1 y_2^{-1}) \psi[\varpi_F^{-r-S} u^{-1} a(y_1-1)(y_2-1)] dy_1 dy_2.$$

If $r < S$, the integral vanishes by Proposition 2.1.2. For $r \geqslant S$, a change of variables $y = v + \varpi z$, with $v \in U_F/U_F^S \times U_F/U_F^S$, $z \in \mathcal{O}_F \times \mathcal{O}_F$, shows that the integral over $y = (y_1, y_2)$ vanishes unless $v \in U_F/U_F^{r-S} \times U_F/U_F^{r-S}$. If that is the case, we can apply again identity (4.2.6), to obtain (we have replaced r by $r - S$):

$$S_2(a) = W_\chi[i(a)] + q^S \sum_{r=0}^{S-1} \Big(\int_{\substack{U_F^r \times U_F^r \\ \nu[\mathrm{N}(y-1)] \geqslant r+S}} q^r \chi(y) dy - \int_{\substack{U_F^r \times U_F^r \\ \nu[\mathrm{N}(y-1)] \geqslant r+S-1}} q^{r-1} \chi(y) dy \Big).$$

The two integrals can be computed as before by breaking down the integration domains, e.g.

$$\{y \in U_F^r \times U_F^r : \mathrm{N}(y-1) \geqslant r+S\} = U_F^r \backslash U_F^{r+1} \times U_F^S \cup \ldots \cup U_F^S \times U_F^r.$$

Taking into account that η has conductor S, it follows that the second integral is always 0, while the first is nonzero only for $r = S - 1$. We obtain:

$$S_2(a) = W_\chi[i(a)](1 - q^{-1}),$$

which leads to the desired formula as in the previous case. $\square$

4.2.2 *Archimedean case* Assume now that $F = \mathbb{R}$ and that $\psi(x) = e^{2\pi i x}$. The archimedean Rankin-Selberg integral can be computed using the following well-known lemma:

Lemma 4.2.4 Barnes' Lemma. *Assume f_1, f_2 are smooth functions defined on $\mathbb{R}^\times$, at least one of which is even, and such that:*

$$\int_{\mathbb{R}^\times} f_1(a)|a|^{s-1/2} d^\times a = G_1(s+r_1)G_1(s+r_2)$$
$$\int_{\mathbb{R}^\times} f_2(a)|a|^{s-1/2} d^\times a = G_1(s+t_1)G_1(s+t_2),$$

the integrals converging absolutely for large enough $Re(s)$. Then the following identity holds:

$$\int_{\mathbb{R}^\times} f_1(a) f_2(a) |a|^{s-1} d^\times a = \frac{\prod_{i,j=1}^2 G_1(s + r_i + t_j)}{G_1(2s + r_1 + r_2 + t_1 + t_2)}.$$

Recall that $G_1(s) = \pi^{-s/2}\Gamma(s/2)$.

Proof. See [Ja72, p. 77]. $\square$

Consider the setting described in §4.2. Let π be an irreducible representation $G(\mathbb{R})$ with trivial central character, $K = \mathbb{R} + \mathbb{R}$ the split algebra, χ a character of $K^\times$ trivial on $\mathbb{R}^\times$, π_χ the principal series representation attached to χ, and $\Lambda \in \mathbb{R}^\times$, $\Lambda > 0$.

Since χ is trivial on the diagonal $\mathbb{R}^\times$, we can write $\chi = (\chi_1, \chi_1^{-1})$, for χ_1 a character of $\mathbb{R}^\times$. For the global case, it is enough to consider the following restriction on the data above:

Assumption 4.2.5. *The representation π is either discrete of weight $2k > 0$, or a principal series $\pi(\mu_1, \mu_2)$ of weight 0, such that $\mu_1(-1) = \chi_1(-1)$.*

By Proposition 4.1.3, we have that:

$$W_\chi(g;\varphi_1) = W_\chi(g), \quad \text{for } \varphi_1(x_1,x_2) = e^{-\pi(x_1^2+x_2^2)},$$

where W_χ is the Whittaker newform for π_χ. With this choice of φ_1, the Rankin-Selberg integral becomes:

$$\Psi^+(s,\varphi_1,\varphi_2) = \int_{\mathbb{R}^\times\times \mathrm{SO}_2(\mathbb{R})} W_\pi[i(a)k]W_\chi[i(-a)k]|a|^{s-1}f(k;\varphi_2)dkd^\times a, \tag{4.2.8}$$

which is the classical zeta integral denoted by $\Psi[s,W_\pi,W_\chi,f(\cdot;\varphi_2)]$ in §4.2. We have normalized the measure dg on $Z(\mathbb{R})N(\mathbb{R})\backslash G(\mathbb{R})$ in equation (4.2.1) such that:

$$dg = |a|^{-1}d^\times a dk \text{ for the decomposition } G(\mathbb{R}) = Z(\mathbb{R})N(\mathbb{R})T_1(\mathbb{R})\mathrm{SO}_2(\mathbb{R}).$$

Here $d^\times a$ is the multiplicative Lebesgue measure on $T_1(\mathbb{R}) \simeq \mathbb{R}^\times$ and dk is the measure on $SO_2(\mathbb{R})$ of total mass 1.

Proposition 4.2.6. *Let π,π_χ be two representations of $G(\mathbb{R})$ satisfying Assumption 4.2.5, and let $2k\geqslant 0$ be the weight of π. Let $\varphi_2 \in S(\mathbb{R}^2)$ be the weight $-2k$ function denoted by φ_k in Proposition 2.5.5. Then*

$$\Psi^+(s,\varphi_1,\varphi_2) = \chi_1(-1)L(s,\pi_1\times\pi_2)/G_1(2s+2k).$$

Proof. By Proposition 2.5.5, the function $f(\cdot,\varphi_2)$ has weight $-2k$ under the action of $\mathrm{SO}_2(\mathbb{R})$. Hence the integrand in the formula (4.2.8) is right $\mathrm{SO}_2(\mathbb{R})$-invariant, and the desired identity follows easily from Proposition 3.2.1 and Lemma 4.2.4. If π is a principal series $\pi(\mu_1,\mu_2)$, the assumption that $\mu_1(-1)=\chi_1(-1)$ has been made to ensure that the product $W_\pi[i(a)]W_\chi[i(-a)]$ is an even function of a, otherwise $\Psi^+(s,\varphi_1,\varphi_2)$ vanishes. □

4.3 The Rankin-Selberg identity

We now come back to the global setting considered in the beginning of §4. We choose $\Lambda = \lambda N c(\chi)^2$ with λ a prime that splits in K such that $\lambda \equiv -N \pmod{d_K}$. As we shall show in §5.1, the quaternion algebra ramified exactly at the even number of places dividing N which are inert in K, has global Hilbert symbol $(d_K,-\Lambda)$.

Putting together the local computations from the previous section, we obtain the following integral representation for the Rankin L-function:

$$L(s,\pi_f\times\pi_\chi) = M(s)\int_{Z(\mathbb{A})N(\mathbb{A})\backslash G(\mathbb{A})^+} W_f(g)W_\chi(\epsilon g;\varphi_1)f(s,g;\varphi_2)dg,$$

where W_f is the product of the Whittaker newforms for π_f over all places, and the local components of the functions $\varphi_1,\varphi_2\in S(K_\mathbb{A})$ are given in the table of §4.2.1 in the nonarchimedean case (see the last column in that table for the case corresponding to a given finite prime p; if $p|d_K$, take $\alpha\in\mathcal{O}_{K,p}$), and in Proposition 4.2.6 in the archimedean case. The constant $M(s)$ is the product of the local factors given in Propositions 4.2.3 and 4.2.6; we are only interested in its value at $s=1/2$: (recall $\chi_\infty = (\chi_1,\chi_1^{-1})$):

$$M(1/2) = \chi_1(-1)2^\alpha G_1(1+2k)L_{\text{fin}}(1,\omega_K)ND\prod_{p|ND}(1+1/p), \tag{4.3.1}$$

where χ_1 is one of the components of χ_∞ and α is the number of primes dividing d_K.

Let $E(s,g) = E(s,g;\varphi_2)$ be the Eisenstein series formed from the flat section $f(s,g;\varphi_2)$:

$$E(s,g) = \sum_{\gamma\in B(\mathbb{Q})\backslash G(\mathbb{Q})} f(s,\gamma g;\varphi_2).$$

Also let ϕ_f be the automorphic form on $G(\mathbb{A})$ with Whittaker coefficient W_f, and recall that $\theta_\chi(\cdot;\varphi_1)$ is the automorphic form on $G(\mathbb{Q})G(\mathbb{A})^+$ with Whittaker coefficient $W_\chi(\cdot;\varphi_1)$. That is:

$$\phi_f(g) = \sum_{\xi\in\mathbb{Q}^\times} W_f\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix} g\right) \tag{4.3.2}$$

$$\theta_\chi(g;\varphi_1) = C(g) + \sum_{\xi\in \mathrm{N}(K^\times)} W_\chi\left(\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix} g;\varphi_1\right) \tag{4.3.3}$$

where $C(g)$ is the (possibly 0) constant term of θ_χ, which appears since the representation π_χ is not cuspidal in general. The second series is summed over $\mathrm{N}(K^\times)$ rather than over $\mathbb{Q}^\times$ because of Proposition 4.1.2 (ii).

Then an easy modification of the standard Rankin-Selberg argument (e.g. [Bu97, Ch. 3.7]) leads to the following:

Proposition 4.3.1. *With $\varphi_1, \varphi_2, M(s)$ defined above, the following identity holds for* $\mathrm{Re}(s)$ *large enough:*

$$L(s,\pi_f\times\pi_\chi) = M(s)\int_{Z(\mathbb{A})G(\mathbb{Q})^+\backslash G(\mathbb{A})^+} \phi_f(g)\theta_\chi(g;\varphi_1)E(s,g;\varphi_2)dg.$$

Proof. We have to prove the identity:

$$\int_{Z(\mathbb{A})N(\mathbb{A})\backslash G(\mathbb{A})^+} W_f(g)W_\chi(\epsilon g;\varphi_1)f(s,g;\varphi_2)dg = \int_{Z(\mathbb{A})G(\mathbb{Q})^+\backslash G(\mathbb{A})^+} \phi_f(g)\theta_\chi(g;\varphi_1)E(s,g;\varphi_2)dg.$$

For reasons of space we omit the Schwartz functions from the notation. Denote by I_1 the integral appearing in the first line above and by I_2 the integral on the second line. Using the formula for $E(s,g)$ we have (see Notation):

$$\begin{aligned} I_2 &= \int_{Z(\mathbb{A})B(\mathbb{Q})^+\backslash G(\mathbb{A})^+} \phi_f(g)\phi_\chi(g)f(s,g)dg\\ &= \int_{B(\mathbb{A})^+\backslash G(\mathbb{A})^+}\int_{Z(\mathbb{A})B(\mathbb{Q})^+\backslash B(\mathbb{A})^+} \phi_f(bg)\theta_\chi(bg)f(s,bg)dbdg\\ &= \int_{B(\mathbb{A})^+\backslash G(\mathbb{A})^+}\int_{T_1(\mathbb{Q})^+\backslash T_1(\mathbb{A})^+}\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \phi_f(tng)\theta_\chi(tng)f(s,tng)dndtdg. \end{aligned}$$

The superscript $+$ denotes the intersection of the corresponding groups with $G(\mathbb{A})^+$. For the second identity we use the isomorphism $B/Z\simeq T_1N$. The measures on $G(\mathbb{A})^+/B(\mathbb{A})^+$ is normalized in the same way as in the integral I_1 (using the local measures), while the measures on $T_1(\mathbb{A})\simeq\mathbb{A}^\times$ and $N(\mathbb{A})/N(\mathbb{Q})\simeq\mathbb{A}/\mathbb{Q}$ are the standard ones.

Next we replace θ_χ by its Fourier expansion (4.3.3). Since T_1 normalizes N, we have $C(tng) = C(tg)$, $f(s,tng) = f(s,tg)$ for $t\in T_1(\mathbb{A})$, $n\in N(\mathbb{A})$, where $C(g)$ is the constant term in the Fourier expansion. Therefore integrating $C(tng)$ against $\phi_f(tng)$ over $N(\mathbb{Q})\backslash N(\mathbb{A})$ yields 0, and we have:

$$\begin{aligned} I_2 &= \int_{B(\mathbb{A})^+\backslash G(\mathbb{A})^+}\int_{T_1(\mathbb{Q})^+\backslash T_1(\mathbb{A})^+}\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \phi_f(tng)\sum_{\xi\in T_1(\mathbb{Q})^+} W_\chi(\xi tng)f(s,tg)dndtdg\\ &= \int_{B(\mathbb{A})^+\backslash G(\mathbb{A})^+}\int_{T_1(\mathbb{A})^+}\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \phi_f(tng)W_\chi(tng)f(s,tg)dndtdg. \end{aligned}$$

In the last step we have collapsed summation with integration. The inner integral can be written using a change of variables and the fact that W_χ has character ψ under left multiplication by $N(\mathbb{A})$:

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \phi_f(tnt^{-1}tg)W_\chi(tnt^{-1}tg)dn = W_f(\epsilon tg)W_\chi(tg)|t|^{-1}$$

hence we finally obtain:

$$I_2 = \int_{B(\mathbb{A})^+\backslash G(\mathbb{A})^+} \int_{T_1(\mathbb{A})^+} W_f(\epsilon tg) W_\chi(tg) f(s,tg) |t|^{-1} dt dg.$$

But this is just another way of writing the integral I_1 (recalling that the measure on $T_1(\mathbb{A}) \simeq \mathbb{A}^\times$ was normalized by a factor of $|t|^{-1}$ in the local integrals). $\square$

5. The Shimizu Correspondence

Evaluating the formula in Proposition 4.3.1 at $s = 1/2$ (by the principle of analytic continuation) we obtain:

$$L(1/2, \pi_f \times \pi_\chi) = M(1/2) \int_{Z(\mathbb{A})G^+(\mathbb{Q})\backslash G^+(\mathbb{A})} \phi_f(g)\theta_\chi(g;\varphi_1)E(1/2,g;\varphi_2)dg,$$

where $M(1/2)$ is given by (4.3.1). The forms $\theta_\chi, E(1/2,g)$ are theta lifts from the quadratic field K viewed as a vector space with norms $\mathrm{N}_{K/F}, \Lambda\mathrm{N}_{K/F}$ respectively (the second in virtue of the Siegel-Weil formula). Recall that $\Lambda = \lambda N c(\chi)^2$, with an odd prime λ such that $\lambda \equiv -N \pmod{d_K}$, and α is the number of prime divisors of d_K.

This section is organized as follows. In §5.1, we consider the quaternion algebra B ramified at the primes in $S = \{p|N: \ p \text{ inert in } K\}$, and we show that the quadratic space $(B, N_{B/\mathbb{Q}})$ decomposes orthogonally into the two dimensional quadratic spaces of the previous paragraph. In §5.2, we apply the "seesaw identity" to rewrite the Rankin-Selberg integral above in terms of a toric integral of the Shimizu theta lift $\theta_f(x,y)$ on $GSO(B_\mathbb{A}) \simeq B_\mathbb{A}^\times \times B_\mathbb{A}^\times/\mathbb{A}^\times$, of the form ϕ_f. In §5.3, we compute the level of θ_f, thus identifying up to a constant its components belonging to the space of the Jacquet-Langlands lift of π_f. When the character χ is unramified, the constant is determined in §5.4 using a result of T. Watson [Wa02].

5.1 Quaternion algebras

First we show that the quadratic vector spaces $(K, \mathrm{N}_{K/F})$, and $(K, \Lambda\mathrm{N}_{K/F})$ provide an orthogonal decomposition of the quaternion algebra B ramified at the set S of primes dividing N, which are inert in K. Recall that the set S has even cardinality, due to the assumption on the sign in the functional equation for $L(s, \pi_f \times \pi_\chi)$.

Indeed, we claim that the quaternion algebra B has global Hilbert symbol $(d_K, -\Lambda)$, that is:

$$B = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}k$$

with $i^2 = d_K, j^2 = -\Lambda$, and $k = ij = -ji$, where $\Lambda = \lambda N c(\chi)^2$ as above. To check that B is ramified exactly at the primes in S, we compute the local Hilbert symbols $(d_K, -\Lambda)_p$ for all odd primes p and use the product formula (see [Vi80] for the facts about the local Hilbert symbol that we use):

- If $p|N$, p odd, then $(d_K, -\Lambda)_p = (d_K, p)_p$ since N is assumed square free, and the latter symbol is 1 or -1 as p is split or inert in K respectively.
- If $p|d_K$, p odd, then $(d_K, -\Lambda)_p = (p, -\lambda N)_p = 1$ by the assumption that $\lambda \equiv -N \pmod{d_K}$.
- If $p = \lambda$, then $(d_K, -\Lambda)_\lambda = (d_K, \lambda)_\lambda = 1$ because λ splits in K.
- If $p \nmid \lambda N d_K$, p odd, clearly $(d_K, \lambda N c(\chi)^2)_p = 1$.

Note that the quadratic field K embeds into B and the decomposition above becomes:

$$B = K \oplus Kj, \tag{5.1.1}$$

where $j \in B$ is such that $kj = j\bar{k}$ for all $k \in K$, and $\mathrm{N}_{B/\mathbb{Q}}(j) = \Lambda$. The bar denotes the nontrivial automorphism of K over $\mathbb{Q}$. The decomposition (5.1.1) is orthogonal with respect to the quadratic form on B induced from the reduced norm, and we can view K and Kj as two dimensional quadratic spaces over $\mathbb{Q}$ with norm derived from the reduced norm $N_{B/\mathbb{Q}}$ on B.

5.2 The seesaw identity

Let B be the quaternion algebra defined in §5.1. Viewing θ_χ and $E(1/2, g)$ as global theta lifts, we change the order of integration in the Rankin-Selberg integral, a technique which has been formalized by S. Kudla in [Ku83]. The present situation has also considered by B. Roberts in [Ro98], and it can be summarized by the following *seesaw dual pair diagram.*

$$\begin{tikzcd}
{[G(\mathbb{A})^+ \times G(\mathbb{A})^+]} \arrow[drr, dash] & & GSO(B_{\mathbb{A}}) \arrow[dll, dash] \\
G(\mathbb{A})^+ \arrow[u] & & {[GSO(K_{\mathbb{A}}) \times GSO(K_{\mathbb{A}} j)] = H(\mathbb{A})} \arrow[u]
\end{tikzcd}$$

The square brackets around a product of groups indicate the subgroup of elements with the same scale factor (e.g. with the same determinant inside $G(\mathbb{A})^+ \times G(\mathbb{A})^+$). The diagonal lines indicate the dual pairs on which the theta correspondence takes place. The left vertical arrow is diagonal inclusion, and the right vertical arrow is the natural embedding given by viewing $(\mu, \nu) \in [GSO(K_{\mathbb{A}}) \times GSO(K_{\mathbb{A}} j)]$ as the similitude $x + yj \to \mu(x) + \nu(y)j$ of $B(\mathbb{A})$. Note that the similitude (μ, ν) of $B_{\mathbb{A}}$ has the same similitude factor as that of μ or ν.

To make the notation uniform, in this section we denote by r_1, r_2, r_B the Weil representations defined on the groups $[GSO(K_{\mathbb{A}}) \times G(\mathbb{A})^+]$, $[GSO(K_{\mathbb{A}} j) \times G(\mathbb{A})^+]$, and on $[GSO(B_{\mathbb{A}}) \times G(\mathbb{A})^+]$ respectively, as in Section 2.3 (r_2 has been denoted by r_Λ previously). Let $\theta_1, \theta_2, \theta_B$ be the corresponding theta kernels. For simplicity, denote by $H(\mathbb{A})$ the group $[GSO(K_{\mathbb{A}}) \times GSO(K_{\mathbb{A}} j)]$.

Tensoring the decomposition $B = K \oplus Kj$ in equation (5.1.1) with $\mathbb{Q}_p$ at all primes p (including $p = \infty$), we obtain local decompositions:

$$B_p = K_p \oplus K_p j_p, \tag{5.2.1}$$

where $j_p = j$ for $p \neq \lambda$, while if $p = \lambda$ we take $j_\lambda = j/(1, \lambda)$ (recall $\Lambda = \lambda N c(\chi)^2$, with λ split in K). Here and in the sequel we fix embeddings $B \to B_p$ at all places p. Thus j_p is an element of B_p which anticommutes with K_p and whose reduced norm satisfies $\mathrm{N} j_p = \Lambda$, for $p \neq \lambda$, while $N j_\lambda$ a unit in $\mathbb{Z}_\lambda$.

Let $\varphi_1, \varphi_2 \in S(K_{\mathbb{A}})$ be the functions that give rise to θ_χ, $E(1/2, g)$, and which were determined in Section 4.3. Also let $\varphi \in S(B_{\mathbb{A}})$ be the function with local components

$$\varphi_p(x_1 + x_2 j_p) = \varphi_{1,p}(x_1)\varphi_{2,p}(x_2). \tag{5.2.2}$$

The crucial property of the seesaw pair is that the representation r_B decomposes as follows:

$$r_B[(h_1, h_2), g]\varphi(x_1 + x_2 j) = r_1(h_1, g)\varphi_1(x_1) r_2(h_2, g)\varphi_2(x_2),$$

where $(h_1, h_2) \in H(\mathbb{A})$ with similitude factors $\lambda(h_1) = \lambda(h_2) = \det g$. In terms of the theta kernels, this implies:

$$\theta_B((h_1, h_2), g; \varphi_1 \otimes \varphi_2) = \theta_1(h_1, g; \varphi_1)\theta_2(h_2, g; \varphi_2) \tag{5.2.3}$$

The seesaw identity uses this observation together with Fubini's theorem to move the integration from one side of the seesaw to the other.

Consider the theta lift of the form ϕ_f to an automorphic form on $GSO(B_{\mathbb{A}})$ (the adjoint to

"Shimizu's lift"):

$$\theta_f(\sigma;\varphi) = \int_{\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})} \theta_B(\sigma, g_1 g;\varphi)\phi_f(g_1 g)dg_1, \tag{5.2.4}$$

for $\sigma \in GSO(B_{\mathbb{A}})$ and $g \in G(\mathbb{A})$ with $\det g = \lambda(\sigma)$ (here λ denotes the similitude factor). Then the following "seesaw identity" holds:

$$\int_{Z_G(\mathbb{A})G(\mathbb{Q})^+\backslash G(\mathbb{A})^+} \phi_f(g)\theta_\chi(g;\varphi_1)E(1/2, g;\varphi_2)dg = 2\int_{\mathbb{A}^\times H(\mathbb{Q})\backslash H(\mathbb{A})} \theta_f(h;\varphi)\chi(h)dh, \tag{5.2.5}$$

where the group $\mathbb{A}^\times$ on the right hand side is identified with the center of $GSO(B_{\mathbb{A}})$.

First it is convenient to replace the integration domains by $Z_G(\mathbb{A}_{\infty,+})G(\mathbb{Q})^+\backslash G(\mathbb{A})^+$, and by $\mathbb{A}^\times_{\infty,+}H(\mathbb{Q})\backslash H(\mathbb{A})$ (which does not change the integrals since $\mathbb{A}^\times/\mathbb{A}^\times_{\infty,+}\mathbb{Q}^\times$ has volume 1). We have denoted by $\mathbb{A}^\times_{\infty,+}$ the ideles with positive archimedean component and unit nonarchimedean components. Let C denote the compact group $\mathrm{N}_{K/\mathbb{Q}}(\mathbb{A}_K^\times)/\mathbb{A}^\times_{\infty,+}\mathrm{N}_{K/\mathbb{Q}}K^\times$. We use the following exact sequences to normalize the measure on the groups involved:

$$1 \to \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A}) \to Z_G(\mathbb{A}_{\infty,+})G(\mathbb{Q})^+\backslash G(\mathbb{A})^+ \xrightarrow{\det} C \to 1; \tag{5.2.6}$$

$$1 \to H_1(\mathbb{Q})\backslash H_1(\mathbb{A}) \to \mathbb{A}^\times_{\infty,+}H(\mathbb{Q})\backslash H(\mathbb{A}) \xrightarrow{\lambda} C \to 1. \tag{5.2.7}$$

Here λ denotes the similitude homomorphism, and H_1 denotes the subgroup of elements of H with unit similitude factor. Note that $H_1(\mathbb{Q})\backslash H_1(\mathbb{A})$ can be identified with two copies of $\mathrm{SO}(K)\backslash \mathrm{SO}(K_{\mathbb{A}})$.

Fix a measure $d\xi$ on C which is the restricted product of local measures on $\mathrm{N}(K_p)$ such that $\mathrm{N}(U_{K,p})$ has volume 1, and of the multiplicative Lebesgue measure at infinity. It is easy to see that C has unit volume with respect to this measure. In the first exact sequence, let dg be the measure on $Z_G(\mathbb{A}_{\infty,+})G(\mathbb{Q})^+\backslash G(\mathbb{A})^+$ used to derive the Ranking-Selberg formula, and let dg_1 be the measure on $\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})$ such that $dg = dg_1 d\xi$. It follows that dg_1 is the restricted product of local measures on $\mathrm{SL}_2(\mathbb{Q}_p)$ such that $\mathrm{SL}_2(\mathbb{Z}_p)$ has volume 1, and a measure at infinity for which $\mathrm{SO}_2(\mathbb{R})$ has volume 1.

In the second exact sequence, let dh_1 be the measure on $\mathrm{SO}(K)\backslash \mathrm{SO}(K_{\mathbb{A}})$ used in the theta integral for θ_χ, which has total mass $h_K \ln \epsilon_K$, where h_K is the cardinality of the narrow class group of K, and ϵ_K is the smallest power of the fundamental unit which is totally positive (see §4.1). It follows that the measure on $\mathrm{SO}(K)\backslash \mathrm{SO}(K_{\mathbb{A}})$ used in the Siegel-Weil formula is $dh_2 = dh_1/(h_K \ln \epsilon_K)$ (since it is a measure of unit total mass). The measure on $H_1(\mathbb{Q})\backslash H_1(\mathbb{A}) \simeq (\mathrm{SO}(K)\backslash \mathrm{SO}(K_{\mathbb{A}}))^2$ is then $dh_1 dh_2$, and we normalize the measure dh on $\mathbb{A}^\times_{\infty,+}H(\mathbb{Q})\backslash H(\mathbb{A})$ by $dh = dh_1 dh_2 d\xi$. Then the right term of the seesaw identity becomes:

$$\int\limits_C \int\limits_{H_1(\mathbb{Q})\backslash H_1(\mathbb{A})} \int\limits_{\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})} \theta_B[(h_1 h(\xi), h_2 h(\xi)), g_1 g(\xi);\varphi_1\otimes\varphi_2]\cdot$$

$$\cdot\ \phi_f(g_1 g(\xi))\chi(h_1 h(\xi))dg_1 dh_1 dh_2 d\xi,$$

where $h(\xi) \in GSO(K_{\mathbb{A}}) = GSO(K_{\mathbb{A}}j)$, $g(\xi) \in G(\mathbb{A})$ with $\lambda(h(\xi)) = \det g(\xi) = \xi$. Similarly, the left term can be written:

$$2\int\limits_C \int\limits_{\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})} \int\limits_{(\mathrm{SO}(K)\backslash \mathrm{SO}(K_{\mathbb{A}}))^2} \theta_1[h_1 h(\xi), g_1 g(\xi);\varphi_1]\theta_2[h_2 h(\xi), g_1 g(\xi);\varphi_2]\cdot$$

$$\cdot\ \phi_f(g_1 g(\xi))\chi(h_1 h(\xi))dh_1 dh_2 dg_1 d\xi,$$

where the constant 2 in front of the integral is the value of κ in the Siegel-Weil formula (Theorem 2.4.1). The identity now follows from Eq. (5.2.3) by interchanging the order of integration. This is justified, since both sides converge absolutely due to the presence of the rapidly decaying cusp form ϕ_f.

Remark 5.2.1. In the proof of the seesaw identity, we have used a version of the Siegel-Weil formula in which the integration is over $\mathrm{SO}(K)\backslash\mathrm{SO}(K_{\mathbb{A}})$ rather than over the whole orthogonal group. The difference between the two integrals is due to the existence of places p such that $\varphi_p \neq \varphi_p \circ i_p$, where $\varphi = \prod_p \varphi_p$, and i_p is the generator of $\mathrm{Gal}(K_p/\mathbb{Q}_p)$. Indeed, letting S be the (finite) set of such finite places, the two integrals are related as follows:

$$\int_{O(K)\backslash O(K_{\mathbb{A}})} \theta(\sigma h, g; \varphi) d\sigma = \frac{1}{2^{\#S}} \sum_{R\subset S} \int_{\mathrm{SO}(K)\backslash\mathrm{SO}(K_{\mathbb{A}})} \theta(\sigma h, g; \varphi^R) d\sigma,$$

where $\varphi_p^R = \varphi_p \circ i_p$ for $p \in R$, and $\varphi_p^R = \varphi_p$ for $p \notin R$. Since the function φ_2 considered here is invariant under the local involutions i_p for each prime p, it follows that the two integrals agree here. However that is not the case for more general functions φ, as it will be seen in the proof of Theorem 5.3.9.

To interpret the right hand side of Eq. (5.2.5), we identify the right vertical embedding in the seesaw diagram as follows:

$$\begin{array}{ccc} GSO(B_{\mathbb{A}}) & \xleftarrow{\simeq} & B_{\mathbb{A}}^\times \times B_{\mathbb{A}}^\times/\mathbb{A}^\times \\ \uparrow & & \uparrow \\ H(\mathbb{A}) & \xleftarrow{\simeq} & K_{\mathbb{A}}^\times \times K_{\mathbb{A}}^\times/\mathbb{A}^\times \end{array} \qquad (5.2.8)$$

The top isomorphism is given by $(g, g')v = gvg'^{-1}$ for $g, g' \in B^\times, v \in B$, while the bottom one is given by viewing $(x, y) \in K_{\mathbb{A}}^\times \times K_{\mathbb{A}}^\times$ as multiplication by xy^{-1} in $K_{\mathbb{A}}$, and as (left) multiplication by $x\bar{y}^{-1}$ in $K_{\mathbb{A}}j$. Finally, the right vertical arrow is given by the fixed embedding $K_{\mathbb{A}} \hookrightarrow B_{\mathbb{A}}$.

Using the identification (5.2.8), the seesaw identity (5.2.5) becomes:

$$\int_{Z(\mathbb{A})G^+(\mathbb{Q})\backslash G^+(\mathbb{A})} \phi_f(g)\theta_\chi(g;\varphi_1)E(1/2, g; \varphi_2) dg = \frac{2}{h_K \ln \epsilon_K} \int_{(\mathbb{A}^\times K^\times\backslash K_{\mathbb{A}}^\times)^2} \theta_f(x, y; \varphi)\chi(xy^{-1}) dx dy.$$

The measure on $\mathbb{A}^\times K^\times\backslash K_{\mathbb{A}}^\times$ is normalized as in §4.1. The factor $1/h_K \ln \epsilon_K$ appears because of the difference between the measure normalization in the two dimensional theta integrals, as explained above.

5.3 A level computation

Combining the last identity in the previous section with Proposition 4.3.1, and taking into consideration Dirichlet's class number formula[2] $L_{\mathrm{fin}}(1, \omega_K) = h_K \ln \epsilon_K/\sqrt{d_K}$, we have shown that:

$$L(1/2, \pi_f \times \pi_\chi) = \chi_1(-1)\frac{2^{1+\alpha}}{\sqrt{d_K}} G_1(1 + 2k)ND \prod_{p|ND} (1 + 1/p) I(\varphi),$$

where:

$$I(\varphi) = \int_{(\mathbb{A}^\times K^\times\backslash K_{\mathbb{A}}^\times)^2} \theta_f(x, y; \varphi)\chi(xy^{-1}) dx dy. \qquad (5.3.1)$$

Recall that θ_f is the automorphic form on $GSO(B_{\mathbb{A}}) \simeq B_{\mathbb{A}}^\times \times B_{\mathbb{A}}^\times/\mathbb{A}^\times$ defined via the Shimizu correspondence in Eq. (5.2.4), and that χ_1 is one of the two components of χ_∞.

Note at this stage that the form $\theta_f(x, y; \varphi)$ depends on the quadratic field K and on the character χ only via the Schwartz function $\varphi \in S(B_{\mathbb{A}})$ used to define the theta kernel in the integral (5.2.4).

[2] Dirichlet's formula is usually stated in terms of the class number and fundamental unit of K, while here we have written it in terms of the narrow class number h_K and the smallest totally positive power ϵ_K of the fundamental unit.

We shall first express the Schwartz function more intrinsically in terms of certain orders in the quaternion algebra B.

Using the decomposition (5.2.1), define for each $p<\infty$ an order R_p of B_p as follows:

$$R_p = \mathcal{O}_p + \mathcal{O}_p j_p,$$

where we write $\mathcal{O}_p$ for $\mathcal{O}_{K,p}$, the ring of integers in K_p. By our choice of j_p, the order R_p is maximal for $p \nmid D$ and for $p|N$, p inert in K, and it has level ND for $p|D$ and for $p|N$, p split in K (recall that $D = d_K c(\chi)^2$).

For $p|c(\chi)$, the character χ_p extends to a character of the group of units $R_p^\times$ in the order R_p by:

$$\chi_p(k_1 + k_2 j_p) = \chi_p(k_1), \quad \text{for } k_1, k_2 \in \mathcal{O}_p \text{ such that } k_1 + k_2 j_p \in R_p^\times$$

(this makes sense since $\mathrm{N} j_p$ is a unit multiple of $c(\chi)^2$). From the definition (5.2.2) of φ, it follows that:

$$\varphi_p = \begin{cases} 1_{R_p} & \text{if } p \nmid c(\chi), \\ \chi 1_{R_p^\times} & \text{if } p|c(\chi). \end{cases}$$

At $p=\infty$, it is convenient to embed K_∞ into $B_\infty = M_2(\mathbb{R})$ as the diagonal subgroup, and to let $j_\infty = \begin{pmatrix} 0 & \sqrt{\Lambda} \\ -\sqrt{\Lambda} & 0 \end{pmatrix}$ in the archimedean decomposition:

$$B_\infty \simeq M_2(\mathbb{R}) = K_\infty + K_\infty j_\infty. \tag{5.3.2}$$

With this identification the Schwartz function φ_∞ is given by (recall that $\pi_{f,\infty}$ has weight $2k \geqslant 0$):

$$\varphi_\infty \begin{pmatrix} x & y \\ z & t \end{pmatrix} = P_k(y+z) e^{-\pi(x^2+y^2+z^2+t^2)},$$

where P_k is the polynomial in Proposition 2.5.5.

Remark 5.3.1. This decomposition is conjugate to that of Eq. (5.2.1) by a matrix $\gamma_\infty \in G(\mathbb{R})$. To simplify notation, we ignore this matrix in the sequel, with the understanding that if $\tau(x)$ is an archimedean Schwartz function defined using the identification (5.3.2), then it should be read as $\tau(\gamma_\infty^{-1} x \gamma_\infty)$. The matrix γ_∞ will reappear in the final result, Theorem 5.3.9.

Unfortunately the Schwartz function φ does not endow the theta lift $\theta_f(x,y;\varphi)$ with the desired level at primes $p|d_K$, and with the desired weight at infinity (if f has positive weight). Therefore we replace the function $\varphi \in S(B_{\mathbb{A}})$ with a similar one φ', which agrees with φ at all finite places not dividing d_K, and which gives the theta lift $\theta_f(x,y;\varphi')$ a level structure and weight that identifies it uniquely, via Shimizu's theorem. Of course we then have to check how these changes affect the integral (5.3.1), by tracing back the changes to the local Rankin-Selberg integrals, via the seesaw identity.

At infinity, we let φ'_∞ depend on the weight $2k$ of π_f as follows:

$$\varphi'_\infty \begin{pmatrix} x & y \\ z & t \end{pmatrix} = [(x-t) + i(y+z)]^{2k} e^{-\pi(x^2+y^2+z^2+t^2)}.$$

Note that this function agrees with φ_∞ for $k=0$. It is easy to check that $\varphi'_\infty(k_\alpha x k_\beta) = e^{2k\pi i(\beta-\alpha)}\varphi'(x)$, for $k_\alpha, k_\beta \in \mathrm{SO}_2(\mathbb{R})$.

At each prime p dividing d_K, there are two maximal orders $R_p^\pm$ containing R_p:

$$R_p^\pm = \{a + b j_p : a, b \in \delta_p^{-1} \text{ such that } a \equiv \pm b u_p \pmod{\mathcal{O}_{K_p}}\}, \tag{5.3.3}$$

where δ_p denotes the different of the extension $K_p/\mathbb{Q}_p$, and $u_p \in U_{K_p}$ is such that $\mathrm{N}u_p = -\Lambda$ (such a choice of u_p is possible because $\omega_p(-\Lambda) = (d_K, -\Lambda)_p = 1$). The two orders $R_p^{\pm}$ are conjugate; if v_p is a generator of the ideal δ_p^{-1}, then:

$$R_p^- = v_p^{-1} R_p^+ v_p, \tag{5.3.4}$$

which follows from the fact that $\mathrm{Tr}(\delta_p^{-1}) \subset \mathbb{Z}_p$.

We denote by $\widetilde{R}_p$ either of the two maximal orders $R_p^{\pm}$, and we set:

$$\varphi'_p = 1_{\widetilde{R}_p}.$$

Remark 5.3.2. The choice of maximal orders $\widetilde{R}_p$ in the definition of φ'_p at primes $p|d_K$ does not influence the integral $I(\varphi')$ defined in equation (5.3.1). Indeed, if φ'' is defined as φ' with a different choice of orders $\widetilde{R}_p$, then equation (5.3.4) implies that:

$$\theta_f(x, y; \varphi'') = \theta_f(xv, yv; \varphi'),$$

with $v \in \mathbb{A}_K^{\times}$ supported at some of the primes dividing $p|d_K$, for which v_p is a generator of δ_p or of δ_p^{-1}. It is then clear that $I(\varphi') = I(\varphi'')$ by a change of variables.

The unit groups in the orders $R_p, \widetilde{R}_p$ determine a compact subgroup of $B_{\mathbb{A}}^{\times}$, which we denote $\widehat{\mathcal{R}}^{\times}$:

$$\widehat{\mathcal{R}}^{\times} = \prod_{p \nmid d_K} R_p^{\times} \prod_{p|d_K} \widetilde{R}_p^{\times}. \tag{5.3.5}$$

The notation is motivated by the fact that $\widehat{\mathcal{R}}^{\times} \cap B$ is the unit group in an Eichler order $\mathcal{R} \subset B$ of level $Nc(\chi)^2$, such that $\mathcal{R}_p := \mathcal{R} \otimes \mathbb{Q}_p$ is either R_p or $\widetilde{R}_p$.

Remark 5.3.3. Fix isomorphisms $B_p \simeq M_2(\mathbb{Q}_p)$ at the primes p where B splits. We can choose the decomposition (5.1.1) such that $\mathcal{R}_p$ is the standard order of level $Nc(\chi)^2$ in B_p. Indeed, choose a global order $\mathcal{R} \subset B$ whose localizations $\mathcal{R}_p$ are the standard orders above, and choose an algebra embedding $\Psi : K \to B$ such that $\Psi(\mathcal{O}_K) = \Psi(K) \cap \mathcal{R}$ (that this is possible can be checked locally, and it follows from [Vi80]). This embedding can then be extended to a decomposition as in (5.1.1).

Proposition 5.3.4. *The automorphic form $\theta_f(x, y; \varphi')$ has the following level structure:*

$$\theta_f(xk, yk'; \varphi') = \theta_f(x, y; \varphi') \prod_{p|c(\chi)} \chi_p(k_p^{-1} k'_p) \quad \textit{for } k, k' \in \widehat{\mathcal{R}}^{\times}$$

$$\theta_f(xk_\alpha, yk_\beta; \varphi') = \theta_f(x, y; \varphi') e^{2\pi k i(\alpha+\beta)} \quad \textit{for } k_\alpha, k_\beta \in \mathrm{SO}_2(\mathbb{R}),$$

where $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

Proof. Recall that the theta kernel used to define the form θ_f in §5.2 is given by:

$$\theta_B((x, y), g; \varphi') = \sum_{t \in B} r_B[(x, y), g]\varphi'(t),$$

where $(x, y) \in B_{\mathbb{A}}^{\times} \times B_{\mathbb{A}}^{\times}$ and $g \in G(\mathbb{A})$ such that $\mathrm{N}(xy^{-1}) = \det g$. Since ϕ_f is right invariant under the action of $K_0(N)$, it is clear from the definition of θ_f in (5.2.4) that the proposition is an immediate consequence of the following lemma. This lemma can be found in [Wa02], however the proof given here is different in that it only makes use of the Weil representation attached to two dimensional quadratic spaces.

Lemma 5.3.5. *(i) Let $k, k' \in \widehat{\mathcal{R}}^\times$, and choose $g \in K_0(Nc(\chi)^2)$ such that $\mathrm{N}_{B/\mathbb{Q}}(kk'^{-1}) = \det g$. Then:*

$$r_B[(k,k'),g]\varphi'_{\mathrm{fin}} = \prod_{p|c(\chi)} \chi_p^{-1}(kk')\varphi'_{\mathrm{fin}}.$$

(ii) If $k_\alpha, k_\beta \in \mathrm{SO}_2(\mathbb{R})$, then $r_B[(k_\alpha,k_\beta),1]\varphi'_\infty = e^{2\pi ki(\alpha+\beta)}\varphi'_\infty$.

Proof. (i) By the definition of the Weil representation:

$$r_B[(k,k'),g]\varphi = L(k,k')r_B(g_1)\varphi$$

where $L(k,k')\varphi(t) = \varphi(k^{-1}tk')$, and $g_1 \in K_0(Nc(\chi)^2) \cap \mathrm{SL}_2(\mathbb{A})$ (see Notation). To prove the statement, it is enough to show that $r_B(g_1)\varphi' = \varphi'$, as the action of $L(k,k')$ is easy to compute from the definition of φ'_p in terms of the characteristic function of R_p or $\widetilde{R}_p$.

If $p \nmid d_K$, we have $\varphi'_p = \varphi_p$, and the Weil representation r_B decomposes as follows with respect to the decomposition 5.2.1:

$$r_B(h)\varphi_p(x_1 + x_2 j_p) = r_1(h)\varphi_{1,p}(x_1) r_\Lambda(h)\varphi_{2,p}(x_2),$$

where r_1, r_Λ are the Weil representations of SL(2) associated with $(K, \mathrm{N}_{K/\mathbb{Q}}), (K, \Lambda\mathrm{N}_{K/\mathbb{Q}})$ respectively. The conclusion then follows from Proposition 2.5.1.

If $p|d_K$, the function φ'_p is the characteristic function of the maximal order $\widetilde{R}_p$, and no longer decomposes as product of two dimensional functions. Instead:

$$\varphi'_p(x_1 + x_2 j_p) = \sum_{\alpha \in \delta_p^{-1}/\mathcal{O}_{K,p}} 1_{\alpha+\mathcal{O}_p}(x_1) 1_{\alpha u_p^{-1}+\mathcal{O}_p}(x_2), \tag{5.3.6}$$

[for concreteness, we take $\widetilde{R}_p = R_p^+$ with the notations from (5.3.3)]. Denote by $\varphi^\alpha_{1,p}, \varphi^\alpha_{2,p}$ the characteristic functions appearing in the decomposition above, for $\alpha \in \delta_p^{-1}/\mathcal{O}_p$. To check that φ'_p is invariant under $\mathrm{SL}_2(\mathbb{Z}_p)$, it is enough to check it on the generators, the only nontrivial case being the action of $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We compute it by decomposing again the action of r_B on the components of φ'_p into the action of the two dimensional Weil representations, which are easy to compute:

$$r_B(w)\varphi^\alpha_{1,p} \otimes \varphi^\alpha_{2,p}(x_1 + x_2 j_p) = \psi_{K,p}[\bar{\alpha}(x_1 + \Lambda \bar{u}_p^{-1} x_2)] \widehat{1}_{\mathcal{O}_p}(x_1) \widehat{1}_{\mathcal{O}_p}(x_2)$$

where $\psi_K = \psi \circ \mathrm{Tr}_{K/\mathbb{Q}}$, and the hat denotes Fourier transform on K_p with respect to the character $\psi_{K,p}$. The Fourier transform has been computed in the proof of Lemma 2.1.1: $\widehat{1}_{\mathcal{O}_p} = \mu(\mathcal{O}_p) 1_{\delta_p^{-1}}$. Since the character $\psi_{K,p}$ has conductor ideal δ_p^{-1}, it follows that for $x_1, x_2 \in \delta_p^{-1}$ we have:

$$\sum_{\alpha \in \delta_p^{-1}/\mathcal{O}_p} \psi_{K,p}[\bar{\alpha}(x_1 + \Lambda \bar{u}_p^{-1} x_2)] = \begin{cases} 0 & \text{if } x_1 + \Lambda \bar{u}_p^{-1} x_2 \notin \mathcal{O}_p, \\ \#(\mathcal{O}_p/\delta_p) & \text{if } x_1 + \Lambda \bar{u}_p^{-1} x_2 \in \mathcal{O}_p. \end{cases}$$

Since $\mathrm{N}(u_p) = -\Lambda$, and $\mu(\mathcal{O}_p)^{-2} = \#(\mathcal{O}_p/\delta_p)$, it follows that $r_B(w)\varphi'_p = \varphi'_p$ as desired.

(ii) It is easy to check that $\varphi'_\infty(k_\alpha^{-1} t k_\beta) = e^{2\pi ki(\beta+\alpha)}\varphi'_\infty(t)$. □

The level computation allows us to identify the forms in the space of π_f^{JL} appearing in the decomposition of $\theta_f(x,y;\varphi')$ (by Shimizu's Theorem 2.3.4).

Proposition 5.3.6. *There is a nontrivial automorphic form ϕ_f^{JL} on $B^\times(\mathbb{A})$, belonging to the space of π_f^{JL}, such that*

$$\theta_f(x,y;\varphi') = C\overline{\phi_f^{JL}}(x\epsilon)\phi_f^{JL}(y),$$

for a constant C, where $\epsilon = i(-1) \in G(\mathbb{R})$ (see Notation), and the bar denotes complex conjugation. The form ϕ_f^{JL} is uniquely determined up to a constant by the following level structure:

LS. *ϕ_f^{JL} has weight $2k$ at infinity, and it transforms as follows under the action of the compact group $\widehat{\mathcal{R}}^\times$ defined in Eq.* (5.3.5)*:*

$$\phi_f^{JL}(xk) = \prod_{p|c(\chi)} \chi_p(k_p)\phi_f^{JL}(x) \quad \text{for } k \in \widehat{\mathcal{R}}^\times.$$

Moreover, ϕ_f^{JL} is an eigenform of the Hecke operators at all finite primes not dividing ND, with the same eigenvalues as those of ϕ_f.

Remark 5.3.7. If χ is unramified, the Shimizu lift has been computed explicitly by T. Watson [Wa02, Ch. 3], leading also to a value for the constant C in the proposition. We shall review T. Watson's computation in §5.4.

Remark 5.3.8. The form ϕ_f^{JL} has almost the same level structure at finite primes as the toric newform appearing in S.W. Zhang's formula for the central value of the Rankin L-function in the quadratic imaginary case [Zh01]. It differs, however, at the primes p dividing d_K, when our form has full level, while Zhang's toric newform has level d_K, and transforms by χ under the action of $K_p^\times$. It would be best to replace the form ϕ_f^{JL} in Theorem 5.3.9 by a form of level N, independent of χ, in view of applications to nonvanishing results for twists by characters of varying conductor (as in the work of Cornut and Vatsal). This is plausible by comparison with the imaginary case, where a similar result was proved by S.W. Zhang in a later paper [Zh04].

Proof. We prove first the claim that the level structure of the form ϕ_f^{JL} determines it uniquely up to a constant. This is a local statement, which follows from the fact that the space of vectors in the local representations $\pi_{f,p}^{JL}$ having the behavior described above under the action of the corresponding compact subgroups of $B_p^\times$, is one dimensional. For $p \nmid c(\chi)$, this follows from Casselman's work [Ca73], while if $p|c(\chi)$ it follows from work of Zhang [Zh01, Theorem 2.3.5] (if $p|N$, p inert in K, then B_p is a division algebra so Casselman's theorem does not apply; in this case however $\pi_{f,p}^{JL}$ itself is one dimensional). Therefore, ϕ_f^{JL} also lies in a one dimensional subspace of the representation space of π_f^{JL}. The statement about the action of the Hecke operators follows from the strong multiplicity one theorem for automorphic representations of $B^\times(\mathbb{A})$.

By Shimizu's theorem (Theorem 2.3.4), the automorphic form $\theta_f(x, y; \varphi')$ can be written as a linear combination of products of forms on $B^\times(\mathbb{A})$ belonging to the representation space of π_f^{JL}. The level structure of $\theta_f(x, y; \varphi')$ given in Proposition 5.3.4 then implies:

$$\theta_f(x, y; \varphi') = \widetilde{\phi}_f^{JL}(x)\phi_f^{JL}(y),$$

where $\widetilde{\phi}_f^{JL}$ is a form with the same level structure as ϕ_f^{JL}, but with χ replaced by χ^{-1}. Note that the same is true about the form $\overline{\phi_f^{JL}}(x\epsilon)$, and both are eigenforms for the Hecke operators at primes not dividing ND. Therefore they must differ by a constant, by the strong multiplicity one theorem invoked before. See also [Zh01, Thm. 2.4.3], where a similar argument is used to define the notion of toric newform. $\square$

The previous proposition allows us to express the integral $I(\varphi')$ given by (5.3.1), in terms of the linear form l defined in the Introduction. By identifying ϵ with $(-1, 1) \in K_\infty$ via the decomposition (5.3.2), it follows that:

$$I(\varphi') = C\chi_1(-1)|l(\phi_f^{JL})|^2. \tag{5.3.7}$$

It remains to compute how the integral $I(\varphi)$ changes, when we replace the Schwartz function φ by φ'. Tracing back the changes through the seesaw identity and into the local Rankin-Selberg integrals for $p|d_K$ and for $p = \infty$, we arrive at the following theorem:

Theorem 5.3.9. *If the form ϕ_f^{JL} and the constant C are the ones defined in Proposition 5.3.6, we have the identity:*

$$L(1/2, \pi_f \times \pi_\chi) = M \cdot C \cdot |l(\phi_f^{JL})|^2$$

where

$$M = \frac{\beta}{\sqrt{d_K}} 2^{-2k} N c(\chi)^2 \prod_{p | Nc(\chi)} (1 + 1/p),$$

with $\beta = 4$ for $k > 0$, $\beta = 2$ if $k = 0$, and the linear form l is given by:

$$l(\phi) = \int_{\mathbb{A}^\times K^\times \backslash K_{\mathbb{A}}^\times} \phi(x\gamma_\infty)\chi^{-1}(x)dx.$$

The presence of the matrix $\gamma_\infty \in G(\mathbb{R})$ is explained in Remark 5.3.1.

Proof. First we note that if the central value of the L-function is zero, the linear form l vanishes by Waldspurger's result mentioned, and thus the theorem is trivial. Otherwise, the constant C is clearly nonzero.

Because of Eq. (5.3.7) and the formula for the central value given in the beginning of §5.3, we only need to check how the integral $I(\varphi)$ given by equation (5.3.1) changes when we replace φ by φ'. That amounts, via the seesaw identity, to re-computing the local integrals encountered already in §4.2:

$$\Psi_p^+(s, \varphi_{1,p}, \varphi_{2,p}) = \iint\limits_{K_0(1)_p^+ \times \mathrm{N}(K_p^\times)} W_{f,p}[i(t)k] W_{\chi,p}[i(-t)k; \varphi_{1,p}] f_p(s, k; \varphi_{2,p}) |t|^{s-1} dt dk, \qquad (5.3.8)$$

for $p | d_K$ and $p = \infty$, when we replace the components $\varphi_{1,p}, \varphi_{2,p}$ of φ_p by the components of φ'_p. The situation is complicated by the fact that φ'_p is no longer a product of two dimensional Schwartz functions, but a linear combination of such products.

First assume p is a prime dividing d_K, and denote by δ_p the different ideal of the quadratic extension $K_p/\mathbb{Q}_p$. Assume for concreteness $\widetilde{R}_p = R_p^+$ with the notations of equation (5.3.3). Then φ'_p decomposes into two dimensional components as in Eq. (5.3.6), and for $\alpha \in \delta_p^{-1}/\mathcal{O}_{K,p}$ we have to compute $\Psi_p^+(s, \varphi_{1,p}^\alpha, \varphi_{2,p}^\alpha)$ (see the paragraph following Eq. (5.3.6) for the notations). This has already been done in Proposition 2.5.1, and we have:

$$\sum_{\alpha \in \delta_p^{-1}/\mathcal{O}_p} \Psi_p^+(s, \varphi_{1,p}^\alpha, \varphi_{2,p}^\alpha) = L_p(s, \pi_f \times \pi_\chi).$$

Globally, let $T = \prod_{p|d_K} \delta_p^{-1}/\mathcal{O}_p$ (a direct product), and for each element $t = (t_p)_{p|d_K} \in T$, let $\varphi_1^t, \varphi_2^t \in S(K_{\mathbb{A}})$ be two Schwartz functions that agree with φ_1, φ_2 for all $p \nmid d_K$, while for $p | d_K$:

$$(\varphi_1^t)_p = \varphi_{1,p}^{t_p}, \quad (\varphi_2^t)_p = \varphi_{2,p}^{t_p}.$$

Then the previous identity together with the Ranking-Selberg identity over $G(\mathbb{A})^+$ (Proposition 4.3.1) yield:

$$\begin{aligned} L(s, \pi_f \times \pi_\chi) &= M'(s) \sum_{t \in T} \Psi^+(s, \varphi_1^t, \varphi_2^t) \\ &= M'(s) \sum_{t \in T} \int_{Z(\mathbb{A})G(\mathbb{Q})^+ \backslash G(\mathbb{A})^+} \phi_f(g) \theta_\chi(g; \varphi_1^t) E(s, g; \varphi_2^t) dg, \end{aligned}$$

where $M'(1/2) = \chi_1(-1) G_1(1 + 2k) N c(\chi)^2 \prod_{p|Nc(\chi)} (1 + 1/p)$.

Before applying the seesaw identity, we have to take into account the fact that $\varphi_{2,p}^t$ is not always invariant under the generator i_p of $\mathrm{Gal}(K_p/\mathbb{Q}_p)$, and hence integrating over the whole orthogonal group in the Siegel-Weil formula for $E(1/2,g;\varphi_2^t)$ is not the same as integrating over the special orthogonal group. By Remark 5.2.1, we have instead:

$$E(1/2,g;\varphi_2^t)=\frac{2}{2^{\#S}}\sum_{R\subset S}\int_{\mathrm{SO}(K)\backslash\mathrm{SO}(K_{\mathbb{A}})}\theta_2[\sigma h,g;(\varphi_2^t)^R]d\sigma,$$

where S denotes the set of primes diving d_K. By applying the seesaw identity $2^{\#S}d_K$ times we obtain:

$$L(1/2,\pi_f\times\pi_\chi)=2M'(1/2)\sum_{t\in T}\frac{1}{2^{\#S}}\sum_{R\subset S}I[\varphi_1^t\otimes(\varphi_2^t)^R],$$

where $I(\varphi)$ is given by equation (5.3.1) for $\varphi\in S(B_{\mathbb{A}})$. For a fixed set of primes $R\subset S$, define the function $\varphi^R\in S(B_{\mathbb{A}})$ as follows:

$$\varphi^R=\sum_{t\in T}\varphi_1^t\otimes(\varphi_2^t)^R.$$

Note that φ^R equals φ' locally at each finite place, except at the primes $p\in R$, when φ_p^R is the characteristic function of the order R_p^-. By Remark 5.3.2, we conclude that $I(\varphi^R)=I(\varphi'_{\mathrm{fin}}\varphi_\infty)$ independent of R, and we have:

$$L(1/2,\pi_f\times\pi_\chi)=2M'(1/2)I(\varphi'_{\mathrm{fin}}\varphi_\infty).$$

Assume now that $p=\infty$, and assume that the weight of f is $2k>0$. We need to compute the change in $I(\varphi)$ when φ is replaced by $\varphi_{\mathrm{fin}}\varphi'_\infty$. This is done by decomposing the four variable function φ'_∞ into a sum of functions which have weights $(2j,2k-2j)$ under the two dimensional Weil representations associated with the decomposition (5.3.2). It is convenient to denote by $Q_k(X,Y)$ the polynomial appearing in the definition of φ'_∞, that is:

$$Q_k(X,Y)=(X+iY)^{2k}.$$

Write $Q_k=Q_k^e+iQ_k^o$, where Q_k^e, Q_k^o are the real and imaginary parts of Q_k, respectively (the first is an even, while the second is an odd polynomial in X and Y). We will see that the odd polynomial Q_k^o will not give any contribution in the local Rankin-Selberg integral, while the even part decomposes as follows in terms of the polynomials P_j defined in Proposition 2.5.5:

$$Q_k^e(X,Y)=\frac{(2k)!}{(4\pi)^k k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}P_j(X)P_{k-j}(Y).$$

To prove this identity, we integrate both sides over $\mathbb{R}^2$ against the kernel $e^{-\pi\Lambda_1X^2}e^{-\pi\Lambda_1Y^2}dXdY$, using Lemma 2.5.4 on the left hand side, and Proposition 2.5.7 on the right hand side; the identity then becomes the binomial formula for expanding $(1/\Lambda_1-1/\Lambda_2)^k$. Note that the constant appearing in front of the sum equals $G_1(1+2k)$.

The previous identity shows that, with respect to the decomposition of B_∞ fixed in Eq. (5.3.2), the four dimensional function φ' decomposes as follows in terms of two dimensional functions:

$$\varphi'_\infty\begin{pmatrix}x&y\\z&t\end{pmatrix}=G_1(1+2k)\sum_{j=0}^{k}(-1)^j\binom{k}{j}\varphi_1^{(j)}(x,t)\varphi_\Lambda^{(k-j)}(y/\sqrt{\Lambda},z/\sqrt{\Lambda})+\text{ odd part,}\tag{5.3.9}$$

where $\varphi_\Lambda^{(j)}(x_1,x_2)\in S(K_\infty)$ is the weight $-2j$ function under the Weil representation r_Λ at infinity given in Proposition 2.5.5, while "odd part" denotes the sum of terms whose two dimensional components are products of an odd polynomial and the Gaussian in two variables. We let $\varphi_\infty^{(j)}(t_1+t_2j_\infty)=\varphi_1^{(j)}(t_1)\varphi_\Lambda^{(k-j)}(t_2)$ denote the j-th term in the sum above, with respect to the decomposition (5.3.2); note that for $j=0$ we recover the original function φ_∞. We also let $\varphi^{(j)}=\varphi_{\mathrm{fin}}\varphi_\infty^{(j)}$.

The odd part in the decomposition above does not contribute to the Rankin-Selberg integral (5.3.8), since $f_\infty(s,g;\varphi^o_\infty)=0$ for all functions $\varphi^o_\infty \in S(K_\infty)$ which are a product of an odd polynomial and the two dimensional Gaussian. Therefore we only have to compute the terms $I(\varphi^{(j)})$, which reduces to computing the archimedean integrals $\Psi^+_\infty(s,\varphi_1^{(j)},\varphi_\Lambda^{(k-j)})$, by the seesaw identity. We shall see that although each such integral is fairly complicated, their sum evaluated at $s=1/2$ simplifies to an elementary function of k.

By Proposition 2.5.5, the functions $\varphi_1^{(j)},\varphi_\Lambda^{(k-j)}$ have weights j, $k-j$ under the two dimensional Weil representations r_1 and r_Λ respectively, hence the integral (5.3.8) becomes:

$$\Psi^+_\infty(s,\varphi_1^{(j)},\varphi_\Lambda^{(k-j)})=\int_{\mathbb{R}^\times} W_{f,\infty}[i(t)]W_{\chi,\infty}[i(-t);\varphi_1^{(j)}]|t|^{s-1}d^\times t.$$

The values of $W_{\chi,\infty}$ on $T_1(\mathbb{R})$ can be computed using Eq. (4.1.6). Let $\chi_\infty=(\chi_1,\chi_1^{-1})$, with $\chi_1=|\ |^r\mathrm{sgn}^m$ for $r\in\mathbb{C}$, $m\in\{0,1\}$. We only need to consider $t>0$ in the following formula, since $W_{f,\infty}[i(t)]$ vanishes for $t<0$:

$$W_{\chi,\infty}[i(-t);\varphi_1^{(j)}]=\chi_1(-1)t^{1/2}\sum_{i=0}^{j}\frac{(-4\pi)^i i!}{(2i)!}\binom{j}{i}\int_{\mathbb{R}^\times}(tx+1/x)^{2i}(tx^2)^{-r}e^{-\pi(t^2x^2+x^{-2})}d^\times x$$

The integrals above can be computed in terms of Bessel functions, using formula (3.2.1), after a change of variables $u=tx^2$, and after using the binomial formula:

$$W_{\chi,\infty}[i(-t);\varphi_1^{(j)}]=2\chi_1(-1)t^{1/2}\sum_{i=0}^{j}\frac{(-4\pi)^i i!}{(2i)!}\binom{j}{i}t^i\sum_{l=-i}^{i}\binom{2i}{i+l}J_{l-r}(2\pi t).$$

We compute $\Psi^+_\infty(s,\varphi_1^{(j)},\varphi_\Lambda^{(k-j)})$ by using Barnes' lemma, for which we need the value of the integral:

$$\int_0^\infty W_{\chi,\infty}[i(-t);\varphi_1^{(j)}]t^{s-1/2}d^\times t=\frac{\chi_1(-1)}{2}\sum_{i=0}^{j}\sum_{l=-i}^{i}\frac{(-4\pi)^i i!}{(2i)!}\binom{j}{i}\binom{2i}{i+l}\cdot$$
$$\cdot G_1(s+i-l+r)G_1(s+i+l-r).$$

This follows from formula (3.2.2). Recall that $G_1(s)=\pi^{-s/2}\Gamma(s/2)$.

Since $W_{f,\infty}$ is a Whittaker newform, its Mellin transform equals the L-function of $\pi_{f,\infty}$, and moreover $W_{f,\infty}[i(t)]=0$ for $t<0$ (see Proposition 3.2.1 and the Corollary following it). Hence we can apply Barnes' Lemma 4.2.4 to conclude:

$$\Psi^+_\infty(s,\varphi_1^{(j)},\varphi_\Lambda^{(k-j)})=\chi_1(-1)\sum_{i=0}^{j}\sum_{l=-i}^{i}\frac{(-4\pi)^i i!}{(2i)!}\binom{j}{i}\binom{2i}{i+l}\frac{\prod G_1[s+i+k\pm(l-r)\pm 1/2]}{G_1(2s+2i+2k)},$$

where here and in the sequel, the product is taken over all combinations of plus and minus signs. In order to get the whole contribution corresponding to φ'_∞, we sum over j with weights given by the decomposition (5.3.9); denoting by $S_k(s)$ the resulting sum, we obtain:

$$S_k(s)=\chi_1(-1)G_1(1+2k)\sum_{j=0}^{k}\sum_{i=0}^{j}\sum_{l=-i}^{i}(-1)^j\binom{k}{j}\frac{(-4\pi)^i i!}{(2i)!}\binom{j}{i}\binom{2i}{i+l}\cdot$$
$$\cdot\frac{\prod G_1[s+i+k\pm(l-r)\pm 1/2]}{G_1(2s+2i+2k)}.$$

It is remarkable that the resulting triple sum simplifies considerably. We interchange the first two sums, and sum over j, using the identity:

$$\sum_{j=i}^{k}(-1)^{j-i}\binom{k}{j}\binom{j}{i} = \begin{cases} 0 & \text{if } i < k, \\ 1 & \text{if } i = k. \end{cases}$$

Using the identity $G_2(s) = G_1(s)G_1(s+1)$, we obtain:

$$S_k(s) = \chi_1(-1)\sum_{l=-k}^{k}\binom{2k}{k+l}\frac{\prod G_2[s-1/2+2k\pm(l-r)]}{G_1(2s+4k)}.$$

We relate the last expression to the L-factor $L_\infty(s, \pi_f \times \pi_\chi) = \prod G_2(s-1/2+k\pm r)$ using the recurrence relation $G_2(s+1) = \frac{s}{2\pi}G_2(s)$:

$$S_k(s) = \chi_1(-1)L_\infty(s,\pi_f\times\pi_\chi)\frac{(2k)!(2\pi)^{-2k}}{G_1(2s+4k)}\sum_{l=-k}^{k}\binom{s-3/2-r+2k+l}{k+l}\binom{s-3/2+r+2k-l}{k-l},$$

where the binomial coefficient $\binom{x}{n}$ is defined as usually for $x \in \mathbb{C}$, $n \in \mathbb{Z}$, $n > 0$. The sum over l equals the binomial coefficient $\binom{2s-2+4k}{2k}$, by a standard combinatorics identity[3]. Since we are interested in the central value, we let $s = 1/2$, and use the formula for $G_1(1+4k)$ quoted above. We finally have:

$$S_k(1/2) = \chi_1(-1)2^{2k-1}L_\infty(1/2,\pi_f\times\pi_\chi),$$

whence the factor 2^{1-2k} in the theorem, when $k > 0$. $\square$

5.4 Determining the constant

It remains to find the constant C appearing in Theorem 5.3.9. This has been done by T. Watson in [Wa02, Ch. 3], using an explicit computation of the Shimizu correspondence, and we merely have to translate that result to the present setting.

In order to apply T. Watson's result, we assume henceforth that χ is unramified. Then the function φ' from the previous section can be easily expressed in terms of the Schwartz function used in [Wa02], which we shall denote φ^w. It is convenient to denote by N^- the product of the primes dividing N that are inert in K, that is the discriminant of the quaternion algebra B.

For each finite prime p, the function φ^w_p is a multiple of φ', depending on the measure normalization on $\mathrm{SL}_2(\mathbb{Q}_p)$ used in the integral (5.2.4) to define the form θ_f. Explicitly, recall that the measure on $\mathrm{SL}_2(\mathbb{Q}_p)$ is normalized using the local version of the exact sequence Eq. (5.2.6), and with respect to this measure we have $\mu[K_0(N)_p \cap \mathrm{SL}_2(\mathbb{Q}_p)] = \mu[K_0(N)_p^+]$. We respect to this measure, we have:

$$\varphi^w_p = \frac{1}{\mu[K_0(N)_p^+]}\cdot\begin{cases} \varphi' & \text{if } p \nmid N^-, \\ \frac{p-1}{p+1}\varphi' & \text{if } p|N^-. \end{cases}$$

Note that $\mu[K_0(N)_p^+]$ is either 1 (if $p \nmid N$), or $1/(1+p)$ (if $p|N$).

At infinity, the function φ^w_∞ is given by [recall $\epsilon = i(-1) \in G(\mathbb{R})$]:

$$\varphi^w_\infty = \frac{1}{4^k\mu[\mathrm{SO}_2(\mathbb{R})]}r_B[(\epsilon,1),\epsilon]\varphi'_\infty,$$

---

[3] If $x, y \in \mathbb{C}$, and c is a positive integer, then:

$$\sum_{m+n=c}\binom{x+m}{m}\binom{y+n}{n} = \binom{x+y+c+1}{c}.$$

and the measure on $\mathrm{SO}_2(\mathbb{R})$ has total volume 1. Even though the definition of our measure on $\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})$ differs from that in [Wa02] (there it is a Tamagawa measure), the theta lift $\theta_f(x,y;\varphi^w)$ is the same as in that paper.

Therefore we can use Theorem 1 from section 3.2.2 in [Wa02], which computes the theta lift $\theta_{\overline{f}}$ of the form $\overline{\phi_f}$ in terms of the form ϕ_f^{JL} from Proposition 5.3.6:

$$\theta_{\overline{f}}(x,y;\varphi^w) = \frac{\|\phi_f\|^2}{\|\phi_f^{JL}\|^2}\overline{\phi_f^{JL}}(x)\phi_f^{JL}(y),$$

where the norms are with respect to the Petersson inner product, normalized by using Tamagawa measures on $G(\mathbb{A})$ and $B^\times(\mathbb{A})$, as in [Wa02, §2.2]. This identity is proved by first showing that $\theta_{\overline{f}} = C\overline{\phi_f^{JL}}\phi_f^{JL}$ for a constant C, as in the proof of Proposition 5.3.6. The constant is then computed via the adjoint identity, relating the Petersson inner products of lifts in both directions (again with respect to Tamagawa measures on $G(\mathbb{A})$ and $GSO(B_\mathbb{A})$):

$$(\theta_H, \phi_f)_{G(\mathbb{A})} = (H, \overline{\theta}_{\overline{f}})_{GSO(B_\mathbb{A})},$$

where $H(x,y) = \phi_f^{JL}(x)\overline{\phi_f^{JL}}(y)$ is a form on $GSO(B_\mathbb{A})$, and θ_H denotes the theta lift of H to a form on $G(\mathbb{A})$, with respect to the same Schwartz function φ^w as before. The form θ_H can be computed explicitly by identifying its Whittaker coefficients, and it equals $\|\phi_f^{JL}\|^2\phi_f$. It follows that $C = \|\phi_f\|^2/\|\phi_f^{JL}\|^2$ as desired.

It remains to express the form $\theta_f(x,y;\varphi')$ in terms of the form $\theta_{\overline{f}}(x,y;\varphi^w)$:

$$\begin{aligned}\theta_{\overline{f}}(x,y;\varphi^w) &= \int_{\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})} \theta_B[(x,y), g_1g;\varphi^w]\overline{\phi}_f(g_1g)dg_1\\ &= M'\int_{\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})} \theta_B[(x\epsilon,y), g_1g\epsilon;\varphi']\overline{\phi}_f(g_1g)dg_1\\ &= M'\theta_f(x\epsilon,y;\varphi'),\end{aligned}$$

where $M' = \frac{1}{4^k\mu[K_0(N)^+]}\prod_{p|N^-}\frac{p-1}{p+1}$, and the last equality follows from the fact that $\overline{\phi}_f(g\epsilon) = \phi_f(g)$, as ϕ is the adelization of a newform with real Fourier coefficients. Therefore we have:

$$\theta_f(x,y;\varphi') = \frac{1}{M'}\frac{\|\phi_f\|^2}{\|\phi_f^{JL}\|^2}\overline{\phi_f^{JL}}(x\epsilon)\phi_f^{JL}(y).$$

Theorem 5.3.9 becomes:

THEOREM 5.4.1. *Assuming the character χ is unramified, we have the following explicit formula:*

$$L(1/2,\pi_f\times\pi_\chi) = \frac{\beta}{\sqrt{d_K}}\prod_{p|N^-}\frac{p+1}{p-1}\frac{\|\phi_f\|^2}{\|\phi_f^{JL}\|^2}|l(\phi_f^{JL})|^2,$$

where $\beta = 4$ if $k > 0$ and $\beta = 2$ if $k = 0$. The linear form l has been defined in Theorem 5.3.9, and the form ϕ_f^{JL} is a form of level N and weight $2k$ belonging to the space of π_f^{JL}, as in Proposition 5.3.6.

6. Classical formulation

For applications, it is useful to rewrite the main formula in terms of the classical newform f. We only consider here the case when χ is unramified, and the quaternion algebra B is the matrix algebra, that is when all the primes diving N split in K. This case contains many features of the general case, and it is the case employed in [BD05].

6.1 Geodesic cycles

We start by exhibiting a decomposition of $M_2(\mathbb{Q})$ into quadratic spaces in Eq. (5.1.1) such that the local orders used to define the level of π_f^{JL} are the standard orders of level N in $M_2(\mathbb{Q}_p)$ (see Remark 5.3.3).

To define the embedding $\Psi : K \to M_2(\mathbb{Q})$, choose $a, b, c \in \mathbb{Z}$ such that the following conditions are satisfied:

$$a^2 + bc = d_K, \quad 2N|c, \ 2|b, \ \gcd(a, b/2, c/2) = 1. \tag{6.1.1}$$

Such a choice is possible because all primes dividing N split in K by assumption. We then have an embedding $\Psi : K \to M_2(\mathbb{Q})$ given by

$$\Psi(\sqrt{d_K}) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \mathbf{i},$$

which satisfies $\Psi(K) \cap M_0(N) = \Psi(\mathcal{O}_K)$, where $M_0(N)$ is the set of matrices with integer entries whose lower left entry is divisible by N. By Remark 5.3.3, Ψ extends to a decomposition:

$$M_2(\mathbb{Q}) = K + K\mathbf{j}.$$

as in Eq. (5.1.1) and the orders R_p, $\widetilde{R}_p$ that give the level of ϕ_f^{JL} are the standard orders of level N in $M_2(\mathbb{Q}_p)$.

At the archimedean place, the embedding $\Psi \otimes \mathbb{R}$ is not the diagonal embedding considered in (5.3.2), rather it is the conjugate of that one by the matrix[4] $\gamma_\infty = \begin{pmatrix} a + \sqrt{d_K} & a - \sqrt{d_K} \\ c & c \end{pmatrix}$ (see Remark 5.3.1). Without loss of generality, we can assume $c > 0$, so that γ_∞ has positive determinant.

Note that the form ϕ_f^{JL} in Theorem 5.4.1 can be taken to be ϕ_f, hence the Petersson norms cancel out in the statement of the theorem, since both norms are computed with respect to Tamagawa measures. The task at hand is to rewrite the integral:

$$l(\phi_f) = \int_{\mathbb{A}^\times K^\times \backslash \mathbb{A}_K^\times} \phi_f(\Psi_{\mathbb{A}}(x)\gamma_\infty)\chi^{-1}(x)dx, \tag{6.1.2}$$

in terms of the modular form f. In order to avoid confusion, we denote by $\Psi_{\mathbb{A}}, \Psi_{\text{fin}}, \Psi_\infty$ the embeddings obtained from Ψ by tensoring with $\mathbb{A}$, with $\mathbb{A}_{\text{fin}}$, and with $\mathbb{R}$ respectively.

Let H_K denote the narrow class group of K, that is the group of integral ideals modulo principal ideals having a totally positive generator. The measure used in Eq. (6.1.2) has been normalized using the decomposition (4.1.2), and, since $\Psi_{\mathbb{A}}(\widehat{\mathcal{O}}_K^\times) \subset K_0(N)$ by the choice of embedding, the integral (6.1.2) becomes:

$$l(\phi_f) = \sum_{\mathfrak{a} \in H_K} \chi^{-1}(\mathfrak{a}) \int_{\epsilon_K^{\mathbb{Z}} \backslash \mathbb{R}_+^\times} \phi_f[\Psi_{\text{fin}}(\mathfrak{a})\Psi_\infty(t, t^{-1})\gamma_\infty] d^\times t,$$

where $\mathfrak{a}$ runs through a set of idele representatives for H_K. Recall that ϵ_K is the smallest totally positive power of the fundamental unit of K.

Fix a choice of $i = \sqrt{-1}$. In terms of the decomposition:

$$\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q})Z(\mathbb{A})K_0(N)\mathrm{GL}_2^+(\mathbb{R}), \tag{6.1.3}$$

the form ϕ_f is related to the classical modular form f as follows:

$$\phi_f(g) = 2f(g_\infty i)j(g_\infty, i)^{-2k}, \text{ for } g = \gamma z k g_\infty, \tag{6.1.4}$$

[4] Strictly speaking, we need a matrix γ_∞ that also conjugates $\mathbf{j}$ to the matrix j_∞ in (5.3.2), but it is easy to see that such a matrix would equal the chosen γ_∞, up to a scaling of its rows. It is easy to check that such a scaling has no effect on the computations below.

where $j(A,z) = (cz+d)(\det A)^{-1/2}$ is the automorphy factor for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$. The factor of 2 is due to the fact that f is normalized such that its first nonzero Fourier coefficient is 1 (if $k > 0$), while the first Fourier coefficient of ϕ_f is 2, due to the normalization of the Whittaker newforms. If f is a weight zero Maass form, we use this formula to normalize it.

We change variables $z = \Psi_\infty(t, t^{-1})\gamma_\infty i = \gamma_\infty i t^2$ in the integrals above. As t varies in the interval $[1, \epsilon_K]$, the complex number z varies between $z_\Psi = \gamma_\infty i$ and $\Psi(\epsilon_K) z_\Psi$, along the semicircle on the upper half-plane, connecting the real numbers $\frac{a \pm \sqrt{d_K}}{c}$. We call this path the *geodesic cycle* corresponding to the embedding Ψ. The terminology is justified by the fact that $\Psi(\epsilon_K) \in \Gamma_0(N)$, so the endpoints of the path become identified on the modular curve $\mathcal{H}/\Gamma_0(N)$.

In terms of the new variable z we have:

$$d^\times t = \frac{\sqrt{d_K}}{-cz^2 + 2az + b} dz \tag{6.1.5}$$

$$j(\Psi_\infty(t, t^{-1})\gamma_\infty, i)^2 = \frac{2\sqrt{d_K} i}{-cz^2 + 2az + b}. \tag{6.1.6}$$

Note that $2d^\times t$ is the hyperbolic differential along the semicircle parameterized by $z = \gamma_\infty i t^2$. Let $g_\mathfrak{a} \in \mathrm{GL}_2^+(\mathbb{R})$ be the *inverse* of the archimedean component of $\Psi_{\mathrm{fin}}(\mathfrak{a})$ with respect to the decomposition (6.1.3) (the archimedean component is not unique, but all choices give the same value for the integrals below). Using the identity $j(AB, z) = j(A, Bz) j(B, z)$, after the change of variables the integral becomes:

$$l(\phi_f) = \sqrt{d_K}^{1-k} i^{-k} \sum_{\mathfrak{a} \in H_K} \chi^{-1}(\mathfrak{a}) \int_{z_\Psi}^{\Psi(\epsilon_K) z_\Psi} f[g_\mathfrak{a}^{-1} z] j(g_\mathfrak{a}, g_\mathfrak{a}^{-1} z)^{2k} Q_\Psi(z, 1)^{k-1} dz, \tag{6.1.7}$$

where Q_Ψ is the binary quadratic form of discriminant d_K associated to the embedding Ψ as follows:

$$Q_\Psi(x, y) = -\frac{c}{2} x^2 + axy + \frac{b}{2} y^2.$$

Note that the variable z in the integral above describes a portion of the geodesic semicircle connecting the roots $\frac{a \pm \sqrt{d_K}}{c}$ of the quadratic polynomial $Q_\psi(z, 1)$.

The group $\mathrm{GL}_2^+(\mathbb{R})$ acts on the right on quadratic forms in the usual way:

$$(Q \cdot g)(x, y) := \frac{1}{\det g} Q(Ax + By, Cx + Dy), \quad \text{for } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}).$$

After a change of coordinates $z = g_\mathfrak{a} z'$, the integral in Eq. (6.1.7) corresponding to the idele $\mathfrak{a}$, which we denote $M(\mathfrak{a})$, becomes:

$$M(\mathfrak{a}) = \int_{z_\mathfrak{a}}^{(g_\mathfrak{a}^{-1} \Psi g_\mathfrak{a})(\epsilon_K) z_\mathfrak{a}} f(z') [(Q_\Psi \cdot g_\mathfrak{a})(z', 1)]^{k-1} dz', \tag{6.1.8}$$

where $z_\mathfrak{a} = g_\mathfrak{a}^{-1} z_\Psi$, and the integral is over part of the geodesic semicircle connecting the two real roots of the quadratic polynomial $(Q_\psi \cdot g_\mathfrak{a})(z, 1)$.

Next we choose a system of representatives for the narrow class field H_K that will alow us to compute the matrices $g_\mathfrak{a}$ explicitly. Let $\mathfrak{p}_1 = 1, \mathfrak{p}_2, \ldots, \mathfrak{p}_h$ be a set of ideal representatives for Hilbert class group of K, such that $\mathfrak{p}_i$, $2 \leqslant i \leqslant h$, are prime ideals that split in K, dividing primes $p_i \in \mathbb{Z}$ which are coprime to $2Nc$. Then a set of ideal representatives for the narrow class group H_K is:

$$S_K = \begin{cases} \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_h\} & \text{if } h_K = h, \\ \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_h\} \cup \{\sqrt{d_K}\mathfrak{p}_1, \sqrt{d_K}\mathfrak{p}_2, \ldots, \sqrt{d_K}\mathfrak{p}_h\} & \text{if } h_K = 2h\ . \end{cases}$$

For an ideal α of K, denote by $\widehat{\alpha}$ the corresponding finite idele (which is well defined modulo $\widehat{\mathcal{O}_K}$). We shall take as idele class representatives $\mathfrak{a}$ in Eq. (6.1.7) the ideles $\widehat{\alpha}$, for $\alpha \in S_K$.

In order to compute the matrices $g_{\mathfrak{a}}$, for each $2 \leqslant i \leqslant h$, let $a_i \in \mathbb{Z}$ such that:

$$a_i \equiv a \pmod{c}, \quad a_i^2 \equiv d_K \pmod{p_i}. \tag{6.1.9}$$

To make the notation uniform, set $a_1 = a$, $p_1 = 1$. We claim that for one of the two choices of a_i (mod d_K) satisfying (6.1.9), we have (this holds trivially for $i = 1$ as well):

$$\mathfrak{p}_i = [(a_i + \sqrt{d_K})/2, p_i], \tag{6.1.10}$$

where $[u, v]$ denotes the oriented ideal generated over $\mathbb{Z}$ by $u, v \in \mathcal{O}_K$. Indeed, the ideal $p_i\mathcal{O}_K$ factorizes as follows:

$$p_i\mathcal{O}_K = [(a_i + \sqrt{d_K})/2, p_i][p_i, (a_i - \sqrt{d_K})/2],$$

since both sides have norm p_i^2, and the congruences (6.1.9) ensure that the right hand side is contained in the left hand side (using the fact that c is even, so a, a_i, d_K have the same parity).

For $1 \leqslant i \leqslant h$, define the matrices:

$$\gamma_i = \begin{pmatrix} p_i & (a - a_i)/c \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{Z}). \tag{6.1.11}$$

Since $\mathfrak{p}_i$ is split, we have:

$$\Psi_{\text{fin}}(\widehat{\mathfrak{p}_i}) = \begin{pmatrix} \dfrac{p_i + 1}{2} + \dfrac{p_i - 1}{2\sqrt{d_K}}a & \dfrac{p_i - 1}{2\sqrt{d_K}}b \\ \dfrac{p_i - 1}{2\sqrt{d_K}}c & \dfrac{p_i + 1}{2} - \dfrac{p_i - 1}{2\sqrt{d_K}}a \end{pmatrix}_{p_i},$$

where $\sqrt{d_K}$ is viewed as a unit of $\mathbb{Q}_{p_i} = K_{\mathfrak{p}_i}$ via a fixed embedding $K \to \mathbb{Q}_{p_i}$. The index p_i indicates that the matrix is to be viewed as belonging to $\mathrm{GL}_2(\mathbb{Q}_{p_i}) \subset \mathrm{GL}_2(\mathbb{A})$. With γ_i defined as above, the congruence $a_i^2 \equiv d_K \pmod{p_i}$ implies that

$$\Psi_{\text{fin}}(\widehat{\mathfrak{p}_i}) = (\gamma_i)_{p_i} k, \text{ with } k \in K_0(N)_{p_i}.$$

Since $\gamma_i \in M_2(\mathbb{Q})$, the matrix $\Psi_{\text{fin}}(\widehat{\mathfrak{p}_i})$ factors as follows with respect to the decomposition (6.1.3):

$$\Psi_{\text{fin}}(\widehat{\mathfrak{p}_i}) = \gamma_i k'(\gamma_i^{-1})_\infty,$$

where $k' \in K_0(N)$, which means that we can take $\mathfrak{g}_{\mathfrak{a}} = \gamma_i$, for $\mathfrak{a} = \widehat{\mathfrak{p}_i}$.

Similarly, letting $\epsilon = i(-1) \in G(\mathbb{R})$ (see Notation), we can take:

$$g_{\mathfrak{a}} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \gamma_i \epsilon, \quad \text{for } \mathfrak{a} = \widehat{\sqrt{d_K}\mathfrak{p}_i}.$$

As $\mathfrak{a}$ runs through the chosen set of idele representatives for the narrow class group, we shall show that the embeddings $g_{\mathfrak{a}}^{-1}\Psi g_{\mathfrak{a}}$ and the quadratic forms $Q_\Psi \cdot g_{\mathfrak{a}}$ run through a system of representatives for oriented embeddings, and Heegner forms respectively, of level N, modulo the action of $\Gamma_0(N)$. Towards this goal, in the next section we recall the connection between optimal embeddings, Heegner forms, and ideal classes in the narrow class group of K.

6.2 Optimal embeddings and Heegner forms

As in the previous section, let K be a real quadratic field and N an integer coprime with the discriminant d_K of K. We assume that all prime divisors of N split in K, and we fix a choice of square root a_0 of d_K modulo $4N$.

An algebra embedding $\alpha : K \to M_2(\mathbb{Q})$ given by:

$$\alpha(\sqrt{d_K}) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

is called *optimal* of level N if $\alpha(K) \cap M_0(N) = \alpha(\mathcal{O}_K)$, where $M_0(N)$ is the set of matrices with entries in $\mathbb{Z}$ which are upper triangular modulo N. It is easy to check that α is optimal if and only if $a, b, c \in \mathbb{Z}$ satisfy the conditions (6.1.1). Such embeddings exist if and only if all prime divisors of N split in K (otherwise condition (6.1.1) cannot be satisfied), which we are assuming here.

An embedding α is called oriented, with respect to the given choice of $a_0^2 \equiv d_K \pmod{4N}$, if $a \equiv a_0 \pmod{2N}$. The group $\mathrm{GL}_2^+(\mathbb{R})$ acts on embeddings by conjugation, and the subgroup $\Gamma_0(N)$ of matrices of determinant 1 in $M_0(N)^\times$ fixes the set of oriented optimal embeddings of level N.

Next we define *Heegner forms*. A quadratic form with integer coefficients $Q(x, y) = Cx^2 + Axy + By^2$ of discriminant d_K is called Heegner of level N if $N|C$, and $A \equiv a_0 \pmod{2N}$. The group $\mathrm{GL}_2^+(\mathbb{R})$ acts on quadratic forms on the right as described in the last section, and $\Gamma_0(N)$ fixes the set of Heegner forms of level N. The form Q is called *primitive* if its coefficients do not have any nontrivial common divisor. The action of $\Gamma_0(N)$ takes primitive forms to primitive forms, and we assume from now on that all Heegner forms are primitive.

To each embedding $\alpha : K \to M_2(\mathbb{Q})$ with $\alpha(\sqrt{d_K}) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ one can associate a quadratic form Q_α as in the last section:

$$Q_\alpha(x, y) = -\frac{c}{2}x^2 + axy + \frac{b}{2}y^2.$$

This correspondence is compatible with the action of $\mathrm{GL}_2^+(\mathbb{R})$ on both sides, and oriented optimal embeddings of level N correspond bijectively to Heegner forms of level N.

Let us denote by $\mathcal{E}_N$, $\mathcal{F}_N$ the set of oriented optimal embeddings, respectively Heegner forms of level N. Their number, modulo the action of $\Gamma_0(N)$, equals h_K, the cardinality of the narrow class group H_K, independent of N. For $N = 1$, this is just the Gauss correspondence between narrow ideal classes and primitive quadratic forms of discriminant d_K. For an arbitrary N, the connection between Heegner forms of level N and ordinary primitive forms of discriminant d_K is discussed in great detail in [GKZ, §I.1]. A concise account of the facts needed here is given in [Da92], from where we have taken the terminology "Heegner forms."

We first recall the correspondence between optimal embeddings (or Heegner forms), and narrow ideal classes. We denote by $[u, v]$ the oriented ideal of $\mathcal{O}_K$ generated over $\mathbb{Z}$ by u and v, whose norm is $|u\overline{v} - v\overline{u}|/\sqrt{d_K}$. Recall that the ideal $[u, v]$ is called oriented if $u\overline{v} - v\overline{u} > 0$.

PROPOSITION 6.2.1. *There is a bijection:*

$$I : \mathcal{E}_N/\Gamma_0(N) \to H_K$$

that sends an embedding $\alpha \in \mathcal{E}_N$ *with* $\alpha(\sqrt{d_K}) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ *to the class of the oriented ideal*

$$I(\alpha) = \begin{cases} [(a + \sqrt{d_K})/2, c/2] & \text{if } c > 0, \\ [\sqrt{d_K}(a + \sqrt{d_K})/2, \sqrt{d_K}c/2] & \text{if } c < 0. \end{cases}$$

Proof. Let $\mathcal{F}$ be the set of primitive forms of discriminant d_K. The natural map $i : \mathcal{F}_N/\Gamma_0(N) \to \mathcal{F}/\Gamma_0(N)$ is a bijection, by [Da92], Proposition 1.4, while the map $I' : \mathcal{F}/\Gamma_0(N) \to H_K$, acting as in the statement of the proposition on the form $-x^2c/2 + axy + y^2b/2$, is a bijection–see for example [He23, Thm. 153]. The statement now follows from the fact that Heegner forms and optimal embeddings are in one to one correspondence. $\square$

For any matrix $A \in M_2(\mathbb{R})$, let A^* denote the conjugate of A by the matrix $\epsilon = i(-1)$. We extend this definition to embeddings α by defining $\alpha^*(\sqrt{d_K}) = [\alpha(\sqrt{d_K})]^*$. If α is an oriented optimal embedding of level N, so is α^*, and α and α^* are congruent modulo $\Gamma_0(N)$ if and only if $h_K = h$, with h the cardinality of the Hilbert class group of K.

Let Ψ be the embedding fixed in the previous section, and recall that $\Psi(\sqrt{d_K}) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, with $c > 0$. In the previous section we have chosen a system of representatives S_K for the narrow class group of K, in terms of which we have defined matrices $\gamma_i \in M_2(\mathbb{Z})$, for $i = 1, \ldots, h$. Keeping the notations from the end of last section, we have the following explicit description of a distinct set of oriented optimal embeddings of level N.

Proposition 6.2.2. *(i) The map $E : S_K \to \mathcal{E}_N/\Gamma_0(N)$ defined as follows:*

$$E(\mathfrak{p}_i) = \gamma_i^{-1}\Psi\gamma_i$$
$$E(\sqrt{d_K}\mathfrak{p}_i) = E(\mathfrak{p}_i)^* = (\gamma_i^*)^{-1}\Psi^*\gamma_i^* \quad (\text{if } h_K = 2h)$$

is a bijection.

(ii) More precisely, if $I : \mathcal{E}_N/\Gamma_0(N) \to H_K$ is the correspondence defined in the previous proposition, then we have:

$$I \circ E(\alpha) = [I(\Psi)][\alpha] \text{ for all } \alpha \in S_K,$$

where $[\alpha]$ denotes the class of the ideal α in the narrow class group.

Proof. For $1 \leqslant i \leqslant h$, the image of $\sqrt{d_K}$ under the embedding $E(\mathfrak{p}_i)$ defined above is (recall that $a_1 = a$, $p_1 = 1$):

$$E(\mathfrak{p}_i)(\sqrt{d_K}) = \begin{pmatrix} a_i & b_i \\ cp_i & -a_i \end{pmatrix}, \text{ with } b_i = \frac{d_K - a_i^2}{cp_i}.$$

Clearly $2N|cp_i$, and $b_i \in 2\mathbb{Z}$ because of the congruences (6.1.9). Moreover $a_i \equiv a \equiv a_0 \pmod{2N}$, hence the embeddings $E(\mathfrak{p}_i)$, and therefore $E(\sqrt{d_K}\mathfrak{p}_i)$, are oriented optimal embeddings.

To show they are distinct modulo conjugation by $\Gamma_0(N)$, it is enough to prove the second part of the proposition, which would imply that the given embeddings map to distinct ideal classes under the map I of Proposition 6.2.1. We have to show that

$$I \circ E(\mathfrak{p}_i) = [I(\Psi)][\mathfrak{p}_i],$$

which, by the explicit formula (6.1.10), reduces to proving the following identity between oriented ideals (recall also that we have assumed $c > 0$):

$$[(a_i + \sqrt{d_K})/2, cp_i/2] = [(a + \sqrt{d_K})/2, c/2] \cdot [(a_i + \sqrt{d_K})/2, p_i].$$

Indeed, both sides have the same ideal norm, and it can be easily checked, using the congruences (6.1.9), that the ideal on the right side is contained in the ideal on the left side. For example, $p_i(a + \sqrt{d_K}) = p_i(a_i + \sqrt{d_K}) + p_i(a - a_i)$, and since $a \equiv a_i \pmod{c}$, the last term belongs to the ideal on the left side, etc.

Similarly,

$$I \circ E(\sqrt{d_K}\mathfrak{p}_i) = [I(\Psi)][\sqrt{d_K}\mathfrak{p}_i],$$

hence the map $I \circ E : S_K \to H_K$ is simply translation by the ideal class $[I(\Psi)]$, hence it is a bijection. It follows that the map E is a bijection as well. □

6.3 A classical formula

Proposition 6.2.2 shows that, as α runs through the set of representatives S_K for the narrow ideal class group, the embedding $g_{\widehat{\alpha}}^{-1}\Psi g_{\widehat{\alpha}}$ appearing in the integral (6.1.8) runs through the set of representatives $\{E(\alpha) : \alpha \in S_K\}$ for $\mathcal{E}_N/\Gamma_0(N)$. Therefore, formula (6.1.7) becomes:

$$l(\phi_f) = \sqrt{d_K}^{1-k} i^k \cdot \sum_{\alpha \in S_K} \chi^{-1}(\widehat{\alpha}) \int_{z_\alpha}^{E(\alpha)(\epsilon_K) z_\alpha} f(z) Q_{E(\alpha)}(z,1)^{k-1} dz,$$

where the integral is over part of the semicircle in the upper half plane connecting the real roots of $Q_{E(\alpha)}(z,1)$. The matrix $E(\alpha)(\epsilon_K) \in \Gamma_0(N)$ is an automorph of the quadratic form $Q_{E(\alpha)}$, that is a generator (modulo torsion) of the rank 1 subgroup of $\mathrm{SL}_2(\mathbb{Z})$ fixing the quadratic form; denote it by $M_{E(\alpha)}$. Concretely, if $\epsilon_K = m + n\sqrt{d_K}$, with $m, n \in \mathbb{Z}/2$, and $Q_\Psi(x,y) = -x^2c/2 + axy + by^2/2$, then:

$$M_\Psi = \begin{pmatrix} m+na & nb \\ nc & m-na \end{pmatrix}.$$

Note that the previous formula for $l(\phi_f)$ depends on the choice of embedding Ψ used to define the map $E : S_K \to \mathcal{E}_N/\Gamma_0(N)$. However, its absolute value does not, due to the second part of Proposition 6.2.2, and to the fact that χ is unitary. Viewing χ as a character of the narrow class group H_K, and identifying narrow ideal classes with oriented optimal embeddings by Proposition 6.2.1, we arrive at the following formula:

$$|l(\phi_f)|^2 = d_K^{1-k} \left| \sum_{\Psi \in \mathcal{E}_N/\Gamma_0(N)} \chi^{-1}(\Psi) \int_{z_\Psi}^{M_\Psi z_\Psi} f(z) Q_\Psi(z,1)^{k-1} dz \right|^2,$$

where the integral is over part of the semicircle connecting the real roots of Q_Ψ. If f is a holomorphic form of weight $2k > 0$, the integrals above do not depend on the base point z_Ψ, since $M_\Psi \in \Gamma_0(N)$. If f is a weight zero Maass form, the value of the base point becomes important, and it can be extracted from the preceding computation:

$$z_\Psi = \begin{cases} \frac{a+i\sqrt{d_K}}{c} & \text{if } c > 0, \\ \frac{a-i\sqrt{d_K}}{c} & \text{if } c < 0, \end{cases} \quad \text{for } \Psi(\sqrt{d_K}) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

Theorem 5.4.1 becomes in this case:

Theorem 6.3.1. *Let K be a real quadratic field of discriminant d_K, and f a newform of even weight $2k \geqslant 0$, trivial central character, and square free level N, coprime to d_K. Assume that all the primes dividing N split in K. If χ is a character of the narrow class group of K, the central value of the (completed) Rankin L-series $L(s, \pi_f \times \pi_\chi)$ is given by:*

$$L(1/2, \pi_f \times \pi_\chi) = \frac{\beta}{d_K^{k-1/2}} \left| \sum_{\Psi \in \mathcal{E}_N} \chi^{-1}(\Psi) \int_{z_\Psi}^{M_\Psi z_\Psi} f(z) Q_\Psi(z,1)^{k-1} dz \right|^2,$$

where the sum is over oriented optimal embeddings modulo conjugation by $\Gamma_0(N)$. Recall that $\beta = 4$, unless f has weight 0, when $\beta = 2$.

Remark 6.3.2. When f is a Maass form of weight zero, normalized by means of Eq. (6.1.4), the differential $\frac{\sqrt{d_K}dz}{Q_\Psi(z)}$ appearing in Theorem 6.3.1 is the hyperbolic arc length differential over the geodesic arc between z_Ψ and $M_\Psi z_\Psi$.

Remark 6.3.3. When f is a holomorphic form of weight $2k > 0$, the geodesic cycle integrals can be easily computed, being values of the period mapping between f and the weight $2k$ modular symbols: $Q_\Psi^{k-1}\{0, M_\Psi 0\}$. Using the `PeriodMapping` algorithm implemented by W. Stein in MAGMA, and an algorithm implemented by T. Dokchitser [Do02] in PARI for computing the special values, we have checked that the formula in Theorem 6.3.1 is exact up to more than 10 decimal digits for a range of forms f of weight 2, 4, 6 and 8, taking for χ the trivial character.

6.4 Connection with the Birch and Swinnerton-Dyer Conjecture

For a weight 2 newform f, the central value of the *finite* Rankin L-series can be written in terms of the differential $\omega_f = 2\pi i f(z)dz$ on the compactified Riemann surface $X = \overline{\mathcal{H}/\Gamma_0(N)}$:

$$L_{\text{fin}}(1/2, \pi_f \times \pi_\chi) = \frac{1}{\sqrt{d_K}} \left| \int_{\alpha_\chi} \omega_f \right|^2,$$

where $\alpha_\chi \in H_1(X, \mathbb{Z}) \otimes \mathbb{C}$ is the complex valued one-cycle:

$$\alpha_\chi = \sum_{\Psi \in \mathcal{E}_N/\Gamma_0(N)} \chi^{-1}(\Psi)\gamma_\Psi,$$

and where $\gamma_\Psi \in H_1(X, \mathbb{Z})$ is the homology class of the closed geodesic on X obtained by projecting the geodesic arc between z_Ψ and $M_\Psi z_\Psi$.

Assume now that f is the newform associated to an elliptic curve E of conductor N defined over $\mathbb{Q}$. For simplicity, assume E is a strong Weil curve for a modular parameterization $\Pi : X \to E$, and the Manin constant of this modular parameterization is 1,[5] so that the pullback of a Neron differential ω_E is ω_f. If $\chi = 1$, the L-function $L_{\text{fin}}(s, \pi_f \times \pi_1)$ equals the L-function $L(s, E/K)$ of the base change of E to the real quadratic field K. Assuming that the group $E(K)$ is finite, and assuming the Birch and Swinnerton-Dyer conjecture, we compare our formula with that given by the conjecture.

The action of complex conjugation on the cycle $\alpha_1 \in H_1(X, \mathbb{Z})$ is easy to compute, and yields $\alpha_1 \in \{\gamma + \overline{\gamma} : \gamma \in H_1(X, \mathbb{Z})\}$, where the bar denotes complex conjugation. Consider the pushforward $\alpha_K = \Pi_*(\alpha_1) \in H_1(E, \mathbb{Z})$. From the previous observation, it follows that $\alpha_K = m_K \alpha_E$, where $m_K \in \mathbb{Z}$ and α_E is the class of $E(\mathbb{R})$, that is a generator of $\{\gamma + \overline{\gamma} : \gamma \in H_1(E, \mathbb{Z})\}$. By the assumption that the Maning constant is 1, it follows that:

$$L(1/2, E/K) = \frac{\Omega_E^2}{\sqrt{d_K}} m_K^2,$$

where $\Omega_E = \int_{\alpha_E} \omega_E$ is the real period of E.

Comparison with the formula conjectured by Birch and Swinnerton-Dyer yields

$$|m_K| = \frac{\prod_{p|N} c_p}{|E(\mathbb{Q})|} \sqrt{|\text{Sha}(E/K)|},$$

where c_p are the Tamagawa factors for $E/\mathbb{Q}$, and Sha denotes the Tate-Shafarevich group of E. We have used the fact that all the primes dividing N split in K, which implies that the product of the Tamagawa factors for E/K is the square of that for $E/\mathbb{Q}$, and that $E(K) = E(\mathbb{Q})$.

For example, when the conductor N is prime, it is a theorem of Manin that $c_N = |E(\mathbb{Q})|$, and the formula reduces to: $|\text{Sha}(E/K)| = m_K^2$. Since both integers have geometric meaning, this equality suggests a deeper connection between elements of the Tate-Shafarevich group of E over K, and the sum of geodesic cycles α_K attached to K.

6.5 Equidistribution of closed geodesics on the modular curve

As in the imaginary case discussed in [HM04], the formula of Theorem 6.3.1 can be used in conjunction with subconvexity bounds for $L(1/2, \pi \times \pi_\chi)$ when K and the unramified character χ vary, to prove equidistribution results for the closed geodesics appearing in the formula. The following theorem generalizes W. Duke's result on the equidistribution of the cycles γ_a as the discriminant of K goes to infinity [Du88].

[5] This is true for a large class of elliptic curves, and conjectured to be true in general

THEOREM 6.5.1. *Let K run through quadratic fields of narrow class number $h_K \ll d_K^{\delta}$ with $\delta = 1/23042$, and let γ_K be any of the geodesic arcs attached to K. Then γ_K becomes equidistributed on $X_0(N)$, in the sense that for any convex set $\Omega \subset X_0(N)$ with smooth boundary we have:*

$$\frac{|\Omega \cap \gamma_K|}{|\gamma_K|} \to \mu(\Omega),$$

as d_K tends to infinity, where μ is the hyperbolic measure normalized by having total mass 1 and $|\gamma|$ is the hyperbolic length of the curve γ. Assuming Lindelöff's hypothesis instead of the subconvexity result of [HM04], one can replace the exponent δ by $1/4 - \epsilon$, for any $\epsilon > 0$.

REMARK 6.5.2. Since $|\gamma_K| = \ln \epsilon_K$, using Siegel's theorem we can rephrase the theorem as follows: "long" geodesics attached to quadratic fields K become equidistributed *individually* when d_K tends to infinity, where "long" means of length $\gg d_K^{1/2-\delta}$ (or $\gg d_K^{1/4+\epsilon}$ assuming Lindelöff's hypothesis).

REMARK 6.5.3. This is one of the few theorems available in the literature on the equidistribution of *individual* closed geodesics on $X_0(N)$. The assumption on the class number growth seems to be very often satisfied in view of existing heuristics for the growth of class numbers of real quadratic fields [CL82], [Ho84]. Also note that an assumption of this type is necessary, due to the fact that there are individual closed geodesics that do not become equidistributed in the limit, for example the ones attached to the principal ideal class in certain families of real quadratic fields of large class number and small fundamental unit. See [Sa05] for a concrete example and other interesting open problems related to equidistribution of naturally defined sequences of closed geodesics.

Proof. By Weyl's equidistribution criterion, it is enough to check that:

$$\frac{\int_{\gamma_K} \psi(z) ds}{|\gamma_K|} \to 0 \quad \text{as } d_K \to \infty,$$

where ψ runs through a basis of eigenforms for the Laplacian acting on $X_0(N)$ and ds is the hyperbolic arc differential. Let us parameterize the optimal embeddings Ψ and the quadratic forms Q_Ψ by ideal classes $\mathfrak{a} \in H_K$, and let $\gamma_{\mathfrak{a}}$ be the geodesic arc from z_Ψ to $M_\Psi z_\Psi$ for $\Psi = \Psi_{\mathfrak{a}}$. If the geodesic arc γ_K corresponds to the ideal class $\mathfrak{c}$, then by the orthogonality relation for characters of H_K we have:

$$\frac{\int_{\gamma_K} \psi(z) ds}{|\gamma_K|} = \frac{1}{h_K \ln \epsilon_K} \sum_{\chi \in \widehat{H}_K} \sum_{\mathfrak{a} \in H_K} \chi(\mathfrak{a})^{-1} \int_{\gamma_{\mathfrak{a}\mathfrak{c}}} \psi(z) ds. \tag{6.5.1}$$

We show below that subconvexity bounds for L-functions from [HM04] and [DFI2] imply that the interior sums are $\ll d_K^{1/2-\delta'}$ for $\delta' = 1/23041$, and that Lindelöff's hypothesis implies they are $\ll d_K^{1/4+\epsilon}$ for every $\epsilon > 0$. The conclusion then follows from Remark 6.5.2.

We treat the discrete spectrum first. A basis for it consists of:

$$\{f_q(dz) : q|N, d|(N/q), f_q \text{ Maass newform of level } q\},$$

so it is enough to take $\psi(z) = f_q(dz)$. A change of variables $t = dz$, together with the fact that $i(d)\Gamma_0(N)i(d)^{-1} \subset \Gamma_0(q)$ for $d|(N/q)$, shows that (see Remark 6.3.2):

$$\sum_{\Psi \in \mathcal{E}_N} \chi^{-1}(\Psi) \int_{\gamma_\Psi} f_q(dz) \frac{dz}{Q_\Psi(z,1)} = \sum_{\Psi \in \mathcal{E}_q} \chi^{-1}(\Psi) \int_{\gamma_{\Psi_d}} f_q(t) \frac{dt}{Q_{\Psi_d}(t,1)}$$

where $\Psi_d \in \mathcal{E}_q$ is the embedding $i(d)\Psi i(d)^{-1}$. It is easy to see that the map from $\mathcal{E}_N$ to $\mathcal{E}_q$ sending Ψ to Ψ_d maps a system of representatives of $\mathcal{E}_N/\Gamma_0(N)$ to a system of representatives of $\mathcal{E}_q/\Gamma_0(q)$, and that the corresponding map on ideals (cf. Prop. 6.2.1) is multiplication by a fixed ideal of $\mathcal{O}_K$ of norm d dividing $d\mathcal{O}_K$ (recall that N is square free and that all of its divisors split in K). It follows from Theorem 6.3.1 that:

$$\sum_{\Psi\in\mathcal{E}_N}\left|\chi^{-1}(\Psi)\int_{\gamma_\Psi} f_q(dz)ds\right|^2=\frac{\sqrt{d_K}}{2}L(1/2,\pi_{f_q}\times\pi_\chi),$$

and the subconvexity bounds for $L(1/2,\pi_{f_q}\times\pi_\chi)$ due to G. Harcos and P. Michel yield the bound $\ll d_K^{1/2-1/5296}$ for the weighted sum, as in Theorem 6 of [HM04].

The continuous spectrum is spanned by Eisenstein of level N, and a similar discussion to [HM04, §6.4] reduces the problem to bounding the interior sum in (6.5.1) when $\psi(z)=E(z,1/2+it)$ with t real, the standard weight zero Eisenstein series for the full modular group. The Weyl sum for this Eisenstein series when $\chi=1$ has been computed by C.L. Siegel [Si61, Ch. II, §3] (compare the first two equations on p.113 and p.115, modified to take into consideration that we are considering ideal classes in the narrow sense). The computation for general χ is similar, and it yields:

$$\sum_{\mathfrak{a}\in H_K}\chi(\mathfrak{a})^{-1}\int_{\gamma_\mathfrak{a}} E(z,s)ds'=d_K^{s/2}\frac{\Gamma^2(s/2)}{\Gamma(s)}L_K(s,\chi)$$

where $L_K(s,\chi)$ is the zeta function of K twisted by χ. The following subconvexity bound is known on the line $\mathrm{Re}(s)=1/2$ [DFI2, Thm. 2.6]: $L_K(s,\chi)\ll|s|^{10}d_K^{1/4-\delta'}$ for $\delta'=1/23041$, and the conclusion follows. $\square$

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