My area of research is in number theory, more specifically the theory of modular forms and automorphic representations. There are three general problems on which I have worked, which are all connected with the modular group.

I. Central values of automorphic L-series

In [1], which is based on my PhD thesis at Harvard University, under the supervision of Benedict Gross, I have considered the Rankin L-function associated with a modular form of even weight $k \geq 0$ and level N, and with a character χ of the narrow class group H_K of a real quadratic field K. The natural setting for defining and studying this L-function is the theory of automorphic representations: to the modular form f there is an associated adelic representation π_f of $\mathrm{GL}_2(\mathbb{A})/\mathrm{GL}_2(\mathbb{Q})$, with $\mathbb{A}$ the adeles of $\mathbb{Q}$, and the character χ induces a dihedral representation π_χ of $\mathrm{GL}_2(\mathbb{A})/\mathrm{GL}_2(\mathbb{Q})$. To this data, Jaquet and Langlands associate the L-function $L(s, \pi_f \times \pi_\chi)$, which has an Euler product where almost all Euler factors at primes p have degree 4 in p^{-s} and satisfies a functional equation relating its value at s and $1-s$.

The main result of [1] is that under certain assumptions on N and the discriminant d_K of K we have an explicit formula for the central value

$$L(1/2, \pi_f \times \pi_\chi) = \frac{\beta}{d_K^{k-1/2}} \left| \sum_{\mathfrak{a} \in H_K} \chi^{-1}(\mathfrak{a}) \int_{\gamma_\mathfrak{a}} f(z) Q_\mathfrak{a}(z,1)^{k-1} dz \right|^2$$

where the sum is over ideals $\mathfrak{a}$ in the narrow class field H_K, $Q_\mathfrak{a}(x,y)$ is a Heegner binary quadratic form of discriminant d_K associated to $\mathfrak{a}$ by Gauss, and γ_a is a primitive geodesic segment on the hyperbolic geodesic on the upper half plane connecting the real roots of $Q_\mathfrak{a}(x,1)$. The constant $\beta = 4$ unless $k = 0$ (that is f is a Maass form) when $\beta = 2$. The proof uses automorphic representations techniques, such as the Weil representation, the Jacquet-Langlands correspondence and the seesaw identity of Kudla.

This formula has important arithmetic and analytic consequences. On the arithmetic side, it was used by Bertolini and Darmon (Annals of Math., 2009) to define a two variable p-adic L-function interpolating the central values of the automorphic L-function $L(s, \pi_f \times \pi_\chi)$. On the analytic side, the growth of the central value when the discriminant d_K goes to infinity can be controlled by subconvexity bounds due to Harcos and Michel (Invent. Math. 2006), leading to equidistribution results for the geodesics $\gamma_\mathfrak{a}$ appearing in the formula. In this way we refine a result of Duke (Invent. Math. 1988), by showing that not only all the geodesics $\{\gamma_a\}_{a \in H_K}$ become equidistributed as $d_K \to \infty$, but individual "long" geodesics become equidistributed as well. This result is of interest to people working at the intersection of dynamical systems and number theory, such as in the work of Einsiedler, Lindenstrauss (recent Fields medalist), Michel, Venkatesh (Duke Math. J. 2009, Annals of Math. 2011), who could not show by ergodic methods alone the equidistribution of individual "long" geodesics.

A related problem we solved in [2] is the determination of spherical vectors for representations of $\mathrm{GL}_2(\mathbb{C})$, which was left open by the seminal work of Jacquet and Langlands on Automorphic forms on GL(2). Part of the interest of our approach is that we treat uniformly the determination of both the archimedean and nonarchimedean spherical vectors, following a suggestion of B.H. Gross. The explicit computation of spherical vectors has found applications in recent work on the subject (mentioned in the list of publications).

II. Periods of modular forms

In [4], together with V. Pasol we develop a theory of period polynomials for modular forms for finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$, generalizing what was known for the full modular group by work of Kohnen and Zagier, ("Modular forms with rational periods", 1984). Besides the space of period polynomials for cusp forms, we introduce the space of (extended) period polynomials of Eisenstein series, and show that the (extended) Petersson product on the space of all modular forms can be expressed as a pairing on the space of (extended) period polynomials,. We thus generalize a formula proved for $\mathrm{SL}_2(\mathbb{Z})$ by Haberland (1983) and Kohnen and Zagier (1984), and refined in a previous paper of the candidate [3]. Among the results of [4] we mention:

1. a theory of Hecke operators on the space of period polynomials of arbitrary modular forms,
2. explicit formulas for the Fourier coefficients of a general Hecke eigenform in terms of period polynomials,
3. numerical computations of period polynomials,
4. convenient inverses of the Eichler-Shimura map, and
5. the determination of the extra relations satisfied by period polynomials of cusp forms, determined by Kohnen and Zagier for $\mathrm{SL}_2(\mathbb{Z})$.

In [5], together with V. Pasol we give an algebraic proof of a relation satisfied by the Hecke operators acting on period polynomials, which comes from the Hecke equivariance of the pairing on period polynomials that corresponds to the Petersson product on modular forms. Interestingly, as a consequence of this proof we discover two "indefinite theta series" which can be seen as indefinite counterparts of Jacobi's theta series associated to the sum of four squares definite quadratic form. As an elementary application, we obtain that when p is an odd prime, the number of integer solutions of

$$x^2 + z^2 - y^2 - t^2 = p, \text{ with } x, z > |y|, |t|,$$

is exactly $p - 3$. This can be seen as an indefinite analogue of Jacobi's result on the number of ways of representing a positive integer as a sum of four squares.

In [6], together with V. Pasol we give a natural extension of the Petersson product to the space of all modular forms for $\Gamma_1(N)$, and weight $k \geq 2$. We show that this extended Petersson product is nondegenerate when $k \geq 3$, but that it could be degenerate for $k = 2$. This fundamental result has consequences to the theory of period polynomials for all modular forms in [4], generalizing a result of Zagier obtained for $\mathrm{SL}_2(\mathbb{Z})$ (J. Fac. Sci. Univ. Tokyo 1981).

During work on these papers, I benefitted from conversations with Don Zagier while visiting the Max Planck Institute in Bonn (in 2008, 2010, 2012, 2013). I am currently working on two papers in collaboration with Don Zagier:

1. An elementary proof of the Eichler-Selberg trace formula for traces of Hecke operators on modular forms for the full modular group. The proof is based on the theory of period polynomials and it goes back to an ideea of Don from 1992, which we make more precise in our joint work.

2. A generalization of the previous approach to give trace formulas for a wide class of double coset operators acting on modular forms for finite index subgroups, based on our generalization of the theory of period polynomials in [4]. For $\Gamma_0(N)$ these trace formulas are the simplest available in the literature for the trace of Hecke and Atkin-Lehner operators, and we make no restrictive assumption on the index of the Hecke and Atkin-Lehner operators.

III. Pair correlation of hyperbolic lattice angles

A problem of great interest in number theory is the statistical behaviour of various sequences, often associated with geometric problems. A striking conjecture of Montgomery (Proc. Symp. Pure Math. 1973), proved partially by Rudnick and Sarnak (Duke Math. J. 1996), is that the zeroes of the Riemann zeta function on the critical line have the same local spacing statistics as that of the Gaussian Unitary Ensemble of random matrix theory. The local spacings of a sequence of numbers–their consecutive spacings (or gap distribution) and n-level correlations–are finer measures of how regular their distribution is in small intervals, than simply knowing that they are uniformly distributed. Unlike uniform distribution, which can be established by well-known techniques such as the Weyl criterion, finding the local spacings of naturally occuring sequences is a difficult problem, and the techniques used vary widely. For example, Elkies and McMullen (Duke Math. J. 2004) compute the gap distribution of the fractional parts of $\{\sqrt{n}\}$ mod 1, using ergodic methods: Ratner's theorem on unipotent flows. Rudnick, Sarnak and Zaharescu (Invent. Math. 2001) consider the fractional parts of $n^2\alpha$ for α irrational satisfying a certain diophantine condition. They conjecture that this sequence exhibits Poissonian local spacings, and partial results are established in this direction using point counting techniques, and the Riemann Hypothesis for curves over finite fields proved by Weil.

In the papers [7], [8] we are interested in the local spacings of angles in hyperbolic lattices. The group $\mathrm{PSL}_2(\mathbb{R})$ acts on the complex upper half plane by fractional linear transformations, and we consider the lattice formed by the hyperbolic geodesics between a fixed point ω and its translates by elements of a discrete, finite covolume subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$. More precisley let M_Q be the set of matrices γ such that the point $\gamma\omega$ is in a ball of radius Q centered at ω, and let $N_Q = \#M_Q$. The angles (with multiplicities) θ_γ made by the geodesics $\omega \to \gamma\omega$ with the vertical through ω have been known to be equidistributed in the interval $[-\pi, \pi]$ since the work of Selberg on the Trace Formula. We consider the limit as Q tends to infinity of the pair correlation measure

$$R_Q(\xi) = \frac{\#\{(\gamma, \gamma') \in M_Q^2 : \gamma \neq \gamma', |\theta_\gamma - \theta'_\gamma| < \xi/N_Q\}}{N_Q}$$

which we denote by $R_2(\xi)$, and the pair correlation function $g_2(\xi) = R_2'(\xi)$.

In [7] we consider the case $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and $\omega = i$, and we show that $R_2(\xi), g_2(\xi)$ exist, and we compute them explicitly. These are exactly the angles made by the reciprocal geodesics on the modular surface studied by Sarnak (1995). This is the first result on the pair correlation of hyperbolic angles available in the literature. Our method is based on couting lattice points in planar regions, and makes use of analytic bounds on Kloosterman sums.

Building on the methods of [7], in [8] we give a different formula for $g_2(\xi)$, which we conjecture that it holds for every lattice Γ in $\mathrm{PSL}_2(\mathbb{R})$ and point ω on the upper half plane. We prove this formula for Γ the full modular group and ω an elliptic point.

Instead of counting pairs (γ, γ') in the definition of the pair correlation measure, our approach is based on fixing a matrix M, counting pairs $(\gamma, \gamma M)$ and summing over M. Thus the formula for $g_2(\xi)$ takes the shape of a infinite series over $M \in \Gamma$. The same approach could be succesful in the pair correlation problem for other groups as well.

Our results have been received with considerable interest. The preprint [7] has been already cited twice (see the list of publications), and a proof of the conjecture in [8] has been proposed in a very recent preprint by Kelmer and Kontorovich (arXiv:1308.0754), using spectral methods.

References

[1] A.A. Popa, *Central values of Rankin L-series over real quadratic fields.* Compositio Math. 142 (2006), 811-866

[2] A.A. Popa, *Whittaker newforms for archimedean representations of GL(2).* J. of Number Theory 128/6 (2008), 1637-1645

[3] A.A. Popa, *Rational decomposition of modular forms.* Ramanujan J. of Math. 26/3 (2011), 419-435

[4] V. Pasol, A.A. Popa, *Modular forms and period polynomials*. Proc. London Math. Soc., Online first February 2013, DOI 10.1112/plms/pdt003

[5] V. Pasol, A.A. Popa, *An algebraic property of Hecke operators and two indefinite theta series*. Forum Math., Online first February 2013, DOI 10.1515/forum-2012-0114

[6] V. Pasol, A.A. Popa, *On the Petersson scalar product of arbitrary modular forms*. Proc. Amer. Math. Soc., to appear (arXiv:1204.0502)

[7] F.P. Boca, V. Pasol, A.A. Popa, A. Zaharescu, *Pair correlation of angles between reciprocal geodesics on the modular surface*. Preprint (2011), arXiv:1102.0328

[8] F.P. Boca, A.A. Popa, A. Zaharescu, *Pair correlation of hyperbolic lattice angles*. Preprint (2013), arXiv:1302.5067