PAIR CORRELATION OF HYPERBOLIC LATTICE ANGLES

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Abstract. Let ω be a point in the upper half plane, and let Γ be a discrete, finite covolume subgroup of $\mathrm{PSL}_2(\mathbb{R})$. We conjecture an explicit formula for the pair correlation of the angles between geodesic rays of the lattice $\Gamma\omega$, intersected with increasingly large balls centered at ω. We prove this conjecture for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and ω an elliptic point.

Contents

1. Introduction

The statistics of spacings measure the fine structure of sequences of real numbers, going beyond the classical Weyl uniform distribution. Originating in work of physicists on random matrices [20, 8], spacing statistics are conveniently expressed as the convergence of certain measures, called level correlations, and respectively level spacing measures. In the past decades these notions have received significant attention in many areas of mathematical physics, analysis, probability, and number theory. For most sequences of interest it is usually very challenging to prove the existence and describe the limiting spacing measures, such as the gap distribution or pair correlation, even when their existence is experimentally predicted.

One class of interesting sequences studied in recent years arises from the angular distribution of lattice points. In the Euclidean scenery one such question is: for a given point $\alpha \in \mathbb{R}^2$, describe

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the statistics of the increasing sequence of finite sets

$$\left\{ \frac{\mathbf{m}+\alpha}{|\mathbf{m}+\alpha|} : \mathbf{m} \in \mathbb{Z}^2 \setminus \{-\alpha\}, |\mathbf{m}+\alpha| < R \right\} \subset S^1, \quad \text{with } R \to \infty,$$

representing the directions of points in the affine lattice $\alpha + \mathbb{Z}^2$ with observer located at the origin. When $\alpha \notin \mathbb{Q}^2$ the gap distribution of this sequence was proved by Marklof and Strömbergsson [15] to coincide with the gap distribution of the sequence ($\sqrt{n}$ mod 1), previously computed by Elkies and McMullen [9]. In this situation we are not aware of any result concerning the pair correlation for an individual α. When $\alpha \in \mathbb{Z}^2$ the limiting gap distribution was computed in [2]. A major difference between the two situations is that the gap distribution density, always supported in $[0, \infty)$, vanishes on $\big[0, \frac{1}{3\pi^2}\big]$ when $\alpha \in \mathbb{Z}^2$ and is equal to $\frac{6}{\pi^2}$ on $[0, \frac{1}{2}]$ when $\alpha \notin \mathbb{Q}^2$. This kind of repulsion can be used to explicitly compute the pair correlation when $\alpha \in \mathbb{Z}^2$. When the disk is replaced by the square $[-R, R]^2$ and one only considers primitive lattice points, the pair correlation was computed in [4].

In the hyperbolic situation, the lattice $\mathbb{Z}^2$ is replaced with a lattice (discrete subgroup of finite covolume) Γ in $\mathrm{PSL}_2(\mathbb{R})$. We consider the angles between geodesic rays ($\omega \to \gamma\omega$) in the upper half plane $\mathbb{H}$, connecting a fixed point $\omega \in \mathbb{H}$ with the (finitely many) points $\gamma\omega$ in its Γ-orbit, lying in increasingly large hyperbolic balls. These angles are well-known to be uniformly distributed (see, e.g., [16]) and their uniform distribution in angular sectors can be made effective [1, 6, 7, 11, 12, 13, 17, 18].

A first step in the study of the pair correlation of directions of hyperbolic lattice points was completed in [5], where we treated the case $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and $\omega = i$, establishing a formula for the pair correlation density $g_2(\xi)$ that involves two terms. The first term is a series over the set of matrices M with nonnegative entries of an explicit function of ξ depending only on the Hilbert-Schmidt norm of M, while the second term is a finite sum involving volumes of bodies defined in terms of the triangle transformation introduced in [3]. In this paper we extend the approach introduced in [5], conjecturing an explicit formula for the pair correlation density $g_2(\xi)$ for arbitrary Γ and ω, and we prove the formula for the modular group and ω an elliptic point. Remarkably, we find that $g_2(\xi)$ equals the diagonal value at ω of an explicit automorphic kernel.

To state the results we introduce some notation and definitions. Let $\omega = u + iv \in \mathbb{H}$ and let Γ be a discrete, finite covolume subgroup of $\mathrm{PSL}_2(\mathbb{R})$. For $\gamma \in \Gamma$, define $\|\gamma\| := v\sqrt{2\cosh d(\omega, \gamma\omega)}$, where $d(z_1, z_2)$ denotes the hyperbolic distance between two points $z_1, z_2 \in \mathbb{H}$. Let B_Q^{tot} be the number of matrices $\gamma \in \Gamma$ in the ball $\|\gamma\| \leqslant Q$, so that asymptotically $B_Q^{\mathrm{tot}} \sim \frac{3}{v^2} Q^2$. We are interested in the pair correlation density function

$$g_2(\xi) = \frac{dR_2(\xi)}{d\xi}, \text{ where } R_2(\xi) = \lim_{Q\to\infty} \frac{\mathcal{R}_Q(\xi)}{B_Q^{\mathrm{tot}}}, \text{ and}$$

$$\mathcal{R}_Q(\xi) = \#\left\{ (\gamma, \gamma') \in \Gamma^2 : \gamma' \neq \gamma, \|\gamma\| \leqslant Q, \|\gamma'\| \leqslant Q \ , \ 0 \leqslant \frac{1}{2\pi}\big(\theta_{\gamma'} - \theta_\gamma\big) \leqslant \frac{\xi}{B_Q^{\mathrm{tot}}} \right\}.$$

The approach and computations from this paper provide evidence toward the following:

Conjecture 1. *Let Γ be a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ with fundamental domain of finite area V_Γ. The pair correlation measure $R_2(\xi)$ exists on $[0,\infty)$, and is given by a C^1 function expressed as a series of three dimensional volumes. Its density g_2 is given by the formula*

$$g_2\left(\frac{\xi}{V_\Gamma}\right) = \frac{V_\Gamma}{\pi\xi^2} \sum_{M\in\Gamma} f_\xi\big(\ell(M)\big), \tag{1.1}$$

where $\ell(M) = d(\omega, M\omega)$ *and* $f_\xi(\ell)$ *is the continuous function defined for* $\ell \geqslant 0$ *and* $\xi > 0$ *by*

$$f_\xi(\ell) = \begin{cases} \ln\left(\frac{\cosh\ell+\sinh\ell}{\cosh\ell+\sqrt{\sinh^2\ell-\xi^2}}\right) & \text{if } \xi \leqslant 2\sinh\left(\frac{\ell}{2}\right), \\ \ln\left(\frac{(\cosh\ell+\sinh\ell)(1+\xi^2)}{(\cosh\ell+\sqrt{\sinh^2\ell-\xi^2})^2}\right) & \text{if } 2\sinh\left(\frac{\ell}{2}\right) \leqslant \xi \leqslant \sinh\ell, \\ \ln\left(\cosh\ell+\sinh\ell\right) = \ell & \text{if } \sinh\ell \leqslant \xi. \end{cases} \tag{1.2}$$

Since the series above is absolutely convergent, by l'Hospital we also deduce the conjectural formula:

$$g_2(0) = \frac{V_\Gamma}{\pi} \sum_{M\in\Gamma, \ell(M)>0} \frac{1}{e^{2\ell(M)}-1}. \tag{1.3}$$

For the elliptic points for $\mathrm{PSL}_2(\mathbb{Z})$ we prove the conjecture, using extra symmetries of the hyperbolic lattices centered at i and $\rho = e^{\pi i/3}$.

Theorem 1. *Conjecture 1 and formula* (1.3) *hold for* $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ *and for* ω *one of the elliptic points* i *or* ρ *(with* $V_\Gamma = \pi/3$*).*

The conjecture has also been verified numerically for a few congruence subgroups $\Gamma_0(N)$ and a few points ω. In Figure 1, we compare the pair correlation function given by (1.1) with the actual pair correlation function computed by counting the pairs in the definition, for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and for a few choices of ω. To count the pairs (γ, γ') in the definition of $\mathcal{R}_Q(\xi)$, we first reduce to a half ball $|\gamma\omega|, |\gamma'\omega| \leqslant k$ as explained in Section 3.

When $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and ω is one of the elliptic points i or respectively ρ, each angle in the definition of $\mathcal{R}_Q(\xi)$ is counted a number of times equal to the order of the stabilizer of ω in Γ, namely 2 or respectively 3 times. Therefore it is more natural to consider the pair correlation measure $\mathcal{R}_Q^{\mathrm{el}}$ defined as $\mathcal{R}_Q$ with the condition $\gamma \neq \gamma'$ replaced by $\gamma\omega \neq \gamma'\omega$. Denoting by g_2^{el} the corresponding pair correlation functions, we have $g_2^{\mathrm{el}}(\xi) = g_2(e_\omega \xi)$ where e_ω is the cardinality of the stabilizer of ω, so that $g_2^{\mathrm{el}} = g_2$ if ω is not an elliptic point. For $\omega = i$, the function g_2^{el} is identical with the pair correlation function found in [5], but the formula here is entirely explicit for all ξ.

Formula (1.1) relates the pair correlation of hyperbolic lattice angles with the length spectrum of the lattice. For example, the spikes in the graphs in Figure 1 occur at values of ξ related in a straightforward way to the length spectrum. Assuming that lattices centered at different points in a half[1] fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})$ have different length spectra, it would follow that the distribution of lattice angles determines the point ω in a half fundamental domain.

A common feature of pair correlation density functions, encountered also for the pair correlation of Farey fractions [4], is that they tend to one at infinity. We expect that the same is true for the function in Conjecture 1, namely that if Γ is a Fuchsian group of the first kind with fundamental domain of finite hyperbolic area V_Γ, then

$$\lim_{\xi\to\infty} \frac{V_\Gamma}{\pi\xi^2} \sum_{M\in\Gamma} f_\xi\big(d(\omega, \gamma\omega)\big) = 1, \tag{1.4}$$

where the function $f_\xi(\ell)$ is defined in Conjecture 1. The standard way of proving an asymptotic formula such as (1.4) is by using the spectral decomposition of the automorphic kernel $K_\xi(z,w) = \sum_{\gamma\in\Gamma} k_\xi(u(\gamma z, w))$, where $k_\xi(u(z,w)) = f_\xi(d(z,w))$, and we attempt to spectrally decompose the kernel K_ξ in Appendix B. However, we encounter technical problems due to the fact that the kernel

[1]It is shown in Section 3 that the pair correlation functions for the lattices centered at ω and $-\overline{\omega}$ are equal.

$k_\xi(u)$ is not smooth at two points, and as a consequence we can only prove that its Selberg-Harish-Chandra transform $h_\xi(t)$ satisfies $h_\xi(t) \ll (1+|t|)^{-2}$ for real t. This is an epsilon short of the decay $(1+|t|)^{-2-\epsilon}$ for some $\epsilon > 0$, required for absolute convergence in the spectral decomposition of K_ξ, and therefore we leave the proof of (1.4) for later work.

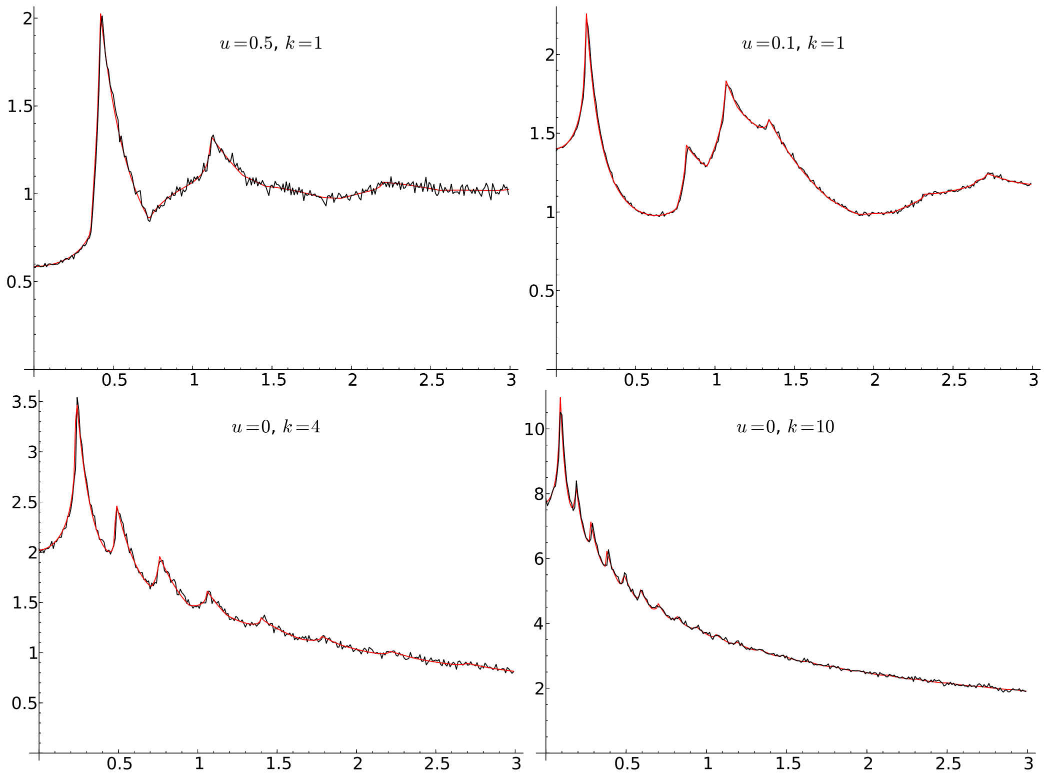

FIGURE 1. The pair correlation functions g_2^{el} for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and $\mathrm{Re}(\omega) = u$, $|\omega| = k$ computed using (1.1) (smooth line) and by counting pairs in the definition (jagged line) with $Q = 1000$ in the first two plots, $Q = 2000$ in the third, and $Q = 5000$ in the fourth.

For $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and $\omega = i$, the pair correlation function $g_2^{\mathrm{el}}(\xi)$ is the same as that of angles made by reciprocal geodesics on the modular surface, namely the closed geodesics passing through the projection of i on the modular surface. Reciprocal geodesics were first studied by Fricke and Klein [10], and more recently by Sarnak [19]. In Appendix A we similarly describe the arithmetic and geometry of closed geodesics passing through the projection of ρ on the modular surface. While reciprocal geodesics always consist of two loops, one tracing the other in the opposite direction, we show that a geodesic on the modular surface passing through the image of ρ consists of one, two, or four closed loops. The precise situation depends on the arithmetic properties of the discriminant attached to the geodesic.

We now sketch the main steps in the proof of Theorem 1, while also describing the organization of the paper. In the remaining of the introduction and throughout the paper we take $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, and keep ω mostly arbitrary. For technical reasons, we assume that $\mathrm{Re}(\omega)$ and $|\omega|^2$ are rational. An important role is played by the set $\mathfrak{S}$ of matrices with nonnegative entries, distinct from the identity.

Step 1. As in [5], our approach is based on computing the pair correlation of the quantities $\Psi(\gamma) = u + v\tan\left(\frac{\theta_\gamma}{2}\right)$ by first approximating them with $\Phi(\gamma) = \mathrm{Re}(\gamma\omega)$. The reason for preferring the function $\Phi(\gamma)$ is explained by Lemma 1, where we show that $\Phi(\gamma) - \Phi(\gamma M) = \Xi_M(c,d)$ with a function Ξ_M depending only on the lower row (c,d) of γ. In Section 3 we reduce to angles in the hyperbolic half balls for which $|\gamma\omega| < |\omega|$, and we show in Section 4 that the sets $\{\Psi(\gamma)\}$ and $\{\Phi(\gamma)\}$ have the same pair correlation.

Step 2. To compute the pair correlation of $\{\Phi(\gamma)\}$, we estimate the number $\mathcal{R}_Q^\Phi(\xi)$ of pairs $(\gamma,\gamma') \in \Gamma^2$ with $|\gamma\omega|, |\gamma'\omega| < k$, $\|\gamma\|, \|\gamma'\| \leqslant Q$, and $0 \leqslant \Phi(\gamma) - \Phi(\gamma') < \frac{\xi}{Q^2}$ as follows:

$$\mathcal{R}_Q^\Phi(\xi) = \frac{1}{2} \sum_{M\in\Gamma\setminus\{I\}} \mathcal{N}_{M,Q}(\xi), \tag{1.5}$$

where $\mathcal{N}_{M,Q}(\xi)$ is the cardinality of the set

$$\mathcal{S}_{M,Q}(\xi) := \left\{ \gamma \in \Gamma : |\Phi(\gamma M) - \Phi(\gamma)| \leqslant \frac{\xi}{Q^2},\ |\gamma\omega|, |\gamma M\omega| < k,\ \|\gamma\|, \|\gamma M\| \leqslant Q \right\}. \tag{1.6}$$

Replacing b in terms of a,c,d, for $M \in \mathfrak{S}$ we show that $\mathcal{N}_{M,Q}(\xi)$ is asymptotic as $Q \to \infty$ to the cardinality $\widetilde{\mathcal{N}}_{M,Q}(\xi)$ of the set $\widetilde{\mathcal{S}}_{M,Q}(\xi)$ of integer triples (a,c,d) such that

$$\begin{cases} |a| \leqslant kc \leqslant \widetilde{Q}, \quad |d| \leqslant \widetilde{Q}, \quad ad \equiv 1 \pmod{c}, \quad |\Xi_M(c,d)| \leqslant \frac{\xi}{Q^2}, \\ \max\{c^2k^2 + d^2 + 2cdu, c^2X_M + d^2Y_M + 2cdZ_M\} \leqslant \frac{Q^2c^2}{a^2+k^2c^2-2acu}, \end{cases} \tag{1.7}$$

with $\widetilde{Q} = \frac{Q\sqrt{k}}{v\sqrt{k-|u|}}$ and X_M, Y_M, Z_M defined in (2.1). This approximation holds for fixed $M \in \mathfrak{S}$ with an explicit error term (Lemma 9 (iii)), but in order to control the error when when summing the series (1.5) we need to replace $\widetilde{\mathcal{N}}_{M,Q}(\xi)$ by the cardinality $\widetilde{\mathcal{N}}^+_{M,Q}(\xi)$ of the subset $\widetilde{\mathcal{S}}^+_{M,Q}(\xi) \subset \widetilde{\mathcal{S}}_{M,Q}(\xi)$ consisting of triples (a,c,d) as above with $d > 0$.

Step 3. Using estimates for number of points in hyperbolic regions based on bounds on Kloosterman sums, we show in Lemmas 10 and 11 that for $M \in \mathfrak{S}$, $\widetilde{\mathcal{N}}_{M,Q}(\xi) \sim \frac{Q^2}{\zeta(2)}\mathrm{Vol}(S_{M,\xi})$, and a similar estimate holds also when summing $\widetilde{\mathcal{N}}^+_{M,Q}(\xi)$ over $M \in \mathfrak{S}$. The region $S_{M,\xi}$ consists of triples $(x,y,z) \in \left[0, \frac{1}{v\sqrt{k(k-|u|)}}\right] \times [-\frac{\sqrt{k}}{v\sqrt{k-|u|}}, \frac{\sqrt{k}}{v\sqrt{k-|u|}}] \times [-k,k]$ for which

$$|\Xi_M(x,y)| \leqslant \xi, \quad \max\{x^2k^2 + y^2 + 2xyu, x^2X_M + y^2Y_M + 2xyZ_M\} \leqslant \frac{1}{k^2+z^2-2uz}. \tag{1.8}$$

Step 4. Using extra symmetries of the hyperbolic lattice in the case $\omega = i$ (Section 8) and $\omega = \rho$ (Section 10), we show that the summation range in (1.5) can be reduced to a subset of $\mathfrak{S}$. Moreover, using repulsion arguments involving the Farey tessellation, we can define finite subsets $\widetilde{\mathcal{F}}(\xi) \in \Gamma$ such that for $M \in \mathfrak{S}\setminus\widetilde{\mathcal{F}}(\xi)$ the quantities $\widetilde{\mathcal{N}}_{M,Q}(\xi)$ can be expressed only in terms of $\widetilde{\mathcal{N}}^+_{M^\dagger,Q}(\xi)$ for appropriate $M^\dagger \in \mathfrak{S}$. Therefore we place ourselves in the situations analysed in Steps 2 and 3, obtaining

$$\mathcal{R}_Q^\Phi(\xi) \sim \frac{Q^2}{2\zeta(2)} \sum_{M\in\Gamma} \mathrm{Vol}(S_{M,\xi}). \tag{1.9}$$

Step 5. The resulting volumes are expressed in closed form as integrals in Section 9 for ω arbitrary, and passing to the pair correlation function R_2 of the angles $\{\theta_\gamma\}$ we obtain the formula in Conjecture 1, and finish the proof of Theorem 1.

The difficulty in proving Conjecture 1 for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and general ω resides in the estimates of Steps 2 and 3, where we use positivity of some of the entries of the matrices involved to obtain good control on the error in lattice point counting. It would be interesting to find a proof that involves the spectral theory of the hyperbolic Laplacian. For the problem of equidistribution of the geodesic angles, such an approach is provided for example by the estimate for exponential sums in [11, Theorem 4] (restated in a more transparent way in [6, Theorem 1.1]). For the pair correlation problem we would need a suitable extension of these results, instead of the lattice point counting arguments used in this paper. Such an approach could potentially generalize to Fuchsian groups of the first kind.

2. Preliminary computations

To each $\omega = u + iv \in \mathbb{H}$ with $|\omega| = k$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, we associate the quantities

$$\begin{gathered} X = |a\omega + b|^2, \quad Y = |c\omega + d|^2, \quad Z = Y\,\mathrm{Re}(\gamma\omega) = ac|\omega|^2 + bd + u(ad + bc), \\ T = \|\gamma\|^2 := X + |\omega|^2 Y - 2uZ. \end{gathered} \tag{2.1}$$

Setting $\Delta = 2v^2$ and $\epsilon_T = \frac{T - \sqrt{T^2 - \Delta^2}}{\Delta}$, we have

$$XY - Z^2 = v^2, \quad \cosh d(\omega, \gamma\omega) = \frac{T}{\Delta}, \tag{2.2}$$

$$\sin\theta_\gamma = 2v \frac{Z - uY}{\sqrt{T^2 - \Delta^2}}, \quad \cos\theta_\gamma = \frac{\Delta Y - T}{\sqrt{T^2 - \Delta^2}}, \quad \tan\left(\frac{\theta_\gamma}{2}\right) = \frac{1}{v}\frac{Z - uY}{Y - \epsilon_T}. \tag{2.3}$$

The point $\gamma\omega$ is completely determined by the "coordinates" $(X, Y, Z) = (X_\gamma, Y_\gamma, Z_\gamma)$, as its hyperbolic polar coordinates are so determined.

The x-intercept $\Psi(\gamma)$ of the oriented geodesic $\omega \to \gamma\omega$ is given by

$$\Psi(\gamma) = u + v\tan\left(\frac{\theta_\gamma}{2}\right) = \frac{Z_\gamma - u\epsilon_T}{Y_\gamma - \epsilon_T}.$$

Since $\epsilon_T = e^{-d(\omega,\gamma\omega)} \to 0$, a better behaved quantity which approximates $\Psi(\gamma)$ well is

$$\Phi(\gamma) := \frac{Z_\gamma}{Y_\gamma} = \mathrm{Re}(\gamma\omega).$$

For $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$, let $\ell(M) = d(\omega, M\omega)$, and let the angle θ_M be defined as for γ. For $c, d \in \mathbb{R}$, let $c\omega + d = re^{i\theta}$, and define

$$\Xi_M(c, d) := -\frac{v}{r^2} \frac{\sin(\theta_M - 2\theta)}{\coth \ell(M) + \cos(\theta_M - 2\theta)}. \tag{2.4}$$

Lemma 1. *For $M \in \mathrm{SL}_2(\mathbb{R})$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, we have*

$$\Phi(\gamma) - \Phi(\gamma M) = \Xi_M(c, d).$$

Proof. We compute $\mathrm{Re}(\gamma\omega - \gamma M\omega)$ using the KAK decomposition for M and the NAK decomposition for γ, both centered at ω. Denote

$$a(y) = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \in A, \quad n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N, \quad k(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in K$$

and let $\alpha = n(u)a(v)$, so that $\alpha i = \omega$ and α maps the vertical through i onto the vertical through ω. Let $z := \alpha^{-1}\gamma\omega$, so

$$\gamma\omega = \alpha z = u + vz, \quad z = x + iy. \tag{2.5}$$

We have

$$\alpha^{-1}\gamma\alpha = n(x)a(y)k(\theta), \quad \text{with } c\omega + d = re^{i\theta}, r = y^{-1/2}.$$

Let θ'_M be the angle between $i \to i\infty$ and $i \to \alpha^{-1}M\alpha i$. Since α maps $i \to \alpha^{-1}M\alpha i$ onto $\omega \to M\omega$, we have

$$\theta'_M = \pi - \theta_M, \quad d(i, \alpha^{-1}M\alpha i) = \ell(M).$$

Consequently

$$\alpha^{-1}M\alpha = k\big(\theta'_M/2\big)a\big(e^{\ell(M)}\big)k\big(\theta''\big)$$

for some θ'', and taking $\nu = \theta + \frac{1}{2}\theta'_M$ we obtain

$$\alpha^{-1}\gamma M\omega = n(x)a(y)k(\theta + \theta'_M/2)a(e^{\ell(M)})i = x + y\frac{ie^{\ell(M)}\cos\nu - \sin\nu}{ie^{\ell(M)}\sin\nu + \cos\nu}.$$

The equation above and (2.5) imply

$$\operatorname{Re}(\gamma M\omega) = \operatorname{Re}(\gamma\omega) + \frac{v}{r^2}\operatorname{Re}\left(\frac{ie^{\ell(M)}\cos\nu - \sin\nu}{ie^{\ell(M)}\sin\nu + \cos\nu}\right).$$

The real part in the last expression equals the second fraction in (2.4), and the claim follows. □

A direction calculation yields

$$\Xi_M(c,d) = \frac{cd(k^2Y_M - X_M) + c^2(k^2Z_M - uX_M) + d^2(uY_M - Z_M)}{(c^2k^2 + d^2 + 2cdu)(c^2X_M + d^2Y_M + 2cdZ_M)}.$$

3. Reduction to angles with $|\gamma\omega| < |\omega|$

We next show that the pair correlation function is determined only by the angles θ_γ with $|\gamma\omega| < |\omega| = k$, justifying the assumption made in the introduction. When $k \neq 1$, our proof is conditional upon the formula in Conjecture 1.

We denote by $\mathfrak{R}_Q^{\mathrm{tot}}$ the set of $\gamma \in \Gamma$ with $\|\gamma\| < Q$, of cardinality B_Q^{tot}, and let $\mathfrak{R}_Q$, respectively $\mathfrak{R}_Q^{>}$ denote the subsets for which $|\gamma\omega| < k$, respectively $|\gamma\omega| > k$, of cardinalities B_Q, respectively $B_Q^{>}$. It is shown in Lemma 8 that $B_Q \sim B_Q^{>} \sim \frac{3}{\Delta}Q^2$. Let

$$\mathcal{R}_Q^{\mathrm{tot}}(\xi) = \#\left\{(\gamma,\gamma') \in (\mathfrak{R}_Q^{\mathrm{tot}})^2 : \gamma' \neq \gamma,\ 0 \leqslant \frac{1}{2\pi}\big(\theta_{\gamma'} - \theta_\gamma\big) \leqslant \frac{\xi}{B_Q^{\mathrm{tot}}}\right\},$$

$$R_2^{\mathrm{tot}}(\xi) = \lim_{Q\to\infty}\frac{\mathcal{R}_Q^{\mathrm{tot}}(\xi)}{B_Q^{\mathrm{tot}}}, \qquad g_2^{\mathrm{tot}} = \frac{dR_2^{\mathrm{tot}}}{d\xi},$$

and define similarly $R_2^{>}$, $g_2^{>}$. Since $B_Q \sim B_Q^{>} \sim \frac{1}{2}B_Q^{\mathrm{tot}}$, we have that $R_2^{\mathrm{tot}} = \frac{1}{2}(R_2 + R_2^{>})$, $g_2^{\mathrm{tot}} = \frac{1}{2}(g_2 + g_2^{>})$.

Let $s = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Since we will compare the hyperbolic lattices centered at the points ω and $s\omega$, in the following two paragraphs only we attach subscripts to all notation to denote this dependence, eg $\|\gamma\|_\omega$, $(B_Q)_\omega$ etc.

The map $\gamma \mapsto s\gamma s$ induces the mapping $\gamma\omega \mapsto s\gamma\omega$, taking the part of the lattice $\Gamma\omega$ with $|\gamma\omega| > |\omega|$, bijectively and conformally onto the part of the lattice $\Gamma s\omega$ with $|\gamma s\omega| < |s\omega|$. Note that

$$k^2(X_{s\gamma s})_{s\omega} = (X_\gamma)_\omega, \quad k^2(Y_{s\gamma s})_{s\omega} = (Y_\gamma)_\omega, \quad k^2(Z_{s\gamma s})_{s\omega} = -(Z_\gamma)_\omega,$$
$$k^4\|s\gamma s\|_{s\omega} = \|\gamma\|_\omega, \quad (\theta_{s\gamma s})_{s\omega} = -(\theta_\gamma)_\omega,$$

yielding

$$(B_Q^{>})_\omega = (B_{Q/k^2})_{s\omega}, \quad (\mathcal{R}_Q^{>})_\omega = (\mathcal{R}_{Q/k^2})_{s\omega}, \quad R_2^{>}(\xi)_\omega = R_2(\xi)_{s\omega}, \quad g_2^{>}(\xi)_\omega = g_2(\xi)_{s\omega}.$$

We conclude that $g_2(\xi)_\omega^{\mathrm{tot}} = \frac{1}{2}\big(g_2(\xi)_\omega + g_2(\xi)_{s\omega}\big)$.

Assuming now that $g_2(\xi)_\omega$ is given by the series in Conjecture 1, we observe that the application $M \to sMs$ rearranges the terms of the series for $g_2(\xi)_\omega$ into the terms of the series for $g_2(\xi)_{s\omega}$ because the summands only depend on $d(\omega, M\omega)$. Therefore $g_2(\xi)_\omega = g_2(\xi)_{s\omega}$, and hence we have

$$g_2^{\mathrm{tot}}(\xi) = g_2(\xi) = g_2^{>}(\xi)$$

(dropping the subscripts ω since the basepoint of the lattice is fixed).

When $k = 1$, one can see directly that $g_2(\xi) = g_2^{>}(\xi)$, because of an extra symmetry of the hyperbolic lattice. Keeping ω arbitrary, let $\widetilde{\gamma} = \eta\gamma\eta$ for $\eta = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\widetilde{\omega} = \frac{\omega}{k^2}$. Since

$$(X_{\widetilde{\gamma}})_{\widetilde{\omega}} = \frac{(Y_\gamma)_\omega}{k^2}, \quad (Y_{\widetilde{\gamma}})_{\widetilde{\omega}} = \frac{(X_\gamma)_\omega}{k^2}, \quad (Z_{\widetilde{\gamma}})_{\widetilde{\omega}} = \frac{(Z_\gamma)_\omega}{k^2},$$

we have $|\gamma\omega| < k \iff |\widetilde{\gamma}\widetilde{\omega}| > k$ and

$$\big(\Xi_{\widetilde{M}}(y,x)\big)_{\widetilde{\omega}} = -\big(\Xi_M(x,y)\big)_\omega.$$

The angles $(\theta_\gamma)_\omega$ and $(\theta_{\widetilde{\gamma}})_{\widetilde{\omega}}$ are related as in the following lemma, which shows directly that $g_2(\xi) = g_2^{>}(\xi)$ when $|\omega| = 1$ (that is $\widetilde{\omega} = \omega$).

Lemma 2. *Let $\beta \in (0,\pi)$ be the polar angle of $\omega = ke^{i\beta}$. We have the relation:*

$$(\theta_\gamma)_\omega + (\theta_{\widetilde{\gamma}})_{\widetilde{\omega}} = 2\beta, \tag{3.1}$$

namely the angle between the circle $|z| = k$ and $\omega \to \gamma\omega$ is the same as the angle between the circle $|z| = k^{-1}$ and $\widetilde{\omega} \to \widetilde{\gamma}\widetilde{\omega}$.

Proof. When $\omega = ki$ the claim is immediate from (2.3). In general, let $\alpha = \left(\begin{smallmatrix} ak & bk^2 \\ b & ak \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ for $a = \sqrt{\frac{k+v}{2vk^2}}$, $b = \sqrt{\frac{k-v}{2vk^2}}\,\mathrm{sgn}\, u$. Then α fixes the circle $|z| = k$ and takes ki to ω, while $\widetilde{\alpha}$ fixes the circle $|z| = k^{-1}$ and takes $k^{-1}i$ to $\widetilde{\omega}$, therefore

$$\angle(\omega \to \gamma\omega, \omega \to k) = \angle(ki \to \alpha^{-1}\gamma\alpha ki, ki \to k).$$

Likewise one has

$$\angle(\widetilde{\omega} \to \widetilde{\gamma}\widetilde{\omega}, \widetilde{\omega} \to k^{-1}) = \angle(k^{-1}i \to \widetilde{\alpha}^{-1}\widetilde{\gamma}\widetilde{\alpha}k^{-1}i, k^{-1}i \to k^{-1}),$$

and the last angles in both equalities are equal by the case $\omega = ki$ already proved.

Alternatively, equality (3.1) can be checked by direct computation, using the formula for $\tan(a + b)$. ∎

From the lemma, combined with $g_2(\xi)_\omega = g_2(\xi)_{s\omega}$ we also deduce that the pair correlation functions for the hyperbolic lattices centered at $\omega = u + iv$ and $s\widetilde{\omega} = -u + iv$ are equal. This shows that we can restrict ourselves, whenever convenient, to points ω in the half fundamental domain for Γ given by

$$|\omega| \leqslant 1, \quad \mathrm{Re}(\omega) \geqslant 0, \quad |\omega - 1| \leqslant 1. \tag{3.2}$$

4. The coincidence of the pair correlations of Φ and Ψ

Since the pair correlation of the lattices centered at ω and $\gamma_0\omega$ is the same for $\gamma_0 \in \Gamma = \mathrm{PSL}_2(\mathbb{Z})$, in this section we assume without loss of generality that ω lies in a specific fundamental domain for the action of Γ on the upper half plane. Namely, we assume that $0 \leqslant \mathrm{Re}(\omega) \leqslant 1$ and $\left|\omega - \frac{1}{2}\right| > \frac{1}{2}$, that is

$$0 \leqslant u \leqslant 1, \quad k^2 > u. \tag{4.1}$$

We also need to assume that $u, k^2 \in \mathbb{Q}$, which is needed in the proof of Lemma 3.

Next we show that Φ, Ψ have the same pair correlation. As in Section 3, let $\mathfrak{R}_Q$ be the set of $\gamma \in \Gamma$ with $|\gamma\omega| < k$ and $\|\gamma\| \leqslant Q$. Consider

$$\mathcal{R}_Q^{\Phi}(\xi) := \# \left\{ (\gamma, \gamma') \in (\mathfrak{R}_Q)^2 \ : \ \gamma \neq \gamma', \quad 0 \leqslant \Phi(\gamma) - \Phi(\gamma') < \frac{\xi}{Q^2} \right\}$$

and the likewise defined $\mathcal{R}_Q^{\Psi}(\xi)$.

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, $\|\gamma\| \leqslant Q$ and X, Y, Z, T be the quantities defined in the beginning of Section 2. By the results of Section 3 we can restrict to those $\gamma \in \Gamma$ such that $|\gamma\omega| < k$, that is $X < k^2 Y$. In this case we have

$$\frac{T}{Y} = k^2 + \frac{X}{Y} - 2u\frac{Z}{Y} < 2k^2 + 2k|u| \ll 1,$$

and employing $|Z - uY| \ll T$, a consequence of the first formula (2.3), we have

$$|\Phi(\gamma) - \Psi(\gamma)| = \frac{\left|u - \frac{Z}{Y}\right|}{\epsilon_T^{-1} Y - 1} \ll \frac{1}{Y^2} \ll \frac{1}{T^2} = \frac{1}{\|\gamma\|^4}. \tag{4.2}$$

Using $XY = Z^2 + v^2$ we have $T + 2uZ = X + k^2 Y \geqslant 2k\sqrt{XY} > 2k|Z|$, hence $|Z| < \frac{T}{2(k-|u|)}$. It follows that $\max\{X, k^2 Y\} < T + 2uZ < \frac{kT}{k-|u|}$, and in particular

$$X, Y, |Z| \ll \|\gamma\|^2. \tag{4.3}$$

Since $Y > v^2 \max\{c^2, \frac{d^2}{k^2}\}$, $X > v^2 \max\{a^2, \frac{b^2}{k^2}\}$, we also have $\|\gamma\|_\infty \ll \|\gamma\|$, or more precisely:

$$|a|, |d| < \frac{\|\gamma\|\sqrt{k}}{v\sqrt{k - |u|}}, \quad |c| < \frac{\|\gamma\|}{v\sqrt{k(k-|u|)}}, \quad |b| < \frac{\|\gamma\| k\sqrt{k}}{v\sqrt{k-|u|}}. \tag{4.4}$$

To compare the quantities $\Phi(\gamma)$ and $\Psi(\gamma)$, it will be important to show that they both lie in a certain Farey interval associate to γ. For $\Phi(\gamma)$ we can use a geometric argument to determine this interval. Recall that $\Phi(\gamma) = \mathrm{Re}(\gamma\omega)$. Looking at the images under γ of the geodesics $u \to \omega \to \infty$ and $0 \to \omega \to \frac{k^2}{u}$, if follows that

$$\Phi(\gamma) \in \big(\gamma u, \gamma\infty\big) \cap \big(\gamma 0, \gamma(k^2/u)\big) := J_\gamma^0 \tag{4.5}$$

(the endpoints of the intervals are not necessarily ordered increasingly). Since $k^2 Y > X$, we can assume that $cd(ck^2 + du)(cu + d) \neq 0$ at the expense of ignoring a finite number of matrices, which does not affect the pair correlation. To determine J_γ^0 explicitly from (4.5) there are four cases to consider, and in each one we also define a Farey interval J_γ containing J_γ^0 (using assumption (4.1) on ω and assuming $c > 0$):

1. $d > 0$. Then $J_\gamma^0 = \left(\frac{au+b}{cu+d}, \frac{ak^2+bu}{ck^2+du}\right) \subseteq \left(\frac{b}{d}, \frac{a}{c}\right) =: J_\gamma$.
2. $d < 0$, $cu + d > 0$. Then $ck^2 + du > 0$, $c + d > 0$, $J_\gamma^0 = \left(\frac{ak^2+bu}{ck^2+du}, \frac{a}{c}\right) \subseteq \left(\frac{a+b}{c+d}, \frac{a}{c}\right) =: J_\gamma$.
3. $d < 0$, $ck^2 + du < 0$. Then $cu + d < 0$, $c + d < 0$, $J_\gamma^0 = \left(\frac{b}{d}, \frac{au+b}{cu+d}\right) \subseteq \left(\frac{-b}{-d}, \frac{-(a+b)}{-(c+d)}\right) =: J_\gamma$.
4. $d < 0$, $ck^2 + du > 0 > cu + d$. Then $J_\gamma^0 = \left(\frac{ak^2+bu}{ck^2+du}, \frac{au+b}{cu+d}\right) \subseteq \left(\frac{a}{c}, \frac{-b}{-d}\right) =: J_\gamma$.

With this definition of J_γ, we have the following asymptotic result for $\Psi(\gamma)$.

Lemma 3. *Assume ω satisfies* (4.1) *and let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ with $c > 0$. Assume also that $u, k^2 \in \mathbb{Q}$. There exists $T_0 = T_0(\omega)$ such that*

$$\Phi(\gamma), \Psi(\gamma) \in J_\gamma, \quad \textit{whenever } T > T_0.$$

Proof. For $\Phi(\gamma)$ the statement was already proved (for all T).

Using $cZ - aY = -cu - d$, $dZ - bY = ck^2 + du$, we infer

$$\Psi(\gamma) - \frac{a}{c} = \frac{-\frac{cu+d}{c} + \epsilon_T\big(u - \frac{a}{c}\big)}{Y - \epsilon_T}, \qquad \Psi(\gamma) - \frac{b}{d} = \frac{\frac{ck^2+du}{d} + \epsilon_T\big(\frac{b}{d} - u\big)}{Y - \epsilon_T},$$

$$\Psi(\gamma) - \frac{a+b}{c+d} = \frac{\frac{c(k^2-u)-d(1-u)}{c+d} + \epsilon_T\big(\frac{a+b}{c+d} - u\big)}{Y - \epsilon_T}.$$

We discuss only the second difference, since the analysis is similar for the others. We have $\big|\frac{ck^2+du}{d}\big| \gg T^{-1/2}$, since the numerator is bounded from below as a result of the rationality assumption on u and k^2, and $|d| \leqslant \|\gamma\|_\infty \ll T^{1/2}$. Since the term involving ϵ_T is $\ll T^{-1}$, it follows that the sign of $\Psi(\gamma) - \frac{b}{d}$ is the same as that of $\frac{ck^2+du}{d}$, and similarly for the other two cases, leading to the desired result about Ψ. $\square$

Proposition 4. *For each $\beta \in \big(\frac{1}{2}, 1\big)$ we have*

$$\mathcal{R}_Q^\Psi(\xi) < \mathcal{R}_Q^\Phi\big(\xi + K_1 Q^{2-4\beta}\big) + K_2 Q^{1+\beta} \log Q,$$

for some constants $K_1, K_2 > 0$ depending only on ξ. The same equality holds with Φ, Ψ interchanged.

Proof. Let $\mathcal{N}_{Q,\beta}^{>}(\xi)$, respectively $\mathcal{N}_{Q,\beta}^{<}(\xi)$, be defined as for $\mathcal{R}_Q^\Psi(\xi)$, with the additional condition $\min\{\|\gamma\|, \|\gamma'\|\} > Q^\beta$, and respectively $\|\gamma\| < Q^\beta$. We trivially have

$$\mathcal{R}_Q^\Psi(\xi) \leqslant \mathcal{N}_{Q,\beta}^{>}(\xi) + 2\mathcal{N}_{Q,\beta}^{<}(\xi).$$

The estimate (4.2) shows that $\mathcal{N}_{Q,\beta}^{>}(\xi) \leqslant \mathcal{R}_Q^\Phi(\xi + K_1 Q^{2-4\beta})$, the constant K_1 being twice the implicit constant in (4.2).

To show that $\mathcal{N}_{Q,\beta}^{<}(\xi) = O_\xi(Q^{1+\beta}\log Q)$, we follow the same proof as that of Proposition 2 in [5]. Because of (4.4), at the expense of counting more pairs we can replace the set $\mathfrak{R}_Q$ in the definition of $\mathcal{N}_{Q,\beta}^{<}(\xi)$ with the set

$$\mathfrak{R}_Q' := \Big\{\gamma \in \Gamma : |\gamma\omega| < k, \|\gamma\|_\infty \leqslant \widetilde{Q} := \frac{Q\sqrt{k}}{v\sqrt{k-|u|}}\Big\}.$$

Lemma 3 shows that $\Psi(\gamma), \Phi(\gamma)$ lie between Farey fractions determined by $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. More precisely, if $\mathfrak{R}_Q''$ denotes the subset of $\mathfrak{R}_Q'$ consisting of matrices with $cd > 0$, and let $I_\gamma = \big(\frac{b}{d}, \frac{a}{c}\big)$. Then for each $\gamma \in \mathfrak{R}_Q'$ we have that

$$\Psi(\gamma), \Phi(\gamma) \in I_{\gamma'}, \text{ for } \gamma' \in \mathfrak{R}_Q'', \tag{4.6}$$

with $\gamma' = \gamma$ in Case 1, $\gamma' = \left(\begin{smallmatrix} a & a+b \\ c & c+d \end{smallmatrix}\right)$ in Case 2, $\gamma' = \left(\begin{smallmatrix} a+b & b \\ c+d & d \end{smallmatrix}\right)$ in Case 3, and $\gamma' = \left(\begin{smallmatrix} -b & a \\ -d & c \end{smallmatrix}\right)$ in Case 4 (see the four cases before the statement of the lemma). Note that in all four cases, $I_{\gamma'} \subset [0,k]$ or $I_{\gamma'} \subset [-k, 0]$, since $\Phi(\gamma) \in (-k, k)$. Clearly each $\gamma' \in \mathfrak{R}_Q''$ is associated with one, two, or three such pairs $\Psi(\gamma), \Phi(\gamma)$ for $\gamma \in \mathfrak{R}_Q'$.

The proof now follows the same pattern as that of Proposition 2 in [5], using (4.6) above instead of (4.3) there, after further dividing $\mathfrak{R}_Q''$ into the subsets of those γ with $I_\gamma \subset [-k, 0]$, and of those γ with $I_\gamma \subset [0, k]$. For each subset the analysis of the associated Farey tessellation formed by the intervals I_γ is the same as in [5]. $\square$

5. The Farey tessellation and repulsion

In the previous section we associated to each $\gamma \in \Gamma$ an interval J_γ between two consecutive Farey points, such that $\Phi(\gamma) \in J_\gamma$. We also associate to γ the geodesic arc on the upper half plane connecting the endpoints of the Farey interval, which is part of the well known Farey tessellation. The purpose of this section is to quantify the statement that there is repulsion between $\Phi(\gamma), \Phi(\gamma')$, if the intervals associated to γ, γ' are disjoint.[2]

Lemma 5. *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $\gamma' = \begin{pmatrix} e & a \\ f & c \end{pmatrix} \in \Gamma$ *with* $\frac{a}{c} < \frac{e}{f}$ *and* $c, d, f > 0$. *Then there exists* $K \in \mathbb{N}$ *such that*

$$\gamma \begin{pmatrix} K & 1 \\ -1 & 0 \end{pmatrix} = \gamma'.$$

Moreover, if $\max\{Y_\gamma, Y_{\gamma'}\} \leqslant Q^2$ *and we assume* $u \geqslant 0$ *and* $k \leqslant 1$, *then*

$$\Phi(\gamma') - \Phi(\gamma) \geqslant \frac{Kk^4}{Q^2}.$$

Proof. Since the matrices γ and $\gamma' s$ have the same first column, there exists $K \in \mathbb{Z}$ such that $\gamma\left(\begin{smallmatrix} 1 & -K \\ 0 & 1 \end{smallmatrix}\right) = \gamma' s$, which implies the desired equality. The fact that $K > 0$ follows from $\frac{e}{f} = \frac{aK-b}{cK-d} > \frac{a}{c}$.

A direct calculation provides

$$\begin{aligned} \Phi(\gamma') - \Phi(\gamma) &= \frac{K\big(c^2(k^4+2u^2) + k^2 df + cu(2d + k^2 f)\big) + cd(1-k^4) + u\big(k^2 df + c^2(1+k^2) - d^2\big)}{(k^2c^2 + d^2 + 2ucd)(k^2 f^2 + c^2 + 2ucf)} \\ &\geqslant \frac{K\alpha(c,d,f)}{Y_\gamma Y_{\gamma'}} \geqslant \frac{Kk^4}{Q^2}, \end{aligned}$$

where we denoted by $\alpha(c,d,f)$ the coefficient of K on the first line, and for the first inequality we used $u \geqslant 0$, $k \leqslant 1$ and $\frac{c}{d} > K \geqslant 1 > \frac{1}{\sqrt{1+k^2}}$. If $d \geqslant f$, then we have $\alpha(c,d,f) \geqslant k^4 Y_{\gamma'}$, and if $f \geqslant d$, then $\alpha(c,d,f) \geqslant k^2 Y_\gamma$, which, together with $\max\{Y_\gamma, Y_{\gamma'}\} \leqslant Q^2$, proves the second inequality. ☐

For each $\xi > 0$ consider the finite set

$$\mathcal{F}(\xi) := \bigcup_{\ell \geqslant 1} \left\{ M = \gamma_1 \cdots \gamma_\ell : \gamma_j = \left(\begin{smallmatrix} K_j & 1 \\ -1 & 0 \end{smallmatrix}\right), K_j \in \mathbb{N}, K_1 + \cdots + K_\ell \leqslant \frac{4}{k^4}\xi \right\}. \tag{5.1}$$

Lemma 6. *Assume* $u \geqslant 0, k \leqslant 1$. *Suppose* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \gamma' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $c, d, c', d' > 0$, $\frac{a}{c} \leqslant \frac{b'}{d'}$, *and* $\Phi(\gamma') - \Phi(\gamma) \leqslant \frac{\xi}{Q^2}$ *for some* $Q \geqslant \max\{c, d, c', d'\}$. *Then*

(i) $\gamma' = \gamma M$ *for some* $M \in \mathcal{F}(\xi)$.

(ii) *Furthermore, if* $M = \gamma_1 \cdots \gamma_\ell$ *is as in* (5.1), *then*

$$\gamma\gamma_1 \cdots \gamma_j = \left(\begin{smallmatrix} a_j & a_{j-1} \\ q_j & q_{j-1} \end{smallmatrix}\right),$$

with $q_1, \ldots, q_\ell \in \{1, 2, \ldots, Q\}$.

Proof. As $Q \geqslant \max\{c, d'\}$ the fractions $\frac{a}{c}$ and $\frac{b'}{d'}$ belong to the set $\mathcal{F}_Q$ of "extended" Farey fractions $\frac{a}{q}$ with $(a,q) = 1$ and $1 \leqslant q \leqslant Q$. Let $\frac{a}{c} = \frac{a_0}{q_0} < \frac{a_1}{q_1} < \cdots < \frac{a_\ell}{q_\ell} = \frac{b'}{d'}$ be the elements in $\mathcal{F}_Q$ between $\frac{a}{c}$ and $\frac{b'}{d'}$. By Lemma 5 there are positive integers K_1 and $K_{\ell+1}$ such that

$$\begin{pmatrix} a_1 & a_0 \\ q_1 & q_0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} K_1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \ldots \quad, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a_\ell & a_{\ell-1} \\ q_\ell & q_{\ell-1} \end{pmatrix} \begin{pmatrix} K_{\ell+1} & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.2}$$

[2]Recall that two Farey arcs are nonintersecting, so the corresponding intervals are either disjoint or one contains the other.

The recursion relations for consecutive Farey fractions $\frac{q_{j-1}+q_{j+1}}{q_j} = K_j = \frac{a_{j-1}+a_{j+1}}{a_j}$, $j = 2, \dots, \ell-1$, $\ell \geqslant 2$, (5.2), and the consequence $\Phi(\gamma') - \Phi(\gamma) \geqslant \frac{1}{Q^2}(K_1 + \dots + K_{\ell+1})$ of Lemma 5 yield both (i) and (ii). Note that $Y_\gamma, Y_{\gamma'} \leqslant Q^2(k^2+1+2u) \leqslant 4Q^2$ under the assumptions on c, d, c', d', k, u. □

6. The case when $\ell(M)$ is large

In this section we generalize Lemma 13 of [5]. Let $\mathcal{S}^+_{M,Q}(\xi)$, respectively $\mathcal{S}^-_{M,Q}(\xi)$, denote the subsets of the set $\mathcal{S}_{M,Q}(\xi)$ defined in (1.6) consisting of matrices with $c, d > 0$, respectively $c > 0 > d$. Denote by $\mathcal{N}^\pm_{M,Q}(\xi)$ the cardinality of $\mathcal{S}^\pm_{M,Q}(\xi)$.

Lemma 7. *Assume ω is in the half fundamental domain given by (3.2), and $u, k^2 \in \mathbb{Q}$. Suppose $\beta_0 \in \left(\frac{2}{3}, 1\right)$, $M \in \Gamma$ has nonnegative entries and $\max\{X_M, Y_M\} \geqslant Q^{2\beta_0}$. There exists $Q_0 = Q_0(\xi, \omega)$ independent on M such that $\mathcal{N}^+_{M,Q}(\xi) = 0$ for $Q \geqslant Q_0$.*

Proof. We show that the region $\Omega_{M,Q}(\xi)$ of $(c,d) \in (0,\infty)^2$ for which

$$|\Xi_M(c,d)| \leqslant \frac{\xi}{Q^2}, \qquad v^2 \max\{Y_\gamma, Y_{\gamma M}\} \leqslant Q^2,$$

contains no coprime integer lattice points. This plainly gives $\mathcal{N}^+_{M,Q}(\xi) = 0$.

Suppose there is $(c,d) \in \Omega_{M,Q}(\xi) \cap \mathbb{Z}^2$. Write $c\omega + d = re^{i\theta}$, and let $X = X_M$, $Y = Y_M$, $Z = Z_M$ and $T = T_M$ be given by (2.1). With $U_M = \coth \ell(M) = 1 + O\left(\frac{1}{T^2}\right)$, the inequalities in the definition of $\Omega_{M,Q}(\xi)$ can be described as

$$\frac{v}{\xi} \frac{|\sin(\theta_M - 2\theta)|}{U_M + \cos(\theta_M - 2\theta)} \leqslant \frac{r^2}{Q^2} \leqslant \min\left\{ \frac{1}{v^2}, \frac{2}{\sqrt{T^2-\Delta^2}(U_M + \cos(\theta_M - 2\theta))} \right\}. \tag{6.1}$$

Since $\sin\theta > 0, \cos\theta > 0$ we can take $\theta \in \left(0, \frac{\pi}{2}\right)$. Denoting $\delta_M = \frac{\theta_M}{2} - \theta$, from the first and last fraction in (6.1) we have $|\sin 2\delta_M| \ll \frac{1}{T}$. Therefore δ_M is close to 0, or to $\pm\frac{\pi}{2}$. When δ_M is close to 0 we have $|\tan\delta_M| \ll |\delta_M| \ll |\sin 2\delta_M| \ll \frac{1}{T}$.

When δ_M is close to $\pm\frac{\pi}{2}$ we similarly have $|\delta_M \mp \frac{\pi}{2}| \ll \frac{1}{T}$, which we claim is impossible. From

$$\frac{|\tan\delta_M|}{1 + \frac{U_M - 1}{1+\cos 2\delta_M}} = \frac{|\sin 2\delta_M|}{U_M + \cos 2\delta_M} \leqslant \frac{\xi}{v^3} \tag{6.2}$$

it suffices to bound $\frac{U_M-1}{1+\cos 2\delta_M}$ from above, which would imply $|\tan\delta_M| \ll \xi$, thus contradicting $|\delta_M \mp \frac{\pi}{2}| \ll \frac{1}{T}$. Next we consider the two cases that can occur.

Case I: $Z > uY$. Since $u, k^2 \in Q$, it follows from (2.3) that $\sin\theta_M \gg \frac{1}{T}$. Since $\cos\theta, \sin\theta > 0$ and $\theta_M \in (0,\pi)$, at least one of $\cos 2\delta_M + \cos\theta_M$ or $\cos 2\delta_M - \cos\theta_M$ must ne nonnegative, showing that $\cos 2\delta_M \geqslant -|\cos\theta_M|$, and so we have

$$1 + \cos 2\delta_M \geqslant 1 - |\cos\theta_M| = 1 - \sqrt{1-\sin^2\theta_M} \gg \tfrac{1}{T^2}.$$

Since $U_M - 1 \ll \frac{1}{T^2}$, it follows that $\frac{U_M-1}{1+\cos 2\delta_M} \ll 1$, proving the claim.

Case II: $Z < uY$. In this case $\sin\theta_M < 0$. We also have $\cos\theta_M > 0$ by (2.3), since

$$\Delta Y - T = 2uZ - X + (v^2-u^2)Y > uZ + (v^2-u^2)Y - v^2/Y > 0$$

for Q large enough, where we used $XY - Z^2 = v^2$, and $v > u$ for ω in the region given by (3.2). In this case $\delta_M \in \left(-\frac{3\pi}{4}, 0\right)$, so we have to show that $|\delta_M + \frac{\pi}{2}| \ll \frac{1}{T}$ leads to contradiction.

From $|\sin 2\delta_M| \ll \frac{1}{T}$ we have $|\tan\delta_M| \gg T$, and (6.2) gives $1 \gg T\frac{1+\cos 2\delta_M}{U_M+\cos 2\delta_M}$. This leads (for large T) to $\frac{1}{T^2} \gg U_M - 1 \gg T(1+\cos 2\delta_M)$, and therefore to $U_M + \cos 2\delta_M \ll \frac{1}{T^2}$. Back to (6.2), we infer $|\sin 2\delta_M| \ll \frac{1}{T^2}$, leading in turn to

$$T^2 \ll |\tan\delta_M| = \frac{\left|\tan\left(\frac{\theta_M}{2}\right) - \frac{cv}{cu+d}\right|}{\left|1+\frac{cv}{cu+d}\tan\left(\frac{\theta_M}{2}\right)\right|} < \frac{1+\frac{v}{u}}{\left|1+\frac{cv}{cu+d}\tan\left(\frac{\theta_M}{2}\right)\right|},$$

where we used $\theta_M \in \left(-\frac{\pi}{2}, 0\right)$. This further gives

$$\left|1+\frac{cv}{cu+d}\tan\left(\theta_M/2\right)\right| = \left|1+\frac{cv}{cu+d}\left(\frac{1}{v}\left(\frac{Z}{Y}-u\right)+O\left(\frac{1}{TY}\right)\right)\right| \ll_\omega \frac{1}{T^2},$$

and so because $u, k^2 \in \mathbb{Q}$ we infer

$$1 \ll_\omega |(cu+d)Y + c(Z-uY)| \ll_\omega (cu+d)Y\left(\frac{1}{T^2}+\frac{1}{TY}\right). \tag{6.3}$$

As $|u| < 1$, (6.3) and (4.3) now yield $cu+d \gg T \gg Q^{2\beta_0}$, which contradicts $cu+d \ll Q$.

We have thus shown that $|\tan\delta_M| \ll \frac{1}{T}$. Next we consider two cases, $\tan\theta < v$ and $\tan\theta > v$.

Case (A): $\tan\theta < v$. Recall that $\Psi(M) = u + v\tan\left(\frac{\theta_M}{2}\right)$, and we also have $u + v\tan\theta = \frac{ck^2+du}{cu+d}$. Since $|\delta_M| \ll \frac{1}{T}$, $\tan\left(\frac{\theta_M}{2}\right)$ is also bounded, leading to

$$\left|\Psi(M) - \frac{ck^2+du}{cu+d}\right| = v\left|\tan\left(\theta_M/2\right) - \tan\theta\right| = v|\tan\delta_M|\left|1+\tan\theta\tan\left(\theta_M/2\right)\right| \ll \xi Q^{-2\beta_0}.$$

Since $\Psi(M) \ll 1$, we have $Z \ll Y$, and from $XY - Z^2 = v^2$ we conclude $X \ll Y$. From (4.2) it follows that $|\Psi(M) - \Phi(M)| \ll Y^{-2} \ll Q^{-4\beta_0}$, so $\left|\frac{Z}{Y} - \frac{ck^2+du}{cu+d}\right| \ll Q^{-2\beta_0}$. On the other hand,

$$\frac{A}{C} - \frac{Z}{Y} = \frac{D+uC}{CY} \ll \frac{1}{Y} \ll Q^{-2\beta_0},$$

where we assumed without loss of generality $C \geqslant D$ (if $D \geqslant C$, then use $\frac{B}{D}$ instead of $\frac{A}{C}$). We conclude that

$$\left|\frac{A}{C} - \frac{ck^2+du}{cu+d}\right| \ll Q^{-2\beta_0}. \tag{6.4}$$

If nonzero, the left hand side of (6.4) is $\gg \frac{1}{C(cu+d)}$, using the rationality assumption $u, k^2 \in \mathbb{Q}$. From $\tan\theta < v$ and $u < 1$ it follows that $c \ll d$, so $C(cu+d) \ll \sqrt{d^2Y} < \sqrt{Y_{\gamma M}} \ll Q$, obtaining a contradiction. It remains that $\frac{A}{C} = \frac{ck^2+du}{cu+d} = \frac{Mc+Nd}{Pc+Rd}$ with $M, N, P, R \in \mathbb{N}$ constants, so $d \gg Pc + Rd \geqslant C \gg \sqrt{Y}$ (using $C \geqslant D$). It follows that $Q^2 > Y_{\gamma M} > d^2Y \gg Y^2 \gg Q^{4\beta_0}$, which again provides a contradiction.

Case (B): $\tan\theta > v$. We have $\frac{u}{v} < \cot\theta < \frac{1}{v}$, and since $|\delta_M| \ll \frac{1}{T}$ it follows that $\cot\left(\frac{\theta_M}{2}\right)$ is bounded as well. Consequently

$$\left|\frac{1}{\Psi(M)} - \frac{cu+d}{ck^2+du}\right| = v|\tan\delta_M|\frac{|1+\cot(\theta)\cot(\theta_M/2)|}{|(u\cot(\theta)+v)(u\cot(\theta_M/2)+v)|} \ll Q^{-2\beta_0}.$$

In this case $\Psi(M) \gg 1$ implies $Z \gg Y$, so $X \gg Z \gg Y$, and from $XY - Z^2 = v^2$ we have $Z \gg Q^{\beta_0}$. Taking into account (4.3) we arrive at

$$\left|\frac{1}{\Psi(M)} - \frac{1}{\Phi(M)}\right| = \epsilon_T\frac{\left|1-u\frac{Y}{Z}\right|}{Z-u\epsilon_T} \ll \frac{1}{TZ} \ll Q^{-3\beta_0}.$$

Assuming $A > B$ (the other case is similar), we infer

$$\frac{Y}{Z} - \frac{C}{A} = \frac{D + uC}{AZ} \ll \frac{\sqrt{Y}}{\sqrt{X}Z} \ll \frac{1}{X} \ll Q^{-2\beta_0},$$

and thus $\left|\frac{C}{A} - \frac{cu+d}{ck^2+du}\right| \ll Q^{-2\beta_0}$. This leads to a contradiction for large Q as before. □

7. Approximating the number of lattice points in planar regions by volumes

In this section we approximate $\mathcal{N}_{M,Q}(\xi)$ for $M \in \mathfrak{S}$ by volumes of three dimensional regions, where $\mathfrak{S} \subset \Gamma$ is the set of matrices with nonnegative entries, distinct from the identity. Using the result of the previous section, we also show that the sum of $\mathcal{N}^+_{M,Q}(\xi)$ over subsets of $M \in \mathfrak{S}$ can be approximated by the corresponding sum of volumes.

To count points in two dimensional regions we use Lemma 12 of [5]. The prototype for its application in the present setting is given in the following simpler counting problem. By well known asymptotics for the number of points in expanding hyperbolic balls we have $B_Q^{\rm tot} \sim \frac{6Q^2}{\Delta}$ (for the notation see Section 3). In the next lemma we show that in half balls we have half this number of points.

Lemma 8. *Let $B_Q = \#\mathfrak{R}_Q$ be as defined in Section 3, with $k \geqslant 1$. Then*

$$B_Q = \frac{3Q^2}{\Delta} + O_\varepsilon\big(Q^{11/6+\varepsilon}\big),$$

and so $B_Q \sim \frac{1}{2}B_Q^{\rm tot}$.

Proof. Replacing $b = \frac{ad-1}{c}$, the condition $|\gamma\omega| < k$ is equivalent to

$$\left(k^2 - \frac{a^2}{c^2}\right)(c^2k^2 + d^2 + 2cdu) + \frac{2ad-1}{c^2} + \frac{2au}{c} > 0. \tag{7.1}$$

A direct calculation shows that $\|\gamma\| \leqslant Q$ is equivalent to

$$\left(k^2 + \frac{a^2}{c^2} - \frac{2au}{c}\right)(c^2k^2 + d^2 + 2cdu) + \frac{2du}{c} + \frac{1-2ad}{c^2} + \left(2u - \frac{2a}{c}\right)u \leqslant Q^2. \tag{7.2}$$

Fix $\alpha = \frac{13}{18}$ and let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathfrak{R}_Q$ with $c > 0$. The contribution of the matrices γ with $|c| < Q^\alpha$ or $|d| < Q^\alpha$ to the error term is $\ll Q^{1+\alpha}$, so we can assume $c > Q^\alpha$ and $|d| > Q^\alpha$.

We show that the matrices γ with $\frac{a^2}{c^2} > k^2$ contribute negligibly to B_Q. From (7.1) and $c^2k^2 + d^2 + 2cdu \geqslant c^2v^2$, it follows that for such γ we have $0 < \frac{a^2}{c^2} - k^2 \ll \frac{1}{c^2}\big(\frac{Q^2}{c^2} + \frac{Q}{c}\big) \ll Q^{2-4\alpha}$, so $\frac{|a|}{c} \in [k, k+m]$ with $m \ll \sqrt{k^2 + KQ^{2-4\alpha}} - k \ll Q^{1-2\alpha}$. Since $|a|, c \ll Q$ from the equidistribution of the Farey fractions $\mathcal{F}_Q$ in intervals I of length $|I| > Q^{-\delta}$ with $\delta = 2\alpha - 1 \in (0,1)$ it follows that the number of pairs (a,c) is $\ll Q^{3-2\alpha}$ as long as $\alpha \in \big(\frac{1}{2}, 1\big)$. Since the number of values d can take is $\ll \frac{Q}{c} < Q^{1-\alpha}$, there are $\ll Q^{4-3\alpha}$ such matrices as long as $A > \frac{19}{30}$, so they can be absorbed in the error term.

Therefore we can assume $|a| \leqslant kc$, and the condition $|\gamma\omega| < k$ is satisfied except for a negligible number of matrices. Via (7.2), the condition $\|\gamma\| \leqslant Q$ can be replaced without affecting the asymptotics by

$$\left(k^2 + \frac{a^2}{c^2} - \frac{2au}{c}\right)(k^2c^2 + d^2 + 2cdu) \leqslant Q^2.$$

Therefore we can apply Lemma 11 in [5], with $L = \frac{5}{6}$, to the set

$$\Omega_c = \left\{(a,d) \in [-kc, kc] \times [-\widetilde{Q}, \widetilde{Q}] : (k^2c^2 + a^2 - 2uca)(k^2c^2 + d^2 + 2ucd) \leq Q^2c^2\right\}$$

(recall $\widetilde{Q} = \frac{Q\sqrt{k}}{v\sqrt{(k-|u|)}}$), and conclude that

$$B_Q = \sum_{c=1}^{\widetilde{Q}/k} \frac{\varphi(c)}{c} \frac{\operatorname{Area}(\Omega_c)}{c} + O_\varepsilon\big(Q^{11/6+\varepsilon}\big).$$

Möbius summation and a change of variables $a = cz, d = Qy, c = Qx$ then gives

$$B_Q = \frac{Q^2}{\zeta(2)} \operatorname{Vol}(V_Q) + O_\varepsilon\big(Q^{11/6+\varepsilon}\big),$$

where V_Q denotes the set of points $(x, y, z) \in \big[0, \frac{\widetilde{Q}}{kQ}\big] \times \big[-\frac{\widetilde{Q}}{Q}, \frac{\widetilde{Q}}{Q}\big] \times [-k, k]$ such that $(k^2x^2 + y^2 + 2uxy)(k^2 + z^2 - 2uz) \leqslant 1$. The substitution $x\omega + y = re^{i\theta}$, $z = v \tan t + u$, then yields

$$\operatorname{Vol}(V_Q) = \int_{\beta/2-\pi/2}^{\beta/2} \int_0^\pi \int_0^{\frac{\cos t}{v}} \frac{r}{\cos^2 t}\, dr d\theta dt = \frac{\pi^2}{2\Delta},$$

which concludes the proof. □

Next we seek to replace inequalities defining the set $\mathcal{S}_{M,Q}(\xi)$ in (1.6) with simpler ones, involving only the entries (a, c, d) or (b, c, d) of the matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Using

$$X_{\gamma M} = a^2 X_M + b^2 Y_M + 2ab Z_M, \quad Y_{\gamma M} = c^2 X_M + d^2 Y_M + 2cd Z_M,$$
$$Z_{\gamma M} = ac X_M + bd Y_M + (ad + bc) Z_M,$$

and substituting $b = \frac{ad-1}{c}$, we find

$$\|\gamma M\|^2 = \left(k^2 + \frac{a^2}{c^2} - 2u\frac{a}{c}\right) Y_{\gamma M} + \frac{Y_M + 2(uc-a)(dY_M + cZ_M)}{c^2}, \tag{7.3}$$

$$|\gamma M \omega| < k \iff \left(k^2 - \frac{a^2}{c^2}\right) Y_{\gamma M} + \frac{2a(dY_M + cZ_M) - Y_M}{c^2} > 0.$$

The previous formulas lead us to consider the cardinality $\widetilde{\mathcal{N}}_{M,Q}(\xi)$ of the set $\widetilde{\mathcal{S}}_{M,Q}(\xi)$ of integer triples (a, c, d) satisfying (1.7). Let $\widetilde{\mathcal{S}}^+_{M,Q}(\xi)$, respectively $\widetilde{\mathcal{S}}^-_{M,Q}(\xi)$, be the subsets of $\widetilde{\mathcal{S}}_{M,Q}(\xi)$ for which $c, d > 0$, respectively $c > 0 > d$.

Lemma 9. *Assume $u, k^2 \in \mathbb{Q}$.*

(i) *There is a constant $K = K(\xi) > 0$ such that, for every Q, the number of pairs $(\gamma, \gamma') \in \Gamma^2$ with $\|\gamma\|, \|\gamma'\| \leqslant Q$, $\min\{|\gamma\omega|, |\gamma'\omega|\} < k < \frac{|a|}{c}$ or $\max\{|\gamma\omega|, |\gamma'\omega|\} > k > \frac{|a|}{c}$, and*

$$|\Phi(\gamma') - \Phi(\gamma)| \leqslant \frac{\xi}{Q^2}, \tag{7.4}$$

is at most K.

(ii) *For each $\alpha \in (0, 1)$ the following asymptotic estimates hold:*

$$\sum_{M \in \mathfrak{S}} \mathcal{N}^+_{M,Q}(\xi) \leqslant \sum_{M \in \mathfrak{S}} \widetilde{\mathcal{N}}^+_{M,Q(1+O(Q^{-\alpha/2}))}(\xi) + O(Q^{1+\alpha}), \tag{7.5}$$

$$\sum_{M \in \mathfrak{S}} \widetilde{\mathcal{N}}^+_{M,Q}(\xi) \leqslant \sum_{M \in \mathfrak{S}} \mathcal{N}^+_{M,Q(1+O(Q^{-\alpha/2}))}(\xi) + O(Q^{1+\alpha})$$

(iii) *For $M \in \mathfrak{S}$ we have individually*

$$\mathcal{N}_{M,Q}(\xi) \leqslant \widetilde{\mathcal{N}}_{M,Q(1+O(Q^{-1}))}(\xi) + O(Q^{1+\alpha}), \quad \widetilde{\mathcal{N}}_{M,Q}(\xi) \leqslant \mathcal{N}_{M,Q(1+O(Q^{-1}))}(\xi) + O(Q^{1+\alpha}).$$

Proof. (i) Assume first $\frac{|a|}{c} > k > |\gamma\omega|$ or $\frac{|a|}{c} < k < |\gamma\omega|$. Since $\Psi(\gamma)$ is the x-intercept of the geodesics from ω to $\gamma\omega$, it follows that $|\gamma\omega| < k$ if and only if $|\Psi(\gamma)| < k$, so we have (assuming $a > 0$, the other case being similar with k replaced by $-k$ below)

$$\left|\frac{a}{c} - k\right| < \left|\Psi(\gamma) - \frac{a}{c}\right| < \left|\Phi(\gamma) - \frac{a}{c}\right| + \left|\Psi(\gamma) - \Phi(\gamma)\right| \ll \frac{1}{c^2},$$

by (4.2) and $\left|\Phi(\gamma) - \frac{a}{c}\right| = \frac{1}{c^2}\frac{|u+d/c|}{(u+d/c)^2+v^2} \ll \frac{1}{c^2}$. Since $k \in \mathbb{Q}$, there are finitely many such pairs (a, c), and repeating the argument with $\frac{a}{c}$ replaced by $\frac{b}{d}$ we obtain that there are finitely many such matrices γ (also using the fact that there are finitely many γ with k between $\frac{|a|}{c}$ and $\frac{|b|}{d}$). From (7.4) and the fact that $\Phi(\gamma) \in \mathbb{Q}$, there are also finitely many matrices γ' satisfying the assumptions.

Finally assume $\frac{|a|}{c} > k > |\gamma'\omega|$, the remaining case being similar. Then $|\Phi(\gamma')| < k < \frac{|a|}{c}$, and as before we have (assuming $a > 0$, otherwise replace k by $-k$)

$$\left|\frac{a}{c} - k\right| < \left|\Phi(\gamma') - \frac{a}{c}\right| < \left|\Phi(\gamma) - \frac{a}{c}\right| + |\Phi(\gamma) - \Phi(\gamma')| \ll \frac{1}{c^2},$$

and we conclude as in the previous case.

(ii) Next we look at pairs $(\gamma, \gamma' = \gamma M)$ satisfying (7.4), with $|a| \leqslant kc, \|\gamma\|, \|\gamma M\| \leqslant Q$, estimating their contribution to the left-hand side of (7.5) according to whether $Y_\gamma < Q^{2\alpha}$ or $Y_\gamma \geqslant Q^{2\alpha}$. By part (i) we can assume $X_\gamma \ll Y_\gamma$, and by (7.3) we have $\|\gamma\| \ll Q^\alpha$ or $\|\gamma\| \gg Q^\alpha$, respectively in the two cases.

Assume first $Y_\gamma < Q^{2\alpha}$. With the Farey interval J_γ associated to γ defined before Lemma 3, the Farey intervals J_γ and $J_{\gamma M}$ are either disjoint or one contains the other. Since each Farey interval is associated with at most three matrices $\gamma \in \Gamma$, it follows as in the proof of [5, Proposition 2] that the number of pairs $(\gamma, \gamma M)$ is $\ll Q^{1+\alpha} \ln Q$.

Therefore we are left to consider pairs $(\gamma, \gamma M)$ with $Y_\gamma \geqslant Q^{2\alpha}$, $\frac{|a|}{c}, |\gamma\omega|, |\gamma M\omega| \leqslant k$, and with $\|\gamma\|, \|\gamma M\| \ll Q$. These conditions are satisfied by γ in either $\mathcal{S}_{M,Q}(\xi)$ or $\widetilde{\mathcal{S}}_{M,Q}(\xi)$, as we can assume $X_\gamma \ll Y_\gamma$, $X_{\gamma M} \ll Y_{\gamma M}$ by part (i). Without loss of generality we assume $c > d > 0$ (otherwise substitute a in terms of b, c, d in the left side of (7.3)), and show that

$$\frac{|(2a - uc)(dY_M + cZ_M) - Y_M|}{c^2} \ll \frac{Q^2}{c} \leqslant Q^{2-\alpha}. \tag{7.6}$$

The inequality $\|\gamma M\| \ll Q$ plainly gives $Y_M \ll \frac{Q^2}{d^2}$. Employing also $(dY_M + cZ_M)^2 + v^2c^2 = Y_{\gamma M}Y_M$ we arrive at (7.6). By (7.3) the claim follows.

(iii) By the proof of part (ii), it remains to consider pairs $(\gamma, \gamma M)$ with $Y_\gamma \geqslant Q^\alpha$, $|\gamma\omega|, |\gamma M\omega| \leqslant k$, and with $d < 0$. Without loss of generality we can assume $c > |d|$ (otherwise substitute a in terms of b, c, d in the left side of (7.3)), and (7.6) follows trivially since M is fixed, with the upper bound being now $Q^{-\alpha}$. □

Lemma 10. *For any $M \in \mathfrak{S}$, uniformly in M and ξ,*

$$\widetilde{\mathcal{N}}_{M,Q}(\xi) = \frac{Q^2}{\zeta(2)} \operatorname{Vol}(S_{M,\xi}) + O_\varepsilon(Q^{11/6+\varepsilon}).$$

Proof. $\widetilde{\mathcal{N}}_{M,Q}(\xi)$ represents the sum over $c \in \{1, \dots, Q\}$ of the number of integer lattice points (a, d) with $ad \equiv 1 \pmod{c}$ in the region $\Omega = \Omega_{M,Q,c}(\xi)$ of points $(a, d) \in [-kc, kc] \times [-\widetilde{Q}, \widetilde{Q}]$ for which $|\Xi_M(c, d)| \leqslant \frac{\xi}{Q^2}$ and $\left(1 + \frac{a^2}{c^2} - 2u\frac{a}{c}\right) \max\{k^2c^2 + d^2 + 2cdu, c^2X_M + d^2Y_M + 2cdZ_M\} \leqslant Q^2$. Applying

Lemma 11 of [5] with $q = c$, $\operatorname{Area}(\Omega) \leqslant cQ$, $\operatorname{length}(\partial\Omega) \ll Q$, and $L = c^{5/6}$, we find

$$\widetilde{\mathcal{N}}_{M,Q}(\xi) = \sum_{c=1}^{Q} \left(\frac{\varphi(c)}{c^2} \operatorname{Area}\left(\Omega_{M,Q,c}(\xi)\right) + O_\varepsilon(Qc^{-1/6+\varepsilon}) \right).$$

Möbius summation applied to the function $h(c) = \frac{1}{c}\operatorname{Area}(\Omega_{M,Q,c}(\xi))$ with $\|h\|_\infty \leqslant Q$, and the change of variables $(c, u, v) = (Qx, Qxz, Qy)$, yield

$$\begin{aligned}\widetilde{\mathcal{N}}_{M,Q}(\xi) &= \frac{1}{\zeta(2)} \int_0^Q A\big(\Omega_{M,Q,c}(\xi)\big)\, \frac{dc}{c} + O_\varepsilon(Q^{11/6+\varepsilon}) \\ &= \frac{Q^2}{\zeta(2)} \operatorname{Vol}(S_{M,\xi}) + O_\varepsilon(Q^{11/6+\varepsilon}).\end{aligned}$$

□

When restricting to the subset $\widetilde{\mathcal{S}}^+_{M,Q}(\xi)$, we have the following better estimate.

Lemma 11. *Let $\mathfrak{S}' \subset \mathfrak{S}$ be any subset. For each $\beta_0 \in \left(\frac{2}{3}, 1\right)$ the following estimate holds (uniformly in ξ on compacts)*

$$\sum_{M \in \mathfrak{S}'} \widetilde{\mathcal{N}}^+_{M,Q}(\xi) = \frac{Q^2}{\zeta(2)} \sum_{\substack{M \in \mathfrak{S}' \\ X_M, Y_M \leqslant Q^{2\beta_0}}} \operatorname{Vol}(S^+_{M,\xi}) + O_\xi(Q^{(11+\beta_0)/6}). \tag{7.7}$$

Proof. The contribution of $c \leqslant d$ to $\widetilde{\mathcal{N}}^+_{M,Q}(\xi)$ is

$$\mathcal{N}'_{M,Q}(\xi) = \sum_{c \leqslant Q/(v\sqrt{X_M})} \#\big\{(a,d) \in \mathbb{Z}^2 \cap \Omega'_{M,Q,c}(\xi),\ ad \equiv 1 \pmod{c}\big\},$$

where $\Omega = \Omega'_{M,Q,c}(\xi)$ denotes the set of points $(a,d) \in [-kc, kc] \times [c, \widetilde{Q}]$ with

$$\max\{k^2c^2 + d^2 + 2cdu, c^2X_M + d^2Y_M + 2cdZ_M\} \leqslant \frac{c^2}{a^2 + k^2c^2 - 2acu} Q^2, \quad |\Xi_M(c,d)| \leqslant \frac{\xi}{Q^2}.$$

Applying Lemma 11 of [5] to Ω with $\operatorname{Area}(\Omega) \leqslant \frac{cQ}{v\sqrt{Y_M}}$, $\operatorname{length}(\partial\Omega) \ll c + \frac{Q}{v\sqrt{Y_M}} \leqslant d + \frac{Q}{v\sqrt{Y_M}} \leqslant \frac{2Q}{v\sqrt{Y_M}}$, $L = c^{5/6} \leqslant c \leqslant d \leqslant \frac{Q}{v\sqrt{Y_M}}$, we find

$$\mathcal{N}'_{M,Q}(\xi) = \sum_{c \leqslant Q/(v\sqrt{X_M})} \left(\frac{\varphi(c)}{c^2} \operatorname{Area}\left(\Omega'_{M,Q,c}(\xi)\right) + O_\varepsilon\left(\frac{Qc^{-1/6+\varepsilon}}{\sqrt{Y_M}} \right) \right). \tag{7.8}$$

A choice of the pair (m_1, m_2) where $m_1 = \max\{A, B\}$, $m_2 = \max\{C, D\}$ determines M uniquely, and $\max\{v^2C^2, \frac{v^2}{k^2}D^2\} < Y_M$, with similar inequalities for A, B, X_M. According to Lemma 7 we should sum over M with $\min\{X_M, Y_M\} \leqslant Q^{2\beta_0}$. Thus the total contribution of the error term in (7.8) to (7.7) is

$$\ll \sum_{\substack{m_1 \ll Q^{\beta_0} \\ m_2 \ll Q^{\beta_0}}} \frac{Q}{m_2} \sum_{c \ll Q/m_1} c^{-1/6+\varepsilon} \ll Q \log Q \sum_{m_1 \ll Q^{\beta_0}} \left(\frac{Q}{m_1}\right)^{5/6+\varepsilon} \ll_\varepsilon Q^{(11+\beta_0)/6+3\varepsilon}.$$

Möbius summation over c applied to the function $h(c) = \frac{1}{c}\operatorname{Area}(\Omega_{M,Q,c}(\xi))$ with $\|h\|_\infty \leqslant \frac{Q}{\sqrt{C^2+D^2}}$, the change of variables $(c, a, d) = (Qx, Qxz, Qy)$, $(x, y, z) \in [0, \frac{1}{v\sqrt{k(k-|u|)}}] \times [0, \frac{\sqrt{k}}{v\sqrt{k-|u|}}] \times [-k, k]$, and Lemma 7 provide

$$\begin{aligned}\sum_{M\in\mathfrak{S}}\mathcal{N}'_{M,Q}(\xi) &= \sum_{\substack{M\in\mathfrak{S}\\ X_M,Y_M\leqslant Q^{2\beta_0}}}\ \sum_{c\leqslant Q/(v\sqrt{X_M})}\frac{\varphi(c)}{c}h(c)+O_{\varepsilon,\xi}(Q^{(11+\beta_0)/6+\varepsilon})\\ &= \frac{1}{\zeta(2)}\sum_{\substack{M\in\mathfrak{S}\\ X_M,Y_M\leqslant Q^{2\beta_0}}}\left(\int_0^{\frac{Q}{v\sqrt{X_M}}}\operatorname{Area}\left(\Omega'_{M,Q,c}(\xi)\right)\frac{dc}{c}+O\left(\frac{Q\log Q}{\sqrt{X_M}}\right)\right)+O_{\varepsilon,\xi}(Q^{(11+\beta_0)/6+\varepsilon}) \qquad (7.9)\\ &= \frac{Q^2}{\zeta(2)}\sum_{\substack{M\in\mathfrak{S}\\ X_M,Y_M\leqslant Q^{2\beta_0}}}\operatorname{Vol}(S'_{M,\xi})+O_{\varepsilon,\xi}(Q^{1+\beta_0}\log^2 Q+Q^{(11+\beta_0)/6+\varepsilon}),\end{aligned}$$

where $S'_{M,\xi}$ denotes the subset of $S^+_{M,\xi}$ with the additional condition $x\leqslant y$.

The contribution of $d<c$ to $\widetilde{\mathcal{N}}^+_{M,Q}(\xi)$, denoted $\mathcal{N}''_{M,Q}(\xi)$, is estimated similarly, by summing first over $d\leqslant \frac{Q}{v\sqrt{Y_M}}$, then over pairs (b,c) with $bc\equiv -1 \pmod d$. We obtain the same equation as (7.9) with $S'_{M,\xi}$ replaced by $S''_{M,\xi}=S^+_{M,\xi}\setminus S'_{M,\xi}$, thus proving (7.7). □

Corollary 12. *For each $\beta_0\in\left(\frac{2}{3},1\right)$ the following estimate holds:*

$$\sum_{M\in\mathfrak{S}}\widetilde{\mathcal{N}}^+_{M,Q}(\xi)=\frac{Q^2}{\zeta(2)}\sum_{M\in\mathfrak{S}}\operatorname{Vol}(S^+_{M,\xi})+O_\xi(Q^{(11+\beta_0)/6}).$$

Proof. This follows from Lemma 11 and estimate (7.21) in [5]. □

8. Extra symmetries for $\omega=i$

In this section we assume $\omega=i$ and make use of extra symmetries of the hyperbolic lattice centered at i. For each matrix $M=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in\Gamma$ consider the cardinality $\mathcal{N}_{M,Q}(\xi)$ of the set $\mathcal{S}_{M,Q}(\xi)$ defined in (1.6). We first show that in the expression (1.5) we can restrict ourselves to matrices M having positive entries, and then we show that except for finitely many such matrices M we can restrict ourselves to counting only elements $\gamma\in\mathcal{S}_{M,Q}(\xi)$ with positive elements in the second row. Thus we can make use of results of the previous sections to estimate the quantity $\mathcal{R}^\Phi_Q(\xi)$.

Recall the set $\mathfrak{S}$ of matrices $M\neq I$ with nonnegative entries, and denote $s=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$. The signs of the entries of sM, Ms, and sMs show that $\Gamma\setminus\{I,s\}$ is partitioned into $\mathfrak{S}\cup s\mathfrak{S}\cup\mathfrak{S}s\cup s\mathfrak{S}s$. The equalities $\Phi(gs)=\Phi(g)$, $si=i$, and $\|g\|=\|gs\|$ yield

$$\mathcal{S}_{Ms,Q}(\xi)=\mathcal{S}_{M,Q}(\xi),\qquad \mathcal{S}_{sM,Q}(\xi)=\mathcal{S}_{M,Q}(\xi)s,$$

$$\mathcal{N}_{M,Q}(\xi)=\mathcal{N}_{Ms,Q}(\xi)=\mathcal{N}_{sM,Q}(\xi)=\mathcal{N}_{sMs,Q}(\xi).$$

Therefore (1.5) becomes

$$\mathcal{R}^\Phi_Q(\xi)=2\sum_{M\in\mathfrak{S}}\mathcal{N}_{M,Q}(\xi).$$

Let $\mathcal{S}^+_{M,Q}(\xi)$, respectively $\mathcal{S}^-_{M,Q}(\xi)$, denote the subsets of $\mathcal{S}_{M,Q}(\xi)$ consisting of matrices with $c,d>0$, respectively $c>0>d$. Denote by $\mathcal{N}^\pm_{M,Q}(\xi)$ the cardinality of $\mathcal{S}^\pm_{M,Q}(\xi)$.

Recall the finite set $\mathcal{F}(\xi)$ defined in (5.1) and let

$$\widetilde{\mathcal{F}}(\xi):=\mathcal{F}(\xi)\cup s\mathcal{F}(\xi)\cup s\mathcal{F}(\xi)^{-1}.$$

Lemma 13. (i) *For any $M \in \mathfrak{S}$ the mapping $\gamma \mapsto \gamma M s^{-1}$ defines a bijection between the sets*

$$\mathcal{S}^{-,*}_{M,Q}(\xi) = \Big\{ \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{S}^{-}_{M,Q}(\xi) : \tfrac{B}{D} < -\tfrac{d}{c} < \tfrac{A}{C} \Big\}$$

and $\mathcal{S}^{+}_{M^t,Q}(\xi)$.

(ii) *If $M \in \mathfrak{S} \setminus \widetilde{\mathcal{F}}(\xi)$, then $\mathcal{S}^{-}_{M,Q}(\xi) = \mathcal{S}^{-,*}_{M,Q}(\xi)$, so that $\mathcal{N}^{-}_{M,Q}(\xi) = \mathcal{N}^{+}_{M^t,Q}(\xi)$.*

Proof. (i) Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{S}^{-,*}_{M,Q}(\xi)$ and $\gamma M s^{-1} = \left(\begin{smallmatrix} * & * \\ \tilde{c} & \tilde{d} \end{smallmatrix}\right)$. On one hand $c > 0$ and $\frac{B}{D} < \frac{-d}{c} < \frac{A}{C}$ imply $\tilde{c} = -cB - dD > 0, \tilde{d} = cA + dC > 0$, while $\tilde{c} > 0$ and $\tilde{d} > 0$ imply $c = \tilde{c}C + \tilde{d}D > 0$ and $d = -\tilde{c}A - \tilde{d}B < 0$. On the other hand, utilizing also $\|\tilde{\gamma}\| = \|\gamma M\|$, $sM^{-1}s^{-1} = M^t$, $\|\gamma M s^{-1} M^t\| = \|\gamma\|$, we conclude that the map above is a bijection.

(ii) Suppose, by contradiction, that there exists $\gamma \in \mathcal{S}^{-}_{M,Q}(\xi) \setminus \mathcal{S}^{-,*}_{M,Q}(\xi)$. Setting

$$\gamma M = \left(\begin{smallmatrix} aA+bC & aB+bD \\ cA+dC & cB+dD \end{smallmatrix}\right) = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) = \gamma',$$

notice that $Q \geqslant \max\{c, -d, |c'|, |d'|\}$ and two cases can occur:

1) $\frac{-d}{c} < \frac{B}{D} < \frac{A}{C}$. Then $c' > 0$, $d' > 0$, and $\frac{b'}{d'} < \frac{a'}{c'} \leqslant \frac{a}{c} < \frac{-b}{-d}$. By Lemma 6, $\gamma s^{-1} = \gamma' M_0$ with $M_0 \in \mathcal{F}(\xi)$, so $M = (M_0 s)^{-1}$, contradiction.

2) $\frac{B}{D} < \frac{A}{C} < \frac{-d}{c}$. Then $c' < 0$, $d' < 0$, and $\frac{a}{c} < \frac{-b}{-d} \leqslant \frac{b'}{d'} < \frac{a'}{c'}$. Lemma 6 gives $-\gamma' = \gamma s^{-1} M_0$ with $M_0 \in \mathcal{F}(\xi)$, so $M = -s^{-1} M_0$, contradiction. □

Remark 14. The analogue of Lemma 13 (ii) holds for $\widetilde{\mathcal{S}}^{-}_{M,Q}(\xi)$ in place of $\mathcal{S}^{-}_{M,Q}(\xi)$ (see the notation preceding Lemma 9). Namely there is no matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \widetilde{\mathcal{S}}^{-}_{M,Q}(\xi)$ with either $-\frac{d}{c} < \frac{B}{D}$ or $\frac{A}{C} < \frac{-d}{c}$. Indeed, referring to the notation in the proof of Lemma 13 (ii), we have $c'^2 + d'^2 = c^2 X_M + d^2 Y_M + 2cdZ_M \leqslant Q^2$, so that $Q \geqslant \max\{c, -d, |c'|, |d'|\}$, and the rest of the proof goes through unchanged.

Consider now the region $S_{M,\xi}$ in (1.8), which in the present case $\omega = i$ becomes the region of triples $(x, y, z) \in [0,1] \times [-1,1]^2$ for which[3]

$$|\Xi_M(x,y)| \leqslant \xi, \quad \max\{x^2 + y^2, x^2 X_M + y^2 Y_M + 2xyZ_M\} \leqslant \frac{1}{1+z^2}. \tag{8.1}$$

Consider also the subsets $S^{+}_{M,\xi}$, $S^{-}_{M,\xi}$, $S^{-,*}_{M,\xi}$ of $S_{M,\xi}$ defined, respectively, by $y > 0$, $y < 0$, $y < 0$ and $\frac{B}{D} < \frac{-y}{x} < \frac{A}{C}$. As in Lemma 13, the mapping $(x,y) \mapsto (x,y)Ms^{-1} = (\tilde{x}, \tilde{y})$ defines a diffeomorphism between the sets $S^{-,*}_{M,\xi}$ and $S^{+}_{M^t,\xi}$, showing in particular that

$$\mathrm{Vol}(S^{-,*}_{M,\xi}) = \mathrm{Vol}(S^{+}_{M^t,\xi}), \qquad \forall M \in \mathfrak{S} \setminus \widetilde{\mathcal{F}}(\xi). \tag{8.2}$$

Lemma 15. *If $M \in \mathfrak{S} \setminus \widetilde{\mathcal{F}}(\xi)$, then* $\mathrm{Vol}(S^{-}_{M,\xi}) = \mathrm{Vol}(S^{-,*}_{M,\xi})$.

Proof. Suppose by contradiction that the set $S^{-}_{M,\xi} \setminus S^{-,*}_{M,\xi}$ contains an interior point (x_0, y_0, z_0), that is $(x_0, y_0, z_0) \in (0,1) \times (-1,0) \times (-1,1)$ satisfies both (8.1) and $\frac{-y}{x} \in \big(0, \frac{B}{D}\big) \cup \big(\frac{A}{C}, \infty\big)$. Now the set of rational points

$$\Omega_Q = \Big\{ \Big(\frac{c}{Q}, \frac{d}{Q}, \frac{a}{Q}\Big) : (c,d) = 1, \quad Q \geqslant c > 0 > d \geqslant -Q, \quad ad \equiv 1 \pmod{c}, \quad |a| < c \Big\}$$

is dense in $D = [0,1] \times [-1,0] \times [-1,1]$; indeed for each parallelepiped $R \subset D$, we can count the number of points in the scaled sets $QR \cap Q\Omega_Q$, as in the proof of Lemma 8, and conclude that $\Omega_Q \cap R$ is dense in R. Therefore for large enough Q, we can find points in Ω_Q arbitrarily

[3]As $X_M, Y_M \geqslant 1$, $Z_M \geqslant 0$, when $y > 0$ the inequality $x^2 + y^2 \leqslant \frac{1}{1+z^2}$ is obsolete.

close to $(x_0, y_0, x_0 z_0)$, and it follows that there exists $(a,c,d) \in \widetilde{\mathcal{S}}_{M,Q}(\xi)$ with $c > 0 > d$ and $\frac{-d}{c} \in \big(0, \frac{B}{D}\big) \cup \big(\frac{A}{C}, \infty\big)$, which contradicts Remark 14. $\square$

Estimates for $\mathcal{N}^{-}_{M,Q}(\xi)$ with $M \notin \widetilde{\mathcal{F}}(\xi)$ are derived from those on $\mathcal{N}^{+}_{M^t,Q}(\xi)$ by Lemma 13. We can now estimate $\sum_{M\in\mathfrak{S}} \mathcal{N}_{M,Q}(\xi)$ by first breaking the sum into sums over $\widetilde{\mathcal{F}}(\xi)$ and over $\mathfrak{S}\backslash\widetilde{\mathcal{F}}(\xi)$; for the first sum we use Lemma 10, while for the second we use $\mathcal{N}^{-}_{M,Q}(\xi) = \mathcal{N}^{+}_{M^t,Q}(\xi)$ and Lemmas 9 and 11. Finally, employing (8.2) and Lemma 15, we find

$$\sum_{M\in\mathfrak{S}} \mathcal{N}_{M,Q}(\xi) = \frac{Q^2}{\zeta(2)} \sum_{M\in\mathfrak{S}} \mathrm{Vol}(S_{M,\xi}) + O_\xi(Q^{(11+\beta_0)/6}). \tag{8.3}$$

To complete the sum to $M \in \Gamma$, note that $\mathrm{Vol}(S_{M,\xi}) = \mathrm{Vol}(S_{Ms,\xi})$. This is also seen to coincide with $\mathrm{Vol}(S_{sM,\xi})$, and thus with $\mathrm{Vol}(S_{sMs,\xi})$, by employing $\Xi_{sM}(y,-x) = \Xi_M(x,y)$ and the change of variable $(x,y) \mapsto (-y,x)$ if $(x,y) \in [0,1]\times[-1,0]$, respectively $(x,y) \mapsto (y,-x) \mapsto (y,-x)$ if $(x,y) \in [0,1]\times[0,1]$. This proves (1.9).

9. A closed form formula for $\mathrm{Vol}(S_{M,\xi})$

In this section we evaluate the volume of the body $S_{M,\xi}$ in (1.8) for *arbitrary* ω, which leads to the formula in Conjecture 1. For $\omega = i$, the proof of this conjecture is based on the results of the previous section.

The volume can be brought in closed form using the substitution

$$x\omega + y = re^{i\theta}, \quad z = v\tan t + u, \tag{9.1}$$

with $\Xi_M(x,y)$ given by the first equation in (2.4), $k^2x^2 + y^2 + 2uxy = r^2$, $x^2X_M + y^2Y_M + 2xyZ_M = r^2 \sinh \ell(M)\big(\coth \ell(M) + \cos(\theta_M - 2\theta)\big)$, $k^2 + z^2 - 2uz = \frac{v^2}{\cos^2 t}$, to

$$\mathrm{Vol}(S_{M,\xi}) = v \int_{\arctan((-k-u)/v)}^{\arctan((k-u)/v)} B_M(\xi,t) \frac{dt}{\cos^2 t} = v \int_{\beta/2-\pi/2}^{\beta/2} B_M(\xi,t)\frac{dt}{\cos^2 t}, \tag{9.2}$$

with $\beta \in (0,\pi)$ such that $\omega = ke^{i\beta}$, where $B_M(\xi,t)$ is the area of the region defined in polar coordinates (r,θ) by

$$\begin{cases} \frac{r}{v}(\sin\theta, k\sin(\beta-\theta)) \in \big[0, \frac{\widetilde{Q}}{kQ}\big] \times \big[-\frac{\widetilde{Q}}{Q}, \frac{\widetilde{Q}}{Q}\big] \\ \dfrac{v}{\xi} \dfrac{|\sin(\theta_M - 2\theta)|}{U_M + \cos(\theta_M - 2\theta)} \leqslant r^2 \leqslant \dfrac{\cos^2 t}{v^2} \min\Big\{1, \dfrac{1}{\sinh \ell(M)(U_M + \cos(\theta_M - 2\theta))}\Big\}, \end{cases} \tag{9.3}$$

with $U_M = \coth \ell(M) = \frac{T}{\sqrt{T^2-\Delta^2}} > 1$, $T = \|M\|^2$.

Using the second condition in (9.3), we have $r^2 \leqslant \frac{1}{v^2} \leqslant \frac{1}{k(k-|u|)}$. Hence the first condition in (9.3) can be replaced by $0 \leqslant \theta \leqslant \pi$, and the area can be expressed in closed form:

$$B_M(\xi,t) = \frac{1}{2v} \int_0^\pi \frac{\Big(\frac{\cos^2 t}{v^2} \min\big\{\frac{1}{\sinh \ell(M)}, U_M + \cos(\theta_M - 2\theta)\big\} - \frac{v}{\xi}|\sin(\theta_M - 2\theta)|\Big)_+}{U_M + \cos(\theta_M - 2\theta)}\, d\theta,$$

with $f_+ = \max\{f, 0\}$.

Since we are interested in the pair correlation of the angles θ_γ, we define

$$\mathcal{R}^\theta_Q(\xi) := \#\Big\{(\gamma,\gamma') \in \mathfrak{R}^2_Q \;:\; \gamma \neq \gamma', \quad 0 \leqslant \theta(\gamma) - \theta(\gamma') < \frac{\xi}{Q^2}\Big\}.$$

Following the approximation arguments from Section 8 of [5], from (8.3) we obtain the following asymptotics:

Proposition 16. *For $\omega = i$ the estimate*

$$\mathcal{R}_Q^\theta(\xi) = \frac{Q^2}{2\zeta(2)} \sum_{M\in\Gamma\setminus\{I\}} B_M(\xi) + O_{\xi,\varepsilon}\big(Q^{47/24+\varepsilon}\big)$$

holds, where

$$B_M(\xi) = v\int_{\beta/2-\pi/2}^{\beta/2} B_M\bigg(\frac{v\xi}{2\cos^2 t}, t\bigg)\frac{dt}{\cos^2 t}.$$

Since $B_M\big(\frac{\xi}{\cos^2 t}, t\big) = B_M(\xi, 0)\cos^2 t$, we find

$$B_M(\xi) = B_M\bigg(\frac{v\xi}{2}, 0\bigg)\frac{\pi v}{2}.$$

Taking derivatives we obtain

$$B_M'(\xi) = \frac{\pi}{2\xi^2}\int_{I_{\xi,M}} \frac{|\sin(\theta_M - 2\theta)|}{U_M + \cos(\theta_M - 2\theta)}\,d\theta,$$

with $I_{\xi,M} = \big\{\theta\in[0,\pi] : |\sin(\theta_M - 2\theta)| \leqslant \frac{\xi}{\Delta}\min\{U_M + \cos(\theta_M-2\theta), \frac{1}{\sinh\ell(M)}\}\big\}$ (recall $\Delta = 2v^2$). With $C_M := \sqrt{\frac{T-\Delta}{T+\Delta}} = \tanh\big(\frac{\ell(M)}{2}\big) \in (0,1)$ we have $\sinh\ell(M) = \frac{2C_M}{1-C_M^2}$, $\cosh\ell(M) = \frac{1+C_M^2}{1-C_M^2}$, $U_M - C_M = \frac{1}{\sinh\ell(M)}$, $\sinh\big(\frac{\ell(M)}{2}\big) = \frac{\sqrt{T-2v^2}}{2v} = \frac{C_M}{\sqrt{1-C_M^2}}$. Using the change of variable $u = 2\theta - \theta_M \in [-\pi,\pi]$ the integrand is even on $[-\pi,\pi]$, and so we have

$$B_M'(\Delta\xi) = \frac{\pi}{2\Delta^2\xi^2}\int_{J_{\xi,M}} \frac{\sin u}{U_M + \cos u}\,du,$$

with $J_{\xi,M} = \big(J_{\xi,M}^{(1)} \cup J_{\xi,M}^{(2)}\big)\cap[0,\pi]$, where

$$J_{\xi,M}^{(1)} = \bigg\{u : \cos u \geqslant -C_M,\ \sin u \leqslant \frac{\xi}{\sinh\ell(M)}\bigg\},$$

$$J_{\xi,M}^{(2)} = \big\{u : \cos u \leqslant -C_M,\ \sin u \leqslant \xi(U_M + \cos u)\big\}.$$

A direct calculation provides

$$J_{\xi,M}^{(1)} = \begin{cases} [0, \arccos(-C_M)] & \text{if } \xi \geqslant \sinh\ell(M) = \frac{\sqrt{T^2-\Delta^2}}{\Delta}, \\ \big[0, \arcsin\big(\frac{\xi}{\sinh\ell(M)}\big)\big] \cup \big[\pi - \arcsin\big(\frac{\xi}{\sinh\ell(M)}\big), \arccos(-C_M)\big] & \\ \qquad\qquad \text{if } 2\sinh\big(\frac{\ell(M)}{2}\big) \leqslant \xi \leqslant \sinh\ell(M), & \\ \big[0, \arcsin\big(\frac{\xi}{\sinh\ell(M)}\big)\big] & \text{if } \xi \leqslant 2\sinh\big(\frac{\ell(M)}{2}\big) = \frac{\sqrt{T-\Delta}}{v}. \end{cases}$$

$$J_{\xi,M}^{(2)} = \begin{cases} [\arccos(-C_M), \pi] & \text{if } \xi \geqslant \sinh\ell(M), \\ [\arccos(-C_M), \alpha + \arcsin(U_M\sin\alpha)] \cup [\pi + \alpha - \arcsin(U_M\sin\alpha), \pi] & \\ \qquad\qquad \text{if } 2\sinh\big(\frac{\ell(M)}{2}\big) \leqslant \xi \leqslant \sinh\ell(M), & \\ [\pi + \alpha - \arcsin(U_M\sin\alpha), \pi] & \text{if } \xi \leqslant 2\sinh\big(\frac{\ell(M)}{2}\big). \end{cases}$$

$$J_{\xi,M} = \begin{cases} [0,\pi] & \text{if } \xi \geqslant \sinh\ell(M), \\ \big[0, \arcsin\big(\frac{\xi}{\sinh\ell(M)}\big)\big] \cup \big[\pi - \arcsin\big(\frac{\xi}{\sinh\ell(M)}\big), \alpha + \arcsin(U_M\sin\alpha)\big] & \\ \qquad \cup[\pi + \alpha - \arcsin(U_M\sin\alpha), \pi] \qquad \text{if } 2\sinh\big(\frac{\ell(M)}{2}\big) \leqslant \xi \leqslant \sinh\ell(M), & \\ \big[0, \arcsin\big(\frac{\xi}{\sinh\ell(M)}\big)\big] \cup [\pi + \alpha - \arcsin(U_M\sin\alpha), \pi] \qquad \text{if } \xi \leqslant 2\sinh\big(\frac{\ell(M)}{2}\big), & \end{cases}$$

where $\alpha=\alpha(\xi)=\arcsin\big(\frac{\xi}{\sqrt{\xi^2+1}}\big)\in\big(0,\frac{\pi}{2}\big)$. We obtain

$$B'_M(\Delta\xi)=\frac{\pi}{\Delta^2\xi^2}f_\xi\big(\ell(M)\big),$$

with $f_\xi(\ell)$ the function defined in the introduction. Letting $R_2^\theta(\xi)=\lim_{Q\to\infty}\frac{1}{Q^2}\mathcal{R}_Q^\theta(\xi)$, from Proposition 16 we infer

$$\frac{dR_2^\theta}{d\xi}(\Delta\xi)=\frac{\pi}{2\zeta(2)\Delta^2\xi^2}\sum_{M\in\Gamma}f_\xi\big(\ell(M)\big)$$

(note that $f_\xi(0)=0$ so we can include I in the range of summation). Taking into account that the pair correlation distribution $R_2(\xi)$ in the introduction involves normalized angles, and that $B_Q\sim\frac{3}{\Delta}Q^2$, we have

$$R_2\left(\frac{3\xi}{\pi\Delta}\right)=\frac{\Delta}{3}R_2^\theta(\xi),\quad g_2\left(\frac{3\xi}{\pi}\right)=\frac{\pi\Delta^2}{9}\frac{dR_2^\theta}{d\xi}(\Delta\xi).$$

This leads to the formula for g_2 stated in Conjecture 1, and proves Theorem 1 when $\omega=i$.

10. The case $\omega=\rho$

In the case of the other elliptic point $\omega=\rho$ one can take advantage of other symmetries to prove Conjecture 1. Consider the matrix $w=\left(\begin{smallmatrix}1&-1\\1&0\end{smallmatrix}\right)$ fixing the point ρ. This time we partition the upper half plane $\mathbb{H}$ into three regions which are permuted clockwise by w:

$$\mathbf{I}=\left\{z\in\mathbb{H}:\operatorname{Re}z>\tfrac{1}{2},|z-1|<1\right\},\quad \mathbf{II}=\left\{z\in\mathbb{H}:\operatorname{Re}z<\tfrac{1}{2},|z|<1\right\},$$
$$\mathbf{III}=\{z\in\mathbb{H}:|z-1|>1,|z|>1\}.$$

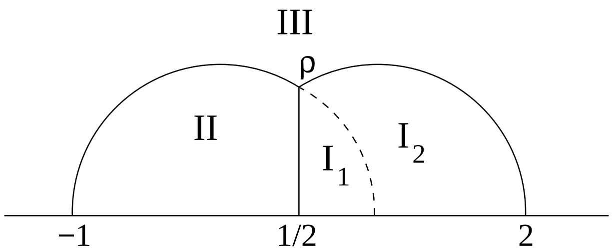

Figure 2. Symmetric geodesics through ρ

The condition that $\gamma\rho$ belongs to one of the three regions is easily stated in terms of (X,Y,Z) given by (2.1), using the relations $\operatorname{Re}(\gamma\rho)=\frac{Z}{Y}$, $|\gamma\rho|^2=\frac{X}{Y}$, $|\gamma\rho-1|^2=1+\frac{X-2Z}{Y}$:

$$\gamma\rho\in\mathbf{I}\iff 2Z>X,\ 2Z>Y,\qquad \gamma\rho\in\mathbf{II}\iff Y>X,\ Y>2Z,$$
$$\gamma\rho\in\mathbf{III}\iff X>Y,X>2Z.$$

Next we determine the restrictions that the condition $\gamma\rho\in\mathbf{I}$ places on the entries of γ. As a consequence of $2Z>Y,2Z>X$, a quick check shows that the entries of γ are nonzero. Since $abcd=bc+(bc)^2>0$, we also have that ac and bd have the same sign. In fact $ac>0$, $bd>0$: if the contrary were true, from $2Z=2(ac+bd)+2ad+1>0$ it would follow $ad>0$ and without loss of generality we can assume $a>0$, $c<0$, $d>0$, $b<0$; from $ad>-ac-bd$ it would then follow that $d>-c$, $a>-b$, which implies $ad-bc>1$, a contradiction. Since $ac>0$, $bd>0$, among the matrices $\gamma,\gamma w,\gamma w^2$ with the same coordinates (X,Y,Z) precisely one has entries of the same sign. Therefore we assume from now on that γ has positive entries whenever $\gamma\rho\in\mathbf{I}$.

Using the substitution $b = \frac{ad-1}{c}$, one checks that

$$2Z > Y \iff 2a - c > \frac{2d+c}{c^2+d^2+cd} \iff 2a > c$$

(the last equivalence follows since the fraction is less than 1 and $2a - c$ is integral). Similarly $2Z > X$ if and only if $2d > b$. In conclusion, the point $\gamma\rho$ belongs to $\mathbf{I}$ if and only if, after perhaps replacing γ by γw or γw^2, we have

$$a, b, c, d > 0, \quad \frac{a}{c} > \frac{1}{2}, \quad \frac{b}{d} < 2. \tag{10.1}$$

Since w permutes the regions $\mathbf{I}$, $\mathbf{II}$ and $\mathbf{III}$ and $w\rho = \rho$, the previous discussion shows that Γ can be partitioned as

$$\Gamma \setminus \{I, w, w^2\} = \bigcup_{r,s\in\{0,1,2\}} w^r \mathfrak{M} w^s,$$

where

$$\mathfrak{M} := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma : \ C, D > 0, \ \frac{1}{2} \leqslant \frac{B}{D} < \frac{A}{C} \leqslant 2 \right\}.$$

We can now rewrite the sum (1.5) as follows. Since $\|gw\| = \|g\|$ and $\Phi(gw) = \Phi(g)$, we have $\mathcal{S}_{Mw^s,Q}(\xi) = \mathcal{S}_{M,Q}(\xi)$. Moreover the map $g \mapsto gw^{-r}$ takes $\mathcal{S}_{M,Q}(\xi)$ bijectively onto $\mathcal{S}_{w^r M,Q}(\xi)$, so $\mathcal{N}_{w^r M w^s,Q}(\xi) = \mathcal{N}_{M,Q}(\xi)$. We infer

$$\mathcal{R}_Q^\Phi = \frac{9}{2} \sum_{M\in\mathfrak{M}} \mathcal{N}_{M,Q}(\xi).$$

To state the equivalent of Lemma 13, we further divide region $\mathbf{I}$ in two regions $\mathbf{I_1}$ and $\mathbf{I_2}$, according as $|z| < 1$ or $|z| > 1$ (see Figure 2). Lemma 2 shows that $\gamma\rho \mapsto \widetilde{\gamma}\rho$ is a bijection of $\mathbf{I_1}$ onto $\mathbf{I_2}$. Let also $\mathfrak{M}_1$, respectively $\mathfrak{M}_2$, denote the subset of $M \in \mathfrak{M}$ with $M\rho \in \mathbf{I_1}$, respectively $M\rho \in \mathbf{I_2}$.

For $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathfrak{M}$, let $\mathcal{S}^{-,1}_{M,Q}(\xi)$, respectively $\mathcal{S}^{-,2}_{M,Q}(\xi)$ be the sets of those $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{S}^-_{M,Q}(\xi)$ for which $\frac{B}{D} < -\frac{d}{c} < \frac{A+B}{C+D}$, respectively $\frac{A+B}{C+D} < -\frac{d}{c} < \frac{A}{C}$. With $\mathcal{F}(\xi)$ as in in (5.1) define

$$\widetilde{\mathcal{F}}(\xi) = \mathcal{F}(\xi) \cup w\mathcal{F}(\xi) \cup w^{-1}\mathcal{F}(\xi) \cup w\mathcal{F}(\xi)^{-1} \cup w^{-1}\mathcal{F}(\xi)^{-1}.$$

Lemma 17. (i) *The map $\gamma \mapsto \gamma M w^{-1}$ is a bijection between $\mathcal{S}^{-,1}_{M,Q}(\xi)$ and $\mathcal{S}^{+}_{wM^{-1}w^2,Q}(\xi)$, and the map $\gamma \mapsto \gamma M w$ is a bijection between $\mathcal{S}^{-,2}_{M,Q}(\xi)$ and $\mathcal{S}^{+}_{w^2M^{-1}w,Q}(\xi)$.*

(ii) *If $M \in \mathfrak{M} \setminus \widetilde{\mathcal{F}}(\xi)$, then $\mathcal{S}^-_{M,Q}(\xi) = \mathcal{S}^{-,1}_{M,Q}(\xi) \cup \mathcal{S}^{-,2}_{M,Q}(\xi)$, so that*

$$\mathcal{N}^-_{M,Q} = \mathcal{N}^+_{wM^{-1}w^2,Q} + \mathcal{N}^+_{w^2M^{-1}w,Q}.$$

The proof is very similar to that of Lemma 13 and we leave it as an exercise for the reader.

Let now $\mathfrak{M}_i(\xi) = \mathfrak{M}_i \setminus \widetilde{\mathcal{F}}(\xi)$, $i \in \{0, 1\}$, and define the sets

$$\mathfrak{S}_1(\xi) = \{wM^{-1}w^2, w^2M^{-1}w^2 : M \in \mathfrak{M}_1(\xi)\}, \quad \mathfrak{S}_2(\xi) = \{w^2M^{-1}w, wM^{-1}w : M \in \mathfrak{M}_2(\xi)\},$$

both easily checked to be contained in $\mathfrak{S}$ (namely they contain matrices with positive entries). From Lemma 17 and the fact that $\mathcal{N}_{Mw^r,Q} = \mathcal{N}_{Mw^r}$ it follows that:

$$\sum_{M\in\mathfrak{M}} \mathcal{N}_{M,Q}(\xi) = \sum_{M\in\mathfrak{M}\cap\widetilde{\mathcal{F}}(\xi)} \mathcal{N}_{M,Q}(\xi) + \sum_{M\in\mathfrak{M}_1(\xi)\cup\mathfrak{M}_2(\xi)} \mathcal{N}^+_{M,Q}(\xi) + \sum_{M\in\mathfrak{S}_1(\xi)\cup\mathfrak{S}_2(\xi)} \mathcal{N}^+_{M,Q}(\xi)$$

Since now we only sum over matrices with positive entries, the approximation arguments employed in the case $\omega = i$ also apply here, with $S_{M,\xi}$ defined in (1.8) and $S^{\pm}_{M,\xi}$ the subset of $S_{M,\xi}$ defined by the additional condition $y > 0$ or $y < 0$, leading to

$$\frac{\zeta(2)}{Q^2} \sum_{M \in \mathfrak{M}} \mathcal{N}_{M,Q}(\xi) \sim \sum_{M \in \mathfrak{M} \cap \widetilde{\mathcal{F}}(\xi)} \operatorname{Vol}(S_{M,\xi}) + \sum_{M \in \mathfrak{M}_1(\xi) \cup \mathfrak{M}_2(\xi)} \operatorname{Vol}(S^{+}_{M,\xi}) + \sum_{M \in \mathfrak{S}_1(\xi) \cup \mathfrak{S}_2(\xi)} \operatorname{Vol}(S^{+}_{M,\xi}).$$

Using equalities analogous to those of Lemma 17, for volumes instead of the number of lattice points, we arrive at

$$\mathcal{R}^{\Phi}_{Q}(\xi) \sim \frac{9Q^2}{2\zeta(2)} \sum_{M \in \mathfrak{M}} \operatorname{Vol}(S_{M,\xi}). \tag{10.2}$$

Finally the sum of volumes in (10.2) can be extended from $\mathfrak{M}$ to Γ since $\operatorname{Vol}(S_{w^r M w^s,\xi}) = \operatorname{Vol}(S_{M,\xi})$. To check this, we use the polar coordinates $x\rho + y = re^{i\theta}$ from (9.1), leading to formula (9.2) for $\operatorname{Vol}(S_{M,\xi})$. Note that the inequalities defining the volume in polar coordinates only depend on $\ell(M)$, r and $\theta_M - 2\theta$, with the restriction $\theta \in [0, \pi]$. Since $\ell(M)$, θ_M only depend on $M\rho$, it follows that $\operatorname{Vol}(S_{Mw^s,\xi}) = \operatorname{Vol}(S_{M,\xi})$.

To show $\operatorname{Vol}(S_{wM,\xi}) = \operatorname{Vol}(S_{M,\xi})$, let $\gamma_{x,y} \in \mathrm{SL}_2(\mathbb{R})$ be any matrix with lower row $(x, y) \neq (0, 0)$; note that $j(\gamma_{x,y}, \rho) := x\rho + y = re^{i\theta}$ in polar coordinates. The transformation $(x, y, z) \mapsto (x', y', z)$ with (x', y') defined by $\gamma_{x',y'} = \gamma_{x,y} w^{-1}$ has $x'\rho + y' = j(\gamma_{x,y}, \rho) j(w^{-1}, \rho) = re^{i(\theta - \pi/3)}$, so in polar coordinates it corresponds to $(r, \theta) \mapsto (r' = r, \theta' = \theta - \pi/3)$. Since $M \mapsto wM$ results in $\theta_M \mapsto \theta_M - 2\pi/3$, and the inequalities (9.3) defining $S_{M,\xi}$ involve only $\theta_M - 2\theta$, it follows that the transformation $(x, y, z) \mapsto (x', y', z)$ above maps the volume $S_{M,\xi}$ onto a volume $S'_{wM,\xi}$, defined like $S_{wM,\xi}$ but with the range $\theta \in [0, \pi]$ replaced by $\theta' \in [-\pi/3, 2\pi/3]$. Since in the formula for $B_M(\xi, t)$ following (9.3) the integrand has period π, we conclude $\operatorname{Vol}(S_{wM,\xi}) = \operatorname{Vol}(S_{M,\xi})$,

This concludes the proof of (1.9). The formula in Theorem 1 follows from the results of Section 9.

Appendix A. Arithmetic description of closed geodesics through ρ

In this appendix we discuss the connection between the hyperbolic lattice centered at ρ and closed geodesics on the modular surface passing through $\Pi(\rho)$ where $\Pi : \mathbb{H} \to \mathbb{H}/\Gamma$ is the projection map. In the case of the hyperbolic lattice centered at i, the corresponding geodesics are the reciprocal geodesics studied by Fricke and Klein [5, Section 2]. We similarly describe the primitive closed geodesics passing through $\Pi(\rho)$, which have an interesting arithmetic structure.

Closed geodesics on the modular surface correspond to conjugacy classes $\{g\}$ of hyperbolic elements $g \in \Gamma$. If $\mathfrak{a}_g \subset \mathbb{H}$ is the axis of g (the semicircle connecting the two fixed points of g on the real axis), then the geodesic corresponding to g on X is $\Pi(z_0 \to gz_0)$ for any fixed point $z_0 \in \mathfrak{a}_g$. We are interested in geodesics passing through $\Pi(\rho)$. Let $\mathcal{R}$ denote the set of conjugacy classes of hyperbolic elements which contain a matrix g whose axis passes through ρ. Let $\mathcal{R}^{\mathrm{prim}} \subset \mathcal{R}$ be the subset of primitive conjugacy classes. We will give an arithmetic description of $\mathcal{R}^{\mathrm{prim}}$.

Let $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a primitive hyperbolic matrix whose axis $\mathfrak{a}_g$ passes through ρ. The fixed points $\lambda > \overline{\lambda}$ of g satisfy the equation

$$C\lambda^2 + (D - A)\lambda - B = 0.$$

Imposing the condition $(\lambda - \overline{\lambda})^2 = |\rho - \lambda|^2 + |\overline{\rho} - \lambda|^2$ we conclude that

$$\rho \in \mathfrak{a}_g \iff D - A = 2(B - C). \tag{A.1}$$

The matrices wgw^2, and w^2gw are also primitive. Their axes are the same as $\mathfrak{a}_g$ rotated by $\pm 2\pi/3$ around ρ, hence the class $\{g\}$ contains a matrix, still denoted by g, with $g\rho \in \mathbf{I}$ (the first region in Section 10). This is equivalent to $\lambda \in \left(\frac{1}{2}, 2\right)$ and $\operatorname{Re}(g\rho) > \frac{1}{2}$, which is further equivalent with A, B, C, D being all positive or all negative.

In conclusion each class $\{h\} \in \mathcal{R}^{\mathrm{prim}}$ contains a matrix g as above with positive entries satisfying (A.1), and so we are left to describe the set of such matrices and determine when two such matrices are conjugate.

The condition (A.1), together with $AD - BC = 1$, implies that $(A+D)^2 - 4(B^2 + C^2 - BC) = 4$. Writing $k = \gcd(B, C)$, $B = kB_0$, $C = kC_0$, $T := A + D$, the pair (T, k) is a solution to Pell's equation

$$T^2 - 4k^2\Delta = 4, \tag{A.2}$$

with $\Delta = B_0^2 + C_0^2 - B_0C_0$. In fact, (T, k) is the minimal positive solutions since g is primitive. Direct computation using (A.1) shows that

$$\cosh d(\rho, g\rho) = \frac{T^2}{2} - 1. \tag{A.3}$$

We are led to define the set

$$\begin{aligned} \mathcal{D}_\rho &:= \{(B_0, C_0) : B_0, C_0 > 0,\ (B_0, C_0) = 1,\ \Delta = B_0^2 + C_0^2 - B_0C_0 \text{ not a square }\} \\ &= \bigcup_{\Delta \in D_\rho} \mathcal{D}_\Delta, \end{aligned} \tag{A.4}$$

where $\mathcal{D}_\Delta$ is the finite subset of pairs (B_0, C_0) as above having fixed Δ. We denoted by $\mathcal{D}_\rho$ the set of possible such Δ, which is the same as the set of positive numbers all of whose prime factors are congruent to 1 mod 3, or the prime 3 appearing to the first power. The cardinality of $\mathcal{D}_\Delta$ is $2^{1+\nu}$ with ν the number of distinct prime factors $p \equiv 1 \pmod 3$ of Δ. We conclude that there is a parametrization

$$\varphi : \mathcal{D}_\rho \to \mathcal{R}^{\mathrm{prim}}, \quad \varphi(B_0, C_0) = \left\{ \begin{pmatrix} \frac{T}{2} - k(B_0 - C_0) & kB_0 \\ kC_0 & \frac{T}{2} + k(B_0 - C_0) \end{pmatrix} \right\},$$

where (T, k) is the smallest positive solution of Pell's equation $T^2 - 4k^2\Delta = 4$ (the minimality of (T, k) ensures that the image of φ consists of primitive conjugacy classes only).

We are left to determine, for each primitive hyperbolic $g \in \Gamma$ satisfying (3.1) and having positive entries, the set of $h \neq g$ with positive entries, having $\{g\} = \{h\}$. Assume therefore that $h = \gamma^{-1} g \gamma$. Then γ maps $\mathfrak{a}_h$ onto $\mathfrak{a}_g$ (as it can be seen by looking at what γ does to the endpoints of $\mathfrak{a}_h, \mathfrak{a}_g$), and we therefore have that $\gamma\rho \in \mathfrak{a}_g$. Replacing γ by $g^n\gamma$ for an appropriate n, we can assume that $\gamma\rho \in (\rho \to g\rho)$. Writing $g = \gamma\gamma'$, it follows that $h = \gamma'\gamma$. Since γ maps $\mathfrak{a}_h$ onto $\mathfrak{a}_g$, and since the point $g\rho = \gamma\gamma'\rho \in (\gamma\rho \to \gamma\gamma'\gamma\rho)$, it follows that $\gamma'\rho \in (\rho \to h\rho)$.

Therefore the number of hyperbolic h with $\{h\} = \{g\}$ and $h\rho \in \mathbf{I}$ is the same as the number of points $\gamma\rho \in (\rho \to g\rho)$ (open geodesic segment) with $\gamma \in \Gamma$ (compare with Lemma 1 of [5]), that is the same as the number of decompositions $g = \gamma\gamma'$ with γ, γ' having positive entries (the positivity follows from $\gamma\rho, \gamma'\rho \in \mathbf{I}$ and (10.1)). We will show that there are 0, 1, or 3 such points, depending on arithmetic conditions on Δ.

Let X, Y, Z be the coordinates (2.1) for such a $\gamma \in \Gamma$, so $X, Y, 2Z \in \mathbb{N}$. We have

$$\gamma\rho \in \mathfrak{a}_g \iff \frac{2Z - X}{2Z - Y} = \frac{B}{C} = \frac{B_0}{C_0},$$

therefore $2Z = X + uB_0$, $2Z = Y + uC_0$. The equation $4XY - 4Z^2 = 3$ becomes, after setting $t = 6Z - 2u(B_0 + C_0)$,

$$t^2 - 4u^2\Delta = 9. \tag{A.5}$$

In terms of solutions (t, u) with $t > 0$ we find

$$2Z = \frac{1}{3}(t + 2u(B_0 + C_0)), \quad X = \frac{1}{3}(t + u(2C_0 - B_0)), \quad Y = \frac{1}{3}(t + u(2B_0 - C_0)). \tag{A.6}$$

If $(3, u) = 1$, the sign of u is determined by the condition $t \equiv 2u(B_0 + C_0) \pmod 3$ which ensures that $X, Y, 2Z \in \mathbb{Z}$. Notice that $X, Y > 0$, so the triple (X, Y, Z) indeed determines a matrix γ with $\gamma\rho \in \mathfrak{a}_g$. By (2.2) we find

$$\cosh d(\rho, \gamma\rho) = \frac{t}{3}. \tag{A.7}$$

We distinguish two types of solutions, depending on whether 3 divides u or not.

Case I: 3 divides u. Letting $u = 3u'$, $t = 3t'$, with (t', u') a solution of

$$t'^2 - 4\Delta u'^2 = 1,$$

we have from (A.6) that $2Z > X, 2Z > Y$ when $u' > 0$, so the point $\gamma\rho$ is on the same side of $\mathfrak{a}_g$ as $g\rho$. Since $d(\rho, \gamma\rho) = t'$, to determine when $\gamma\rho \in (\rho \to g\rho)$ we distinguish two cases : if the minimal positive solution (T, k) of (A.2) has k even, then $2t' = T, 2u' = k$ and $d(\rho, g\rho) = 2d(\rho, \gamma\rho)$, and we find that $\gamma\rho$ is the midpoint of $(\rho \to g\rho)$. On the other hand if k is odd, then the minimal solution (t', u') has $t' = \frac{T^2}{2} - 1$, in which case $\gamma = g$ and there are no points $\gamma\rho$ on $(\rho \to g\rho)$.

Case II: $(3, u) = 1$. In this case we necessarily have $(3, \Delta) = 1$. Writing (A.5) as $N(\alpha) = 9$ with $\alpha = t + 2u\sqrt{\Delta}$, and assuming a solution $\alpha_0 = t_0 + 2u_0\sqrt{\Delta}$ exists with $(3, u_0) = 1$, then all solutions are $\alpha = \alpha_0\big(\frac{T}{2} + k\sqrt{\Delta}\big)$, with (T, k) a solution of (A.2) with k even. The corresponding points $\gamma\rho$ to these solutions lie on the axis $\mathfrak{a}_g$ at distance $\frac{1}{2}d(\rho, g\rho)$ apart if the minimal solution (T, k) of (A.2) has k even, or at distance $d(\rho, g\rho)$ apart if the minimal solution has k odd. In the former case we find two points $\gamma\rho$ on $(\rho \to g\rho)$, at distance $\frac{1}{2}d(\rho, g\rho)$ apart, while in the latter only one. These are distinct from the point found in Case I, since here $\cosh d(\rho, \gamma\rho)$ is not integral by (A.7).

We conclude that there is a partition of the set D_ρ in (A.4):

$$D_\rho = D_\rho^{0,0} \cup D_\rho^{1,0} \cup D_\rho^{0,1} \cup D_\rho^{1,1},$$

where $D_\rho^{0,\epsilon}$, respectively $D_\rho^{1,\epsilon}$ is the subset of Δ for which the minimal positive solution (T, k) of (A.3) has k even, respectively odd, and $D_\rho^{\epsilon,0}$, respectively $D_\rho^{\epsilon,1}$, is the subset of Δ such that (A.5) has a solution with $(3, u) = 1$, respectively it does not have such a solution. From the preceding discussion we conclude that the restriction of the parametrization

$$\varphi : \mathcal{D}_\Delta \to \mathcal{R}_\Delta^{\mathrm{prim}}$$

is 1-1 if $\Delta \in D_\rho^{1,1}$ (no lattice points on $(\rho \to g\rho)$), 2-1 if $\Delta \in D_\rho^{1,0}$ or $\Delta \in D_\rho^{0,1}$ (one point on $(\rho \to g\rho)$), and 4-1 if $\Delta \in D_\rho^{0,0}$ (three points on $(\rho \to g\rho)$). We denoted by $\mathcal{R}_\Delta^{\mathrm{prim}}$ the image of φ restricted to $\mathcal{D}_\Delta$.

Examples. I. $\Delta = 3$, $\mathcal{D}_\Delta = \{(2,1), (1,2)\}$. The minimal solution of $T^2 - 12k^2 = 4$ is $(T, k) = (4, 1)$ with k odd and $3|\Delta$ so $\Delta \in D_\rho^{1,1}$. Therefore there are two conjugacy classes in $\mathcal{R}_3^{\mathrm{prim}}$ with representatives: $\varphi(2,1) = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 3 \end{smallmatrix}\right)$, $\quad \varphi(1,2) = \left(\begin{smallmatrix} 3 & 1 \\ 2 & 1 \end{smallmatrix}\right)$.

II. $\Delta = 7$, $\mathcal{D}_\Delta = \{(3,1), (3,2), (1,3), (2,3)\}$. The minimal solution of $T^2 - 28k^2 = 4$ is $(T, k) = (16, 3)$, and the equation $t^2 - 28u^2 = 9$ has solution $(t, u) = (11, \pm 2)$, so $\Delta \in D_\rho^{1,0}$. From Case II we find

$$g = \varphi(3,1) = \gamma\gamma', \ h = \varphi(3,2) = \gamma'\gamma, \ \text{for } \gamma = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 3 \end{smallmatrix}\right), \gamma' = \left(\begin{smallmatrix} 1 & 4 \\ 1 & 5 \end{smallmatrix}\right),$$

so there are two conjugacy classes in $\mathcal{R}_\Delta^{\mathrm{prim}}$, $\{g\}=\{h\}$ and $\{\widetilde{g}\}=\{\widetilde{h}\}$.
III. $\Delta=21$, $\mathcal{D}_\Delta=\{(5,1),(5,4),(1,5),(4,5)\}$. The minimal solution of $T^2-84k^2=4$ is $(T,k)=(2\cdot 55,12)$ and $3|\Delta$, so $\Delta\in D_\rho^{0,1}$. From Case I we find

$$g=\varphi(5,1)=\gamma\gamma',\ h=\varphi(3,2)=\gamma'\gamma,\ \text{for } \gamma=\left(\begin{smallmatrix}3&4\\5&7\end{smallmatrix}\right),\gamma'=\left(\begin{smallmatrix}1&8\\1&9\end{smallmatrix}\right),$$

so there are two conjugacy classes in $\mathcal{R}_\Delta^{\mathrm{prim}}$, $\{g\}=\{h\}$ and $\{\widetilde{g}\}=\{\widetilde{h}\}$.
IV. $\Delta=13$, $\mathcal{D}_\Delta=\{(4,1),(4,3),(1,4),(3,4)\}$. The minimal solution of $T^2-52k^2=4$ is $(T,k)=(2\cdot 649,180)$, and the equation $t^2-52u^2=9$ has solution $(t,u)=(29,\pm4)$, so $\Delta\in D_\rho^{0,0}$. We have

$$\varphi(4,1)=\left(\begin{smallmatrix}109&720\\180&1189\end{smallmatrix}\right)=\gamma_1\widetilde{\gamma}_1=\gamma_2\gamma_2'=\gamma_3\gamma_3',$$

with $\gamma_1=\left(\begin{smallmatrix}20&3\\33&5\end{smallmatrix}\right),\gamma_2=\left(\begin{smallmatrix}2&1\\3&2\end{smallmatrix}\right),\gamma_2'=\left(\begin{smallmatrix}38&251\\33&218\end{smallmatrix}\right),\gamma_3=\left(\begin{smallmatrix}43&66\\71&109\end{smallmatrix}\right),\gamma_3'=\left(\begin{smallmatrix}1&6\\1&7\end{smallmatrix}\right)$. The first decomposition comes from Case I, while the last two from Case II. We have $\varphi(1,4)=\widetilde{\gamma}_1\gamma_1,\varphi(3,4)=\gamma_2'\gamma_2,\varphi(4,3)=\gamma_3'\gamma_3$, so $\mathcal{R}_{13}^{\mathrm{prim}}$ contains one conjugacy class.

In Case I, the following lemma determines directly the midpoint of $(\rho\to g\rho)$.

Lemma 18. *Let $g=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)$ be a hyperbolic matrix whose axis passes through ρ, having positive entries. Then a matrix $\gamma\in\mathrm{SL}_2(\mathbb{R})$ such that $\gamma\rho$ is the midpoint of $(\rho\to g\rho)$ has coordinates (X,Y,Z) given by (2.1) as follows*

$$X=A+\tfrac12B,\quad Y=D+\tfrac12C,\quad Z=\tfrac12A+B=\tfrac12D+C.$$

Appendix B. Asymptotic behavior of the pair correlation density function

In this appendix we give partial results towards proving the asymptotic formula (1.4), using the standard method of decomposing an automorphic kernel in terms of the spectrum of the hyperbolic Laplacian [14, Ch.12]. Let Γ be a Fuchsian group of the first kind with fundamental domain having area V_Γ, and fix a point $\omega\in\mathbb{H}$.

Recall the function $f_X(r)$ defined in (1.2), namely f_X is the unique continuous function on $[0,\infty)$ with $f_X(0)=0$, which is smooth except at the two points $r_1<r_2$ such that $\sinh(r_1)=X$ and $2\sinh\left(\frac{r_2}{2}\right)=X$, and whose derivative is

$$f_X'(r)=\begin{cases}1 & \text{if } r<r_1,\\ 1-\frac{2\sinh(r)}{\sqrt{\sinh^2(r)-X^2}} & \text{if } r_1<r<r_2,\\ 1-\frac{\sinh(r)}{\sqrt{\sinh^2(r)-X^2}} & \text{if } r>r_2.\end{cases}\tag{B.1}$$

Let $u(z,w)=\frac{|z-w|^2}{4\operatorname{Im}z\operatorname{Im}w}$, so that $\cosh d(z,w)=1+2u(z,w)$, and define a point-pair invariant kernel function k_X for fixed $X>0$ by

$$k_X(u(z,w))=f_X(d(z,w)).$$

Let $K_X(z,w)=\sum_{\gamma\in\Gamma}k_X(u(z,\gamma w))$ be the corresponding automorphic kernel. Conjecture 1 states that the pair correlation density function for angles of the hyperbolic lattice $\Gamma\omega$ is given by

$$g_2\left(\frac{X}{V_\Gamma}\right)=\frac{V_\Gamma}{\pi X^2}K_X(\omega,\omega),$$

and we have to show that

$$K_X(\omega,\omega)\sim\frac{\pi}{V_\Gamma}X^2,\quad \text{as } X\to\infty.\tag{B.2}$$

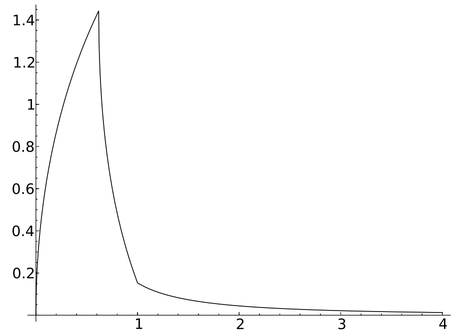

FIGURE 3. The kernel function $k_X(u)$ for $X = 2$.

From now on we fix X and drop the subscript X from the notation. Let $h(t)$ be the Selberg/Harish-Chandra transform of the kernel function $k(u)$, so that we have (see the proof of Theorem 1.16 in [14])

$$h(t) = \int_{\mathbb{H}} k(u(i,z)) y^s d\mu(z),$$

where $s = \frac{1}{2} + it$ and $d\mu(z)$ is the hyperbolic measure. Using hyperbolic polar coordinates $r = d(i,z)$ and φ, we have $y = (\cosh r + \sinh r \cos 2\varphi)^{-1}$ and

$$h(t) = \int_0^{\pi} \int_0^{\infty} f_X(r) \frac{2\sinh r}{(\cosh r + \sinh r \cos 2\varphi)^s} dr d\varphi.$$

Let $F(a,b;c;z) = \sum_{n \geqslant 0} \frac{(a)_n (b)_n}{n! (c)_n} z^n$ be the Gauss hypergeometric function. We have:

$$\int_0^{2\pi} (\cosh r + \sinh r \cos \varphi)^{-s} d\varphi = 2\pi e^{rs} F\left(s, \tfrac{1}{2}; 1; 1 - e^{2r}\right) = 2\pi e^{-rs} F\left(s, \tfrac{1}{2}; 1; 1 - e^{-2r}\right),$$

so that

$$h(t) = 2\pi \int_0^{\infty} f_X(r) \sinh(r) e^{-rs} F\left(s, \tfrac{1}{2}; 1; 1 - e^{-2r}\right) dr. \tag{B.3}$$

By definition the function h is even, which can also be seen from (B.3) via the relation

$$F\left(s, \tfrac{1}{2}; 1; 1-x\right) = \frac{\Gamma(1/2 - s)}{\Gamma(1/2)\Gamma(1-s)} F\left(s, \tfrac{1}{2}; s + \tfrac{1}{2}; x\right) + \frac{\Gamma(s - 1/2)}{\Gamma(1/2)\Gamma(s)} x^{1/2 - s} F\left(1-s, \tfrac{1}{2}; \tfrac{3}{2} - s; x\right)$$

for $0 < x < 1$, $s \neq \frac{1}{2}$.

We have the following estimates for h. Notice that to obtain absolute convergence in the spectral decomposition of the kernel $K_X(z,w)$ and finish the proof of (B.2), we would need $h(t) \ll (1 + |t|)^{-2-\epsilon}$.

Lemma 19. (a) $h\left(\frac{i}{2}\right) = \pi X^2$ *and for* $s = \frac{1}{2} + it \in (0,1)$ *we have* $h(t) \ll X^{\max(2-s, 1+s)}$.

(b) *There exists* $\epsilon > 0$ *such that* $h(t)$ *is holomorphic in the strip* $|\operatorname{Im} t| \leqslant \frac{1}{2} + \epsilon$ *and satisfies* $h(t) \ll \frac{X^{1+\epsilon}}{(1+|t|)^2}$ *for* $t \in \mathbb{R}$.

Proof. (a) For $t = \frac{i}{2}$ we have $s = 0$ and $h\left(\frac{i}{2}\right) = 2\pi \int_0^{\infty} f_X(r) \sinh(r) dr$. Since f_X is continuous, integrating by parts we obtain

$$h\left(\tfrac{i}{2}\right) = -2\pi \int_0^{\infty} f_X'(r) \cosh(r) dr.$$

We break the integral in three parts, at the two points $r_1 < r_2$ where f_X is not smooth. Using (B.1), we find that the three integrals are convergent and can be explicitly computed by the fundamental theorem of calculus, yielding $h\big(\frac{i}{2}\big) = 2\pi\sqrt{\sinh^2(r_2) - \sinh^2(r_1)} = \pi X^2$.

For $s = \frac{1}{2} + it \in (0,1)$, we use formula (B.3) and break the integral in two parts at the point r_1 as in the previous paragraph. For $s \neq \frac{1}{2}$ we have $F\big(s, \frac{1}{2}1, 1, 1\big) = \frac{\Gamma(1/2-s)}{\Gamma(1/2)\Gamma(1-s)}$, while $F\big(\frac{1}{2}, \frac{1}{2}, 1, x\big) \ll -\ln(1-x)$ as $x \to 1$, obtaining

$$\int_0^{r_1} f_X(r)\sinh(r) e^{-rs} F\left(s, \tfrac{1}{2}; 1; 1 - e^{-2r}\right) dr \ll X \ln X.$$

if $s \neq \frac{1}{2}$, with an extra factor of $\ln(X)$ if $s = \frac{1}{2}$.

For $r > r_1$, the integrand in (B.3) is bounded above by $f_X(r)\sinh(r)X^{-s}$, and computing the integral as in the case $s = 0$ yields the claim.

(b) From the identity preceding the lemma, we have $h(t) = h_1(t) + h_1(-t)$ where $(s = \frac{1}{2} + it)$

$$h_1(t) = \frac{2\pi\Gamma(1/2 - s)}{\Gamma(1/2)\Gamma(1-s)} \int_0^\infty f_X(r)\sinh(r) e^{-rs} F\left(s, \tfrac{1}{2}; s + \tfrac{1}{2}; e^{-2r}\right) dr.$$

Using the power series expansion of $\sinh(r)e^{-rs}F\left(s, \frac{1}{2}; s + \frac{1}{2}; e^{-2r}\right)$, integration by parts gives

$$h_1(t) = \frac{\pi\Gamma(1/2 - s)}{\Gamma(1/2)\Gamma(1-s)} \int_0^\infty f_X'(r) \sum_{n \geqslant 0} \frac{(s)_n (1/2)_n}{n!(s+1/2)_n} \left(\frac{e^{-r(2n+s-1)}}{2n+s-1} - \frac{e^{-r(2n+s+1)}}{2n+s+1} \right) dr$$

Denote by $G_s(r)$ the series inside the integral. We proceed as in part (a), breaking the integral at the points $r_1 = r_1(X)$ and $r_2 = r_2(X)$. From the formula (B.1) for $f_X'(r)$, we easily conclude that

$$\int_0^{r_1} f_X'(r) G_s(r) dr \ll \frac{X^{1+\epsilon}}{|s|^2}, \qquad \int_{r_2}^\infty f_X'(r) G_s(r) dr \ll \frac{X^{2\epsilon}}{|s|^2},$$

by integrating by parts, using the series expansion of $G_s(r)$. In the middle range we have

$$\int_{r_1}^{r_2} f_X'(r) G_s(r) dr \ll \frac{X^{1+\epsilon}}{|s|^{3/2}},$$

using that $\int_{r_1}^{r_2} \frac{\sinh(r)}{\sqrt{\sinh^2(r) - \sinh^2(r_1)}} e^{irt} dr \ll \frac{1}{t^{1/2}}$. Since $\frac{\Gamma(1/2-s)}{\Gamma(1-s)} \ll |s|^{-1/2}$, the estimate in part (b) follows. □

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References

[1] F. P. Boca, *Distribution of angles between geodesic rays associated with hyperbolic points*, Quart. J. Math. **58** (2007), 281–295.

[2] F. P. Boca, C. Cobeli, A. Zaharescu, *Distribution of lattice points visible from the origin*, Comm. Math. Phys. **213** (2000), 433–470.

[3] F. P. Boca, C. Cobeli, A. Zaharescu, *A conjecture of R. R. Hall on Farey points*, J. Reine Angew. Mathematik **535** (2001), 207–236.

[4] F. P. Boca, A. Zaharescu *The correlations of Farey fractions*, J. London Math. Soc. **72** (2005), 25–39.

[5] F. P. Boca, V. Pasol, A. A. Popa, A. Zaharescu, *Pair correlation of angles between reciprocal geodesics on the modular surface*, preprint math.NT/1102.0328.

[6] J. Bourgain, A. Kontorovich, P. Sarnak, *Sector estimates for hyperbolic isometries*, Geom. Funct. Anal. **20** (2010), 1175–1200.
[7] F. Chamizo, *Hyperbolic lattice point problems*, Proc. Amer. Math. Soc. **139** (2011), 451–459.
[8] F. Dyson, *Statistical theory of the energy levels of complex systems I,II,III.*, J. Math. Phys. **3** (1962), 140–156, 157–165, 166–175.
[9] N. D. Elkies, C. T. McMullen, *Gaps in* $\sqrt{n}$ mod 1 *and ergodic theory*, Duke Math. J. **123** (2004), 95–139.
[10] R. Fricke, F. Klein, *Theorie der Elliptischen Modulfunktionen, Vol. II*, Leipzig, 1892.
[11] A. Good, *Local Analysis of Selberg's Trace Formula*, Lect. Notes in Math. 1040 (1983).
[12] A. Gorodnik, A. Nevo, *Counting lattice points*, J. Reine Angew. Mathematik **663** (2012), 127–176.
[13] M. N. Huxley, A. A. Zhigljavsky, *On the distribution of Farey fractions and hyperbolic lattice points*, Per. Math. Hungarica **42** (2001), 191–198.
[14] H. Iwaniec, *Spectral Methods of Automorphic Forms*, 2nd ed, Vol 53, Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2002.
[15] J. Marklof, A. Strömbergsson, *The distribution of free path lengths in the periodic Lorentz gas and related lattice point problems*, Annals of Mathematics **172** (2010), 1949–2033.
[16] P. Nicholls, *A lattice point problem in hyperbolic space*, Michigan Math. J. **30** (1983), 273–287.
[17] M. S. Risager, Z. Rudnick, *On the statistics of the minimal solution of a linear diophantine equation and uniform distribution of the real part of orbits in hyperbolic spaces*, Contemporary Mathematics **484** (2009), 187–194.
[18] M. S. Risager, J. L. Truelsen, *Distribution of angles in hyperbolic lattices*, Quart. J. Math. **61** (2010), 117–133.
[19] P. Sarnak, *Reciprocal geodesics*, Clay Mathematics Proceedings, Vol. 7 (2007), 217–237.
[20] E. Wigner, *Random matrices in physics*, SIAM Review **9** (1967), 1–23.

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