arXiv:1207.5766v1 [math.NT] 24 Jul 2012

AN ALGEBRAIC PROPERTY OF HECKE OPERATORS AND TWO INDEFINITE THETA SERIES

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Abstract. We prove an algebraic property of the elements defining Hecke operators on period polynomials associated with modular forms, which implies that the pairing on period polynomials corresponding to the Petersson scalar product of modular forms is Hecke equivariant. As a consequence of this proof, we derive two indefinite theta series identities which can be seen as analogues of Jacobi's formula for the theta series associated with the sum of four squares.

1. Introduction

The action of Hecke operators on period polynomials for the full modular group has been defined algebraically by Choie and Zagier [CZ93] (see also [Za93]). To describe their construction, let M_n be the set of integer matrices of determinant n, modulo $\pm I$, and let $R_n = \mathbb{Q}[M_n]$. Thus $\Gamma_1 = PSL_2(\mathbb{Z})$ acts on R_n by left and right multiplication. Let $M_n^\infty = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) : n = ad, 0 \leqslant b < d \right\}$ be the usual system of representatives for $\Gamma_1 \backslash M_n$, and $T_n^\infty = \sum_{M\in M_n^\infty} M \in R_n$. Then there exist $\widetilde{T}_n, Y_n \in R_n$ such that

$$T_n^\infty(1-S) = (1-S)\widetilde{T}_n + (1-T)Y_n \tag{1.1}$$

where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ are the standard generators of Γ_1. Any such element $\widetilde{T}_n$ gives the action of Hecke operators on period polynomials of cusp forms for the full modular group [CZ93]. In [PP12] we show that the same elements have actions on (multiple) period polynomials of modular forms for finite index subgroups Γ of Γ_1, corresponding to actions of a wide class of double coset operators $\Sigma_n \subset M_n$ on modular forms. Remarkably the elements $\widetilde{T}_n$ are universal, not depending on Γ or the double coset Σ_n.

Elements $\widetilde{T}_n$ satisfying condition (1.1) go back to work of Manin [Ma73]. The element $\widetilde{T}_n$ is unique, up to addition of any element in the right Γ_1-module

$$\mathcal{I} = (1+S)R_n + (1+U+U^2)R_n, \tag{1.2}$$

where $U = TS$ has order 3. Examples of such $\widetilde{T}_n$ are given in [CZ93, Za90]. Another example will be given in an upcoming article of the second author and Don Zagier, leading to a simple proof of the Eichler-Selberg trace formula for modular forms on $\Gamma_1(N)$ with Nebentypus, which generalizes the approach sketched in [Za93] for Γ_1.

In the context of modular symbols, Merel gives many examples of elements $\widetilde{\mathfrak{T}}_n \in R_n$ giving the action of Hecke operators on modular symbols, which satisfy a condition similar to (1.1) [Me94]. It can be shown that their adjoints $\widetilde{T}_n = \widetilde{\mathfrak{T}}_n^\vee$ satisfy (1.1), where for $g \in M_n$ we denote by $g^\vee = g^{-1}\det g$ the adjoint of g, and we apply this notation to all elements of R_n by linearity.

The space of period polynomials is endowed with a pairing corresponding to the Petersson scalar product of modular forms via a generalization of a formula of Haberland. We show in [PP12]

1991 *Mathematics Subject Classification.* 11F25, 11F67, 11F27.

that the Hecke equivariance of this pairing under the action of $\widetilde{T}_n$ is implied by the following algebraic property of the elements $\widetilde{T}_n$.

Theorem 1. *For any element $\widetilde{T}_n \in R_n$ satisfying property* (1.1) *we have*

$$\widetilde{T}_n(T - T^{-1}) + (T^{-1} - T)\widetilde{T}_n^{\vee} \in \mathcal{I} + \mathcal{I}^{\vee}. \tag{1.3}$$

The rest of the paper is devoted to an algebraic proof of the theorem, along the way obtaining other properties of the elements defining Hecke operators, which may be of independent interest. An unexpected outcome of this algebraic approach is the discovery of the following "indefinite theta series" identities, which are proved in Section 3:

$$\sideset{}{'}\sum_{\substack{a,b,c,d\geqslant 0\\ a+b>|d-c|,\ c+d>|a-b|}} q^{ad+bc} + 4\sideset{}{'}\sum_{d\geqslant a>0} aq^{ad} = 3\widetilde{E}_2(q) + \sum_{n>0} q^{n^2}, \tag{1.4}$$

$$\sideset{}{'}\sum_{\substack{x\geqslant|y|,z\geqslant|t|\\ x>|t|,z>|y|}} q^{x^2+z^2-y^2-t^2} + 2\sum_{x>|y|}(x-|y|)q^{x^2-y^2} = \widetilde{E}_2(q) - 2\widetilde{E}_2(q^2) + 4\widetilde{E}_2(q^4) + \sum_{n>0} q^{n^2}, \tag{1.5}$$

where $\widetilde{E}_2(q) = \sum_{n>0}\sigma_1(n)q^n$ is the weight 2 Eisenstein series without the constant term, and the notation $\sum'$ indicates that the terms for which there is equality in the range are summed with weight $1/2$. For example, when p is an odd prime, we obtain that $p-3$ is the number of integer solutions of

$$x^2 + z^2 - y^2 - t^2 = p, \text{ with } x, z > |y|, |t|.$$

This can be seen as an indefinite analogue of Jacobi's result on the number of ways of representing a positive integer as a sum of four squares. It would be interesting to fit these examples into a theory of theta series for indefinite quadratic forms, of the type developed by Zwegers [Zw02] for forms of signature $(r,1)$.

The proof is organized as follows. In §2 we derive a characterization of $\mathcal{I}+\mathcal{I}^{\vee}$ that allows us to reduce (1.3) to a relation involving only T_n^{∞}. This relation is then shown to be equivalent to two relations possessing an extra symmetry besides invariance under taking adjoint, which are proved in §3. The main part of the proof is contained in §3, where we also derive the indefinite theta series identities as immediate corollaries.

2. Preliminary reductions

First we prove a characterization of the set $\mathcal{I}+\mathcal{I}^{\vee}$, based on a similar characterization of $\mathcal{I}$ in [CZ93].

Proposition 2. *Let $A \in R_n = \mathbb{Q}[M_n]$. Then $A \in \mathcal{I}+\mathcal{I}^{\vee}$ if and only if*

$$(1-S)A(1-S) \in (1-T)R_n(1-S) + (1-S)R_n(1-T^{-1}).$$

Proof. The proof is based on the characterization of $\mathcal{I}$ in [CZ93, Lemma 3]:

$$v \in \mathcal{I} \iff (1-S)v \in (1-T)R_n. \tag{2.1}$$

We need a more precise version, which appears in the proof of Lemma 3 in [CZ93]:

Lemma 3 ([CZ93])**.** *Let $A, B \in R_n$ such that $(1-S)A = (1-T)B$. Then there exists $C \in R_n$ such that*

$$A = (1+S)C - SB, \text{ with } SB \in (1+U+U^2)R_n.$$

If $A \in \mathcal{I} + \mathcal{I}^\vee$, the claim of Proposition 2 follows immediately from (2.1) and its adjoint. Assume therefore that $A \in R_n$ satisfies:

$$(1-S)A(1-S) = (1-T)\alpha(1-S) + (1-S)\beta(1-T^{-1})$$

By the adjoint of the relations in Lemma 3 it follows that there exists $C \in R_n$ such that

$$(1-S)A = (1-T)\alpha + C(1+S) + (1-S)\beta S, \tag{2.2}$$

with $(1-S)\beta S \in R_n(1+U+U^2)$. Since $(1 \pm S)^2 = 2(1 \pm S)$, multiplying (2.2) by 2 we get

$$(1-S)\cdot 2A = (1-T)\cdot 2\alpha + C(1+S)(1+S) + (1-S)(1-S)\beta S$$

From (2.2) we can write $C(1+S) = (1-S)\gamma + (1-T)\delta$, therefore:

$$(1-S)[2A - \gamma(1+S) - (1-S)\beta S] \in (1-T)R_n.$$

By (2.1) we conclude

$$2A - \gamma(1+S) - (1-S)\beta S \in \mathcal{I}.$$

Since $(1-S)\beta S \in R_n(1+U+U^2)$, it follows by dividing the last equation by 2 that $A \in \mathcal{I}+\mathcal{I}^\vee$. $\square$

Corollary 4. *Let $\mathcal{J} = (1-T)R_n(1-S) \subset R_n$. Then (1.3) is equivalent to the statement*

$$T_n^\infty ST^{-1}(1-S) + (1-S)TST_n^{\infty\vee} \in \mathcal{J} + \mathcal{J}^\vee. \tag{2.3}$$

We remark that $T_n^\infty A(1-S) \pmod{\mathcal{J}}$ is well-defined modulo multiplication by powers of T on the left, for any $A \in \Gamma_1$, so we can take in (2.3)

$$T_n^\infty = \sum_{n=ad} \sum_{b \pmod d} t_{a,d}(b), \qquad \text{where } t_{a,d}(b) = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}. \tag{2.4}$$

Proof. By Proposition 2 and (1.1), (1.3) is equivalent to the following statement:

$$T_n^\infty(1-S)(T-T^{-1})(1-S) + (1-S)(T^{-1}-T)(1-S)T_n^{\infty\vee} \in \mathcal{J} + \mathcal{J}^\vee.$$

By the remark above, we have $T_n^\infty(1-T)(1-S) \in \mathcal{J}$, so that:

$$\begin{aligned} T_n^\infty(1-S)(T-T^{-1})(1-S) &\equiv T_n^\infty S(T^{-1}-T)(1-S) \\ &\equiv T_n^\infty(ST^{-1} + T^{-1}ST^{-1})(1-S) \\ &\equiv 2T_n^\infty ST^{-1}(1-S) \qquad \pmod{\mathcal{J}} \end{aligned}$$

On the second line we used $STS = T^{-1}ST^{-1}$, and on the third the remark above. $\square$

We need the following notation. Let $\epsilon = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and for $m \in M_n$ denote $m' = \epsilon m \epsilon$. For any matrix $m \in M_n$ we denote by $m^\star \in R_n$ the following linear combination of eight matrices:

$$m^\star = (m - m')(1-S) + (1-S)(m^\vee - m'^\vee) \tag{2.5}$$

and extend this notation to elements of R_n by linearity. Note that $m'^\vee = m^{\vee\prime}$.

In the next proposition we show that (2.3) is equivalent with more symmetric relations, involving only terms of type $m^\star$. The latter will be proved in the next section, using only the following easily checked properties:

$$m^\star + (m'^\vee)^\star + (SmS)^\star + (Sm'^\vee S)^\star = 0 \tag{P1}$$

$$(m')^\star = -m^\star \tag{P2}$$

$$(mS)^\star = -m^\star \tag{P3}$$

$$(T^k m)^\star \equiv m^\star \pmod{\mathcal{J}+\mathcal{J}^\vee} \quad \text{for any integer } k. \tag{P4}$$

Proposition 5. *We have the following congruences* $(\bmod\ \mathcal{J}+\mathcal{J}^\vee)$*:*

$$\sum_{n=ad}\Big[\sum_{-d-\frac{a}{2}<k<d-\frac{a}{2}} \left(\begin{smallmatrix} a+k & -k\\ -d & d\end{smallmatrix}\right) - \delta\big(\tfrac{a}{2}\big)\left(\begin{smallmatrix} d & -d\\ a/2 & a/2\end{smallmatrix}\right)\Big]^\star \equiv 4\left[T_n^\infty U^2(1-S) + (1-S)UT_n^{\infty\vee}\right] \tag{2.6}$$

$$\sum_{n=ad}\Big[2\sum_{-\frac{a}{2}<k<\frac{d-a}{2}} \left(\begin{smallmatrix} d & -d\\ -k & a+k\end{smallmatrix}\right) + \delta\big(\tfrac{a}{2}\big)\left(\begin{smallmatrix} d & -d\\ a/2 & a/2\end{smallmatrix}\right)\Big]^\star \equiv 2\left[T_n^\infty U^2(1-S) + (1-S)UT_n^{\infty\vee}\right], \tag{2.7}$$

where $\delta(x)$ is 1 if $x\in\mathbb{Z}$ and 0 otherwise.

The LHS of (2.6) is congruent to $2(T_n^\infty U^2)^\star$, while the LHS of (2.7) is congruent with $(UT_n^\infty U^2)^\star$, but we will not need these facts in the sequel.

Proof. The summand in the LHS of (2.6) is $(t_{a,d}(-k-a)ST^{-1})^\star$ [with $t_{a,d}(b)$ defined in (2.4)], and we note that k is well-defined modulo d by (P4). We can rewrite the LHS as follows:

$$\sum_{n=ad}\sum_{k \bmod d} 2(t_{a,d}(k)ST^{-1})^\star - \delta\big(\tfrac{a}{2}\big)\Big[\left(\begin{smallmatrix} a/2 & a/2\\ -d & d\end{smallmatrix}\right) + \left(\begin{smallmatrix} d & -d\\ a/2 & a/2\end{smallmatrix}\right)\Big]^\star \equiv 2(T_n^\infty ST^{-1})^\star,$$

because the sum of the terms in square brackets equals

$$\sum_{n=2ad}\Big[\left(\begin{smallmatrix} a & a\\ -d & d\end{smallmatrix}\right) + \left(\begin{smallmatrix} d & -d\\ a & a\end{smallmatrix}\right)\Big]^\star \equiv 0$$

(exchanging a with d in the first matrix and using (P2)).

For any $A\in R_n$, we use the abbreviation $\{A\}+\{\mathrm{adj}\} := A + A^\vee$. We have

$$\begin{aligned}(T_n^\infty ST^{-1})^\star &\equiv \big\{[T_n^\infty ST^{-1} - (T_n^\infty)'ST](1-S)\big\} + \{\mathrm{adj}\}\\ &\equiv \big\{[T_n^\infty ST^{-1} + (T_n^\infty)STS](1-S)\big\} + \{\mathrm{adj}\}\\ &\equiv \big\{2T_n^\infty ST^{-1}(1-S)\big\} + \{\mathrm{adj}\},\end{aligned} \tag{2.8}$$

proving (2.6). On the second line we have used that $(T_n^\infty)'A(1-S)\equiv T_n^\infty A(1-S)$ for any $A\in\Gamma_1(\mathbb{Z})$, by the remark following Corollary 4, while on the third we used $(ST)^3=1$ together with the fact that $T_n^\infty T^{-1}(1-S)\equiv T_n^\infty(1-S) \pmod{\mathcal{J}}$, by the same remark.

The summand in (2.7) is $(S\,t_{a,d}(k)ST^{-1})^\star$, and we have as in (2.8):

$$(S\,t_{a,d}(k)ST^{-1})^\star = \big\{(S\,t_{a,d}(k)ST^{-1} + S\,t_{a,d}(-k-a)ST^{-1})(1-S)\big\} + \{\mathrm{adj}\}.$$

The range $-\frac{a+d}{2}<k<\frac{d-a}{2}$ is invariant under $k\to -k-a$, hence the LHS of (2.7), which we denote L_n, becomes:

$$L_n \equiv \sum_{n=ad}\Big\{2\sum_{-\frac{a+d}{2}<k<\frac{d-a}{2}} S\,t_{a,d}(k)ST^{-1}(1-S)\Big\} + \{\mathrm{adj}\}. \tag{2.9}$$

Now we have:

$$S\,t_{a,d}(k)ST^{-1}(1-S) \equiv t_{a,d}(k)ST^{-1}(1-S) + (1-S)[\,t_{a,d}(a+k) - t_{a,d}(k)]S. \tag{2.10}$$

Since $t_{a,d}(k)ST^{-1}(1-S) \pmod{\mathcal{J}}$ depends only on $k \,(\mathrm{mod}\; d)$ we have:

$$\sum_{-\frac{a+d}{2}<k}^{k<\frac{d-a}{2}} t_{a,d}(k)U^2(1-S) \equiv \sum_{k=0}^{d-1} t_{a,d}(k)U^2(1-S) - \delta\big(\tfrac{d-a}{2}\big)\, t_{a,d}\big(\tfrac{d-a}{2}\big)U^2(1-S). \tag{2.11}$$

Similarly

$$\sum_{-\frac{a+d}{2}<k}^{k<\frac{d-a}{2}} [t_{a,d}(a+k)-t_{a,d}(k)](S-1) \equiv \delta\big(\tfrac{d-a}{2}\big)\big[\, t_{a,d}\big(\tfrac{d-a}{2}\big) - \, t_{a,d}\big(\tfrac{a-d}{2}\big)\big](S-1)$$

Next we observe that

$$\begin{aligned}\sum_{n=ad}\sum_{-\frac{a+d}{2}<k}^{k<\frac{d-a}{2}} \{\, t_{a,d}(k) - \, t_{a,d}(a+k)\} + \{\mathrm{adj}\} \equiv \sum_{n=ad}\Big[\sum_{-\frac{a+d}{2}<k}^{k<\frac{d-a}{2}} [\, t_{a,d}(k) - \, t_{a,d}(a+k)]+ \\ + \sum_{-\frac{a+d}{2}<k}^{k<\frac{a-d}{2}} [\, t_{a,d}(-k) - \, t_{a,d}(-d-k)]\Big] \equiv \sum_{n=ad} \delta\big(\tfrac{d-a}{2}\big)\big[\, t_{a,d}\big(\tfrac{a-d}{2}\big) - \, t_{a,d}\big(\tfrac{d-a}{2}\big)\big].\end{aligned} \tag{2.12}$$

Notice that (2.12) can be conjugated by S, since $S^\vee = S$. Putting together the resulting equation, and (2.12), (2.11), (2.10), (2.9), we obtain:

$$L_n \equiv \{2T_n^\infty ST^{-1}(1-S)\} + \{\mathrm{adj}\} + 2\sum_{n=ad} \delta\big(\tfrac{d-a}{2}\big)E_{a,d}$$

where:

$$\begin{aligned}E_{a,d} = S\big[\, t_{a,d}\big(\tfrac{a-d}{2}\big) - \, t_{a,d}\big(\tfrac{d-a}{2}\big)\big]S + \, t_{a,d}\big(\tfrac{d-a}{2}\big) - \, t_{a,d}\big(\tfrac{a-d}{2}\big) + \\ + \big\{\big[\, t_{a,d}\big(\tfrac{d-a}{2}\big) - \, t_{a,d}\big(\tfrac{a-d}{2}\big)\big](S-1) - \, t_{a,d}\big(\tfrac{d-a}{2}\big)U^2(1-S)\big\} + \{\mathrm{adj}\}.\end{aligned}$$

It is easy to see that $E_{a,d} + E_{d,a} \equiv 0 \pmod{\mathcal{J} + \mathcal{J}^\vee}$, finishing the proof of (2.7). □

3. Main algebraic result and a theta series identity

In this section, we finish the proof of (1.3). The indefinite theta series identities from the introduction follow as a bonus from the proof, and since they are of independent interest we keep this section self-contained.

Recall the notation (2.5) and properties (P1)-(P4), which are the only ingredients needed. For convenience we rewrite the four-term relation:

$$\left(\begin{smallmatrix} a & -b \\ c & d \end{smallmatrix}\right)^\star + \left(\begin{smallmatrix} d & -b \\ c & a \end{smallmatrix}\right)^\star + \left(\begin{smallmatrix} a & -c \\ b & d \end{smallmatrix}\right)^\star + \left(\begin{smallmatrix} d & -c \\ b & a \end{smallmatrix}\right)^\star = 0. \tag{P1}$$

We will apply this relation to matrices in the following set

$$\mathcal{S}_n = \Big\{ \left(\begin{smallmatrix} a & -b \\ c & d \end{smallmatrix}\right) \in M_n : c \geqslant b; \quad d \geqslant a; \quad a+b > d-c > 0 \Big\}.$$

More precisely, for each $\gamma = \left(\begin{smallmatrix} a & -b \\ c & d \end{smallmatrix}\right)$, denote $A_\gamma = \big\{ \left(\begin{smallmatrix} a & -b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} d & -b \\ c & a \end{smallmatrix}\right), \left(\begin{smallmatrix} a & -c \\ b & d \end{smallmatrix}\right), \left(\begin{smallmatrix} d & -c \\ b & a \end{smallmatrix}\right) \big\}$ and define

$$\mathcal{T}_n := \bigcup_{\gamma\in\mathcal{S}_n} A_\gamma = \Big\{ \left(\begin{smallmatrix} a & -b \\ c & d \end{smallmatrix}\right) : \; b+c > |a-d|, \quad \max(a,d) > \max(b,c)\Big\} \tag{3.1}$$

(the union is disjoint), so that we have by construction

$$\sum_{\gamma\in\mathcal{T}_n}\gamma^\star=0. \tag{3.2}$$

It is convenient to further symmetrize the set $\mathcal{T}_n$ by interchanging $a\leftrightarrow b$, $c\leftrightarrow d$. Note that if a,b,c,d satisfy $\max(a,d)\geqslant\max(b,c)$ then

$$b+c>|a-d|\iff c+d>|a-b|,\quad a+b>|c-d|.$$

Defining therefore

$$\begin{aligned}\mathcal{U}_n&:=\mathcal{T}_n\cup\Big\{\left(\begin{smallmatrix}b&-a\\d&c\end{smallmatrix}\right):\left(\begin{smallmatrix}a&-b\\c&d\end{smallmatrix}\right)\in\mathcal{T}_n\Big\}\\ \mathcal{X}_n&:=\Big\{\left(\begin{smallmatrix}a&-b\\c&d\end{smallmatrix}\right):\ c+d>|a-b|,\quad a+b>|c-d|\Big\}\\ \mathcal{V}_n&:=\Big\{\left(\begin{smallmatrix}a&-b\\c&d\end{smallmatrix}\right)\in\mathcal{X}_n:\max(a,d)=\max(b,c)\Big\},\end{aligned} \tag{3.3}$$

we have the disjoint union $\mathcal{X}_n=\mathcal{U}_n\cup\mathcal{V}_n$. By (P2),(P3), (3.2) we have $\sum_{\gamma\in\mathcal{U}_n}\gamma^\star=0$, and it is easily found that

$$\mathcal{V}_n=\bigcup_{\gamma}A_\gamma\qquad\text{over }\gamma=\left(\begin{smallmatrix}a&-b\\c&c\end{smallmatrix}\right)\text{ with }c|n,\ c\geqslant a\geqslant\frac{n}{c}-c \tag{3.4}$$

so that $\sum_{\gamma\in\mathcal{V}_n}\gamma^\star=0$. Consequently

$$\sum_{\gamma\in\mathcal{X}_n}\gamma^\star=0. \tag{3.5}$$

Proposition 6. *For every $n>0$ we have the congruence* (mod $\mathcal{J}+\mathcal{J}^\vee$)*:*

$$\sum_{\gamma\in\mathcal{X}_n}\gamma^\star\equiv\sum_{n=ad}\Big[\sum_{-d-\frac{a}{2}<k<d-\frac{a}{2}}\left(\begin{smallmatrix}a+k&k\\d&d\end{smallmatrix}\right)+2\sum_{-\frac{a}{2}<k<\frac{d-a}{2}}\left(\begin{smallmatrix}d&d\\k&a+k\end{smallmatrix}\right)\Big]^\star. \tag{3.6}$$

By Proposition 5 and (P2), we obtain

$$0=\sum_{\gamma\in\mathcal{X}_n}\gamma^\star\equiv-6\left[T_n^\infty U^2(1-S)+(1-S)UT_n^{\infty\vee}\right]\quad(\mathrm{mod}\ \mathcal{J}+\mathcal{J}^\vee),$$

which, together with Corollary 4, finishes the proof of Theorem 1.

Before proving the proposition, we show that it implies the theta series identity (1.4). For a set of 2 by 2 matrices A denote by $\mathcal{N}(A)$ the number of matrices $\left(\begin{smallmatrix}*&*\\c&d\end{smallmatrix}\right)\in A$ with $cd>0$, minus the number of matrices with $cd<0$, namely:

$$\mathcal{N}(A)=\sum_{\gamma\in A}\mathrm{sgn}(c_\gamma d_\gamma).$$

Corollary 7. *We have the identity:*

$$\sum_{\substack{a,b,c,d\geqslant0\\ a+b>|d-c|,\ c+d>|a-b|}}{}' q^{ad+bc}=3\widetilde{E}_2(q)-2\sum_{n>0}\sigma_{\min}(n)\,q^n+\sum_{n>0}q^{n^2}, \tag{3.7}$$

where $\sigma_{\min}(n)=\sum_{n=ad}\min(a,d)$.

Proof. We claim that $\mathcal{N}(\mathcal{X}_n)$ equals the coefficient of q^n in the LHS of (3.7); on the other hand, going from (3.5) to (3.6) we use only relations (P2)-(P4), which do not change the function $\mathcal{N}$, so $\mathcal{N}(\mathcal{X}_n)$ can be computed exactly from the RHS of (3.6), yielding the coefficient of q^n in the RHS of (3.7), and the proof is finished.

To count $\mathcal{N}(\mathcal{X}_n)$, we start with $\mathcal{T}_n$. Let $\gamma = \left(\begin{smallmatrix} a & -b \\ c & d \end{smallmatrix}\right) \in \mathcal{S}_n$. Clearly $c, d > 0$ and we have three cases:

(1) $a > 0 > b$ or $b > 0 > a$. Then $\mathcal{N}(A_\gamma) = 0$;
(2) $a, b > 0$. Then $\mathcal{N}(A_\gamma) = |A_\gamma|$;
(3) One of a, b is 0. Then the other is positive and $\mathcal{N}(A_\gamma) = \frac{1}{2}|A_\gamma|$.

By (3.1) we conclude that $\mathcal{N}(\mathcal{T}_n) = \#\left\{ \left(\begin{smallmatrix} a & -b \\ c & d \end{smallmatrix}\right) \in \mathcal{T}_n : a, b, c, d \geqslant 0 \right\}$, each matrix with $abcd = 0$ being counted with weight 1/2. The same is obviously true for $\mathcal{U}_n$, and by inspection for $\mathcal{V}_n$ (see (3.4)). We conclude that

$$\mathcal{N}(\mathcal{X}_n) = \#\left\{ \left(\begin{smallmatrix} a & -b \\ c & d \end{smallmatrix}\right) \in \mathcal{X}_n : a, b, c, d \geqslant 0; \begin{array}{c} \text{matrices with } abcd = 0 \text{ are} \\ \text{counted with weight } 1/2 \end{array} \right\}, \tag{3.8}$$

that is $\mathcal{N}(\mathcal{X}_n)$ is the coefficient of q^n in the LHS of (3.7), as claimed. □

Proof of Proposition 6. We decompose $\mathcal{X}_n$ as a disjoint union:

$$\mathcal{X}_n = \mathcal{X}_n^< \cup \mathcal{X}_n^= \cup \mathcal{X}_n^>$$

where $\mathcal{X}_n^<$, $\mathcal{X}_n^=$, and $\mathcal{X}_n^>$ consist of those $\gamma = \left(\begin{smallmatrix} a & -b \\ c & d \end{smallmatrix}\right) \in \mathcal{X}_n$ with $|c| < d$, $c = d$, and $c > |d|$ respectively. Using (P2)-(P4), we will show that $\sum_{\gamma \in \mathcal{X}_n} \gamma^\star$ reduces to (3.6).

If $\left(\begin{smallmatrix} x & k \\ d & d \end{smallmatrix}\right) \in \mathcal{X}_n^=$, we have $n = ad$, $x = a + k$ and $2d > |a + 2k|$, hence

$$\sum_{\gamma \in \mathcal{X}_n^=} \gamma^\star = \sum_{n=ad} \sum_{-d-\frac{a}{2} < k < d - \frac{a}{2}} \left(\begin{smallmatrix} a+k & k \\ d & d \end{smallmatrix}\right)^\star. \tag{3.9}$$

Since $m \to m'S$ takes $\mathcal{X}_n^<$ bijectively onto $\mathcal{X}_n^>$, we have by (P2), (P3)

$$\sum_{\gamma \in \mathcal{X}_n^<} \gamma^\star = \sum_{\gamma \in \mathcal{X}_n^>} \gamma^\star. \tag{3.10}$$

It remains to calculate $\sum_{\gamma \in \mathcal{X}_n^<} \gamma^\star$.

For $m \in M_n$, denote by $\{m\}$ the equivalence class of m in $\mathcal{X}_n^<$ modulo multiplication by powers of T on the left, namely $\{m\} := \Gamma_{1\infty} m \cap \mathcal{X}_n^<$ where $\Gamma_{1\infty} = \{T^k : k \in \mathbb{Z}\}$. Consider the involution

$$f : \Gamma_{1\infty} \backslash \mathcal{X}_n^< \to \Gamma_{1\infty} \backslash \mathcal{X}_n^<, \quad f(\{m\}) = \{m'\}.$$

We show that f is bijective, and the classes $\{m\}$, $m \in \mathcal{X}_n$ have one or two elements. Let $m = \left(\begin{smallmatrix} a & -b \\ c & d \end{smallmatrix}\right) \in \mathcal{X}_n^<$, and let $T^{-k} m' = \left(\begin{smallmatrix} A & -B \\ C & D \end{smallmatrix}\right)$. Then

$$T^{-k} m' \in \mathcal{X}_n \iff k > 1 + \frac{b-a}{c+d}, \quad \left| k - \frac{a+b}{d-c} \right| < 1.$$

Since $c + d > |b - a|$, $a + b > d - c > 0$, it is clear that there exist one or two values of k for which $T^{-k} m' \in \mathcal{X}_n$, and therefore f is bijective (since it is an involution). Moreover,

$$|\{m'\}| = \begin{cases} 1 & \text{if } d - c | a + b \quad (k = \frac{a+b}{d-c}); \\ 1 & \text{if } d - c \nmid a + b,\ b \geqslant a,\ \frac{a+b}{d-c} < 2 \quad (k = 2); \\ 2 & \text{otherwise .} \end{cases} \tag{3.11}$$

Since $d-c|a+b \iff A=B$, and $a=b \iff D-C|A+B$, and $b>a$, $\frac{a+b}{d-c}<2 \iff B>A$, $\frac{A+B}{D-C}<2$, there are four possibilities:

(1) $d-c\nmid a+b,\ a\neq b \iff D-C\nmid A+B, A\neq B$. Then $|\{m\}|=|\{m'\}|$.
(2) $d-c|a+b,\ a=b \iff D-C|A+B, A=B$. Then $|\{m\}|=|\{m'\}|=1$.
(3) $d-c\nmid a+b,\ a=b \iff D-C|A+B, A\neq B$. Then

$$|\{m\}|=1,\ |\{m'\}|=\begin{cases}1 & \text{if } \frac{a+b}{d-c}<2\\ 2 & \text{if } \frac{a+b}{d-c}>2\end{cases}.$$

(4) $d-c|a+b,\ a\neq b \iff D-C\nmid A+B, A=B$. Then

$$|\{m'\}|=1,\ |\{m\}|=\begin{cases}1 & \text{if } \frac{A+B}{D-C}<2\\ 2 & \text{if } \frac{A+B}{D-C}>2\end{cases}.$$

Summing over two copies of $\mathcal{X}_n^<$ divided into classes $\{m\}$ and respectively $\{m'\}$, everything cancels by (P2) except for matrices m' in case (3) with $\frac{a+b}{d-c}>2$, and matrices m in case (4) for which $\frac{A+B}{D-C}>2$. The two sums are equal and we obtain

$$2\sum_{\gamma\in\mathcal{X}_n^<}\gamma^\star = 2\sum_{\gamma}\gamma^\star,\quad \text{over } \gamma=\left(\begin{smallmatrix}a & a\\ -c & d\end{smallmatrix}\right), d>|c|, d-c\nmid 2a, \frac{2a}{d-c}>2.$$

Writing $n=ar$, so that $c+d=r$, we obtain (after substituting $-c$ for c)

$$\sum_{\gamma\in\mathcal{X}_n^<}\gamma^\star \equiv \sum_{n=ar}\sum_{\substack{0<r+2c<a\\ r+2c\nmid 2a}}\left(\begin{smallmatrix}a & a\\ c & c+r\end{smallmatrix}\right).$$

By (3.9), (3.10), the proof of (3.6) is finished once we show that the same sum, but with the congruence condition replaced by $r+2c|2a$, yields 0 (mod $\mathcal{J}+\mathcal{J}^\vee$). By (P2), (P4), it is enough to show that

$$\left(\begin{smallmatrix}a & a\\ c & c+r\end{smallmatrix}\right)=\begin{pmatrix}a'-tc' & -a'+t(c'+r')\\ -c' & c'+r'\end{pmatrix} \iff r+2c|2a,\ r'+2c'|2a',$$

where $n=ar=a'r'$, $t\in\mathbb{Z}$, and $0<r+2c<a$, $0<r'+2c'<a'$. Indeed the equality of the matrices implies $t=\frac{2a}{r+2c}=\frac{2a'}{r'+2c'}$, proving the equivalence above.

□

We end this section with a proof of (1.5), which is entirely similar to that of Corollary 7. Let $\mathcal{S}_n',\cdots,\mathcal{X}_n'$ be the sets defined like $\mathcal{S}_n,\cdots,\mathcal{X}_n$ but with the extra parity conditions $a\equiv d \pmod 2$, $b\equiv c \pmod 2$.

Proposition 8. *a) If n is odd then*

$$\sum_{\gamma\in\mathcal{X}_n'}\gamma^\star \equiv \sum_{n=ad}2\sum_{-\frac{a}{2}<k<\frac{d-a}{2}}\left(\begin{smallmatrix}d & d\\ k & a+k\end{smallmatrix}\right)^\star. \tag{3.12}$$

b) If n is even then

$$\sum_{\gamma\in\mathcal{X}_n'}\gamma^\star \equiv \sum_{\substack{n=ad\\ 2|a}}\sum_{\substack{-d-\frac{a}{2}<k<d-\frac{a}{2}\\ k\equiv d \pmod 2}}\left(\begin{smallmatrix}a+k & k\\ d & d\end{smallmatrix}\right)^\star+\sum_{\substack{n=ad\\ 2|a,2|d}}2\sum_{-\frac{a}{2}<k<\frac{d-a}{2}}\left(\begin{smallmatrix}d & d\\ k & a+k\end{smallmatrix}\right)^\star. \tag{3.13}$$

Proof. The proof is very similar to that of Proposition 6, and we prove only part a) leaving b) as an exercise for the reader. Assuming n odd and referring to the proof of Proposition 6, the parity condition implies that $a \not\equiv b$, $c \not\equiv d \pmod 2$, so that $\mathcal{X}_n'^{=}$ is empty, and only cases (1) and (4) can occur for $m \in \mathcal{X}_n'^{<}$. The integer k such that $T^{-k}m' \in \mathcal{X}_n'^{<}$ must be even, so in case (1) we have $|\{m\}| = |\{m'\}| = 1$, while in case (4) we have $|\{m'\}| = 0$, because $\frac{a+b}{d-c}$ is odd in that case. Denoting by $\mathcal{X}_n'^{(4)}$ the set of $m = \left(\begin{smallmatrix} a & -b \\ c & d \end{smallmatrix}\right) \in \mathcal{X}_n'$, with $d > |c|, d-c|a+b$, it follows that

$$\sum_{\gamma \in \mathcal{X}_n'^{<}} \gamma^\star \equiv \sum_{m \in \mathcal{X}_n'^{(4)}} m^\star$$

Writing $d - c = r > 0, a + b = lr$, from $a(d-c) + c(a+b) = n$ we have that $n = rs$ with $s \in \mathbb{Z}$ and $a + lc = s$, so $T^l m = \left(\begin{smallmatrix} s & s \\ c & c+r \end{smallmatrix}\right)$. The condition $m \in \mathcal{X}_n'^{<}$ implies that

$$l > 1, l \text{ odd, and } \left| l - \frac{2s}{2c+r} \right| < 1,$$

so there is a unique such m for each $c > \frac{-r}{2}$ such that $\frac{2s}{2c+r} > 2$, and $2c + r \nmid 2s$. Consequently:

$$\sum_{\gamma \in \mathcal{X}_n'^{<}} \gamma^\star \equiv \sum_{n=rs} \sum_{\substack{\frac{-r}{2} < c < \frac{s-r}{2} \\ 2c+r \nmid 2s}} \left(\begin{smallmatrix} s & s \\ c & c+r \end{smallmatrix}\right)^\star .$$

The proof is finished by observing that the divisibility condition can be removed, exactly as in the previous case. ☐

Corollary 9. *We have the identity*

$$\sum_{\substack{x \geqslant |y|, z \geqslant |t| \\ x > |t|, z > |y|}}{}' q^{x^2+z^2-y^2-t^2} = \widetilde{E}_2(q) - 2\widetilde{E}_2(q^2) + 4\widetilde{E}_2(q^4) - 2\sum_{n>0} \sigma_{\min}^{\mathrm{ev}}(n)\, q^n + \sum_{n>0} q^{n^2}, \tag{3.14}$$

where $\sigma_{\min}^{\mathrm{ev}}(n) = \sum_{n=ad, 2|(d-a)} \min(a,d)$.

Proof. Arguing as before, we have that $\mathcal{N}(\mathcal{X}_n')$ is given by (3.8) with $\mathcal{X}_n$ replaced by $\mathcal{X}_n'$. Making a substitution $a = x+y, d = x-y, b = z+t, c = z-t$, the conditions $a,b,c,d \geqslant 0$ become $x \geqslant |y|$, $z \geqslant |t|$, while $a+b > |c-d|, c+d > |a-b|$ become $x > |t|, z > |y|$. Therefore $\mathcal{N}(\mathcal{X}_n')$ is the coefficient of q^n in the LHS of (3.14).

As before, we can count $\mathcal{N}(\mathcal{X}_n')$ from Proposition 8. When n is odd, (3.12) gives

$$\mathcal{N}(\mathcal{X}_n') = \sigma(n) - 2\sigma_{\min}(n) + \delta(\sqrt{n}).$$

If n is even, (3.13) yields (the first term comes from the first sum, and the second from the second sum)

$$\mathcal{N}(\mathcal{X}_n') = \left[\sigma\left(\tfrac{n}{2}\right) - \tau\left(\tfrac{n}{4}\right)\right] + \left[2\sigma\left(\tfrac{n}{4}\right) + \tau\left(\tfrac{n}{4}\right) - 2\sigma_{\min}^{\mathrm{ev}}(n) + \delta\left(\tfrac{\sqrt{n}}{2}\right)\right],$$

where $\tau(n)$ is the number of divisors of n and we adopt the convention that an arithmetic function is zero on non-integers. This is exactly the coefficient of q^n in the RHS of (3.14) ☐

The relations in §3 have been checked numerically for small n using MAGMA [Mgm].

Acknowledgments. Part of this work was completed at the Max Planck Institute in Bonn, which provided financial support and a great working environment. We would like to thank Don Zagier for inspiring conversations. The first author was partially supported by the CNCSIS grant PN-II-ID-PCE-2012-4-0376. The second author was partially supported by the European Community grant PIRG05-GA-2009-248569.

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