ON THE PETERSSON SCALAR PRODUCT OF ARBITRARY MODULAR FORMS

VICENTIU PASOL, ALEXANDRU A. POPA

Abstract. We consider a natural extension of the Petersson scalar product to the entire space of modular forms of integral weight $k \geqslant 2$ for a finite index subgroup of the modular group. We show that Hecke operators have the same adjoints with respect to this inner product as for cusp forms, and we show that the Petersson product is nondegenerate for $\Gamma_1(N)$ and $k > 2$. For $k = 2$ we give examples when it is degenerate, and when it is nondegenerate.

1. Introduction

Let Γ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, let $k \geqslant 2$ be an integer, and denote by $M_k(\Gamma)$, $S_k(\Gamma)$ the spaces of modular forms, respectively cusp forms of weight k for Γ. For $f, g \in M_k(\Gamma)$, at least one of which is a cusp form, the Petersson scalar product is defined by

$$(f, g) = \frac{1}{[\Gamma_1 : \overline{\Gamma}]} \int_{\Gamma\backslash\mathcal{H}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}.$$

where $\Gamma_1 = \mathrm{PSL}_2(\mathbb{Z})$ and $\overline{\Gamma}$ denotes the projectivisation of Γ. An extension to $M_k(\Gamma_1)$ was given by Zagier [Za81], using a renormalized integral over a fundamental domain for $\Gamma_1\backslash\mathcal{H}$. In this note we use the same approach to extend the Petersson product to all of $M_k(\Gamma)$. We show that the extended Petersson product has the same equivariance properties under the action of Hecke operators as the usual one, and for $\Gamma = \Gamma_1(N)$ we show that it is nondegenerate when $k > 2$. When $k = 2$, we find somewhat surprisingly that it can be degenerate, and we give examples when it is nondegenerate and when it is degenerate.

Our motivation comes from the theory of period polynomials associated to modular forms. In [PP12], we generalize Haberland's formula by showing that the extended Petersson product of arbitrary modular forms is given by a pairing on their (extended) period polynomials. The nondegeneracy of the extended Petersson product is then needed to show that $M_k(\Gamma)$ is isomorphic to the plus and minus parts of the space of period polynomials of all modular forms, extending the classical Eichler-Shimura isomorphism. For $\Gamma_0(N)$, the Hecke equivariance is used to show that the pairing on extended period polynomials is also Hecke equivariant.

Other approaches to extending the Petersson inner product to all modular forms are given by Chiera [Ch07], and by Deitmar and Diamantis [DD09]. An adelic version of the renormalization method is given by Michel and Venkatesh [MV10, Ch. 4.3].

2. Extended Petersson scalar product

We give three equivalent definitions of the extended Petersson product following [Za81]. The first definition appears naturally in [PP12], where we show that the Petersson product of $f, g \in$

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$M_k(\Gamma)$ can be computed in terms of a pairing on the period polynomials of f, g, generalizing a formula proved by Haberland for $S_k(\Gamma_1)$.

For the first definition assume that Γ is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and let $C_\Gamma = [\Gamma_1 : \overline{\Gamma}]$. Let $\mathcal{F}$ be the fundamental domain $\{z \in \mathcal{H} : |z| \geqslant 1, |\mathrm{Re}\, z| \leqslant 1/2\}$ for Γ_1, and for $T > 1$ let $\mathcal{F}_T$ be the truncated domain for which $\mathrm{Im}\, z < T$. Let

$$\mathrm{Tr}(f\overline{g})(z) = \sum_A f|A(z)\overline{g}|A(z),$$

where here and below sums over A are over complete system of representatives for $\overline{\Gamma}\backslash\Gamma_1$.[1] The function $y^k\mathrm{Tr}(f\overline{g})$ is a Γ_1-invariant, renormalizable function in the sense of [Za81], satisfying $y^k\mathrm{Tr}(f\overline{g})(z) = a_0(f,g)y^k + O(y^{-K})$ for all K, with $a_0(f,g) = \sum_A a_0(f|A)\overline{a_0(g|A)}$. Therefore we can define for $f, g \in M_k(\Gamma)$ the scalar product

$$(2.1)\qquad \begin{aligned}(f,g) &= \frac{1}{C_\Gamma}\lim_{T\to\infty}\Big[\int_{\mathcal{F}_T} y^k\mathrm{Tr}(f\overline{g})d\mu - a_0(f,g)\frac{T^{k-1}}{k-1}\Big]\\ &= \frac{1}{C_\Gamma}\int_{\mathcal{F}}[y^k\mathrm{Tr}(f\overline{g})(z) - a_0(f,g)E(z,k)]d\mu\end{aligned}$$

where $E(z,s) = \sum_{\gamma\in\Gamma_{1\infty}\backslash\Gamma_1}\mathrm{Im}(\gamma z)^s$ is the weight 0 Eisenstein series ($\Gamma_{1\infty}$ is the stabilizer of the cusp ∞), and $d\mu = \frac{dxdy}{y^2}$ is the $SL_2(\mathbb{R})$-invariant measure.

The next version can be defined for an arbitrary Fuchsian group of the first kind. Let $\mathcal{D}$ be a fundamental domain for Γ, and for $s \in \mathbb{C}$ let $\mathcal{A}_s(\Gamma)$ be the space of (weight 0) automorphic functions for Γ, which are eigenforms of the hyperbolic Laplacian with eigenvalue $s(s-1)$. For any function $F_{f,g} \in \mathcal{A}_k(\Gamma)$ such that $f(z)\overline{g(z)}y^k - F_{f,g}(z)$ vanishes at all cusps (an example will be given shortly), we have:

$$(2.2)\qquad (f,g) = \frac{\pi}{3\mathrm{Vol}\ \mathcal{D}}\int_{\mathcal{D}}[f(z)\overline{g(z)}y^k - F_{f,g}(z)]d\mu$$

The right side is independent of $F_{f,g}$: if $F'_{f,g}$ is another choice, the difference $F_{f,g} - F'_{f,g} \in \mathcal{A}_k(\Gamma)$ is a cusp form, so its integral over $\mathcal{D}$ vanishes.

Assume now that Γ is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, so that $\mathrm{Vol}\ \mathcal{D} = \frac{\pi}{3}C_\Gamma$. Decomposing $\mathcal{D} = \cup_A A\mathcal{F}$, (2.2) becomes

$$(f,g) = \frac{1}{C_\Gamma}\int_{\mathcal{F}}\Big[y^k\mathrm{Tr}(f\overline{g})(z) - \sum_A F_{f,g}(Az)\Big]d\mu.$$

Since both $\sum_A F_{f,g}(Az)$ and $a_0(f,g)E(z,k)$ belong to $\mathcal{A}_k(\Gamma_1)$, and they have the same behaviour at infinity as $y^k\mathrm{Tr}(f\overline{g})$ it follows as before that the last equation agrees with (2.1).

To give an example of $F_{f,g}$ as above, let $\mathcal{S} \subset \mathbb{P}^1(\mathbb{Q})$ be a complete set of inequivalent cusps of Γ. For $\mathfrak{a} \in \mathbb{P}^1(\mathbb{Q})$ fix $\sigma_\mathfrak{a} \in PSL_2(\mathbb{R})$ with $\sigma_\mathfrak{a}\infty = \mathfrak{a}$. Let $\Gamma_\mathfrak{a}$ be the subgroup of Γ of elements fixing $\mathfrak{a}$. Define the weight 0 Eisenstein series associated with the cusp $\mathfrak{a}$ by:

$$E_\Gamma^\mathfrak{a}(z,s) = \sum_{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma}\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z)^s$$

[1] If k is odd, there is a sign ambiguity in defining $f|A$ for A in $\Gamma_1 = \mathrm{PSL}_2(Z)$, but the ambiguity dissapears when considering the product $f|A\overline{g}|A$.

which converges absolutely for $\mathrm{Re}s > 1$, and belongs to $\mathcal{A}_s(\Gamma)$. From the Fourier expansion of $E_\Gamma^{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s)$ (see [Iw02], Theorem 3.4), it follows that

$$E_\Gamma^{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s) = \delta_{\mathfrak{a}\mathfrak{b}} y^s + \varphi_{\mathfrak{a}\mathfrak{b}}(s) y^{1-s} + O\big((1 + y^{-\mathrm{Re}s}) e^{-2\pi y}\big)$$

with $\delta_{\mathfrak{a}\mathfrak{b}} = 1$ if $\mathfrak{a} = \mathfrak{b}$ and 0 otherwise, and $\varphi_{\mathfrak{a}\mathfrak{b}}(s)$ an explicit function. Assuming that the fundamental domain $\mathcal{D}$ is chosen such that its vertices on the boundary of $\mathcal{H}$ are precisely a complete set of representatives for the cusps of Γ, it follows that the linear combination

$$F_{f,g}(z) = \sum_{\mathfrak{a} \in \mathcal{S}} a_0(f|\sigma_{\mathfrak{a}}) a_0(\overline{g|\sigma_{\mathfrak{a}}}) E_\Gamma^{\mathfrak{a}}(z, k) \in \mathcal{A}_k(\Gamma)$$

is such that $f(z)\overline{g(z)}y^k - F_{f,g}(z)$ vanishes at all cusps, so $F_{f,g}$ is a valid choice in (2.2).

Lastly, assuming Γ is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, from (2.2) with the choice of $F_{f,g}$ as in the previous paragraph we have

$$(f, g) = \frac{\pi}{3}(4\pi)^{-k}\Gamma(k)\mathrm{Res}_{s=k} L(s, f, \overline{g}). \tag{2.3}$$

This identity is well known if f, g are cuspidal when it goes back to Rankin. If both f, g are noncuspidal, it follows from extending to Γ the Rankin-Selberg method developed in [Za81] for the full modular group. Since the generalization is straightforward, we omit the details.

3. Adjoints of Hecke operators

In this section we show that Hecke operators have the same adjoints with respect to the extended Petersson product on $M_k(\Gamma)$ as with respect to the one on $S_k(\Gamma)$. The proof copies the classical one given in [Sh71], using the definition of (f, g) in (2.2). We assume Γ is a Fuchsian subgroup of $\mathrm{SL}_2(\mathbb{R})$ of the first kind, namely a subgroup acting discretely on $\mathcal{H}$ and of finite covolume.

Let $\tilde{\Gamma}$ consist of elements α of $\mathrm{GL}_2(\mathbb{R})$ such that $\alpha\Gamma\alpha^{-1}$ is commensurable with Γ. For $\alpha \in \tilde{\Gamma}$, let $\Gamma = \cup_{i=1}^r (\Gamma \cap \alpha^{-1}\Gamma\alpha)\gamma_i$ (disjoint union). Then $\alpha\gamma_i$ is a complet system of representatives for $\Gamma\backslash\Gamma\alpha\Gamma$ and the action of the Hecke operator associated with the coset $\Gamma\alpha\Gamma$ is defined on $f \in M_k(\Gamma)$ by

$$f|[\Gamma\alpha\Gamma] = n^{k-1} \sum_{i=1}^r f|_k \alpha\gamma_i$$

where $n = \det\alpha$ and $f|_k\gamma(z) = f(\gamma z) j(\gamma, z)^{-k}$ for $\gamma \in \mathrm{GL}_2(\mathbb{R})$ (note that the stroke operator is normalized differently than by Shimura).

Proposition 3.1. *The adjoint of the operator* $[\Gamma\alpha\Gamma]$ *is* $[\Gamma\alpha^\vee\Gamma]$ *namely for* $f, g \in M_k(\Gamma)$

$$(f|[\Gamma\alpha\Gamma], g) = (f, g|[\Gamma\alpha^\vee\Gamma])$$

where $\alpha^\vee = \alpha^{-1}\det\alpha$.

Proof. The proof is identical to that of eq. (3.4.5) in [Sh71], except that a term involving $F_{f,g}$ has to be subtracted at each step, and one has to use repeatedly the fact that $F_{f,g}$ can be replaced by any other function in $\mathcal{A}_k(\Gamma)$ having the same behaviour at the cusps. ∎

4. Nondegeneracy

We show that the Petersson product is nondegenerate on $M_k(\Gamma)$ for $\Gamma=\Gamma_1(N)$ and for $k>2$. For $k=2$ and $\Gamma=\Gamma_0(N)$ or $\Gamma=\Gamma_1(N)$, we give examples when it is degenerate or nondegerate. For the proof, we compute explicitly the determinant of the matrix of the Petersson product, with respect to a basis of Hecke eigenforms for $\mathcal{E}_k(\Gamma)$, using formula (2.3). The degeneracy when $k=2$ is somewhat surprising, and we give an alternate proof for $\Gamma=\Gamma_0(6)$ in [PP12, Sec. 7], using period polynomials and a generalization of Haberland's formula.

Let ψ,φ be primitive characters of conductors c_ψ, c_φ with $c_\psi c_\varphi|N$ and $\psi\varphi(-1)=(-1)^k$, and let t be a divisor of $N/(c_\psi c_\varphi)$. When $k>2$, a basis of $\mathcal{E}_k(\Gamma)$ consists of the Eisenstein series $E_k^{\psi,\varphi,t}(z)=E_k^{\psi,\varphi}(tz)$ where

$$E_k^{\psi,\varphi}(z)=\frac{\delta(\psi)}{2}L(1-k,\varphi)+\sum_{n=1}^{\infty}\sigma_{k-1}^{\psi,\varphi}(n)q^n$$

where $\sigma_{k-1}^{\psi,\varphi}(n)=\sum_{n=ad}\psi(a)\varphi(d)d^{k-1}$, and $\delta(\psi)$ is 1 if $\psi=\mathbf{1}$ (the character of conductor 1), and zero otherwise. When $k=2$, the same elements form a basis, with the series $E_k^{\mathbf{1},\mathbf{1},t}$ replaced by $E_k^{\mathbf{1},\mathbf{1}}(z)-tE_k^{\mathbf{1},\mathbf{1}}(tz)$ for $t>1$, and with $E_k^{\mathbf{1},\mathbf{1},1}$ removed. For χ a character mod N, a basis of the space of Eisenstein series for $\Gamma_0(N)$ with character χ consists of those $E_k^{\psi,\varphi,t}$ for which $\chi=\psi\varphi$. These Eisenstein series are Hecke eigenforms for the operators T_n with $(n,N)=1$ [DS05].

For $(n,N)=1$, the adjoint of T_n with respect to the Petersson scalar product is the operator $T_n^*=<n>^{-1}T_n$, with $<n>$ the diamond operator. On the basis above $<n>$ acts by

$$<n>E_k^{\psi,\varphi,t}=\psi(n)\varphi(n)E_k^{\psi,\varphi,t},$$

and Proposition 3.1 shows that $\left(E_k^{\psi,\varphi,t},E_k^{\psi',\varphi',t'}\right)=0$ unless $\psi=\varphi',\varphi=\psi'$. Therefore the Petersson product is nondegenerate on $\mathcal{E}_k(\Gamma)$ if and only if for every pair ψ,φ as above, the matrix

$$M_{\psi,\varphi}=[(E_k^{\psi,\varphi,t},E_k^{\varphi,\psi,t'})]_{t,t'}$$

is nonsingular, where the rows and columns are indexed by divisors t,t' of $N/(c_\psi c_\varphi)$ (with $t,t'\neq 1$ if $k=2$ and $\psi=\varphi=\mathbf{1}$).

We compute the entries of $M_{\psi,\varphi}$ with the aid of (2.3). Assuming $(k,\psi,\varphi)\neq(2,\mathbf{1},\mathbf{1})$, we have $L(s,E_k^{\psi,\varphi})=L(s,\psi)L(s-k+1,\varphi)$, which has an Euler product. Using [Sh76, Lemma 1] we get

$$L(s,E_k^{\psi,\varphi,t},\overline{E_k^{\varphi,\psi,t'}})=\frac{L(s,\psi\overline{\varphi})L(s-2k+2,\varphi\overline{\psi})L(s-k+1,\psi\overline{\psi})L(s-k+1,\varphi\overline{\varphi})}{L(2s-2k+2,\psi\varphi\overline{\psi}\overline{\varphi})}\cdot$$
$$\cdot (drr')^{-s}\prod_{p^e||rr'}\frac{X_p(e,s)}{1-\psi\varphi\overline{\psi}\overline{\varphi}(p)p^{2k-2-2s}}$$

where $d=(t,t')$ and $t=dr$, $t'=dr'$, and $X_p(e,s)$ is a polynomial of degree $\leqslant 2$ in p^{-s}, given below. The product is over primes $p|rr'$, with $p^e|rr', p^{e+1}\nmid rr'$, and it equals 1 if $rr'=1$. Note that $L(s,\psi\overline{\psi})$, $L(s,\varphi\overline{\varphi})$ have simple poles at $s=1$, and $L(s,\varphi\overline{\psi})$ has a zero at $s=2-k$. Denoting by $R_{\psi,\varphi}$ the residue at $s=k$ of the fraction on the first line, it follows from (2.3) that

$$M_{\psi,\varphi}=\frac{\pi}{3}(4\pi)^{-k}\Gamma(k)R_{\psi,\varphi}M_k^{\psi,\varphi}(L) \tag{4.1}$$

where $L = N/(c_\psi c_\varphi)$ and $M_s^{\psi,\varphi}(L)$ is the matrix whose rows and columns are indexed by divisors t, t' of L with the entry corresponding to t, t' equal to

$$m(t,t') = (drr')^{-s} \prod_{p^e \| rr'} \frac{X_p(e,s)}{1 - \psi\varphi\overline{\psi'\varphi'}(p)p^{2k-2-2s}}. \tag{4.2}$$

When $k > 2$ we have $R_{\psi,\varphi} \neq 0$, since $L(s, \varphi\overline{\psi})$ has a simple zero at $s = 2-k$ (recall $\psi\varphi(-1) = (-1)^k$). Therefore the Petersson product is nondegenerate if and only if the matrix $M_k^{\psi,\varphi}(L)$ is nonsingular for all choices ψ, φ as above. When $k = 2$, one can have $R_{\psi,\varphi} = 0$, as $L(s, \varphi\overline{\psi})$ may have a zero of order at least two at $s = 0$ when $\varphi\overline{\psi}$ is not primitive. We discuss in more detail the case $k = 2$ at the end of this section, and we now proceed to compute $\det M_s^{\psi,\varphi}(L)$, assuming only $(k, \psi, \varphi) \neq (2, \mathbf{1}, \mathbf{1})$. We fix ψ, φ and let $M_s(L) = M_s^{\psi,\varphi}(L)$ for brevity.

Let $\alpha = \psi(p)$, $\alpha' = \varphi(p)p^{k-1}$, and $\beta = \overline{\varphi}(p)$, $\beta' = \overline{\psi}(p)p^{k-1}$ be the local factors in the Euler product of $L(s, E_k^{\psi,\varphi})$, and $L(s, \overline{E_k^{\varphi,\psi}})$ respectively, and

$$a(p^n) = \frac{\alpha^{n+1} - \alpha'^{n+1}}{\alpha - \alpha'}, \ b(p^n) = \frac{\beta^{n+1} - \beta'^{n+1}}{\beta - \beta'}$$

(if $\alpha = \alpha' = 0$, then $a(p^n) = 0$, $n > 0$). If $p^e \| r$ we have

$$X_p(e,s) = a(p^e) - a(p^{e-1})b(p)\alpha\alpha' p^{-s} + a(p^{e-2})(\alpha\alpha')^2\beta\beta' p^{-2s} \tag{4.3}$$

while if $p^e \| r'$ interchange α, α' with β, β' and $a(p^i)$ with $b(p^i)$ in the definition of $X_p(e,s)$. We use the convention $a(n) = 0$ if $n \notin \mathbb{Z}$.

The next lemma reduces the computation of $\det M_s(L)$ to the case L is a prime power.

Lemma 4.1. *With the notations as above, consider L_1, L_2 two relatively prime numbers. Then:*

$$\det(M_s(L_1L_2)) = \det(M_s(L_1))^{\sigma_0(L_2)} \cdot \det(M_s(L_2))^{\sigma_0(L_1)},$$

where $\sigma_0(L)$ denotes the number of divisors of L.

Proof. If t_1t_2, $t_1't_2'$ are two divisors of L_1L_2, with $t_i, t_i' | L_i$, it follows from (4.2) that $m(t_1t_2, t_1't_2') = m(t_1, t_1')m(t_2, t_2')$, so the matrix $M(L_1L_2)$ is the Kronecker product of the matrices $M(L_1)$, $M(L_2)$ and the conclusion follows. ☐

Lemma 4.2. *For p a prime and $n \geqslant 1$, let $C_{p,s} = p^{\frac{n(n+1)}{2}s}$ and $y = p^{k-1-s}$. We have:*

$$C_{p,s} \det M_s(p^n) = \begin{cases} 1 & \text{if } \varphi(p) = \psi(p) = 0 \\ (1-y)^n & \text{if exactly one of } \varphi(p), \psi(p) \text{ is } 0 \\ \frac{(1-y)^{n-1}}{(1+y)^{n+1}}\left(1 - \frac{\alpha}{\alpha'}y\right)^n\left(1 - \frac{\alpha'}{\alpha}y\right)^n & \text{if } \psi(p)\varphi(p) \neq 0 \end{cases}$$

with $\alpha = \psi(p)$, $\alpha' = \varphi(p)p^{k-1}$.

Proof. For $0 \leqslant i, j \leqslant n$, denote by $m(i,j) = m(p^i, p^j)$ in (4.2). The matrix $M_s(p^n)$ has elements $1, p^{-s}, \dots, p^{-ns}$ on the diagonal. We rescale it by multiplying the i-th line by p^{is}, $0 \leqslant i \leqslant n$, which multiplies its determinant by $C_{p,s}$, and we denote by $M_s'(p^n)$ the resulting matrix, having $\det M_s'(p^n) = C_{p,s} \det M_s(p^n)$. Note that the matrix $M_s'(p^n)$ has constant entries on all diagonals parallel to the main diagonal, and $m'(i,i) = 1$, where $m'(i,j)$ are its entries, $0 \leqslant i,j \leqslant n$.

When $\varphi(p) = \psi(p) = 0$, the matrix $M_s'(p^n)$ is the identity, and the first formula follows.

When exactly one of $\psi(p), \varphi(p)$ is 0, the off diagonal elements are $m'(i+e,i) = a^e$, $m'(i,i+e) = b^e p^{-se}$, $e \geqslant 0$, with $a = a(p), b = b(p)$ as in (4.3), $ab = p^{k-1}$. The determinant is easy to compute, by subtracting from line i the quantity a times the previous line, starting with $i = n, n-1, \dots 1$. The resulting matrix will be diagonal, of determinant $(1 - abp^{-s})^n = (1-y)^n$.

Assume now that $\psi(p)\varphi(p) \neq 0$. By (4.3), for $e \geqslant 1$ we have $m'(i+e,i) = X(e)$, $m'(i,i+e) = p^{-se}Y(e)$, where $X(e)$ is given by

$$X(e) = \frac{a(p^e) - a(p^{e-1})a(p)y + a(p^{e-2})\alpha\alpha' y^2}{1-y^2}, \quad e \geqslant 1,$$

with $a(p^{-1}) = 0$, and $Y(e)$ given by the same formula as $X(e)$ with a interchanged with b and α, α' with β, β'. We set $X(0) = 1$, so that $m'(i+e,i) = X(e)$ for $e \geqslant 0$.

Since $\{a(p^e)\}$ satisfies the recurrence $a(p^e) - a(p^{e-1})a(p) + a(p^{e-2})\alpha\alpha' = 0$, the same is true about $X(e)$, namely

$$X(e) - X(e-1)a(p) + X(e-2)\alpha\alpha' = 0, \quad e \geqslant 3.$$

In fact one checks that the recurrence holds for $e = 2$ as well, with $X(0) = 1$, and also for $e = 1$, with $X(-1) = p^{-s}Y(1)$ (using $\alpha\alpha' b(p) = p^{k-1}a(p)$ and recalling $y = p^{k-1-s}$). Starting with $i = n, n-1, \dots, 2$, we subtract from the i-th line a multiple $a(p)$ of the $(i-1)$-th line, and add a multiple $\alpha\alpha'$ of the $(i-2)$-th line. The resulting matrix will be upper-triangular, except for the entry $m'(1,0) = X(1)$, so

$$\det M'_s(p^n) = [1 - X(1)Y(1)p^{-s}][1 - a(p)Y(1)p^{-s} + \alpha\alpha' Y(2)p^{-2s}]^{n-1},$$

which is easily seen to equal the expression in the statement. □

Corollary 4.3. *a) If $k > 2$ we have $\det M_k(L) \neq 0$.*

b) If $k = 2$ we have $\det M_k(L) = 0$ if and only if $\varphi(p) = \psi(p) \neq 0$.

From the discussion above we conclude:

Theorem 4.4. *Let $\Gamma = \Gamma_1(N)$.*

a) If $k > 2$ then the extended Petersson product on $M_k(\Gamma)$ is nondegenerate.

b) The extended Petersson product on $M_2(\Gamma)$ is nondegenerate if N is prime. It is degenerate: if N is divisible by p^2q with $p \neq q$ primes; or if N is divisible by pq with $p \neq q$ primes such that q is not a primitive residue mod p.

Proof. Part a) was already proved above.

For part b), assume $p^2q|N$. We take $\psi = \varphi$ characters of conductor p. Then $\psi(q) = \varphi(q) \neq 0$, and Corollary 4.3 shows that $\det M_2^{\psi,\varphi}(N/p^2) = 0$, so the Petersson product is degenerate.

Assuming $pq|N$ with p, q as in the statement, it follows that there is a primitive character ψ mod p with $\psi(q) = 1$. Taking $\varphi = \mathbf{1}$, Corollary 4.3 shows that $\det M_2^{\psi,\varphi}(N/p) = 0$, so the Petersson product is degenerate.

When $N = p$ is prime, the L-function $L(s, E_2^{\mathbf{1},\mathbf{1},p}) = \zeta(s)\zeta(s-1)(1 - \frac{1}{p^{s-1}})$ has an Euler product. Then $\mathrm{Res}_{s=k} L(s, E_2^{\mathbf{1},\mathbf{1},p}, \overline{E_2^{\mathbf{1},\mathbf{1},p}})$ can be computed as before, and it is nonzero. Also if ψ is a primitive character of conductor p, $R_{\psi,\mathbf{1}} \neq 0$ with the notation of (4.1), so the Petersson product is nondegenerate in this case. □

Note that part a) implies that the Petersson product is nondegenerate on $M_k(\Gamma_0(N))$ for $k > 2$. To investigate what happens for $k = 2$ and $\Gamma = \Gamma_0(N)$, we now consider the case $k = 2, \psi = \varphi = \mathbf{1}$. Denote $E_2 = E_2^{\mathbf{1},\mathbf{1}}$, and for $t > 1$ let $E_2^t(z) = E_2(z) - tE_2(tz)$. We have

$$L(s, E_2) = \sum_{n\geqslant 1} \frac{a(n)}{n^s} = \zeta(s)\zeta(s-1),$$

with $a(p) = 1 + p$ for p prime, and $L(s, E_2^t) = \sum_{n\geqslant 1} \frac{a(n)-ta(n/t)}{n^s}$. It follows that $L(s, E_2^t, E_2^t)$ is a sum of four Rankin L-functions with an Euler product, and we have as before

$$L(s, E_2^t, E_2^{t'}) = \frac{\zeta(s)\zeta(s-1)^2\zeta(2-s)}{\zeta(2s-2)} \cdot m_s(t,t') \tag{4.4}$$

where, after writing $t = dr, t = dr'$ with $d = (t,t')$, $y_p = p^{1-s}$, we have

$$m_s(t,t') = 1 + \frac{tt'}{(drr')^s} \prod_{p^e||rr'} \frac{X_p(e,s)}{1-y_p^2} - t^{1-s} \prod_{p^e||t} \frac{X_p(e,s)}{1-y_p^2} - t'^{1-s} \prod_{p^e||t'} \frac{X_p(e,s)}{1-y_p^2}$$

with $X_p(e,s)$ as in (4.3) with $\alpha = \beta = 1, \alpha' = \beta' = p$.

Theorem 4.5. *Let $\Gamma = \Gamma_0(N)$ with $N > 1$ square-free. Then the Petersson product is degenerate on $M_2(\Gamma)$, unless N is prime when it is nondegenerate.*

Proof. A basis for the space $\mathcal{E}_2(\Gamma)$ consists of the Eisenstein series $E_2^{\psi,\varphi,t}$ with $c_\varphi = c_\psi = c$, $\psi\varphi = 1_c$ (the principal character of conductor c), and $t|(N/c^2)$, so when N is square-free, only the case $\psi = \varphi = \mathbf{1}$ is possible. Since $\frac{X_p(1,s)}{1-y_p^2} = p\gamma_s(p)$, with $\gamma_s(p) = \frac{1+p^{-1}}{1+p^{1-s}}$, setting $\gamma_s(u) = \prod_{p|u} \gamma_s(p)$ for u square-free and $\gamma_s(1) = 1$, we have

$$m_s(t,t') = 1 + (drr')^{2-s}\gamma_s(r)\gamma_s(r') - (dr)^{2-s}\gamma_s(r)\gamma_s(d) - (dr')^{2-s}\gamma_s(r')\gamma_s(d)$$

Since $\gamma_2(p) = 1$ for every p, it follows that $m_2(t,t') = 0$, so the L-function (4.4) has a simple pole at $s = 2$ with residue equal to $\zeta(0)$ times the quantitity

$$m'(t,t') = \frac{dm_s(t,t')}{ds}\Big|_{s=2}.$$

Therefore the Petersson product is nondegerate if and only if the matrix $M(N)$, with entries $m'(t,t')$ indexed by divisors t, t' of N with $t, t' > 1$, is nonsingular.

Since $\frac{d\gamma_s(p)}{ds}\big|_{s=2} = \frac{\ln(p)}{1+p}$ and $\gamma_2(p) = 1$, we have $\frac{d\gamma_s(u)}{ds}\big|_{s=2} = \sum_{p|u} \frac{\ln(p)}{1+p}$ for u squarefree, and

$$m'(t,t') = \begin{cases} 0 & \text{if } (t,t') = 1 \\ \sum_{p|d} \frac{p-1}{p+1} \ln(p) & \text{if } d = (t,t') > 1. \end{cases}$$

Then the lines indexed by primes $p \neq q$ add up to the line indexed by pq, showing that the determinant is 0, so the pairing is degenerate unless $N = p$ is prime. ☐

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Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, RO-014700 Bucharest, Romania

E-mail: vicentiu.pasol@imar.ro

E-mail: alexandru.popa@imar.ro