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New competition phenomena in Dirichlet problems

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Abstract

We study the multiplicity of nonnegative solutions to the problem,

$$-\Delta u = \lambda a(x)u^p + f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{P$_\lambda$}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $f : [0, \infty) \to \mathbb{R}$ oscillates near the origin or at infinity, and $p > 0$, $\lambda \in \mathbb{R}$. While oscillatory right-hand sides usually produce infinitely many distinct solutions, an additional term involving u^p may alter the situation radically. Via a direct variational argument we fully describe this phenomenon, showing that the number of distinct non-trivial solutions to problem (P$_\lambda$) is strongly influenced by u^p and depends on λ whenever one of the following two cases holds:

- $p \leqslant 1$ and f oscillates near the origin;
- $p \geqslant 1$ and f oscillates at infinity (p may be critical or even supercritical).

The coefficient $a \in L^\infty(\Omega)$ is allowed to change its sign, while its size is relevant only for the threshold value $p = 1$ when the behaviour of $f(s)/s$ plays a crucial role in both cases. Various H_0^1- and L^∞-norm estimates of solutions are also given.

Résumé

On étudie la multiplicité des solutions non négatives du problème,

$$-\Delta u = \lambda a(x)u^p + f(u) \quad \text{dans } \Omega, \qquad u = 0 \quad \text{sur } \partial\Omega, \tag{P$_\lambda$}$$

où Ω est un domaine borné et régulier de $\mathbb{R}^N$, $f : [0, \infty) \to \mathbb{R}$ est une fonction oscillante au voisinage de l'origine ou à l'infini, et $p > 0$, $\lambda \in \mathbb{R}$. Tant que les terms non linéaires oscillants produisent d'habitude une infinité de solutions distinctes, le terme supplémentaire u^p altére radicalement la situation. Par une méthode variationelle directe, nous décrivons complètement cette situation et nous montrons que le nombre des solutions non trivialles et distinctes du problème (P$_\lambda$) est influencé par le terme u^p, si

- $p \leqslant 1$ et f est oscillatoire au voisinage de l'origine ; ou
- $p \geqslant 1$ et f est oscillatoire à l'infini (p peut être critique ou surcritique).

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La grandeur du coefficient $a \in L^\infty(\Omega)$ est essentielle seulement dans le cas $p = 1$, tandis que le comportement du $f(s)/s$ et aussi important dans les deux cas considerés. Cette méthode permet d'obtenir quelques estimations des solutions dans H_0^1 et L^∞.

Keywords: Dirichlet problem; Oscillatory nonlinearity; Critical and supercritical growth; Indefinite potential; Variational method

1. Introduction and main results

This paper deals with the problem:

$$\begin{cases} -\Delta u = \lambda a(x)u^p + f(u) & \text{in } \Omega, \\ u \geqslant 0,\ u \not\equiv 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ is a smooth bounded domain with boundary $\partial\Omega$, $a \in L^\infty(\Omega)$, $f : [0, \infty) \to \mathbb{R}$ is a continuous function, while $p > 0$ and $\lambda \in \mathbb{R}$ are some parameters. The purpose of this paper is to study the number and behaviour of solutions to problem (P_λ), where f *oscillates* near the *origin* or at *infinity*. We premise a *strong competition* between the term involving u^p and the oscillatory nonlinearity f.

Before starting our detailed analysis, we notice that competition phenomena for related problems have been widely studied recently. For instance, Cârstea, Ghergu and Rădulescu [5] studied the combined effects of *asymptotically linear* and *singular* nonlinearities of a Lane–Emden–Fowler type elliptic problem; Ambrosetti, Brézis and Cerami [3], De Figueiredo, Gossez and Ubilla [6,7], considered the case of *concave–convex* nonlinearities in (P_λ). In the latter cases (i.e., [3,6,7]), the sublinear term u^p and the superlinear term $f(u) = u^q$ compete with each other, where $0 \leqslant p < 1 < q \leqslant (N+2)/(N-2) = 2^* - 1$. As a consequence of this competition, problem (P_λ) has at least two positive solutions for small $\lambda > 0$ and no positive solution for large λ. Since $u \mapsto -\Delta u$ is a linear map, the above statement is well reflected by the algebraic equation,

$$s = \lambda s^p + s^q, \quad s > 0. \tag{$E_{p,q}^\lambda$}$$

Indeed, there exists a $\lambda^* > 0$ such that for every $\lambda \in (0, \lambda^*)$ equation $(E_{p,q}^\lambda)$ has two solutions, $(E_{p,q}^{\lambda^*})$ has one solution, and for $\lambda > \lambda^*$ equation $(E_{p,q}^\lambda)$ has no solution.

Equations involving oscillatory terms usually give infinitely many distinct solutions, see Kristály, Moroşanu and Tersian [8], Obersnel and Omari [10], Omari and Zanolin [11], Saint Raymond [12]. However, surprising facts may occur even in simple cases; indeed, if $p = 1$ and we consider the oscillatory function $f(s) = f_\mu(s) = \mu \sin s$ $(\mu \in \mathbb{R})$, problem (P_λ) has only the trivial solution whenever $(|\lambda| \cdot \|a\|_{L^\infty} + |\mu|)\lambda_1(\Omega) < 1$, where $\lambda_1(\Omega)$ denotes the principal eigenvalue of $-\Delta$ on $H_0^1(\Omega)$, and $\|\cdot\|_{L^\infty}$ is the $L^\infty(\Omega)$-norm. Consequently, our first task is to identify some classes of functions which have a suitable oscillatory behaviour and produce infinitely many distinct solutions for (P_λ). Then, we investigate the influence of u^p on the oscillatory nonlinearities.

In the sequel, we state our main results, treating separately the two cases, i.e., when f oscillates near the *origin,* and at *infinity,* respectively. The coefficient $a \in L^\infty(\Omega)$ is allowed to be *indefinite* (i.e., it may change its sign), suggested by several recent works, including Alama and Tarantello [1,2], Berestycki, Cappuzzo Dolcetta and Nirenberg [4], De Figueiredo, Gossez and Ubilla [6,7], Servadei [13].

1.1. Oscillation near the origin

Let $f \in C([0, \infty), \mathbb{R})$ and $F(s) = \int_0^s f(t)\,dt$, $s \geqslant 0$. We assume:

(f_1^0) $-\infty < \liminf_{s \to 0^+} \frac{F(s)}{s^2}$; $\limsup_{s \to 0^+} \frac{F(s)}{s^2} = +\infty$,
(f_2^0) $l_0 := \liminf_{s \to 0^+} \frac{f(s)}{s} < 0$.

Remark 1.1. Hypotheses (f_1^0) and (f_2^0) imply an oscillatory behaviour of f near the origin. Let $\alpha, \beta, \gamma \in \mathbb{R}$ be such that $0 < \alpha < 1 < \alpha + \beta$, and $\gamma \in (0, 1)$. The function $f_0 : [0, \infty) \to \mathbb{R}$ defined by $f_0(0) = 0$ and $f_0(s) = s^\alpha(\gamma + \sin s^{-\beta})$, $s > 0$, verifies (f_1^0) and (f_2^0), respectively.

Theorem 1.1 *(Case $p \geqslant 1$). Assume $a \in L^\infty(\Omega)$ and let $f \in C([0,\infty),\mathbb{R})$ satisfies (f_1^0) and (f_2^0). If*

(a) *either $p = 1$ and $\lambda a(x) < \lambda_0$ a.e. $x \in \Omega$ for some $0 < \lambda_0 < -l_0$,*
(b) *or $p > 1$ and $\lambda \in \mathbb{R}$ is arbitrary,*

then there exists a sequence $\{u_i^0\}_i \subset H_0^1(\Omega)$ of distinct weak solutions of problem (P_λ) such that

$$\lim_{i\to\infty} \|u_i^0\|_{H_0^1} = \lim_{i\to\infty} \|u_i^0\|_{L^\infty} = 0. \tag{1}$$

Remark 1.2.

(i) If $l_0 = -\infty$, then (a) holds for *every* $\lambda \in \mathbb{R}$. For instance, this may happen for f_0 from Remark 1.1.
(ii) Notice that $p > 1$ may be *critical* or even *supercritical* in Theorem 1.1(b). Having a suitable nonlinearity oscillating near the origin, Theorem 1.1 roughly says that the term defined by $s \mapsto s^p$ $(s \geqslant 0)$ does not affect the number of distinct solutions of (P_λ) whenever $p > 1$; this is also the case for certain values of $\lambda \in \mathbb{R}$ when $p = 1$. A similar relation may be stated as before for both the equation $(E_{p,q}^\lambda)$ and the elliptic problem involving concave–convex nonlinearities. Namely, the thesis of Theorem 1.1 is nicely illustrated by the equation,
$$s = \lambda s^p + f_0(s), \quad s \geqslant 0, \tag{E_0}$$
where f_0 is the function appearing in Remark 1.1. Since $l_0 = -\infty$, for *every* $\lambda \in \mathbb{R}$ and $p \geqslant 1$, equation (E_0) has *infinitely many* distinct positive solutions.

On the other hand, this phenomenon dramatically changes when $p < 1$. In this case, the term $s \mapsto s^p$ $(s \geqslant 0)$ may compete with the function f_0 near the origin such that the number of distinct solutions of (E_0) becomes *finite* for many values of λ; this fact happens when $0 < p < \alpha$ (α is the number defined in Remark 1.1). However, the number of distinct solutions to (E_0) becomes greater and greater if $|\lambda|$ gets smaller and smaller as a simple (graphical) argument shows.

In the language of our Dirichlet problem (P_λ), the latter statement is perfectly described by the following result:

Theorem 1.2 *(Case $0 < p < 1$). Assume $a \in L^\infty(\Omega)$. Let $f \in C([0,\infty),\mathbb{R})$ satisfies (f_1^0) and (f_2^0), and $0 < p < 1$. Then, for every $k \in \mathbb{N}$, there exists $\lambda_k^0 > 0$ such that (P_λ) has at least k distinct weak solutions $\{u_{1,\lambda}^0, \dots, u_{k,\lambda}^0\} \subset H_0^1(\Omega)$ whenever $\lambda \in [-\lambda_k^0, \lambda_k^0]$. Moreover,*

$$\|u_{i,\lambda}^0\|_{H_0^1} < i^{-1} \quad \text{and} \quad \|u_{i,\lambda}^0\|_{L^\infty} < i^{-1} \quad \text{for any } i = \overline{1,k};\ \lambda \in [-\lambda_k^0, \lambda_k^0]. \tag{2}$$

1.2. Oscillation at infinity

Let $f \in C([0,\infty),\mathbb{R})$. We assume:

(f_1^∞) $-\infty < \liminf_{s\to\infty} \frac{F(s)}{s^2}$; $\limsup_{s\to\infty} \frac{F(s)}{s^2} = +\infty$,
(f_2^∞) $l_\infty := \liminf_{s\to\infty} \frac{f(s)}{s} < 0$.

Remark 1.3. Hypotheses (f_1^∞) and (f_2^∞) imply an oscillatory behaviour of f at infinity. Let $\alpha, \beta, \gamma \in \mathbb{R}$ be such that $1 < \alpha$, $|\alpha - \beta| < 1$, and $\gamma \in (0,1)$. Then, the function $f_\infty : [0,\infty) \to \mathbb{R}$ defined by $f_\infty(s) = s^\alpha(\gamma + \sin s^\beta)$ verifies the hypotheses (f_1^∞) and (f_2^∞), respectively.

The counterpart of Theorem 1.1 can be stated as follows:

Theorem 1.3 *(Case $p \leqslant 1$). Assume $a \in L^\infty(\Omega)$. Let $f \in C([0,\infty),\mathbb{R})$ satisfies (f_1^∞) and (f_2^∞) with $f(0) = 0$. If*

(a) *either $p = 1$ and $\lambda a(x) < \lambda_\infty$ a.e. $x \in \Omega$ for some $0 < \lambda_\infty < -l_\infty$,*
(b) *or $p < 1$ and $\lambda \in \mathbb{R}$ is arbitrary,*

then there exists a sequence $\{u_i^\infty\}_i \subset H_0^1(\Omega)$ *of distinct, weak solutions of problem* (P_λ) *such that*

$$\lim_{i\to\infty}\|u_i^\infty\|_{L^\infty}=\infty. \tag{3}$$

Remark 1.4. If

$$\sup_{s\in[0,\infty)}\frac{|f(s)|}{1+s^{2^*-1}}<\infty, \tag{4}$$

then we also have $\lim_{i\to\infty}\|u_i^\infty\|_{H_0^1}=\infty$ in Theorem 1.3. For details, see Section 4.

Remark 1.5. A similar observation can be made as in Remark 1.2. Indeed, when f oscillates at infinity, Theorem 1.3 shows that the term defined by $s\mapsto s^p$ $(s\geqslant 0)$ does not affect the number of distinct solutions of (P_λ) whenever $p<1$. This is also the case for certain values of $\lambda\in\mathbb{R}$ when $p=1$. A similar phenomenon occurs in the equation,

$$s=\lambda s^p+f_\infty(s),\quad s\geqslant 0, \tag{E_∞}$$

where f_∞ is the function defined in Remark 1.3. Since $l_\infty=-\infty$, for *every* $\lambda\in\mathbb{R}$ and $p\leqslant 1$, equation (E_∞) has *infinitely many* distinct positive solutions.

On the other hand, when $p>1$, the term $s\mapsto s^p$ $(s\geqslant 0)$ may dominate the function f_∞ at infinity. In particular, when $\alpha<p$, the number of distinct solutions of (E_∞) may become *finite* for many values of λ (here, α is the number defined in Remark 1.3). The positive finding is that the number of distinct solutions for (E_∞) increases whenever $|\lambda|$ decreases to zero.

In view of this observation, we obtain a natural counterpart of Theorem 1.2.

Theorem 1.4 *(Case $p>1$). Assume $a\in L^\infty(\Omega)$. Let $f\in C([0,\infty),\mathbb{R})$ satisfies (f_1^∞) and (f_2^∞) with $f(0)=0$, and $p>1$. Then, for every $k\in\mathbb{N}$, there exists $\lambda_k^\infty>0$ such that (P_λ) has at least k distinct weak solutions $\{u_{1,\lambda}^\infty,\dots,u_{k,\lambda}^\infty\}\subset H_0^1(\Omega)$ whenever $\lambda\in[-\lambda_k^\infty,\lambda_k^\infty]$. Moreover,*

$$\|u_{i,\lambda}^\infty\|_{L^\infty}>i-1\quad \textit{for any } i=\overline{1,k};\ \lambda\in[-\lambda_k^\infty,\lambda_k^\infty]. \tag{5}$$

Remark 1.6. If f verifies (4) and $p\leqslant 2^*-1$ in Theorem 1.4, then

$$\|u_{i,\lambda}^\infty\|_{H_0^1}>i-1\quad \text{for any } i=\overline{1,k};\ \lambda\in[-\lambda_k^\infty,\lambda_k^\infty].$$

For details, see also Section 4.

We conclude this section by stating a result for a model problem which involves concave–convex nonlinearities and an oscillatory term. We consider the problem:

$$\begin{cases}-\Delta u=\lambda u^p+\mu u^q+f(u),\ u\geqslant 0 & \text{on } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases} \tag{$\mathrm{P}_{\lambda,\mu}$}$$

where $0<p<1<q$, and $\lambda,\mu\in\mathbb{R}$. The following result proves that the number of solutions $(\mathrm{P}_{\lambda,\mu})$ is influenced

(a) by the sublinear term when f oscillates near the origin (with no effect of the superlinear term); and alternatively,
(b) by the superlinear term when f oscillates at infinity (with no effect of the sublinear term).

More precisely, applying Theorems 1.2 and 1.4, we have the:

Theorem 1.5. *Let $f\in C([0,\infty),\mathbb{R})$ and $0<p<1<q$.*

(a) *If (f_1^0) and (f_2^0) hold, then for every $k\in\mathbb{N}$ and $\mu\in\mathbb{R}$, there exists $\lambda_{k,\mu}>0$ such that $(\mathrm{P}_{\lambda,\mu})$ has at least k distinct weak solutions in $H_0^1(\Omega)$ whenever $\lambda\in[-\lambda_{k,\mu},\lambda_{k,\mu}]$.*
(b) *If (f_1^∞) and (f_2^∞) hold with $f(0)=0$, then for every $k\in\mathbb{N}$ and $\lambda\in\mathbb{R}$, there exists $\mu_{k,\lambda}>0$ such that $(\mathrm{P}_{\lambda,\mu})$ has at least k distinct weak solutions in $H_0^1(\Omega)$ whenever $\mu\in[-\mu_{k,\lambda},\mu_{k,\lambda}]$.*

2. An auxiliary result

In this section we consider the problem:

$$\begin{cases} -\Delta u + K(x)u = h(x,u), \ u \geqslant 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_h^K}$$

and assume that

(H_K) $K \in L^\infty(\Omega)$, $\operatorname{essinf}_\Omega K > 0$,
(H_h^1) $h : \Omega \times [0,\infty) \to \mathbb{R}$ is a Carathéodory function, $h(x,0) = 0$ for a.e. $x \in \Omega$, and there is $M_h > 0$ such that $|h(x,s)| \leqslant M_h$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$,
(H_h^2) there are $0 < \delta < \eta$ such that $h(x,s) \leqslant 0$ for a.e. $x \in \Omega$ and all $s \in [\delta, \eta]$.

We extend the function h by $h(x,s) = 0$ for a.e. $x \in \Omega$ and $s \leqslant 0$. We introduce the energy functional $\mathcal{E} : H_0^1(\Omega) \to \mathbb{R}$ associated with problem (P_h^K), defined by:

$$\mathcal{E}(u) = \frac{1}{2}\|u\|_{H_0^1}^2 + \frac{1}{2}\int_\Omega K(x)u^2\,dx - \int_\Omega H\big(x,u(x)\big)\,dx, \quad u \in H_0^1(\Omega),$$

where $H(x,s) = \int_0^s h(x,t)\,dt$, $s \in \mathbb{R}$. Due to hypothesis (H_h^1), it is easy to see that $\mathcal{E}$ is well-defined. Moreover, standard arguments show that $\mathcal{E}$ is of class C^1 on $H_0^1(\Omega)$.

Finally, considering the number $\eta \in \mathbb{R}$ from (H_h^2), we introduce the set,

$$W^\eta = \big\{u \in H_0^1(\Omega): \ \|u\|_{L^\infty} \leqslant \eta\big\}.$$

Since $h(x,0) = 0$, then 0 is clearly a solution of (P_h^K). In the sequel, under some general assumptions, we guarantee the existence of a (possible trivial) weak solution of (P_h^K) which is indispensable in our further investigations (see Sections 3 and 4).

Theorem 2.1. *Assume that* (H_K), (H_h^1), (H_h^2) *hold. Then*

(i) *the functional $\mathcal{E}$ is bounded from below on W^η and its infimum is attained at some $\tilde u \in W^\eta$,*
(ii) *$\tilde u(x) \in [0,\delta]$ for a.e. $x \in \Omega$,*
(iii) *$\tilde u$ is a weak solution of* (P_h^K).

Proof. (i) Due to (H_h^1) and by using Hölder's and Poincaré's inequalities, the functional $\mathcal{E}$ is bounded from below on the whole space $H_0^1(\Omega)$. In addition, one can easily see that $\mathcal{E}$ is sequentially weak lower semicontinuous and the set W^η is convex and closed in $H_0^1(\Omega)$, thus weakly closed. Combining these facts, there is an element $\tilde u \in W^\eta$ which is a minimum point of $\mathcal{E}$ over W^η.

(ii) Let $A = \{x \in \Omega: \ \tilde u(x) \notin [0,\delta]\}$ and suppose that $m(A) > 0$. Here and in the sequel, $m(\cdot)$ denotes the Lebesgue measure. Define the function $\gamma : \mathbb{R} \to \mathbb{R}$ by $\gamma(s) = \min(s_+, \delta)$, where $s_+ = \max(s, 0)$. Now, set $w = \gamma \circ \tilde u$. Since γ is a Lipschitz function and $\gamma(0) = 0$, the theorem of Marcus–Mizel [9] shows that $w \in H_0^1(\Omega)$. Moreover, $0 \leqslant w(x) \leqslant \delta$ for a.e. Ω. Consequently, $w \in W^\eta$.

We introduce the sets $A_1 = \{x \in A: \ \tilde u(x) < 0\}$ and $A_2 = \{x \in A: \ \tilde u(x) > \delta\}$. Thus, $A = A_1 \cup A_2$, and we have that $w(x) = \tilde u(x)$ for all $x \in \Omega \setminus A$, $w(x) = 0$ for all $x \in A_1$, and $w(x) = \delta$ for all $x \in A_2$. Moreover, we have:

$$\begin{aligned} \mathcal{E}(w) - \mathcal{E}(\tilde u) &= \frac{1}{2}\big[\|w\|_{H_0^1}^2 - \|\tilde u\|_{H_0^1}^2\big] + \frac{1}{2}\int_\Omega K(x)\big[w^2 - \tilde u^2\big] - \int_\Omega \big[H(x,w) - H(x,\tilde u)\big] \\ &= -\frac{1}{2}\int_A |\nabla \tilde u|^2 + \frac{1}{2}\int_A K(x)\big[w^2 - \tilde u^2\big] - \int_A \big[H(x,w) - H(x,\tilde u)\big]. \end{aligned}$$

Since $\operatorname{essinf}_\Omega K > 0$, one has,

$$\int_A K(x)\big[w^2-\tilde{u}^2\big]=-\int_{A_1} K(x)\tilde{u}^2+\int_{A_2} K(x)\big[\delta^2-\tilde{u}^2\big]\leqslant 0.$$

Due to the fact that $h(x,s)=0$ for all $s\leqslant 0$, one has,

$$\int_{A_1}\big[H(x,w)-H(x,\tilde{u})\big]=0.$$

By the mean value theorem, for a.e. $x\in A_2$, there exists $\theta(x)\in[\delta,\tilde{u}(x)]\subseteq[\delta,\eta]$ such that

$$H\big(x,w(x)\big)-H\big(x,\tilde{u}(x)\big)=H(x,\delta)-H\big(x,\tilde{u}(x)\big)=h\big(x,\theta(x)\big)\big(\delta-\tilde{u}(x)\big).$$

Thus, on account of (H_h^2), one has,

$$\int_{A_2}\big[H(x,w)-H(x,\tilde{u})\big]\geqslant 0.$$

Consequently, every term of the expression $\mathcal{E}(w)-\mathcal{E}(\tilde{u})$ is non-positive. On the other hand, since $w\in W^\eta$, then $\mathcal{E}(w)\geqslant\mathcal{E}(\tilde{u})=\inf_{W^\eta}\mathcal{E}$. So, every term in $\mathcal{E}(w)-\mathcal{E}(\tilde{u})$ should be zero. In particular,

$$\int_{A_1} K(x)\tilde{u}^2=\int_{A_2} K(x)\big[\tilde{u}^2-\delta^2\big]=0.$$

Due to (H_K), we necessarily have $m(A)=0$, contradicting our assumption.

(iii) Let us fix $v\in C_0^\infty(\Omega)$ and let $\varepsilon_0=(\eta-\delta)/(\|v\|_{C_0}+1)>0$. Define the function $E:[-\varepsilon_0,\varepsilon_0]\to\mathbb{R}$ by $E(\varepsilon)=\mathcal{E}(\tilde{u}+\varepsilon v)$ with $\varepsilon\in[-\varepsilon_0,\varepsilon_0]$. Due to (ii), for every $\varepsilon\in[-\varepsilon_0,\varepsilon_0]$, the element $\tilde{u}+\varepsilon v$ belongs to the set W^η. Consequently, due to (i), one has $E(\varepsilon)\geqslant E(0)$ for every $\varepsilon\in[-\varepsilon_0,\varepsilon_0]$. Since E is differentiable at 0 and $E'(0)=0$ it follows that $\langle\mathcal{E}'(\tilde{u}),v\rangle=0$. Since $v\in C_0^\infty(\Omega)$ is arbitrary, and the set $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, we obtain that $\tilde{u}$ is a weak solution of (P_h^K). □

We conclude this section by constructing a special function which will be useful in the proof of our theorems. In the sequel, let $B(x_0,r)\subset\Omega$ be the N-dimensional ball with radius $r>0$ and center $x_0\in\Omega$. For $s>0$, define:

$$z_s(x)=\begin{cases}0, & \text{if } x\in\Omega\setminus B(x_0,r),\\ s, & \text{if } x\in B(x_0,r/2),\\ \frac{2s}{r}(r-|x-x_0|), & \text{if } x\in B(x_0,r)\setminus B(x_0,r/2).\end{cases}\tag{6}$$

It is clear that $z_s\in H_0^1(\Omega)$. Moreover, we have $\|z_s\|_{L^\infty}=s$, and

$$\|z_s\|_{H_0^1}^2=\int_\Omega|\nabla z_s|^2=4r^{N-2}\big(1-2^{-N}\big)\omega_N s^2\equiv C(r,N)s^2>0,\tag{7}$$

where ω_N is the volume of $B(0,1)\subset\mathbb{R}^N$.

Notation. For every $\eta>0$, we define the truncation function $\tau_\eta:[0,\infty)\to\mathbb{R}$ by $\tau_\eta(s)=\min(\eta,s)$, $s\geqslant 0$.

3. Proofs of Theorems 1.1 and 1.2

Since the parts (a) and (b) of Theorem 1.1 will be treated simultaneously, we consider again the problem from the previous section,

$$\begin{cases}-\Delta u+K(x)u=h(x,u),\ u\geqslant 0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}\tag{P_h^K}$$

where the potential $K:\Omega\to\mathbb{R}$ fulfills (H_K). The function $h:\Omega\times[0,\infty)\to\mathbb{R}$ is Carathéodory, and we assume:

(H_0^0) $h(x,0)=0$ for a.e. $x\in\Omega$, and there exists $s_0>0$ such that

$$\sup_{s\in[0,s_0]}\left|h(\cdot,s)\right|\in L^\infty(\Omega);$$

(H_1^0) $-\infty<\liminf_{s\to 0^+}\frac{H(x,s)}{s^2}$ and $\limsup_{s\to 0^+}\frac{H(x,s)}{s^2}=+\infty$ uniformly for a.e. $x\in\Omega$; here, $H(x,s)=\int_0^s h(x,t)\,dt$;

(H_2^0) there are two sequences $\{\delta_i\}$, $\{\eta_i\}$ with $0<\eta_{i+1}<\delta_i<\eta_i$, $\lim_{i\to\infty}\eta_i=0$, and $h(x,s)\leqslant 0$ for a.e. $x\in\Omega$ and for every $s\in[\delta_i,\eta_i]$, $i\in\mathbb{N}$.

Theorem 3.1. *Assume* (H_K), (H_0^0), (H_1^0) *and* (H_2^0) *hold. Then there exists a sequence* $\{u_i^0\}_i\subset H_0^1(\Omega)$ *of distinct weak solutions of problem* (P_h^K) *such that*

$$\lim_{i\to\infty}\|u_i^0\|_{H_0^1}=\lim_{i\to\infty}\|u_i^0\|_{L^\infty}=0. \tag{8}$$

Proof. Without any loss of generality, we may assume that $\{\delta_i\}_i$, $\{\eta_i\}_i\subset(0,s_0)$, where $s_0>0$ comes from (H_0^0). For every $i\in\mathbb{N}$, define the truncation function $h_i:\Omega\times[0,\infty)\to\mathbb{R}$ by,

$$h_i(x,s)=h\big(x,\tau_{\eta_i}(s)\big), \tag{9}$$

and let $\mathcal{E}_i:H_0^1(\Omega)\to\mathbb{R}$ be the energy functional associated with the problem ($P_{h_i}^K$). Let $H_i(x,s)=\int_0^s h_i(x,t)\,dt$.

Due to hypotheses (H_0^0) and (H_2^0), the function h_i verifies the assumptions of Theorem 2.1 for every $i\in\mathbb{N}$ with $[\delta_i,\eta_i]$. Consequently, for every $i\in\mathbb{N}$, there exists $u_i^0\in W^{\eta_i}$ such that

$$u_i^0 \text{ is the minimum point of the functional } \mathcal{E}_i \text{ on } W^{\eta_i}, \tag{10}$$

$$u_i^0(x)\in[0,\delta_i]\quad\text{for a.e. } x\in\Omega, \tag{11}$$

$$u_i^0 \text{ is a weak solution of } \big(P_{h_i}^K\big). \tag{12}$$

Due to (9), (11) and (12), u_i^0 is a weak solution not only for ($P_{h_i}^K$) but also for the problem (P_h^K).

Now, we prove that there are infinitely many distinct elements in the sequence $\{u_i^0\}_i$. To see this, we first prove that

$$\mathcal{E}_i\big(u_i^0\big)<0\quad\text{for all } i\in\mathbb{N}; \tag{13}$$

$$\lim_{i\to\infty}\mathcal{E}_i\big(u_i^0\big)=0. \tag{14}$$

The left part of (H_1^0) implies the existence of some $l_0^h>0$ and $\zeta\in(0,\eta_1)$ such that

$$\operatorname{essinf}_{x\in\Omega}H(x,s)\geqslant -l_0^h s^2\quad\text{for all } s\in(0,\zeta). \tag{15}$$

Let $L_0^h>0$ be large enough so that

$$\frac{1}{2}C(r,N)+\left(\frac{1}{2}\|K\|_{L^\infty}+l_0^h\right)m(\Omega)<L_0^h(r/2)^N\omega_N, \tag{16}$$

where $r>0$ and $C(r,N)>0$ come from (7). Taking into account the right part of (H_1^0), there is a sequence $\{\tilde{s}_i\}_i\subset(0,\zeta)$ such that $\tilde{s}_i\leqslant\delta_i$ and

$$\operatorname{essinf}_{x\in\Omega}H(x,\tilde{s}_i)>L_0^h\tilde{s}_i^2\quad\text{for all } i\in\mathbb{N}. \tag{17}$$

Let $i\in\mathbb{N}$ be a fixed number and let $z_{\tilde{s}_i}\in H_0^1(\Omega)$ be the function from (6) corresponding to the value $\tilde{s}_i>0$. Then $z_{\tilde{s}_i}\in W^{\eta_i}$, and on account of (7), (17) and (15), one has:

$$\begin{aligned}
\mathcal{E}_i(z_{\tilde{s}_i}) &= \frac{1}{2}\|z_{\tilde{s}_i}\|^2_{H^1_0} + \frac{1}{2}\int_\Omega K(x)z^2_{\tilde{s}_i} - \int_\Omega H_i\big(x, z_{\tilde{s}_i}(x)\big)\,dx \\
&= \frac{1}{2}C(r,N)\tilde{s}_i^2 + \frac{1}{2}\int_\Omega K(x)z^2_{\tilde{s}_i} \\
&\quad - \int_{B(x_0,r/2)} H(x,\tilde{s}_i)\,dx - \int_{B(x_0,r)\setminus B(x_0,r/2)} H\big(x, z_{\tilde{s}_i}(x)\big)\,dx \\
&\leqslant \left[\frac{1}{2}C(r,N) + \frac{1}{2}\|K\|_{L^\infty}m(\Omega) - L_0^h(r/2)^N\omega_N + l_0^h m(\Omega)\right]\tilde{s}_i^2.
\end{aligned}$$

Consequently, using (10) and (16), we obtain that

$$\mathcal{E}_i\big(u_i^0\big) = \min_{W^{\eta_i}} \mathcal{E}_i \leqslant \mathcal{E}_i(z_{\tilde{s}_i}) < 0 \tag{18}$$

which proves in particular (13). Now, let us prove (14). For every $i \in \mathbb{N}$, by using the mean value theorem, (9), (H_0^0) and (11), we have:

$$\mathcal{E}_i\big(u_i^0\big) \geqslant -\int_\Omega H_i\big(x, u_i^0(x)\big)\,dx \geqslant -\Big\|\sup_{s\in[0,s_0]}\big|h(\cdot,s)\big|\Big\|_{L^\infty} m(\Omega)\delta_i.$$

Due to $\lim_{i\to\infty}\delta_i = 0$, the above inequality and (18) leads to (14).

On account of (9) and (11), we observe that

$$\mathcal{E}_i\big(u_i^0\big) = \mathcal{E}_1\big(u_i^0\big) \quad \text{for all } i \in \mathbb{N}.$$

Combining this relation with (13) and (14), we see that the sequence $\{u_i^0\}_i$ contains infinitely many distinct elements.

It remains to prove relation (8). The former limit easily follows by (11), i.e. $\|u_i^0\|_{L^\infty} \leqslant \delta_i$ for all $i \in \mathbb{N}$, combined with $\lim_{i\to\infty}\delta_i = 0$. For the latter limit, we use (18), (H_0^0), (9) and (11), obtaining for all $i \in \mathbb{N}$ that

$$\frac{1}{2}\big\|u_i^0\big\|^2_{H^1_0} \leqslant \frac{1}{2}\big\|u_i^0\big\|^2_{H^1_0} + \frac{1}{2}\int_\Omega K(x)\big(u_i^0\big)^2 < \int_\Omega H_i\big(x, u_i^0(x)\big) \leqslant \Big\|\sup_{s\in[0,s_0]}\big|h(\cdot,s)\big|\Big\|_{L^\infty} m(\Omega)\delta_i$$

which concludes the proof of Theorem 3.1. □

Proof of Theorem 1.1. (a) *Case* $p = 1$. Let $\lambda \in \mathbb{R}$ as in the hypothesis, i.e., $\lambda a(x) < \lambda_0$ a.e. $x \in \Omega$ for some $0 < \lambda_0 < -l_0$. Let us choose $\tilde{\lambda}_0 \in (\lambda_0, -l_0)$, and

$$K(x) = \tilde{\lambda}_0 - \lambda a(x) \quad \text{and} \quad h(x,s) = \tilde{\lambda}_0 s + f(s) \quad \text{for all } (x,s) \in \Omega \times [0,\infty). \tag{19}$$

Note that $\operatorname{essinf}_\Omega K \geqslant \tilde{\lambda}_0 - \lambda_0 > 0$, so (H_K) is satisfied. Due to (f_1^0) and (f_2^0), we have $f(0) = 0$. Thus, (H_0^0) clearly holds. Moreover, since $H(x,s)/s^2 = \tilde{\lambda}_0/2 + F(s)/s^2$, $s > 0$, hypothesis (f_1^0) implies (H_1^0). Finally, since $l_0 < -\tilde{\lambda}_0$, there exists a sequence $\{s_i\}_i \subset (0,1)$ converging to 0 such that $f(s_i)/s_i < -\tilde{\lambda}_0$ for all $i \in \mathbb{N}$. Consequently, by using the continuity of f, we may choose two sequences $\{\delta_i\}_i, \{\eta_i\}_i \subset (0,1)$ such that $0 < \eta_{i+1} < \delta_i < s_i < \eta_i$, $\lim_{i\to\infty}\eta_i = 0$, and $\tilde{\lambda}_0 s + f(s) \leqslant 0$ for all $s \in [\delta_i, \eta_i]$ and $i \in \mathbb{N}$. Therefore, (H_2^0) holds too. It remains to apply Theorem 3.1, observing that (P_h^K) is equivalent to problem (P_λ) via the choice (19).

(b) *Case* $p > 1$. Let $\lambda \in \mathbb{R}$ be arbitrary fixed. Let us also fix a number $\lambda_0 \in (0, -l_0)$ and choose

$$K(x) = \lambda_0 \quad \text{and} \quad h(x,s) = \lambda a(x)s^p + \lambda_0 s + f(s) \quad \text{for all } (x,s) \in \Omega \times [0,\infty). \tag{20}$$

Clearly, (H_K) is satisfied. Since $a \in L^\infty(\Omega)$, a simple argument yields that (H_0^0) also holds. Moreover, since $p > 1$ and $H(x,s)/s^2 = \lambda a(x)s^{p-1}/(p+1) + \lambda_0/2 + F(s)/s^2$, $s > 0$, hypothesis (f_1^0) implies (H_1^0). Note that for a.e $x \in \Omega$ and every $s \in [0,\infty)$, we have:

$$h(x,s) \leqslant |\lambda| \cdot \|a\|_{L^\infty} s^p + \lambda_0 s + f(s) \equiv \tilde{h}_0(s). \tag{21}$$

Due to (f_2^0), $\liminf_{s\to 0^+} \frac{\tilde{h}_0(s)}{s} = \lambda_0 + l_0 < 0$. In particular, there exists a sequence $\{s_i\}_i \subset (0,1)$ converging to 0 such that $\tilde{h}_0(s_i) < 0$ for all $i \in \mathbb{N}$. Consequently, by using the continuity of $\tilde{h}_0$, we can choose two sequences $\{\delta_i\}_i, \{\eta_i\}_i \subset (0,1)$ such that $0 < \eta_{i+1} < \delta_i < s_i < \eta_i$, $\lim_{i\to\infty} \eta_i = 0$, and $\tilde{h}_0(s) \leqslant 0$ for all $s \in [\delta_i, \eta_i]$ and $i \in \mathbb{N}$. Therefore, by using (21), hypothesis (H_2^0) holds. Now, we can apply Theorem 3.1; problem (P_h^K) is equivalent to problem (P_λ) through the choice (20). In both cases (i.e., (a) and (b)), relation (1) is implied by (8). This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. The proof is divided into four steps.

Step 1. Let $\lambda_0 \in (0, -l_0)$. On account of (f_1^0), there exists a sequence $\{s_i\}_i \subset (0,1)$ converging to 0, such that $f(s_i)/s_i < -\lambda_0$. For every $\lambda \in \mathbb{R}$ define the functions $h^\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}$ by:

$$h^\lambda(x,s) = \lambda a(x) s^p + \lambda_0 s + f(s) \quad \text{for all } (x,s) \in \Omega \times [0,\infty),$$

$$\tilde{h}(\lambda, s) = |\lambda| \cdot \|a\|_{L^\infty} s^p + \lambda_0 s + f(s) \quad \text{for all } s \in [0,\infty).$$

Since $\tilde{h}(0, s_i) = \lambda_0 s_i + f(s_i) < 0$ and due to the continuity of $\tilde{h}$, we can choose three sequences $\{\delta_i\}_i, \{\eta_i\}_i, \{\lambda_i\}_i \subset (0,1)$ such that $0 < \eta_{i+1} < \delta_i < s_i < \eta_i$, $\lim_{i\to\infty} \eta_i = 0$, and for every $i \in \mathbb{N}$,

$$\tilde{h}(\lambda, s) \leqslant 0 \quad \text{for all } \lambda \in [-\lambda_i, \lambda_i] \text{ and } s \in [\delta_i, \eta_i]. \tag{22}$$

Clearly, we may assume that

$$\delta_i \leqslant \min\left\{ i^{-1}, 2^{-1} i^{-2} \left[1 + \|a\|_{L^1} + m(\Omega) \max_{s\in[0,1]} \left| f(s) \right| \right]^{-1} \right\}, \quad i \in \mathbb{N}. \tag{23}$$

Since $h^\lambda(x,s) \leqslant \tilde{h}(\lambda, s)$ for a.e. $x \in \Omega$ and all $(\lambda, s) \in \mathbb{R} \times [0,\infty)$, on account of (22), for every $i \in \mathbb{N}$, we have:

$$h^\lambda(x,s) \leqslant 0 \quad \text{for a.e. } x \in \Omega \text{ and all } \lambda \in [-\lambda_i, \lambda_i],\ s \in [\delta_i, \eta_i]. \tag{24}$$

For every $i \in \mathbb{N}$ and $\lambda \in [-\lambda_i, \lambda_i]$, let $h_i^\lambda : \Omega \times [0,\infty) \to \mathbb{R}$ be defined by:

$$h_i^\lambda(x,s) = h^\lambda\big(x, \tau_{\eta_i}(s)\big), \tag{25}$$

and $K(x) = \lambda_0$. Let $\mathcal{E}_{i,\lambda} : H_0^1(\Omega) \to \mathbb{R}$ be the energy functional associated with $(\mathrm{P}^K_{h_i^\lambda})$, i.e.,

$$\mathcal{E}_{i,\lambda}(u) = \frac{1}{2}\|u\|^2_{H_0^1} + \frac{1}{2}\int_\Omega K(x) u^2 - \int_\Omega \left(\int_0^{u(x)} h_i^\lambda(x,s)\, ds \right) dx. \tag{26}$$

Then, for every $i \in \mathbb{N}$ and $\lambda \in [-\lambda_i, \lambda_i]$, the function h_i^λ verifies the hypotheses of Theorem 2.1; see (24) for $(\mathrm{H}^2_{h_i^\lambda})$. Therefore, for every $i \in \mathbb{N}$ and $\lambda \in [-\lambda_i, \lambda_i]$:

$$\text{there exists } u^0_{i,\lambda} \in W^{\eta_i} \quad \text{such that} \quad \mathcal{E}_{i,\lambda}\big(u^0_{i,\lambda}\big) = \min_{W^{\eta_i}} \mathcal{E}_{i,\lambda}, \tag{27}$$

$$u^0_{i,\lambda}(x) \in [0, \delta_i] \quad \text{for a.e. } x \in \Omega, \tag{28}$$

$$u^0_{i,\lambda} \text{ is a weak solution of } \big(\mathrm{P}^K_{h_i^\lambda}\big). \tag{29}$$

Due to the definition of the functions h_i^λ and K, $u^0_{i,\lambda}$ is a weak solution not only for $(\mathrm{P}^K_{h_i^\lambda})$, see (25), (28) and (29), but also for our initial problem (P_λ) once we guarantee that $u^0_{i,\lambda} \not\equiv 0$.

Step 2. For $\lambda = 0$, the function $h_i^\lambda = h_i^0$ verifies the hypotheses of Theorem 3.1; more precisely, h_i^0 is precisely the function appearing in (9) and $\mathcal{E}_i := \mathcal{E}_{i,0}$ is the energy functional associated with problem $(\mathrm{P}^K_{h_i^0})$. Consequently, besides (27)–(29), the elements $u_i^0 := u^0_{i,0}$ also verify:

$$\mathcal{E}_i\left(u_i^0\right) = \min_{W^{\eta_i}} \mathcal{E}_i \leqslant \mathcal{E}_i(z_{\tilde{s}_i}) < 0 \quad \text{for all } i \in \mathbb{N}, \tag{30}$$

where $z_{\tilde{s}_i} \in W^{\eta_i}$ come from the proof of Theorem 3.1, see (18).

Step 3. Let $\{\theta_i\}_i$ be a sequence with negative terms such that $\lim_{i\to\infty}\theta_i = 0$. On account of (30), up to a subsequence, we may assume that

$$\theta_i < \mathcal{E}_i\left(u_i^0\right) \leqslant \mathcal{E}_i(z_{\tilde{s}_i}) < \theta_{i+1}. \tag{31}$$

Let

$$\lambda_i' = \frac{(p+1)(\theta_{i+1} - \mathcal{E}_i(z_{\tilde{s}_i}))}{\|a\|_{L^1}+1} \quad \text{and} \quad \lambda_i'' = \frac{(p+1)(\mathcal{E}_i(u_i^0) - \theta_i)}{\|a\|_{L^1}+1}, \quad i \in \mathbb{N}.$$

Fix $k \in \mathbb{N}$. On account of (31),

$$\lambda_k^0 = \min\left(\lambda_1, \dots, \lambda_k, \lambda_1', \dots, \lambda_k', \lambda_1'', \dots, \lambda_k''\right) > 0.$$

Then, for every $i \in \{1, \dots, k\}$ and $\lambda \in [-\lambda_k^0, \lambda_k^0]$ we have:

$$\begin{aligned}
\mathcal{E}_{i,\lambda}\left(u_{i,\lambda}^0\right) &\leqslant \mathcal{E}_{i,\lambda}(z_{\tilde{s}_i}) \quad \bigl(\text{see (27)}\bigr) \\
&= \frac{1}{2}\|z_{\tilde{s}_i}\|_{H_0^1}^2 - \frac{\lambda}{p+1}\int_\Omega a(x) z_{\tilde{s}_i}^{p+1} - \int_\Omega F\bigl(z_{\tilde{s}_i}(x)\bigr)\,dx \\
&= \mathcal{E}_i(z_{\tilde{s}_i}) - \frac{\lambda}{p+1}\int_\Omega a(x) z_{\tilde{s}_i}^{p+1} \\
&< \theta_{i+1} \quad \bigl(\text{see the choice of } \lambda_i' \text{ and } \tilde{s}_i \leqslant \delta_i < 1\bigr),
\end{aligned}$$

and taking into account that $u_{i,\lambda}^0$ belongs to W^{η_i}, and u_i^0 is the minimum point of $\mathcal{E}_i$ over the set W^{η_i}, see relation (30), we have:

$$\begin{aligned}
\mathcal{E}_{i,\lambda}\left(u_{i,\lambda}^0\right) &= \mathcal{E}_i\left(u_{i,\lambda}^0\right) - \frac{\lambda}{p+1}\int_\Omega a(x)\left(u_{i,\lambda}^0\right)^{p+1} \\
&\geqslant \mathcal{E}_i\left(u_i^0\right) - \frac{\lambda}{p+1}\int_\Omega a(x)\left(u_{i,\lambda}^0\right)^{p+1} \\
&> \theta_i \quad \bigl(\text{see the choice of } \lambda_i'' \text{ and (28)}\bigr).
\end{aligned}$$

In conclusion, for every for every $i \in \{1, \dots, k\}$ and $\lambda \in [-\lambda_k^0, \lambda_k^0]$ we have:

$$\theta_i < \mathcal{E}_{i,\lambda}\left(u_{i,\lambda}^0\right) < \theta_{i+1} < 0,$$

thus

$$\mathcal{E}_{1,\lambda}\left(u_{1,\lambda}^0\right) < \cdots < \mathcal{E}_{k,\lambda}\left(u_{k,\lambda}^0\right) < 0.$$

But, $u_{i,\lambda}^0 \in W^{\eta_1}$ for every $i \in \{1, \dots, k\}$, so $\mathcal{E}_{i,\lambda}(u_{i,\lambda}^0) = \mathcal{E}_{1,\lambda}(u_{i,\lambda}^0)$, see relation (25). Therefore, from above, we obtain that for every $\lambda \in [-\lambda_k^0, \lambda_k^0]$,

$$\mathcal{E}_{1,\lambda}\left(u_{1,\lambda}^0\right) < \cdots < \mathcal{E}_{1,\lambda}\left(u_{k,\lambda}^0\right) < 0 = \mathcal{E}_{1,\lambda}(0).$$

These inequalities show that the elements $u_{1,\lambda}^0, \dots, u_{k,\lambda}^0$ are distinct (and non-trivial) whenever $\lambda \in [-\lambda_k^0, \lambda_k^0]$.

Step 4. It remains to prove conclusion (2). The former relation follows directly by (28) and (23). To check the latter, we observe that for every $i \in \{1, \dots, k\}$ and $\lambda \in [-\lambda_k^0, \lambda_k^0]$,

$$\mathcal{E}_{1,\lambda}\left(u_{i,\lambda}^0\right) = \mathcal{E}_{i,\lambda}\left(u_{i,\lambda}^0\right) < \theta_{i+1} < 0.$$

Consequently, for every $i \in \{1, \dots, k\}$ and $\lambda \in [-\lambda_k^0, \lambda_k^0]$, by a mean value theorem we obtain:

$$\begin{aligned}\frac{1}{2}\|u_{i,\lambda}^0\|_{H_0^1}^2 &< \frac{\lambda}{p+1}\int_\Omega a(x)\big(u_{i,\lambda}^0\big)^{p+1} + \int_\Omega F\big(u_{i,\lambda}^0(x)\big)\,dx\\ &\leqslant \left[\frac{1}{p+1}\|a\|_{L^1} + m(\Omega)\max_{s\in[0,1]}\big|f(s)\big|\right]\delta_i \quad \big(\text{see (28) and } \delta_i, \lambda_k^0 \leqslant 1\big)\\ &< 2^{-1}i^{-2} \quad \big(\text{see (23)}\big)\end{aligned}$$

which concludes the proof of Theorem 1.2. □

4. Proofs of Theorems 1.3 and 1.4

In order to prove Theorems 1.3 and 1.4 we follow more or less the technique of the previous section. However, for completeness, we give all the details. We consider again the problem (P_h^K), where the Carathéodory function $h : \Omega \times [0,\infty) \to \mathbb{R}$ fulfills

(H_0^∞) $h(x,0) = 0$ for a.e. $x \in \Omega$, and for every $s \geqslant 0$,

$$\sup_{t\in[0,s]}\big|h(\cdot,t)\big| \in L^\infty(\Omega);$$

(H_1^∞) $-\infty < \liminf_{s\to\infty}\frac{H(x,s)}{s^2}$ and $\limsup_{s\to\infty}\frac{H(x,s)}{s^2} = +\infty$ uniformly for a.e. $x \in \Omega$; here, $H(x,s) = \int_0^s h(x,t)\,dt$;
(H_2^∞) there are two sequences $\{\delta_i\}$, $\{\eta_i\}$ with $0 < \delta_i < \eta_i < \delta_{i+1}$, $\lim_{i\to\infty}\delta_i = +\infty$, and $h(x,s) \leqslant 0$ for a.e. $x \in \Omega$ and for every $s \in [\delta_i, \eta_i]$, $i \in \mathbb{N}$.

Theorem 4.1. *Assume* (H_K), (H_0^∞), (H_1^∞) *and* (H_2^∞) *hold. Then there exists a sequence* $\{u_i^\infty\}_i \subset H_0^1(\Omega)$ *of distinct weak solutions of problem* (P_h^K) *such that*

$$\lim_{i\to\infty}\|u_i^\infty\|_{L^\infty} = \infty. \tag{32}$$

Proof. For any $i \in \mathbb{N}$, we introduce the truncation function $h_i : \Omega \times [0,\infty) \to \mathbb{R}$ by:

$$h_i(x,s) = h\big(x, \tau_{\eta_i}(s)\big). \tag{33}$$

Let $\mathcal{E}_i : H_0^1(\Omega) \to \mathbb{R}$ be the energy functional associated with problem $(\mathrm{P}_{h_i}^K)$. As before, let $H_i(x,s) = \int_0^s h_i(x,t)\,dt$.

On account of hypotheses (H_0^∞) and (H_2^∞), h_i fulfills the assumptions of Theorem 2.1 for every $i \in \mathbb{N}$ with $[\delta_i, \eta_i]$. Thus, for every $i \in \mathbb{N}$, there is an element $u_i^\infty \in W^{\eta_i}$ such that

$$u_i^\infty \text{ is the minimum point of the functional } \mathcal{E}_i \text{ on } W^{\eta_i}, \tag{34}$$

$$u_i^\infty(x) \in [0, \delta_i] \quad \text{for a.e. } x \in \Omega, \tag{35}$$

$$u_i^\infty \text{ is a weak solution of } \big(\mathrm{P}_{h_i}^K\big). \tag{36}$$

Thanks to (36), (33) and (35), u_i^∞ is also a weak solution for the problem (P_h^K).

We prove that there are infinitely many distinct elements in the sequence $\{u_i^\infty\}_i$. To this end, it is enough to show that

$$\lim_{i\to\infty}\mathcal{E}_i\big(u_i^\infty\big) = -\infty. \tag{37}$$

Indeed, let us assume that in the sequence $\{u_i^\infty\}_i$ there are only finitely many distinct elements, say $\{u_1^\infty, \ldots, u_{i_0}^\infty\}$ for some $i_0 \in \mathbb{N}$. Consequently, due to (33), the sequence $\{\mathcal{E}_i(u_i^\infty)\}_i$ reduces to at most the finite set $\{\mathcal{E}_{i_0}(u_1^\infty), \ldots, \mathcal{E}_{i_0}(u_{i_0}^\infty)\}$, which contradicts relation (37).

Now, we prove (37). By (H_1^∞), there exist $l_\infty^h > 0$ and $\zeta > 0$ such that

$$\operatorname{essinf}_{x\in\Omega} H(x,s) \geqslant -l_\infty^h s^2 \quad \text{for all } s > \zeta. \tag{38}$$

Fix $L_\infty^h > 0$ large enough such that

$$\frac{1}{2}C(r,N) + \left(\frac{1}{2}\|K\|_{L^\infty} + l_\infty^h\right)m(\Omega) < L_\infty^h (r/2)^N \omega_N, \tag{39}$$

where $r > 0$ and $C(r,N) > 0$ are from (7). Due to the right-hand side of (H_1^∞), one can fix a sequence $\{\tilde{s}_i\}_i \subset (0,\infty)$ such that $\lim_{i\to\infty} \tilde{s}_i = \infty$, and

$$\operatorname{essinf}_{x\in\Omega} H(x,\tilde{s}_i) > L_\infty^h \tilde{s}_i^2 \quad \text{for all } i \in \mathbb{N}. \tag{40}$$

Since $\lim_{i\to\infty} \delta_i = \infty$, see (H_2^∞), we can choose a subsequence $\{\delta_{m_i}\}_i$ of $\{\delta_i\}_i$ such that $\tilde{s}_i \leqslant \delta_{m_i}$ for all $i \in \mathbb{N}$. Let $i \in \mathbb{N}$ be fixed and let $z_{\tilde{s}_i} \in H_0^1(\Omega)$ be the function from (6) corresponding to the value $\tilde{s}_i > 0$. Then $z_{\tilde{s}_i} \in W^{\eta_{m_i}}$, and on account of (7), (40) and (38), we have:

$$\begin{aligned}
\mathcal{E}_{m_i}(z_{\tilde{s}_i}) &= \frac{1}{2}\|z_{\tilde{s}_i}\|_{H_0^1}^2 + \frac{1}{2}\int_\Omega K(x) z_{\tilde{s}_i}^2 - \int_\Omega H_{m_i}\big(x, z_{\tilde{s}_i}(x)\big)\,dx \\
&= \frac{1}{2}C(r,N)\tilde{s}_i^2 + \frac{1}{2}\int_\Omega K(x) z_{\tilde{s}_i}^2 - \int_{B(x_0,r/2)} H(x,\tilde{s}_i)\,dx \\
&\quad - \int_{(B(x_0,r)\setminus B(x_0,r/2))\cap\{z_{\tilde{s}_i} > \zeta\}} H\big(x, z_{\tilde{s}_i}(x)\big)\,dx \\
&\quad - \int_{(B(x_0,r)\setminus B(x_0,r/2))\cap\{z_{\tilde{s}_i} \leqslant \zeta\}} H\big(x, z_{\tilde{s}_i}(x)\big)\,dx \\
&\leqslant \left[\frac{1}{2}C(r,N) + \frac{1}{2}\|K\|_{L^\infty} m(\Omega) - L_\infty^h (r/2)^N \omega_N + l_\infty^h m(\Omega)\right]\tilde{s}_i^2 \\
&\quad + \Big\|\sup_{s\in[0,\zeta]} \big|h(\cdot,s)\big|\Big\|_{L^\infty} m(\Omega)\zeta.
\end{aligned}$$

The above estimate, relation (39) and $\lim_{i\to\infty} \tilde{s}_i = \infty$ clearly show that

$$\lim_{i\to\infty} \mathcal{E}_{m_i}(z_{\tilde{s}_i}) = -\infty. \tag{41}$$

On the other hand, using (34), we have:

$$\mathcal{E}_{m_i}\big(u_{m_i}^\infty\big) = \min_{W^{\eta_{m_i}}} \mathcal{E}_{m_i} \leqslant \mathcal{E}_{m_i}(z_{\tilde{s}_i}). \tag{42}$$

Therefore, on account of (41), we have:

$$\lim_{i\to\infty} \mathcal{E}_{m_i}\big(u_{m_i}^\infty\big) = -\infty. \tag{43}$$

Note that the sequence $\{\mathcal{E}_i(u_i^\infty)\}_i$ is non-increasing. Let $i < k$; then, due to (33), we have:

$$\mathcal{E}_i\big(u_i^\infty\big) = \min_{W^{\eta_i}} \mathcal{E}_i = \min_{W^{\eta_i}} \mathcal{E}_k \geqslant \min_{W^{\eta_k}} \mathcal{E}_k = \mathcal{E}_k\big(u_k^\infty\big).$$

Combining this fact with (43), we obtain (37).

Now, we prove (32). Arguing by contradiction assume there exists a subsequence $\{u_{k_i}^\infty\}_i$ of $\{u_i^\infty\}_i$ such that for all $i \in \mathbb{N}$, we have $\|u_{k_i}^\infty\|_{L^\infty} \leqslant M$ for some $M > 0$. In particular, $\{u_{k_i}^\infty\}_i \subset W^{\eta_l}$ for some $l \in \mathbb{N}$. Thus, for every $k_i \geqslant l$, we have:

$$\begin{aligned}
\mathcal{E}_l\big(u_l^\infty\big) &= \min_{W^{\eta_l}} \mathcal{E}_l = \min_{W^{\eta_l}} \mathcal{E}_{k_i} \\
&\geqslant \min_{W^{\eta_{k_i}}} \mathcal{E}_{k_i} = \mathcal{E}_{k_i}\big(u_{k_i}^\infty\big) \\
&\geqslant \min_{W^{\eta_l}} \mathcal{E}_{k_i} \quad \big(\text{cf. hypothesis, } u_{k_i}^\infty \in W^{\eta_l}\big) \\
&= \mathcal{E}_l\big(u_l^\infty\big).
\end{aligned}$$

As a consequence,

$$\mathcal{E}_{k_i}\big(u_{k_i}^{\infty}\big) = \mathcal{E}_l\big(u_l^{\infty}\big) \quad \text{for all } i \in \mathbb{N}. \tag{44}$$

Since the sequence $\{\mathcal{E}_i(u_i^{\infty})\}_i$ is non-increasing, on account of (44), one can find a number $i_0 \in \mathbb{N}$ such that $\mathcal{E}_i(u_i^{\infty}) = \mathcal{E}_l(u_l^{\infty})$ for all $i \geqslant i_0$. This fact contradicts (37), which concludes the proof of Theorem 4.1. □

Proof of Theorem 1.3. (a) *Case* $p = 1$. Let us fix $\lambda \in \mathbb{R}$ as in the hypothesis, i.e., $\lambda a(x) < \lambda_\infty$ a.e. $x \in \Omega$ for some $0 < \lambda_\infty < -l_\infty$. Fix also $\tilde{\lambda}_\infty \in (\lambda_\infty, -l_\infty)$ and let

$$K(x) = \tilde{\lambda}_\infty - \lambda a(x) \quad \text{and} \quad h(x,s) = \tilde{\lambda}_\infty s + f(s) \quad \text{for all } (x,s) \in \Omega \times [0,\infty). \tag{45}$$

It is clear that $\operatorname{essinf}_\Omega K \geqslant \tilde{\lambda}_\infty - \lambda_\infty > 0$, so (H_K) is satisfied. Since $f(0) = 0$, (H_0^∞) holds too. Note that $H(x,s)/s^2 = \tilde{\lambda}_\infty/2 + F(s)/s^2$, $s > 0$; thus, hypothesis (f_1^∞) implies (H_1^∞). Since $l_\infty < -\tilde{\lambda}_\infty$, there is a sequence $\{s_i\}_i \subset (0,\infty)$ converging to $+\infty$ such that $f(s_i)/s_i < -\tilde{\lambda}_\infty$ for all $i \in \mathbb{N}$. By using the continuity of f, we may fix two sequences $\{\delta_i\}_i, \{\eta_i\}_i \subset (0,\infty)$ such that $0 < \delta_i < s_i < \eta_i < \delta_{i+1}$, $\lim_{i\to\infty} \delta_i = \infty$, and $\tilde{\lambda}_\infty s + f(s) \leqslant 0$ for all $s \in [\delta_i, \eta_i]$ and $i \in \mathbb{N}$. Therefore, (H_2^∞) is also fulfilled. Now, we are in the position to apply Theorem 4.1. Throughout the choice (45), (P_h^K) is equivalent to problem (P_λ) which concludes the proof.

(b) *Case* $p < 1$. Let $\lambda \in \mathbb{R}$ be fixed arbitrarily and fix $\lambda_\infty \in (0, -l_\infty)$. Now, let us choose:

$$K(x) = \lambda_\infty \quad \text{and} \quad h(x,s) = \lambda a(x)s^p + \lambda_\infty s + f(s) \quad \text{for all } (x,s) \in \Omega \times [0,\infty). \tag{46}$$

Hypothesis (H_K) is clearly satisfied. Due to the fact that $a \in L^\infty(\Omega)$, hypothesis (H_0^∞) holds too. Since $p < 1$ and $H(x,s)/s^2 = \lambda a(x)s^{p-1}/(p+1) + \lambda_\infty/2 + F(s)/s^2$, $s > 0$, hypothesis (f_1^∞) implies (H_1^∞). For a.e. $x \in \Omega$, and every $s \in [0,\infty)$, we have:

$$h(x,s) \leqslant |\lambda| \cdot \|a\|_{L^\infty} s^p + \lambda_\infty s + f(s) \equiv \tilde{h}_\infty(s). \tag{47}$$

Thanks to (f_2^∞), we have $\liminf_{s\to\infty} \frac{\tilde{h}_\infty(s)}{s} = \lambda_\infty + l_\infty < 0$. Therefore, one can fix a sequence $\{s_i\}_i \subset (0,\infty)$ converging to $+\infty$ such that $\tilde{h}_\infty(s_i) < 0$ for all $i \in \mathbb{N}$. Now, by using the continuity of $\tilde{h}_\infty$, one can fix two sequences $\{\delta_i\}_i, \{\eta_i\}_i \subset (0,\infty)$ such that $0 < \delta_i < s_i < \eta_i < \delta_{i+1}$, $\lim_{i\to\infty} \delta_i = \infty$, and $\tilde{h}_\infty(s) \leqslant 0$ for all $s \in [\delta_i, \eta_i]$ and $i \in \mathbb{N}$. Thus, by using (47), hypothesis (H_2^∞) holds. Now, we can apply Theorem 4.1, observing that problem (P_h^K) is equivalent to problem (P_λ) through the choice (46). Finally, in both cases (i.e., (a) and (b)), (32) implies relation (3). This concludes the proof of Theorem 1.3. □

Proof of Remark 1.4. Assume that (4) holds. By contradiction, let us assume that there exists a bounded subsequence $\{u_{k_i}^\infty\}_i$ of $\{u_i^\infty\}_i$ in $H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is continuously embedded into $L^t(\Omega)$, $t \in [1, 2^*]$, after an elementary estimate, we obtain that the sequence $\{\mathcal{E}_{k_i}(u_{k_i}^\infty)\}_i$ is bounded. Since the sequence $\{\mathcal{E}_i(u_i^\infty)\}_i$ is non-increasing, it will be bounded as well, which contradicts (37). □

Proof of Theorem 1.4. The proof is divided into five steps.

Step 1. Let $\lambda_\infty \in (0, -l_\infty)$. Due to (f_1^∞), we may fix a sequence $\{s_i\}_i \subset (0,\infty)$ converging to ∞, such that $f(s_i)/s_i < -\lambda_\infty$. For every $\lambda \in \mathbb{R}$, let us define two functions $h^\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}$ by:

$$h^\lambda(x,s) = \lambda a(x)s^p + \lambda_\infty s + f(s) \quad \text{for all } (x,s) \in \Omega \times [0,\infty),$$

$$\tilde{h}(\lambda,s) = |\lambda| \cdot \|a\|_{L^\infty} s^p + \lambda_\infty s + f(s) \quad \text{for all } s \in [0,\infty).$$

Note that $\tilde{h}(0,s_i) = \lambda_\infty s_i + f(s_i) < 0$. Due to the continuity of $\tilde{h}$, we can fix three sequences $\{\delta_i\}_i, \{\eta_i\}_i \subset (0,\infty)$ and $\{\lambda_i\}_i \subset (0,1)$ such that $0 < \delta_i < s_i < \eta_i < \delta_{i+1}$, $\lim_{i\to\infty} \delta_i = \infty$, and for every $i \in \mathbb{N}$,

$$\tilde{h}(\lambda,s) \leqslant 0 \quad \text{for all } \lambda \in [-\lambda_i, \lambda_i] \text{ and } s \in [\delta_i, \eta_i]. \tag{48}$$

Without any loss of generality, we may assume that

$$\delta_i \geqslant i, \quad i \in \mathbb{N}. \tag{49}$$

Note that $h^\lambda(x,s) \leqslant \tilde{h}(\lambda,s)$ for a.e. $x \in \Omega$ and all $(\lambda,s) \in \mathbb{R} \times [0,\infty)$. Taking into account of (48), for every $i \in \mathbb{N}$, we have:

$$h^\lambda(x,s) \leqslant 0 \quad \text{for a.e. } x \in \Omega \text{ and all } \lambda \in [-\lambda_i, \lambda_i],\ s \in [\delta_i, \eta_i]. \tag{50}$$

For any $i \in \mathbb{N}$ and $\lambda \in [-\lambda_i, \lambda_i]$, let $h_i^\lambda : \Omega \times [0,\infty) \to \mathbb{R}$ be defined by:

$$h_i^\lambda(x,s) = h^\lambda\big(x, \tau_{\eta_i}(s)\big), \tag{51}$$

and $K(x) = \lambda_\infty$. Let $\mathcal{E}_{i,\lambda} : H_0^1(\Omega) \to \mathbb{R}$ be the energy functional associated with the problem $(\mathrm{P}^K_{h_i^\lambda})$, which is formally the same as in (26). Note that for every $i \in \mathbb{N}$ and $\lambda \in [-\lambda_i, \lambda_i]$, the function h_i^λ fulfills the hypotheses of Theorem 2.1; see (50) for $(\mathrm{H}^2_{h_i^\lambda})$. Consequently, for every $i \in \mathbb{N}$ and $\lambda \in [-\lambda_i, \lambda_i]$,

$$\text{there exists } \tilde{u}_{i,\lambda}^\infty \in W^{\eta_i} \quad \text{with } \mathcal{E}_{i,\lambda}\big(\tilde{u}_{i,\lambda}^\infty\big) = \min_{W^{\eta_i}} \mathcal{E}_{i,\lambda}, \tag{52}$$

$$\tilde{u}_{i,\lambda}^\infty(x) \in [0, \delta_i] \quad \text{for a.e. } x \in \Omega, \tag{53}$$

$$\tilde{u}_{i,\lambda}^\infty \text{ is a weak solution of } \big(\mathrm{P}^K_{h_i^\lambda}\big). \tag{54}$$

On account of the definition of the functions h_i^λ and K, and relations (54) and (53), $\tilde{u}_{i,\lambda}^\infty$ is also a weak solution for our initial problem (P_λ) once we have $\tilde{u}_{i,\lambda}^\infty \not\equiv 0$.

Step 2. Note that for $\lambda = 0$, the function $h_i^\lambda = h_i^0$ verifies the hypotheses of Theorem 4.1; in fact, h_i^0 is the function appearing in (33) and $\mathcal{E}_i := \mathcal{E}_{i,0}$ is the energy functional associated with problem $(\mathrm{P}^K_{h_i^0})$. Denoting $u_i^\infty := \tilde{u}_{i,0}^\infty$, we also have:

$$\mathcal{E}_{m_i}\big(u_{m_i}^\infty\big) = \min_{W^{\eta_{m_i}}} \mathcal{E}_{m_i} \leqslant \mathcal{E}_{m_i}(z_{\tilde{s}_i}), \tag{55}$$

$$\lim_{i\to\infty} \mathcal{E}_{m_i}\big(u_{m_i}^\infty\big) = -\infty, \tag{56}$$

where the special subsequence $\{u_{m_i}^\infty\}_i$ of $\{u_i^\infty\}_i$ and $z_{\tilde{s}_i} \in W^{\eta_{m_i}}$ appear in the proof of Theorem 4.1, see relations (42) and (43), respectively.

Step 3. Let us fix a sequence $\{\theta_i\}_i$ with negative terms such that $\lim_{i\to\infty} \theta_i = -\infty$. Due to (55) and (56), up to a subsequence, we may assume that

$$\theta_{i+1} < \mathcal{E}_{m_i}\big(u_{m_i}^\infty\big) \leqslant \mathcal{E}_{m_i}(z_{\tilde{s}_i}) < \theta_i. \tag{57}$$

For any $i \in \mathbb{N}$, define

$$\lambda_i' = \frac{(p+1)(\theta_i - \mathcal{E}_{m_i}(z_{\tilde{s}_i}))}{\delta_{m_i}^{p+1}(\|a\|_{L^1}+1)} \quad \text{and} \quad \lambda_i'' = \frac{(p+1)(\mathcal{E}_{m_i}(u_{m_i}^\infty) - \theta_{i+1})}{\delta_{m_i}^{p+1}(\|a\|_{L^1}+1)}.$$

Fix $k \in \mathbb{N}$. Thanks to (57),

$$\lambda_k^\infty = \min\big(\lambda_1, \dots, \lambda_k, \lambda_1', \dots, \lambda_k', \lambda_1'', \dots, \lambda_k''\big) > 0.$$

Therefore, for every $i \in \{1,\dots,k\}$ and $\lambda \in [-\lambda_k^\infty, \lambda_k^\infty]$ we have:

$$\begin{aligned}
\mathcal{E}_{m_i,\lambda}\big(\tilde{u}_{m_i,\lambda}^\infty\big) &\leqslant \mathcal{E}_{m_i,\lambda}(z_{\tilde{s}_i}) \quad \big(\text{see (52)}\big) \\
&= \frac{1}{2}\|z_{\tilde{s}_i}\|_{H_0^1}^2 - \frac{\lambda}{p+1}\int_\Omega a(x) z_{\tilde{s}_i}^{p+1} - \int_\Omega F\big(z_{\tilde{s}_i}(x)\big)\,dx \\
&= \mathcal{E}_{m_i}(z_{\tilde{s}_i}) - \frac{\lambda}{p+1}\int_\Omega a(x) z_{\tilde{s}_i}^{p+1} \\
&< \theta_i \quad \big(\text{see the choice of } \lambda_i' \text{ and } \tilde{s}_i \leqslant \delta_{m_i}\big),
\end{aligned}$$

and since $\tilde{u}^{\infty}_{m_i,\lambda}$ belongs to $W^{\eta_{m_i}}$, and $u^{\infty}_{m_i}$ is the minimum point of $\mathcal{E}_{m_i}$ over the set $W^{\eta_{m_i}}$, see relation (55), we have:

$$\begin{aligned}\mathcal{E}_{m_i,\lambda}\big(\tilde{u}^{\infty}_{m_i,\lambda}\big) &= \mathcal{E}_{m_i}\big(\tilde{u}^{\infty}_{m_i,\lambda}\big) - \frac{\lambda}{p+1}\int_\Omega a(x)\big(\tilde{u}^{\infty}_{m_i,\lambda}\big)^{p+1}\\ &\geqslant \mathcal{E}_{m_i}\big(u^{\infty}_{m_i}\big) - \frac{\lambda}{p+1}\int_\Omega a(x)\big(\tilde{u}^{\infty}_{m_i,\lambda}\big)^{p+1}\\ &> \theta_{i+1} \quad \big(\text{see the choice of } \lambda_i'' \text{ and (53)}\big).\end{aligned}$$

Consequently, for every $i \in \{1,\dots,k\}$ and $\lambda \in [-\lambda^{\infty}_k, \lambda^{\infty}_k]$ we have:

$$\theta_{i+1} < \mathcal{E}_{m_i,\lambda}\big(\tilde{u}^{\infty}_{m_i,\lambda}\big) < \theta_i < 0, \tag{58}$$

therefore

$$\mathcal{E}_{m_k,\lambda}\big(\tilde{u}^{\infty}_{m_k,\lambda}\big) < \cdots < \mathcal{E}_{m_1,\lambda}\big(\tilde{u}^{\infty}_{m_1,\lambda}\big) < 0. \tag{59}$$

Note that $\tilde{u}^{\infty}_{m_i,\lambda} \in W^{\eta_{m_k}}$ for every $i \in \{1,\dots,k\}$, so $\mathcal{E}_{m_i,\lambda}(\tilde{u}^{\infty}_{m_i,\lambda}) = \mathcal{E}_{m_k,\lambda}(\tilde{u}^{\infty}_{m_i,\lambda})$, see relation (51). From above, for every $\lambda \in [-\lambda^{\infty}_k, \lambda^{\infty}_k]$, we have:

$$\mathcal{E}_{m_k,\lambda}\big(\tilde{u}^{\infty}_{m_k,\lambda}\big) < \cdots < \mathcal{E}_{m_k,\lambda}\big(\tilde{u}^{\infty}_{m_1,\lambda}\big) < 0 = \mathcal{E}_{m_k,\lambda}(0).$$

In particular, the elements $\tilde{u}^{\infty}_{m_1,\lambda},\dots,\tilde{u}^{\infty}_{m_k,\lambda}$ are distinct (and non-trivial) whenever $\lambda \in [-\lambda^{\infty}_k, \lambda^{\infty}_k]$.

Step 4. Assume that $k \geqslant 2$ and fix $\lambda \in [-\lambda^{\infty}_k, \lambda^{\infty}_k]$. We prove that

$$\big\|\tilde{u}^{\infty}_{m_i,\lambda}\big\|_{L^\infty} > \delta_{m_{i-1}} \quad \text{for all } i \in \{2,\dots,k\}. \tag{60}$$

Let us assume that there exists an element $i_0 \in \{2,\dots,k\}$ such that $\|\tilde{u}^{\infty}_{m_{i_0},\lambda}\|_{L^\infty} \leqslant \delta_{m_{i_0-1}}$. Since $\delta_{m_{i_0-1}} < \eta_{m_{i_0-1}}$, then $\tilde{u}^{\infty}_{m_{i_0},\lambda} \in W^{\eta_{m_{i_0-1}}}$. Thus, on account of (52) and (51), we have:

$$\mathcal{E}_{m_{i_0-1},\lambda}\big(\tilde{u}^{\infty}_{m_{i_0-1},\lambda}\big) = \min_{W^{\eta_{m_{i_0-1}}}} \mathcal{E}_{m_{i_0-1},\lambda} \leqslant \mathcal{E}_{m_{i_0-1},\lambda}\big(\tilde{u}^{\infty}_{m_{i_0},\lambda}\big) = \mathcal{E}_{m_{i_0},\lambda}\big(\tilde{u}^{\infty}_{m_{i_0},\lambda}\big),$$

which contradicts (59). Therefore, (60) holds true.

Step 5. Let $u^{\infty}_{i,\lambda} := \tilde{u}^{\infty}_{m_i,\lambda}$ for any $i \in \{1,\dots,k\}$ and $\lambda \in [-\lambda^{\infty}_k, \lambda^{\infty}_k]$; these elements verify all the requirements of Theorem 1.4. Indeed, since $\mathcal{E}_{m_1,\lambda}(u^{\infty}_{1,\lambda}) = \mathcal{E}_{m_1,\lambda}(\tilde{u}^{\infty}_{m_1,\lambda}) < 0 = \mathcal{E}_{m_1,\lambda}(0)$, then $\|u^{\infty}_{1,\lambda}\|_{L^\infty} > 0$, which proves (5) for $i = 1$. If $k \geqslant 2$, then on account of step 4, (49) and $m_i \geqslant i$, for every $i \in \{2,\dots,k\}$, we have:

$$\big\|u^{\infty}_{i,\lambda}\big\|_{L^\infty} > \delta_{m_{i-1}} \geqslant m_{i-1} \geqslant i-1,$$

i.e., relation (5) holds true. This ends the proof of Theorem 1.4. □

Proof of Remark 1.6. Due to (4), there exists a $C > 0$ such that $|f(s)| \leqslant C(1 + s^{2^*-1})$ for all $s \geqslant 0$. We denote by $S_t > 0$ the Sobolev embedding constant of the continuous embedding $H^1_0(\Omega) \hookrightarrow L^t(\Omega)$, $t \in [1, 2^*]$. Without any loss of generality, we may assume that for every $i \in \mathbb{N}$,

$$\theta_i < -\frac{1}{p+1}\|a\|_{L^\infty} S^{p+1}_{p+1}(i-1)^{p+1} - C\big[S_1(i-1) + S^{2^*}_{2^*}(i-1)^{2^*}\big], \tag{61}$$

where the sequence $\{\theta_i\}_i$ comes from step 3 of the proof of Theorem 1.4.

Fix $\lambda \in [-\lambda^{\infty}_k, \lambda^{\infty}_k]$ and assume that there exists $i_0 \in \{1,\dots,k\}$ such that $\|u^{\infty}_{i_0,\lambda}\|_{H^1_0} \leqslant i_0 - 1$. On account of (58), we have in particular $\mathcal{E}_{m_{i_0},\lambda}(u^{\infty}_{i_0,\lambda}) < \theta_{i_0}$. Consequently, we have:

$$\begin{aligned}\frac{1}{2}\big\|u^{\infty}_{i_0,\lambda}\big\|^2_{H^1_0} &= \mathcal{E}_{m_{i_0},\lambda}\big(u^{\infty}_{i_0,\lambda}\big) + \frac{\lambda}{p+1}\int_\Omega a(x)\big(u^{\infty}_{i_0,\lambda}\big)^{p+1} + \int_\Omega F\big(u^{\infty}_{i_0,\lambda}(x)\big)\,dx\\ &< \theta_{i_0} + \frac{|\lambda|}{p+1}\|a\|_{L^\infty} S^{p+1}_{p+1}\big\|u^{\infty}_{i_0,\lambda}\big\|^{p+1}_{H^1_0}\end{aligned}$$

$$\begin{aligned}
&+ C\big[S_1 \|u_{i_0,\lambda}^\infty\|_{H_0^1} + S_{2^*}^{2^*} \|u_{i_0,\lambda}^\infty\|_{H_0^1}^{2^*}\big] \\
&\leqslant \theta_{i_0} + \frac{1}{p+1}\|a\|_{L^\infty} S_{p+1}^{p+1}(i_0-1)^{p+1} \\
&\quad + C\big[S_1(i_0-1) + S_{2^*}^{2^*}(i_0-1)^{2^*}\big] \quad \big(\lambda_k^\infty \leqslant 1\big) \\
&< 0 \quad \big(\text{see (61)}\big),
\end{aligned}$$

a contradiction. □

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