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Metric characterization of Berwald spaces of non-positive flag curvature

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Abstract

We show the equivalence of the non-positivity of the flag curvature with the non-positive curvature properties of Busemann and Pedersen for (not necessarily reversible) Berwald manifolds. So an analytical property is characterized by synthetic concepts of non-positively curved metric spaces.

MSC: 53C70; 53C23; 53C60

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1. Introduction

In the 1940s, Busemann developed a synthetic geometry on metric spaces. In particular, he axiomatically elaborated a whole theory of non-positively curved metric spaces, which have no differential structure a priori and they possess the essential qualitative geometric properties of Finsler manifolds. These spaces are the so-called G-spaces, see [3, p. 37]. This notion of non-positive curvature requires that in small geodesic triangles the length of a side

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is at least the twice of the geodesic distance of the mid-points of the other two sides, see [3, p. 237]. In 1952, Pedersen [9] introduced a weaker non-positive curvature notion than that of Busemann which is in fact equivalent with the convexity of the (small) capsules (i.e., the loci equidistant to geodesic segments). The curvature notion of Pedersen is enough to prove many of the results obtained by Busemann [3]. However, the real merit of Pedersen's notion relies in the fact that while the Hilbert metric of the interior of a simple, closed and convex curve K in the Euclidean plane clearly satisfies this curvature condition, Busemann's curvature condition holds if and only if K is an ellipse, see Kelly and Straus [6]. Therefore, in the latter case, the Hilbert metric (which is in general a (projective) Finsler metric with constant flag curvature -1) becomes a Riemannian one. This means that, although for Riemannian spaces the non-positivity of the sectional curvature, Pedersen's and Busemann's curvature conditions are mutually equivalent (see [3, Theorem (41.6)]), the non-positivity of the flag curvature of a *generic* Finsler manifold is not enough to guarantee Busemann's property. Basically, we have two possibilities in order to obtain such a characterization for Finsler spaces:

(I) To find a *new* notion of curvature in Finsler geometry such that for an arbitrary Finsler manifold the non-positivity of this curvature is equivalent with the Busemann non-positive curvature condition, as it was proposed by Shen [10, Problem 25];
(II) To keep the flag curvature, but put some restrictive condition on the Finsler metric.

In spite of the fact that (reversible) Finsler manifolds are included in G-spaces, only few results are known which establish a link between the *differential invariants* of a Finsler manifold and the *metric properties* of the induced metric space. In [9] (see also [3, p. 270]), Pedersen restricted his studies to two-dimensional reversible Finsler spaces, showing that if it has convex capsules, then its usual curvature is non-positive; the converse problem remains an open question until now. Using the Cartan connection and suitable modifications of the proof of Pedersen, Moalla [8] extended the above result to Finsler spaces of arbitrary dimension. On the other hand, in [7], the authors and Cs. Varga proved that finite-dimensional *Berwald spaces* of non-positive flag curvature are curved in the sense of Busemann. Therefore, Berwald spaces (i.e., Finsler spaces whose Chern connection coefficients Γ_{ij}^k in natural coordinates depend only on the base point) seem to be the first class of Finsler metrics that are non-positively curved in the sense of Busemann and which are neither flat nor Riemannian.

The above result gave us the expectation to obtain a characterization of the Busemann curvature notion in these special Finsler spaces, which is in accordance with (II); however, (I) seems to be a remarkable, but a very hard problem.

Let (M, F) be a Berwald space, where F is positively (but perhaps not absolutely) homogeneous of degree one and let d_F be the induced (non-reversible) metric by F. The main result of this paper (see Section 2 for the exact notions and formulation) asserts that the following six criteria are mutually equivalent:

- The flag curvature of (M, F) is non-positive;
- (M, d_F) is curved in the sense of Busemann;
- (M, d_F) is curved in the sense of Pedersen in forward manner;

- (M, d_F) is curved in the sense of Pedersen in backward manner;
- The small forward capsules in (M, d_F) are convex sets;
- The small backward capsules in (M, d_F) are convex sets.

This result also includes in particular a partial answer to the question of Pedersen (see [9, p. 87]), i.e., every reversible Berwald space of non-positive flag curvature has convex capsules. Beyond its own interest of our result, we mention that only few notions exist in Finsler geometry which can be compared from the "forward" and "backward" points of view (see Bao et al. [1]).

In the 1950s, Aleksandrov introduced independently another notion of curvature in metric spaces, based on the convexity of the distance function. It is well-known that the condition of Busemann curvature is weaker than the Aleksandrov one, see [5, Corollary 2.3.1]. Nevertheless, in Riemannian spaces the Aleksandrov curvature condition holds if and only if the sectional curvature is non-positive (see [2, Theorem 1A.6]), but in the Finsler case the picture is quite rigid. Namely, if on a reversible Finsler manifold (M, F) the Aleksandrov curvature condition holds (on the induced metric space by (M, F)) then (M, F) it must be Riemannian, see [2, Proposition 1.14].

In the proof of our main result the assumption that the Finsler structure is of Berwald type plays an indispensable role in several times. It would be interesting to examine whether or not the above result works for *a larger class* of Finsler spaces than the Berwald ones, working in the (II) context. As far as the Busemann curvature condition is concerned, we believe that, as in the Aleksandrov case, we face a rigidity result; namely, if the Busemann curvature condition holds on a Finsler manifold (M, F) then it must be of Berwald type. If this is true, the problem (I) would be solved in the following way: among the Finsler manifolds, the Berwald spaces would be the largest class having the property of (I), where the curvature could be chosen to be the flag one.

Shen gave to the authors an example for a specific family of Berwald spaces which is non-Riemannian with non-positive flag curvature, see also [7]. Namely, let (M_0, g) be a two-dimensional Riemannian space of constant sectional curvature $K_g \leq 0$, and ϵ an arbitrary positive constant. Then the Finsler metric on $M = \mathbb{R} \times M_0$ is defined by

$$F(t, x, y; \tau, u, v) = \sqrt{\tau^2 + g_{(x,y)}((u, v), (u, v)) + \epsilon\sqrt{\tau^4 + g^2_{(x,y)}((u, v), (u, v))}},$$

which satisfies the requirements. Moreover, the flag curvature of (M, F) is not constant. Since (M, F) is a non-Riemannian Berwald space, (M, d_F) is not curved in the sense of Aleksandrov, but it is in the sense of Busemann and Pedersen, due to the our result.

2. Basic notions and the main result

2.1. Curvature notions on metric spaces

Let (M, d) be a non-reversible metric space. Since the function d is not necessarily symmetric, we define the *forward* and *backward metric ball*s, respectively, with center

$p \in M$ and radius $r > 0$ as

$$B_p^+(r) = \{q \in M : d(p,q) < r\} \quad \text{and} \quad B_p^-(r) = \{q \in M : d(q,p) < r\}.$$

A continuous curve $\gamma : [a,b] \to M$ with $\gamma(a) = x$, $\gamma(b) = y$ is a *shortest geodesic*, if $l(\gamma) = d(x,y)$, where $l(\gamma)$ denotes the *generalized length* of γ and it is defined by

$$l(\gamma) = \sup\left\{\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : a = t_0 < t_1 < \cdots < t_n = b, \quad n \in \mathbb{N}\right\}.$$

We say that (M,d) is a *locally geodesic (length) space* if for every point $p \in M$ there is a $\rho_p > 0$ such that for every two points $x, y \in B_p^+(\rho_p)$ there exists a shortest geodesic joining them.

Definition 1. A locally geodesic space (M,d) is said to be a *Busemann non-positive curvature space* (shortly, *Busemann NPC space*), if for every $p \in M$ there exists $\rho_p > 0$ such that for any two shortest geodesics $\gamma_1, \gamma_2 : [0,1] \to M$ with $\gamma_1(0) = \gamma_2(0) = x \in B_p^+(\rho_p)$ and with endpoints $\gamma_1(1), \gamma_2(1) \in B_p^+(\rho_p)$ we have

$$2d\left(\gamma_1\left(\frac{1}{2}\right), \gamma_2\left(\frac{1}{2}\right)\right) \leq d(\gamma_1(1), \gamma_2(1)).$$

(We shall say that γ_1 and γ_2 satisfy the *Busemann NPC inequality*).

Let $\gamma : [a,b] \to M$ be a curve and $q \in M$ be fixed arbitrarily. We denote by $\text{dist}(\gamma, q) = \inf\{d(\gamma(t), q) : t \in [a,b]\}$ and $\text{dist}(q, \gamma) = \inf\{d(q, \gamma(t)) : t \in [a,b]\}$.

Definition 2. A locally geodesic space (M,d) is said to be a forward (resp. *backward*) *Pedersen non-positive curvature space* (shortly, forward (resp. *backward*) *Pedersen NPC* space), if for every $p \in M$ there exists $\rho_p > 0$ such that for any two shortest geodesics $\gamma_1, \gamma_2 : [0,1] \to B_p^+(\rho_p)$, the function $f^+ : [0,1] \to \mathbb{R}$ (resp. $f^- : [0,1] \to \mathbb{R}$), defined by

$$f^+(t) = \text{dist}(\gamma_1, \gamma_2(t)) \quad (\text{resp. } f^-(t) = \text{dist}(\gamma_1(t), \gamma_2))$$

is quasiconvex, i.e., for every $t \in [0,1]$

$$f^+(t) \leq \max\{f^+(0), f^+(1)\} \quad (\text{resp. } f^-(t) \leq \max\{f^-(0), f^-(1)\}).$$

Let $\gamma : [a,b] \to M$ be a shortest geodesic and $\alpha > 0$. Attached to γ and α, we define the *forward* and *backward capsule*s, respectively, as

$$\mathcal{C}_\gamma^+(\alpha) = \{q \in M : \text{dist}(\gamma, q) \leq \alpha\} \quad \text{and} \quad \mathcal{C}_\gamma^-(\alpha) = \{q \in M : \text{dist}(q, \gamma) \leq \alpha\}.$$

Let M_0 be a non-empty subset of M. The pair (γ, α) is say to be *forward* (resp. *backward*) M_0-*admissible*, if $\mathcal{C}_\gamma^+(\alpha) \subset M_0$ (resp. $\mathcal{C}_\gamma^-(\alpha) \subset M_0$).

Definition 3. We say that a locally geodesic space (M, d) has *convex forward* (resp. *backward*) *capsules* if for every $p \in M$ there exists $\rho_p > 0$ such that for every forward (resp. backward) $B_p^+(\rho_p)$-admissible pair (γ, α), the set $\mathcal{C}_\gamma^+(\alpha)$ (resp. $\mathcal{C}_\gamma^-(\alpha)$) is convex.

As usually, the convexity of a set means that every two points in it can be uniquely joined by a shortest geodesic and the image of this curve belongs entirely to the set.

2.2. Berwald spaces

In this section, we recall briefly some known facts about Berwald spaces. For details, see [1].

Let M be a n-dimensional C^∞ manifold and $TM = \bigcup_{x\in M} T_xM$ the tangent bundle. If the continuous function $F : TM \to R_+$ satisfies the conditions that it is C^∞ on $TM \setminus \{0\}$; $F(tu) = tF(u)$ for all $t \geq 0$ and $u \in TM$, i.e., F is positively homogeneous of degree one; and the matrix $g_{ij}(u) := (\frac{1}{2}F^2)_{y^iy^j}(u)$ is positive definite for all $u \in TM \setminus \{0\}$, then we say that (M, F) is a *Finsler manifold*.

Let $\gamma : [0, r] \to M$ be a piecewise C^∞ curve. Its *integral length* is defined as

$$L(\gamma) = \int_0^r F(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t.$$

For $x_0, x_1 \in M$ denote by $\Gamma(x_0, x_1)$ the set of all piecewise C^∞ curves $\gamma : [0, r] \to M$ such that $\gamma(0) = x_0$ and $\gamma(r) = x_1$. Define a map $d_F : M \times M \to [0, \infty)$ by

$$d_F(x_0, x_1) = \inf_{\gamma\in\Gamma(x_0,x_1)} L(\gamma).$$

Of course, we have $d_F(x_0, x_1) \geq 0$, where equality holds if and only if $x_0 = x_1$; $d_F(x_0, x_2) \leq d_F(x_0, x_1) + d_F(x_1, x_2)$. In general, since F is only a positive homogeneous function, $d_F(x_0, x_1) \neq d_F(x_1, x_0)$, therefore (M, d_F) is only a non-reversible metric space.

Let π^*TM be the pull-back of the tangent bundle TM by $\pi : TM \setminus \{0\} \to M$. Unlike the Levi–Civita connection in Riemann geometry, there is no unique natural connection in the Finsler case. Among these connections on π^*TM, we choose the *Chern connection* whose coefficients are denoted by Γ_{jk}^i (see [1, p. 38]). This connection induces the *curvature tensor*, denoted by R (see [1, Chapter 3]).

Let $(x, y) \in TM \setminus 0$ and V a section of the pulled-back bundle π^*TM. Then

$$K(y, V) = \frac{g_{(x,y)}(R(V, y)y, V)}{g_{(x,y)}(y, y)g_{(x,y)}(V, V) - [g_{(x,y)}(y, V)]^2}, \tag{1}$$

is the *flag curvature* with flag y and transverse edge V. Here, $g_{(x,y)} := g_{ij(x,y)}\,\mathrm{d}x^i \otimes \mathrm{d}x^j := (\frac{1}{2}F^2)_{y^iy^j}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$ is the Riemannian metric on the pulled-back bundle π^*TM (see [1,

p. 68]). When F is Riemannian, then the flag curvature coincides with the sectional curvature. Let K abbreviate the collection of flag curvatures $\{K(V, W) : 0 \neq V, W \in T_xM, x \in M, V$ and W are not collinear$\}$. We say that the flag curvature of (M, F) is *non-positive* if $K \leq 0$.

A Finsler manifold is of *Berwald type* if the Chern connection coefficients Γ_{ij}^k in natural coordinates depends only on the base point.

The main result of this paper can be formulated as follows.

Theorem 1. *Let (M, F) be a Berwald space where F is positively (but perhaps not absolutely) homogeneous of degree one. The following assertions are equivalent:*

(a) *The flag curvature of (M, F) is non-positive.*
(b) *(M, d_F) is a Busemann NPC space.*
(c) *(M, d_F) is a forward Pedersen NPC space.*
(d) *(M, d_F) is a backward Pedersen NPC space.*
(e) *(M, d_F) has convex forward capsules.*
(f) *(M, d_F) has convex backward capsules.*

The paper is organized as follows. In the next section we recall further results, which will be used throughout the paper. In Section 4, we will prove the implication $(a) \Rightarrow (b)$; in Section 5, the implications $(b) \Rightarrow (c)$ and $(b) \Rightarrow (d)$; in Section 6, the implications $(c) \Rightarrow (e)$ and $(d) \Rightarrow (f)$; in Section 7, the implication $(e) \Rightarrow (a)$ while in the last section the implication $(f) \Rightarrow (a)$.

3. Auxiliary results from Finsler geometry

The Chern connection defines the covariant derivative $D_V U$ of a vector field $U \in \mathfrak{X}(M)$ in the direction $V \in T_pM$. Since, in general, the Chern connection coefficients Γ_{jk}^i in natural coordinates have a directional dependence, we must say explicitly that $D_V U$ is defined with a fixed reference vector. In particular, let $\sigma : [0, r] \to M$ be a smooth curve with velocity field $T = T(t) = \dot{\sigma}(t)$. Suppose that U and W are vector fields defined along σ. We define $D_T U$ with *reference vector* W as

$$D_T U = \left[\frac{\mathrm{d}U^i}{\mathrm{d}t} + U^j T^k (\Gamma_{jk}^i)_{(\sigma, W)}\right] \frac{\partial}{\partial x^i}_{|\sigma(t)}.$$

A curve $\sigma : [0, r] \to M$, with velocity $T = \dot{\sigma}$ is a (*Finslerian*) *geodesic* if

$$D_T \left[\frac{T}{F(T)}\right] = 0, \quad \text{with reference vector } T. \tag{2}$$

If U, V and W are vector fields along a curve σ, which has velocity $T = \dot{\sigma}$, we have the derivative rule

$$\frac{\mathrm{d}}{\mathrm{d}t} g_W(U, V) = g_W(D_T U, V) + g_W(U, D_T V) \tag{3}$$

whenever $D_T U$ and $D_T V$ are with reference vector W and *one* of the following conditions holds:

- U or V is proportional to W, or
- $W = T$ and σ is a geodesic.

A vector field J along a geodesic $\sigma : [0, r] \to M$ (with velocity field T) is said to be a *Jacobi field* if it satisfies the equation:

$$D_T D_T J + R(J, T)T = 0, \tag{4}$$

where R is the curvature tensor. Here, the covariant derivative D_T is defined with reference vector T.

Let $\Sigma : [0, r] \times [-1, 1] \to M$ be a piecewise C^∞ variation of a geodesic $\sigma : [0, r] \to M$, with $\Sigma(\cdot, 0) = \sigma$. Let

$$T = T(t, s) = \frac{\partial \Sigma}{\partial t}, \qquad U = U(t, s) = \frac{\partial \Sigma}{\partial s}$$

the velocities of the *t-curves* (i.e., $\Sigma(t, s = \text{constant})$) and *s-curves* (i.e., $\Sigma(t = \text{constant}, s)$), respectively. The formula for the *first variation of arc length* gives us

$$L'_\Sigma(0) := \frac{\mathrm{d}}{\mathrm{d}s} L(\Sigma(\cdot, s))_{|s=0} = g_T \left(U, \frac{T}{F(T)} \right)_{|s=0} \Bigg|_0^r . \tag{5}$$

The *second variation of arc length* can be expressed as

$$\begin{aligned} L''_\Sigma(0) : &= \frac{\mathrm{d}^2}{\mathrm{d}s^2} L(\Sigma(\cdot, s))_{|s=0} \\ &= \int_0^r \frac{1}{F(T)_{|s=0}} [g_T(D_T U, D_T U) - g_T(R(U, T)T, U)]_{|s=0} \, \mathrm{d}t \\ &\quad + g_T \left(D_U U, \frac{T}{F(T)} \right)_{|s=0} \Bigg|_0^r - \int_0^r \frac{1}{F(T)_{|s=0}} \left(\frac{\partial F(T)}{\partial s} \right)^2_{|s=0} \mathrm{d}t, \end{aligned}$$

where all covariant derivatives are with reference vector $T = T(t, 0)$.

For $p \in M, r > 0$, let $B_p(r) := \{y \in T_p M : F(p, y) < r\}$ be the open *tangent ball* and let $\mathcal{B}_p^+(r)$ and $\mathcal{B}_p^-(r)$ the forward and backward metric balls, respectively, defined by means of d_F. It is well-known that the topology generated by the forward (resp. backward) metric balls coincide with the underlying manifold topology, respectively. Moreover, by the Whitehead's theorem (see [11] or [1, Exercise 6.4.3, p. 164]) and [1, Lemma 6.2.1, p. 146]we can conclude the following useful result (see also [7]).

Proposition 1. *Let* (M, F) *be a Finsler manifold, where* F *is positively* (*but perhaps not absolutely*) *homogeneous of degree one. For every point* $p \in M$ *there exist a small* $\rho_p > 0$ *and* $c_p > 1$ (*depending only on* p) *such that for every pair of points* q_0, q_1 *in* $\mathcal{B}_p^+(\rho_p)$ *we* *have*

$$\frac{1}{c_p} d_F(q_1, q_0) \le d_F(q_0, q_1) \le c_p d_F(q_1, q_0). \tag{6}$$

Moreover, for every real number $k \ge 1$ *and* $q \in \mathcal{B}_p^+(\rho_p/k)$ *the mapping* $\exp_q$ *is* C^1-*diffeomorphism from* $B_q(2\rho_p/k)$ *onto* $\mathcal{B}_q^+(2\rho_p/k)$ *and every pair of points* q_0, q_1 *in* $\mathcal{B}_p^+(\rho_p/k)$ *can be joined by a unique minimal geodesic from* q_0 *to* q_1 *lying entirely in* $\mathcal{B}_p^+(\rho_p/k)$.

A geodesic from q_0 to q_1 is said to be *minimal* if its integral length equals the metric distance $d_F(q_0, q_1)$. Since we are within the Finsler context, the generalized length $l(\gamma)$ and the integral length $L(\gamma)$ of (piecewise) C^∞ curves coincide (see [4, Theorem 2, p. 186]). Therefore, the minimal Finsler geodesic and shortest geodesic notions coincide, too. In particular, our Proposition 1 asserts that every (non-reversible) metric space, induced by a Finsler metric, is a locally geodesic space.

4. (*a*) ⇒ (*b*)

This part is basically included in [7], but for the sake of completeness we give the proof.

Let us fix $p \in M$ and consider $\rho_p > 0$, $c_p > 1$ from our Proposition 1. We will prove that $\rho'_p = \frac{\rho_p}{c_p}$ is a good choice in Definition 1. To do this, let $\gamma_1, \gamma_2 : [0, 1] \to M$ be two (minimal) geodesics with $\gamma_1(0) = \gamma_2(0) = x \in \mathcal{B}_p^+(\rho'_p)$ and $\gamma_1(1), \gamma_2(1) \in \mathcal{B}_p^+(\rho'_p)$. By our Proposition 1, we can construct a unique geodesic $\gamma : [0, 1] \to M$ joining $\gamma_1(1)$ with $\gamma_2(1)$ and $d_F(\gamma_1(1), \gamma_2(1)) = L(\gamma)$. Clearly, $\gamma(s) \in \mathcal{B}_p^+(\rho'_p)$ for all $s \in [0, 1]$ (we applied our Proposition 1 for $k = c_p$). Moreover, $x \in \mathcal{B}^+_{\gamma(s)}(2\rho_p)$. Indeed, by (6), we obtain $d_F(\gamma(s), x) \le d_F(\gamma(s), p) + d_F(p, x) \le c_p d_F(p, \gamma(s)) + \rho'_p \le (c_p + 1)\rho'_p < 2\rho_p$. Therefore, we can define $\Sigma : [0, 1] \times [0, 1] \to M$ by $\Sigma(t, s) = \exp_{\gamma(s)}((1 - t) \cdot \exp^{-1}_{\gamma(s)}(x))$. The curve $t \mapsto \Sigma(1 - t, s)$ is a radial geodesic which joins $\gamma(s)$ with x. Taking into account that (M, F) is of Berwald type, the reverse of $t \mapsto \Sigma(1 - t, s)$, i.e., $t \mapsto \Sigma(t, s)$ is a geodesic too (see [1, Exercise 5.3.3, p. 128]) for all $s \in [0, 1]$. Moreover, $\Sigma(0, 0) = x = \gamma_1(0)$, $\Sigma(1, 0) = \gamma(0) = \gamma_1(1)$. From the uniqueness of the geodesic between x and $\gamma_1(1)$, we have $\Sigma(\cdot, 0) = \gamma_1$. Analogously, we have $\Sigma(\cdot, 1) = \gamma_2$. Since Σ is a geodesic variation (of the curves γ_1 and γ_2), the vector field J_s, defined by $J_s(t) = \frac{\partial}{\partial s}\Sigma(t, s) \in T_{\Sigma(t,s)}M$ is a Jacobi field along $\Sigma(\cdot, s)$, $s \in [0, 1]$ (see [1, p. 130]). In particular, we have $\Sigma(1, s) = \gamma(s)$, $J_s(0) = 0$, $J_s(1) = \frac{\partial}{\partial s}\Sigma(1, s) = \frac{\mathrm{d}\gamma}{\mathrm{d}s}$ and $J_s(\frac{1}{2}) = \frac{\partial}{\partial s}\Sigma(\frac{1}{2}, s)$.

Now, we fix $s \in [0, 1]$. Since $J_s(0) = 0$ and the flag curvature in non-positive, then the geodesic $\Sigma(\cdot, s)$ has no conjugated points. Therefore, $J_s(t) \ne 0$ for $t \in (0, 1]$. Hence, $g_{J_s}(J_s, J_s)(t)$ is well defined for every $t \in (0, 1]$. Moreover

$$F(J_s)(t) := F(\Sigma(t, s), J_s(t)) = [g_{J_s}(J_s, J_s)]^{1/2}(t) \ne 0, \quad \forall t \in (0, 1]. \tag{7}$$

Let T_s the velocity field of $\Sigma(\cdot, s)$. Applying twice formula (3), we obtain

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}[g_{J_s}(J_s, J_s)]^{1/2}(t) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}F(J_s)(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{g_{J_s}(D_{T_s}J_s, J_s)}{F(J_s)}\right](t) \\ &= \frac{\begin{array}{l}[g_{J_s}(D_{T_s}D_{T_s}J_s, J_s) + g_{J_s}(D_{T_s}J_s, D_{T_s}J_s)]\cdot F(J_s) \\ \quad -g^2_{J_s}(D_{T_s}J_s, J_s)\cdot F(J_s)^{-1}\end{array}}{F^2(J_s)}(t) \\ &= \frac{\begin{array}{l}g_{J_s}(D_{T_s}D_{T_s}J_s, J_s)\cdot F^2(J_s) \\ \quad +g_{J_s}(D_{T_s}J_s, D_{T_s}J_s)\cdot F^2(J_s) - g^2_{J_s}(D_{T_s}J_s, J_s)\end{array}}{F^3(J_s)}(t),\end{aligned}$$

where the covariant derivatives (for generic Finsler manifolds) are with reference vector J_s. Since (M, F) is a Berwald space, the Chern connection coefficients do not depend on the direction, the notion of reference vector becomes irrelevant. Therefore, we can use the Jacobi equation (4), concluding that $g_{J_s}(D_{T_s}D_{T_s}J_s, J_s) = -g_{J_s}(R(J_s, T_s)T_s, J_s)$. Using the symmetry property of the curvature tensor (see [1, Exercise 3.9.6, p. 73]), the formula of the flag curvature, and the Schwarz inequality we have

$$\begin{aligned}-g_{J_s}(R(J_s, T_s)T_s, J_s) &= -g_{J_s}(R(T_s, J_s)J_s, T_s) \\ &= -K(J_s, T_s)\cdot[g_{J_s}(J_s, J_s)g_{J_s}(T_s, T_s) - g^2_{J_s}(J_s, T_s)] \geq 0.\end{aligned}$$

For the last two terms of the numerator we apply again the Schwarz inequality and we conclude that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}F(J_s)(t) \geq 0, \quad \text{for all } t \in (0, 1].$$

Since $J_s(t) \neq 0$ for $t \in (0, 1]$, the mapping $t \mapsto F(J_s)(t)$ is C^∞ on $(0, 1]$. From the above inequality and the second order Taylor expansion about $v \in (0, 1]$, we obtain

$$F(J_s)(v) + (t - v)\frac{\mathrm{d}}{\mathrm{d}t}F(J_s)(v) \leq F(J_s)(t) \tag{8}$$

for every $t \in (0, 1]$. Letting $t \to 0$ and $v = 1/2$ in (8), by the continuity of F, we obtain

$$F(J_s)\left(\frac{1}{2}\right) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}F(J_s)\left(\frac{1}{2}\right) \leq 0.$$

Let $v = 1/2$ and $t = 1$ in (8), and adding the obtained inequality with the above one, we conclude that

$$2F\left(\Sigma\left(\frac{1}{2}, s\right), \frac{\partial}{\partial s}\Sigma\left(\frac{1}{2}, s\right)\right) = 2F(J_s)\left(\frac{1}{2}\right) \leq F(J_s)(1) = F\left(\gamma(s), \frac{\mathrm{d}\gamma}{\mathrm{d}s}\right).$$

Integrating the last inequality with respect to s from 0 to 1, we obtain

$$\begin{aligned} 2L\left(\Sigma\left(\frac{1}{2},\cdot\right)\right) &= 2\int_0^1 F\left(\Sigma\left(\frac{1}{2},s\right),\frac{\partial}{\partial s}\Sigma\left(\frac{1}{2},s\right)\right)\mathrm{d}s \\ &\le \int_0^1 F\left(\gamma(s),\frac{\mathrm{d}\gamma}{\mathrm{d}s}\right)\mathrm{d}s = L(\gamma) = d_F(\gamma_1(1),\gamma_2(1)). \end{aligned}$$

Since $\Sigma(\frac{1}{2},0)=\gamma_1(\frac{1}{2})$, $\Sigma(\frac{1}{2},1)=\gamma_2(\frac{1}{2})$ and $\Sigma(\frac{1}{2},\cdot)$ is a C^∞ curve, by the definition of the metric function d_F, we conclude that γ_1 and γ_2 satisfy the Busemann NPC inequality.

5. (b) ⇒ (c) ∧ (d)

Let $p \in M$ be a fixed point and $\rho_p, \rho'_p > 0$ as in Section 4. We will see that ρ'_p is also a good choice in Definition 2. First, we prove that for two arbitrary minimal geodesics $\gamma_1,\gamma_2 : [0,1] \to \mathcal{B}_p^+(\rho'_p)$, the function $f : [0,1] \to \mathbb{R}$, defined by $f(t) := f_{\gamma_1,\gamma_2}(t) = d_F(\gamma_1(t),\gamma_2(t))$ is convex. To this end, it is enough to prove the $\frac{1}{2}$-convexity of f, due to the continuity of the metric function d_F.

Thus, we fix $a, b \in [0,1]$, $a < b$. We define $\tilde{\gamma}_i : [0,1] \to \mathcal{B}_p^+(\rho'_p)$ by $\tilde{\gamma}_i(t) = \gamma_i((b-a)t+a)$, $i \in \{1,2\}$. By [1. Exercise 5.3.2, p. 128], $\tilde{\gamma}_i$ is a minimal geodesic, joining $\gamma_i(a)$ and $\gamma_i(b)$, $i \in \{1,2\}$. Since $\gamma_1(a), \gamma_2(b) \in \mathcal{B}_p^+(\rho'_p)$, there exists a unique minimal geodesic $\gamma : [0,1] \to \mathcal{B}_p^+(\rho'_p)$ which joins $\gamma_1(a)$ with $\gamma_2(b)$. Let $\overline{\gamma}$ and $\overline{\gamma_2}$ the reverse of γ and $\tilde{\gamma}_2$, respectively, i.e., $\overline{\gamma}, \overline{\gamma_2} : [0,1] \to M$, defined by $\overline{\gamma}(t) = \gamma(1-t)$ and $\overline{\gamma_2}(t) = \tilde{\gamma}_2(1-t)$. Since (M,F) is a Berwald space, $\overline{\gamma}$ and $\overline{\gamma_2}$ are geodesics too, which are the shortest ones (they belong to $\mathcal{B}^+_{\gamma_2(b)}(2\rho_p)$). Therefore, they minimize the arc length functional among all piecewise C^∞ curves which join $\gamma_2(b)$ with $\gamma_1(a)$, and $\gamma_2(b)$ with $\gamma_2(a)$, respectively. Now, we apply the Busemann NPC inequality for the geodesic pairs $(\tilde{\gamma}_1,\gamma)$ and $(\overline{\gamma},\overline{\gamma_2})$, respectively. We obtain that

$$2d_F\left(\tilde{\gamma}_1\left(\frac{1}{2}\right),\gamma\left(\frac{1}{2}\right)\right) \le d_F(\tilde{\gamma}_1(1),\gamma(1)) = f(b);$$

$$2d_F\left(\overline{\gamma}\left(\frac{1}{2}\right),\overline{\gamma_2}\left(\frac{1}{2}\right)\right) \le d_F(\overline{\gamma}(1),\overline{\gamma_2}(1)) = f(a).$$

Since $\overline{\gamma}(\frac{1}{2}) = \gamma(\frac{1}{2})$, by the triangle inequality and the above two relations we have

$$\begin{aligned} F\left(\frac{a+b}{2}\right) &= d_F\left(\gamma_1\left(\frac{a+b}{2}\right),\gamma_2\left(\frac{a+b}{2}\right)\right) = d_F\left(\tilde{\gamma}_1\left(\frac{1}{2}\right),\tilde{\gamma}_2\left(\frac{1}{2}\right)\right) \\ &\le d_F\left(\tilde{\gamma}_1\left(\frac{1}{2}\right),\gamma\left(\frac{1}{2}\right)\right) + d_F\left(\overline{\gamma}\left(\frac{1}{2}\right),\overline{\gamma_2}\left(\frac{1}{2}\right)\right) \le \frac{f(a)+f(b)}{2}, \end{aligned}$$

which concludes the $\frac{1}{2}$-convexity of f.

Now, we fix two arbitrary minimal geodesics $\gamma_1, \gamma_2 : [0, 1] \to \mathcal{B}_p^+(\rho_p')$. Let $f^+, f^- : [0, 1] \to \mathbb{R}$, defined, as before, by

$$f^+(t) = \mathrm{dist}(\gamma_1, \gamma_2(t)) \quad \text{and} \quad f^-(t) = \mathrm{dist}(\gamma_1(t), \gamma_2).$$

It is clear that there exist $t_0, t_1 \in [0, 1]$ such that $f^+(0) = d_F(\gamma_1(t_0), \gamma_2(0))$ and $f^+(1) = d_F(\gamma_1(t_1), \gamma_2(1))$. We define $\tilde{\gamma} : [0, 1] \to M$ by $\tilde{\gamma}(t) = \gamma_1((t_1 - t_0)t + t_0)$. Again, due to the fact that (M, F) is a Berwald space, $\tilde{\gamma}$ will be a minimal geodesic (which joins the points $\gamma_1(t_0)$ and $\gamma_1(t_1)$), whose image is contained in that of γ_1. Since the function $t \mapsto d_F(\tilde{\gamma}(t), \gamma_2(t))$ is convex, we have for all $t \in [0, 1]$

$$\begin{aligned} f^+(t) &= \mathrm{dist}(\gamma_1, \gamma_2(t)) \le \mathrm{dist}(\tilde{\gamma}, \gamma_2(t)) \le d_F(\tilde{\gamma}(t), \gamma_2(t)) \le \text{‘}t d_F(\tilde{\gamma}(1), \gamma_2(1)) \\ &\quad +(1 - t) d_F(\tilde{\gamma}(0), \gamma_2(0)) \le (t + 1 - t) \max\{d_F(\tilde{\gamma}(1), \gamma_2(1)), d_F(\tilde{\gamma}(0), \gamma_2(0))\} \\ &= \max\{f^+(1), f^+(0)\}, \end{aligned}$$

i.e., f^+ is quasiconvex. A similar argument shows that f^- is also quasiconvex.

6. (*c*) ⇒ (*e*) and (*d*) ⇒ (*f*)

First, we prove $(c) \Rightarrow (e)$.

Let $p \in M$ be a fixed point and $\rho_p' > 0$ as in Section 4. Let $\gamma : [a, b] \to M$ be a minimal geodesic and $\alpha > 0$ such that (γ, α) is a forward $\mathcal{B}_p^+(\rho_p')$-admissible pair and fix $x, y \in \mathcal{C}_\gamma^+(\alpha)$ arbitrarily. Let $\tilde{\gamma} : [0, 1] \to M$ be the reparametrization of γ, i.e., $\tilde{\gamma}(t) = \gamma((b - a)t + a)$. $\tilde{\gamma}$ will be also a minimal geodesic and $\mathcal{C}_\gamma^+(\alpha) = \mathcal{C}_{\tilde{\gamma}}^+(\alpha)$. Since $\mathcal{C}_{\tilde{\gamma}}^+(\alpha) \subset \mathcal{B}_p^+(\rho_p')$, there exists a unique minimal geodesic $\sigma : [0, 1] \to M$ which joins x and y, and the image of σ belongs to $\mathcal{B}_p^+(\rho_p')$. Since the function $t \mapsto \mathrm{dist}(\tilde{\gamma}, \sigma(t))$ is quasiconvex (cf. (c)), then for all $t \in [0, 1]$

$$\mathrm{dist}(\tilde{\gamma}, \sigma(t)) \le \max\{\mathrm{dist}(\tilde{\gamma}, \sigma(0)), \mathrm{dist}(\tilde{\gamma}, \sigma(1))\} = \max\{\mathrm{dist}(\tilde{\gamma}, x), \mathrm{dist}(\tilde{\gamma}, y)\} \le \alpha.$$

Thus, the curve σ belongs entirely to $\mathcal{C}_{\tilde{\gamma}}^+(\alpha) = \mathcal{C}_\gamma^+(\alpha)$.

In the case of $(d) \Rightarrow (f)$, the argument is similar. We fix arbitrarily a minimal geodesic $\gamma : [a, b] \to M$ and $\alpha > 0$ such that (γ, α) is a backward $\mathcal{B}_p^+(\rho_p')$-admissible pair and choose two points $x, y \in \mathcal{C}_\gamma^-(\alpha)$. Taking $\tilde{\gamma}$ as above, due to $\mathcal{C}_{\tilde{\gamma}}^-(\alpha) \subset \mathcal{B}_p^+(\rho_p')$, there exists a unique minimal geodesic $\sigma : [0, 1] \to M$ lying on x and y, whose image is contained in $\mathcal{B}_p^+(\rho_p')$. By the quasiconvexity of $t \mapsto \mathrm{dist}(\sigma(t), \tilde{\gamma})$ we conclude that σ belongs to $\mathcal{C}_{\tilde{\gamma}}^-(\alpha) = \mathcal{C}_\gamma^-(\alpha)$.

7. (*e*) ⇒ (*a*)

Let $p \in M$ be a fixed point and consider two nonzero, non-collinear vectors in T_pM. Denote them by T_0 and U_0. We will prove that $K(T_0, U_0) \le 0$. To this end, we may suppose that U_0 and T_0 are g_{T_0}-orthogonal (see [1, p. 69]), i.e., $g_{T_0}(T_0, U_0) = 0$.

Let $r \in (0, 1)$ and choose $\delta > 0$ so small that

$$\delta[rF(T_0) + \max\{F(U_0), F(-U_0)\}] < \rho'_p, \tag{9}$$

where ρ'_p is from Section 4. Let us define $\sigma : [0, r] \to M$ by $\sigma(t) = \exp_p(t\delta T_0)$. Clearly, σ is a geodesic which has constant speed $\delta F(T_0)$ and $L(\sigma) = r\delta F(T_0)$. Let $T = T(t), t \in [0, r]$ be the velocity field of σ. We will translate U_0 along σ in a parallel manner, obtaining a vector field $U = U(t)$, i.e.

$$D_T U = 0 \quad \text{and} \quad U(0) = U_0. \tag{10}$$

Since (M, F) is of Berwald type, the reference vector in (10) becomes irrelevant again. Since $D_T T = 0$ (σ is a geodesic with constant speed), by formula (3) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} g_T(T, U) = g_T(D_T T, U) + g_T(T, D_T U) = 0.$$

Therefore, $g_T(T, U) = \text{const.} = g_{T_0}(T_0, U_0) = 0$.

Let $\Sigma : [0, r] \times [-1, 1] \to M$, defined by

$$\Sigma(t, s) = \exp_{\sigma(t)}(s\delta U(t)),$$

which is a variation of σ. From the first variation formula, see (5), we have

$$L'_\Sigma(0) = \frac{1}{\delta F(T_0)} g_T(T, U)\Big|_0^r = 0. \tag{11}$$

Further, let us define $\gamma_t : [-1, 1] \to M$ by $\gamma_t = \Sigma(t, \cdot), t \in [0, r]$. We will prove that γ_t is a minimal geodesic.

To this end, from (10) we observe that the vector field $-U$ is also parallel along σ. Moreover, $F(U(t)) = F(U_0)$ and $F(-U(t)) = F(-U_0)$ along σ, due to Ichijyō's result (see [1, p. 258]). By (9), the geodesic σ belongs to $\mathcal{B}^+_p(\rho'_p)$ and $F(s\delta U(t)) = \delta|s|F(\text{sgn}(s)U(t)) \le \delta \max\{F(U_0), F(-U_0)\} < \rho'_p$ for all $s \in [-1, 1]$. Therefore, $s\delta U(t) \in B_{\sigma(t)}(2\rho'_p)$ for all $(t, s) \in [0, r] \times [-1, 1]$. This implies, in virtue of Proposition 1, that $\exp_{\sigma(t)}$ is a C^1-diffeomorphism from $B_{\sigma(t)}(2\rho'_p)$ into $\mathcal{B}^+_{\sigma(t)}(2\rho'_p)$. In particular, $c^\pm_t : [0, 1] \to M$, defined by $c^\pm_t(s) = \exp_{\sigma(t)}(\pm s\delta U(t))$ will be radial geodesics minimizing distances (which are equals to $\delta F(\pm U_0)$, respectively) among all piecewise C^∞ curves in M that share their endpoints. We observe further that c^+_t coincides $\gamma_{t|[0,1]}$, while c^-_t is the reverse of $\gamma_{t|[-1,0]}$. Therefore, by (9), for fixed $s \in [-1, 1]$, we have

$$\begin{aligned} d_F(p, \gamma_t(s)) &\le d_F(p, \sigma(t)) + d_F(\sigma(t), \gamma_t(s)) \le L(\sigma) + d_F(\gamma_t(0), \gamma_t(s)) \\ &\le L(\sigma) + \max\{d_F(c^+_t(0), c^+_t(|s|)), d_F(c^-_t(0), c^-_t(|s|))\} \\ &= \delta r F(T_0) + \delta|s| \max\{F(U(t)), F(-U(t))\} \\ &\le \delta r F(T_0) + \delta \max\{F(U_0), F(-U_0)\} < \rho'_p. \end{aligned}$$

Thus, γ_t belongs entirely to $\mathcal{B}^+_p(\rho'_p)$. Since c^-_t is a minimal geodesic, its reverse $\gamma_{t|[-1,0]}$ must be also a geodesic. Since γ_t is a C^∞ curve on $[-1, 1]$ which is contained in $\mathcal{B}^+_p(\rho'_p)$, it must be a minimal geodesic.

Let $q \in \mathcal{C}^+_{\gamma_0}(L(\sigma))$ be fixed arbitrarily. Let $s_q \in [-1, 1]$ such that $d_F(\gamma_0(s_q), q) = \text{dist}(\gamma_0, q)$. Moreover, by (9) we have

$$\begin{aligned} d_F(p, q) &\le d_F(p, \gamma_0(s_q)) + d_F(\gamma_0(s_q), q) \\ &\le \max\{d_F(c_0^+(0), c_0^+(|s_q|)), d_F(c_0^-(0), c_0^-(|s_q|))\} + \text{dist}(\gamma_0, q) \\ &\le \delta \max\{F(U_0), F(-U_0)\} + L(\sigma) < \rho'_p. \end{aligned}$$

This implies that $\mathcal{C}^+_{\gamma_0}(L(\sigma)) \subset \mathcal{B}^+_p(\rho'_p)$, i.e., $(\gamma_0, L(\sigma))$ is a forward $\mathcal{B}^+_p(\rho'_p)$-admissible pair.

Since $g_T(T, U) = 0$ (at $t = r$, in particular), the curve γ_r is transversal to σ at the point $\sigma(r) = \gamma_r(0)$. Therefore, γ_r is tangent to the capsule $\mathcal{C}^+_{\gamma_0}(L(\sigma))$, which is a convex set, by the hypothesis. In this way, γ_r will be a supporting line to $\mathcal{C}^+_{\gamma_0}(L(\sigma))$ at $\sigma(r)$, i.e., the curve γ_r does not meet the set $\{q \in M : \text{dist}(\gamma_0, q) < L(\sigma)\}$. This means that $d_F(\gamma_0(u), \gamma_r(s)) \ge \text{dist}(\gamma_0, \gamma_r(s)) \ge L(\sigma)$ for all $u, s \in [-1, 1]$. In particular, from (11) and the Taylor expansion for L_Σ, we have

$$L''_\Sigma(0) \ge 0. \tag{12}$$

For a fixed $t \in [0, r]$, the geodesic $\gamma_{t|[0,1]}$ with velocity $\frac{\partial \Sigma}{\partial s}(t, s)$ has constant speed. Thus, Eq. (2) reduces to $D_{(\partial\Sigma/\partial s)(t,s)} \frac{\partial \Sigma}{\partial s}(t, s) = 0$, $s \in [0, 1]$. In particular, for $s = 0$ we have $\frac{\partial \Sigma}{\partial s}(t, 0) = U(t)$ and $D_{U(t)}U(t) = 0$. Since $t \in [0, r]$ was arbitrarily fixed,

$$D_U U = 0 \text{ along } \sigma. \tag{13}$$

By (10), (12), (13), and the second variation formula, we obtain that

$$\int_0^r g_T(R(U, T)T, U)\, \mathrm{d}t \le 0.$$

Moreover, from (1) and $g_T(T, U) = 0$ we have

$$\int_0^r K(T, U)F^2(T)g_T(U, U)\, \mathrm{d}t \le 0.$$

Since $F^2(T) = F^2(T_0)$ and $g_T(U, U) = \text{const.} = g_{T_0}(U_0, U_0) \ne 0$ (use formula (3)), the above inequality reduces to $\int_0^r K(T, U)\, \mathrm{d}t \le 0$. If $r \to 0$, by the continuity of K we obtain $K(T_0, U_0) = K(T(0), U(0)) \le 0$, which completes the proof.

8. (*f*) ⇒ (*a*)

The proof of this implication is a slight modification of the previous section; thus, we will indicate only the differences.

Let $\beta \in (0, \rho'_p)$ such that $\mathcal{B}^-_p(\beta) \subset \mathcal{B}^+_p(\rho'_p)$, and instead of (9), choose $\delta > 0$ so small that

$$\delta[r(1 + c_p)F(T_0) + c_p \max\{F(U_0), F(-U_0)\}] < \beta, \tag{14}$$

where $c_p > 1$ is from Proposition 1. The construction of σ, Σ and γ_t are the same, obtaining in a similar manner (due to (14)), that γ_t is a minimal geodesic which belongs to $\mathcal{B}_p^+(\rho'_p)$, $t \in [0, r]$.

Let $q \in \mathcal{C}_{\gamma_r}^-(L(\sigma))$ be a fixed point. There is $s_q \in [-1, 1]$ such that $d_F(q, \gamma_r(s_q)) = \text{dist}(q, \gamma_r) \leq L(\sigma) = \delta r F(T_0)$. Since $\gamma_r(0) = \sigma(r)$ and $\gamma_r(s_q)$ belong to $\mathcal{B}_p^+(\rho'_p) \subset \mathcal{B}_p^+(\rho_p)$, by (6) and (14) we have

$$\begin{aligned} d_F(q, p) &\leq d_F(q, \gamma_r(s_q)) + d_F(\gamma_r(s_q), \gamma_r(0)) + d_F(\gamma_r(0), p) \\ &\leq L(\sigma) + c_p[d_F(\gamma_r(0), \gamma_r(s_q)) + d_F(p, \sigma(r))] \\ &\leq L(\sigma) + c_p[\delta \max\{F(U_0), F(-U_0)\} + L(\sigma)] < \beta. \end{aligned}$$

Therefore, $q \in \mathcal{B}_p^-(\beta) \subset \mathcal{B}_p^+(\rho'_p)$. This implies that $\mathcal{C}_{\gamma_r}^-(L(\sigma)) \subset \mathcal{B}_p^+(\rho'_p)$, i.e., $(\gamma_r, L(\sigma))$ is a backward $\mathcal{B}_p^+(\rho'_p)$-admissible pair.

Since the curve γ_0 is transversal to σ at the point $\sigma(0) = \gamma_0(0) = p$ (note that $g_T(T, U) = 0$ at $t = 0$), γ_0 is tangent to the capsule $\mathcal{C}_{\gamma_r}^-(L(\sigma))$, which is a convex set, by hypothesis. Similarly, as in the previous section, we obtain that $d_F(\gamma_0(s), \gamma_r(u)) \geq \text{dist}(\gamma_0, \gamma_r(u)) \geq L(\sigma)$ for all $s, u \in [-1, 1]$; in particular $L''_\Sigma(0) \geq 0$. This completely concludes the proof.

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