J. Differential Equations 249 (2010) 1917–1928

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

On a new class of elliptic systems with nonlinearities of arbitrary growth ☆

Alexandru Kristály

Babeş-Bolyai University, Department of Economics, 400591 Cluj-Napoca, Romania

ARTICLE INFO

Article history:
Received 6 August 2009
Available online 15 May 2010

Dedicated to Professor Gheorghe Morosanu on his 60th birthday

MSC:
35J55
35J50
47J30

Keywords:
Elliptic systems
Nonlinearities of arbitrary growth
Ricceri's variational principle

ABSTRACT

We guarantee the existence of infinitely many different pairs of solutions to the system

$$\begin{cases} -\Delta u = v^p & \text{in } \Omega; \\ -\Delta v = f(u) & \text{in } \Omega; \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < p < \frac{2}{N-2}$, Ω is a bounded domain in $\mathbb{R}^N$ and the continuous nonlinear term f has an unusual oscillatory behavior. The sequence of solutions tends to zero (resp., infinity) with respect to certain norms and the nonlinear term f may enjoy an arbitrary growth at infinity (resp., at zero) whenever f oscillates near zero (resp., at infinity). Our results provide the first applications of Ricceri's variational principle in the theory of coupled elliptic systems.

1. Introduction and results

We consider the elliptic system

$$\begin{cases} -\Delta u = g(v) & \text{in } \Omega; \\ -\Delta v = f(u) & \text{in } \Omega; \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{S}$$

☆ This paper was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences and by Grant CNCSIS PCCE-55/2008 "Sisteme diferentiale in analiza neliniara si aplicatii".

E-mail address: alexandrukristaly@yahoo.com.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2010.05.001

where $\Omega \subset \mathbb{R}^N$ ($N \geqslant 2$) is an open bounded domain with smooth boundary, and $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions.

In the particular case when $g(s) = s^p$, $f(s) = s^q$ ($p, q > 1$) and $N \geqslant 3$ (here and in the sequel, we use the notation $s^\alpha = \mathrm{sgn}(s)|s|^\alpha$, $\alpha > 0$), system (S) has been widely studied replacing the usual criticality notion (i.e., $p, q \leqslant \frac{N+2}{N-2}$) by the so-called "critical hyperbola" which involves both parameters p and q, i.e., those pairs of points $(p, q) \in \mathbb{R}_+^2$ which verify

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N}. \tag{CH}$$

Points (p, q) on this curve meet the typical non-compactness phenomenon of Sobolev embeddings and non-existence of solutions for (S) has been pointed out by Mitidieri [7] and van der Vorst [11] via Pohozaev-type arguments. On the other hand, when

$$1 > \frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}, \tag{1}$$

the existence of nontrivial solutions for (S) has been proven by de Figueiredo and Felmer [2], Hulshof, Mitidieri and van der Vorst [5]. Note that the latter results work also for nonlinearities $g(s) \sim s^p$ and $f(s) \sim s^q$ as $|s| \to \infty$ with (p, q) fulfilling (1). The points verifying (1) form a proper region in the first quadrant of the (p, q)-plane situated below the critical hyperbola (CH). Note that (1) is verified for any $p, q > 1$ whenever $N = 2$.

In spite of the aforementioned results, the whole region below (CH) is far to be understood from the point of view of existence/multiplicity of solutions for (S). By exploiting the Trudinger–Moser inequality and elements from critical point theory, de Figueiredo, do Ó and Ruf [3] considered system (S) when Ω is a bounded domain in $\mathbb{R}^2$, and the nonlinearities f, g have maximal growth like exponential. Moreover, via a Mountain Pass argument, de Figueiredo and Ruf [4] proved the existence of at least one nontrivial solution to the problem

$$\begin{cases} -\Delta u = v^p & \text{in } \Omega; \\ -\Delta v = f(u) & \text{in } \Omega; \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\tilde{S}$}$$

when

$$\begin{cases} 0 < p, & \text{if } N = 2; \\ 0 < p < \frac{2}{N-2}, & \text{if } N \geqslant 3, \end{cases} \tag{$\tilde{1}$}$$

and $f : \mathbb{R} \to \mathbb{R}$ has a suitable superlinear growth at infinity, formulated in terms of the Ambrosetti–Rabinowitz condition. Later on, Salvatore [10] guaranteed via the Pohozaev's fibering method the existence of a whole sequence of solutions to $(\tilde{S})$ in a similar context as [4] assuming in addition that the nonlinear term f is odd. Note that in the latter two papers (i.e., [4] and [10]) *no further growth restriction* is required on the nonlinear term f other than the Ambrosetti–Rabinowitz condition. This latter fact is not surprising taking into account that $(\tilde{1})$ is actually equivalent to

$$1 > \frac{1}{p+1} > 1 - \frac{2}{N},$$

which is nothing but a "degenerate" case of (1) putting formally $q = \infty$, i.e., the growth of f may be arbitrary large.

The aim of the present paper is to complete the works [3,4] and [10] by guaranteeing the existence of infinitely many pairs of distinct solutions to the system $(\tilde{S})$ when $(\tilde{1})$ holds and the nonlinear term

f has an *oscillatory behavior*. Moreover, the nonlinear term f may enjoy an arbitrary growth at infinity (resp., at zero) whenever it oscillates near the origin (resp., at infinity) in a suitable way. In addition, the size of our solutions reflects the oscillatory behavior of the nonlinear term, see relations (2) and (3) below; namely, the solutions are small (resp., large) in L^∞-norm and in a suitably chosen Sobolev space whenever the nonlinearity f oscillates near the origin (resp., at infinity). We emphasize that no symmetry condition is required on f.

In the sequel, we formulate our main results. Before doing that, note that system $(\tilde{S})$ is equivalent to the Poisson equation

$$\begin{cases} -\Delta(-\Delta u)^{\frac{1}{p}} = f(u) & \text{in } \Omega; \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P}$$

The suitable functional space where solutions of (P) is going to be sought is

$$E = W^{2,\frac{p+1}{p}}(\Omega) \cap W_0^{1,\frac{p+1}{p}}(\Omega)$$

endowed with the norm

$$\|u\|_E = \left(\int_\Omega |\Delta u|^{\frac{p+1}{p}}\, dx \right)^{\frac{p}{p+1}}.$$

As we anticipated, two different cases will be considered; namely, the nonlinear term has a suitable oscillatory behavior either near the origin or at infinity.

1.1. Oscillation near the origin

Let $f \in C(\mathbb{R}, \mathbb{R})$ and $F(s) = \int_0^s f(t)\, dt,\ s \in \mathbb{R}$. We assume that:

(H_0^1) $-\infty < \liminf_{s\to 0} \frac{F(s)}{|s|^{\frac{p+1}{p}}} \leqslant \limsup_{s\to 0} \frac{F(s)}{|s|^{\frac{p+1}{p}}} = +\infty$,

(H_0^2) there exist two sequences $\{a_k\}$ and $\{b_k\}$ in $]0, \infty[$ with $b_{k+1} < a_k < b_k$, $\lim_{k\to\infty} b_k = 0$ such that

$$\operatorname{sgn}(s) f(s) \leqslant 0 \quad \text{for every } |s| \in [a_k, b_k], \quad \text{and}$$

(H_0^3) $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\lim_{k\to\infty} \frac{\max_{[-a_k, a_k]} F}{b_k^{\frac{p+1}{p}}} = 0$.

Theorem 1. *Assume that* $(\tilde{1})$ *holds and* $f \in C(\mathbb{R}, \mathbb{R})$ *fulfills* (H_0^1)–(H_0^3). *Then, system* $(\tilde{S})$ *possesses a sequence* $\{(u_k, v_k)\} \subset E \times E$ *of distinct (strong) solutions which satisfy*

$$\lim_{k\to\infty} \|u_k\|_E = \lim_{k\to\infty} \|v_k\|_E = \lim_{k\to\infty} \|u_k\|_\infty = \lim_{k\to\infty} \|v_k\|_\infty = 0. \tag{2}$$

Remark 1. Hypotheses (H_0^1)–(H_0^2) imply an oscillatory behavior of f near the origin while (H_0^3) is a technical assumption which seems to be indispensable in our arguments.

In the sequel, we provide a concrete example when hypotheses (H_0^1)–(H_0^3) are fulfilled. Let $a_k = k^{-k^{k+1}}$ and $b_k = k^{-k^k}$, $k \geqslant 2$, and $a_1 = 1$, $b_1 = 2$. It is clear that $b_{k+1} < a_k < b_k$, $\lim_{k\to\infty} a_k/b_k = 0$, and $\lim_{k\to\infty} b_k = 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(s)=\begin{cases}\varphi_k\left(\frac{s-b_{k+1}}{a_k-b_{k+1}}\right), & s\in[b_{k+1},a_k],\ k\geqslant 1;\\ 0, & s\in\,]a_k,b_k[,\ k\geqslant 1;\\ 0, & s\in\,]-\infty,0];\\ g(s), & s\in[2,\infty[,\end{cases}$$

where $g:[2,\infty[\,\to\mathbb{R}$ is *any* continuous function with $g(2)=0$, and $\varphi_k:[0,1]\to[0,\infty[$ is a sequence of continuous functions such that $\varphi_k(0)=\varphi_k(1)=0$ and there are some positive constants c_1 and c_2 such that

$$c_1\left(b_k^{\frac{2p+2}{p}}-b_{k+1}^{\frac{2p+2}{p}}\right)(a_k-b_{k+1})^{-1}\leqslant\int_0^1\varphi_k(s)\,ds\leqslant c_2\left(b_k^{\frac{p+2}{p}}-b_{k+1}^{\frac{p+2}{p}}\right)(a_k-b_{k+1})^{-1}.$$

Note that $F(s)=0$ for every $s\in\,]-\infty,0]$ and F is non-decreasing on $[0,2]$, while $c_1 b_k^{\frac{2p+2}{p}}\leqslant F(a_k)=\max_{[-a_k,a_k]}F\leqslant c_2 b_k^{\frac{p+2}{p}}$. Due to these inequalities, the hypotheses of Theorem 1 are verified.

1.2. Oscillation at infinity

In this subsection, we state a perfect counterpart of Theorem 1 when the nonlinearity f has an oscillation at infinity. We assume that:

(H^1_∞) $-\infty<\liminf_{|s|\to\infty}\frac{F(s)}{|s|^{\frac{p+1}{p}}}\leqslant\limsup_{|s|\to\infty}\frac{F(s)}{|s|^{\frac{p+1}{p}}}=+\infty$,

(H^2_∞) there exist two sequences $\{a_k\}$ and $\{b_k\}$ in $]0,\infty[$ with $a_k<b_k<a_{k+1}$ and $\lim_{k\to\infty}b_k=\infty$ such that

$$\operatorname{sgn}(s)f(s)\leqslant 0\quad\text{for every } |s|\in[a_k,b_k],\quad\text{and}$$

(H^3_∞) $\lim_{k\to\infty}\frac{a_k}{b_k}=0$ and $\lim_{k\to\infty}\frac{\max_{[-a_k,a_k]}F}{b_k^{\frac{p+1}{p}}}=0$.

Theorem 2. *Assume that* $(\tilde{1})$ *holds and* $f\in C(\mathbb{R},\mathbb{R})$ *fulfills* (H^1_∞)–(H^3_∞). *Then, system* $(\tilde{\mathrm{S}})$ *possesses a sequence* $\{(u_k,v_k)\}\subset E\times E$ *of distinct (strong) solutions which satisfy*

$$\lim_{k\to\infty}\|u_k\|_E=\lim_{k\to\infty}\|v_k\|_E=\lim_{k\to\infty}\|u_k\|_\infty=\lim_{k\to\infty}\|v_k\|_\infty=\infty. \tag{3}$$

Remark 2. Assumptions (H^1_∞)–(H^2_∞) imply an oscillatory behavior of f at infinity. A concrete example is described in the sequel when hypotheses (H^1_∞)–(H^3_∞) are fulfilled. Let $a_k=k^{k^k}$ and $b_k=k^{k^{k+1}}$ $(k\geqslant 2)$ and $a_1=5$, $b_1=10$. Clearly, one has $a_k<b_k<a_{k+1}$, $\lim_{k\to\infty}a_k/b_k=0$, and $\lim_{k\to\infty}b_k=\infty$. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(s)=\begin{cases}\varphi_k\left(\frac{s-b_k}{a_{k+1}-b_k}\right), & s\in[b_k,a_{k+1}],\ k\geqslant 1;\\ 0, & s\in\,]a_k,b_k[,\ k\geqslant 1;\\ g(s), & s\in[-5,5];\\ 0, & s\in\,]-\infty,-5[,\end{cases}$$

where $g:[-5,5]\to\mathbb{R}$ is *any* continuous function with $g(\pm 5)=0$, and $\varphi_k:[0,1]\to[0,\infty[$ is a sequence of continuous functions such that $\varphi_k(0)=\varphi_k(1)=0$ and there are some constants $c_1,c_2>0$ such that

$$c_1\big(b_{k+1}^{\frac{3p+1}{3p}} - b_k^{\frac{3p+1}{3p}}\big)(a_{k+1}-b_k)^{-1} \leqslant \int_0^1 \varphi_k(s)\,ds \leqslant c_2\big(b_{k+1}^{\frac{2p+1}{2p}} - b_k^{\frac{2p+1}{2p}}\big)(a_{k+1}-b_k)^{-1}.$$

Note that $F(s) = 0$ for every $s \in]-\infty, -5]$ and F is non-decreasing on $[5, \infty[$. Moreover, for $k \in \mathbb{N}$ large enough we have

$$c_1\big(b_k^{\frac{3p+1}{3p}} - 10^{\frac{3p+1}{3p}}\big) + \int_0^5 g(s)\,ds \leqslant F(a_k) = \max_{[-a_k,a_k]} F \leqslant c_2\big(b_k^{\frac{2p+1}{2p}} - 10^{\frac{2p+1}{2p}}\big) + \int_0^5 g(s)\,ds.$$

Now, an easy computation shows the hypotheses of Theorem 2 are verified.

The proofs of Theorems 1 and 2 are based on a general variational principle of Ricceri (see [8] and [9]). As far as we know, the present paper gives the first application of Ricceri's variational principle to coupled systems of *non*-gradient type.

The paper is organized as follows. In the next section we first describe the variational framework we are working in, then the abstract form of Ricceri's variational principle is recalled. In Sections 3 and 4 we prove Theorems 1 and 2, respectively, while in the last section we are going to formulate two open problems related to system $(\tilde{S})$.

2. Preliminaries

Due to $(\tilde{1})$ one has $\frac{p+1}{p} > 1 + \frac{N-2}{2} = \frac{N}{2}$, therefore $W^{2,\frac{p+1}{p}}(\Omega) \subset\subset C(\overline{\Omega})$, so

$$E \subset\subset C(\overline{\Omega}). \tag{4}$$

For further use, we denote by $\kappa_0 > 0$ the best embedding constant of $E \subset\subset C(\overline{\Omega})$. The energy functional associated to the Poisson problem (P) is $I : E \to \mathbb{R}$ defined by

$$I(u) = \frac{p}{p+1}\|u\|_E^{\frac{p+1}{p}} - \mathcal{F}(u),$$

where

$$\mathcal{F}(u) = \int_\Omega F\big(u(x)\big)\,dx.$$

Due to (4), the functional I is well defined, is of class C^1 on E and

$$I'(u)(h) = \int_\Omega (-\Delta u)^{\frac{1}{p}}(-\Delta h)\,dx - \int_\Omega f(u)h\,dx, \quad u, h \in E.$$

Note that if $u \in E$ is a critical point of I then it is a weak solution of problem (P); in such a case, the pair $(u, (-\Delta u)^{\frac{1}{p}}) \in E \times E$ is a weak solution of system $(\tilde{S})$. See also [4, Subsection 3.1] and [10, Proposition 2.1]. Moreover, standard regularity arguments show that the pair $(u, (-\Delta u)^{\frac{1}{p}}) \in E \times E$ is actually a strong solution of system $(\tilde{S})$, see [4].

On account of the above facts, in order to prove Theorems 1 and 2, it is enough to find sequences of critical points for the functional I with the required properties, i.e., to fulfill relations (2) and (3). To do that, we apply a general variational principle of Ricceri, see [8] and [9], that can be stated as follows:

Theorem R. *(See [8, Theorem 2.5].) Let X be a reflexive real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous, continuously Gâteaux differentiable functionals. Assume that Ψ is strongly continuous and coercive. For each $s > \inf_X \Psi$, set*

$$\varphi(s) := \inf_{\Psi^s} \frac{\Phi(u) - \inf_{\mathrm{cl}_w \Psi^s} \Phi}{s - \Psi(u)}, \tag{5}$$

where $\Psi^s := \{u \in X\colon\ \Psi(u) < s\}$ and $\mathrm{cl}_w \Psi^s$ is the closure of Ψ^s in the weak topology of X. Furthermore, set

$$\delta := \liminf_{s \to (\inf_X \Psi)^+} \varphi(s), \qquad \gamma := \liminf_{s \to +\infty} \varphi(s). \tag{6}$$

Then, the following conclusions hold.

(A) *If $\delta < +\infty$ then, for every $\lambda > \delta$, either*
  (A1) *there is a global minimum of Ψ which is a local minimum of $\Phi + \lambda\Psi$, or*
  (A2) *there is a sequence $\{u_k\}$ of pairwise distinct critical points of $\Phi + \lambda\Psi$, with $\lim_{k\to+\infty} \Psi(u_k) = \inf_X \Psi$, weakly converging to a global minimum of Ψ.*

(B) *If $\gamma < +\infty$ then, for every $\lambda > \gamma$, either*
  (B1) *$\Phi + \lambda\Psi$ possesses a global minimum, or*
  (B2) *there is a sequence $\{u_k\}$ of critical points of the functional $\Phi + \lambda\Psi$ such that $\lim_{k\to+\infty} \Psi(u_k) = +\infty$.*

In our framework concerning problem (P) (thus, system $(\tilde{S})$), we choose $X = E$, and $\Psi, \Phi : E \to \mathbb{R}$ are defined by

$$\Psi(u) = \|u\|_E^{\frac{p+1}{p}}, \qquad \Phi(u) = -\mathcal{F}(u), \quad u \in E.$$

Standard arguments show that Ψ and Φ are sequentially weakly lower semicontinuous. The energy functional becomes $I = \frac{p}{p+1}\Psi + \Phi$. Moreover the function from (5) takes the form

$$\varphi(s) = \inf_{\|u\|_E^{p+1} < s^p} \frac{\sup\{\mathcal{F}(v)\colon\ \|v\|_E^{p+1} \leqslant s^p\} - \mathcal{F}(u)}{s - \|u\|_E^{\frac{p+1}{p}}}, \quad s > 0.$$

To conclude this section, we are going to construct a special element in the space E which will play a crucial role in our proofs. Let $x_0 \in \Omega$ and $R > 0$ be such that $B(x_0, R) \subset \Omega$; here and in the sequel, $B(x_0, a) = \{x \in \mathbb{R}^N\colon\ |x - x_0| < a\}$, $a > 0$. Let $0 < r < R$ be fixed. We consider the function $w : \Omega \to \mathbb{R}$ defined by

$$w(x) = \frac{\int_{-\infty}^{-|x-x_0|} \alpha(t)\,dt}{\int_{-R}^{-r} \alpha(t)\,dt}, \tag{7}$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is given by

$$\alpha(t) = \begin{cases} e^{\frac{1}{(t+R)(t+r)}}, & \text{if } t \in \,]-R, -r[; \\ 0, & \text{if } t \notin \,]-R, -r[. \end{cases}$$

It is clear that $w \in C_0^\infty(\Omega) \subset E$; moreover, $w \geqslant 0$, $\|w\|_\infty = 1$ and

$$w(x)=\begin{cases}1, & \text{if } x\in B(x_0,r);\\ 0, & \text{if } x\in \Omega\setminus B(x_0,R).\end{cases} \tag{8}$$

Throughout the proofs of Theorems 1 and 2 we will use the following useful observation whose assumptions come from (H_0^2) and (H_∞^2), respectively.

Lemma 1. *Let $\{a_k\},\{b_k\}\subset\,]0,\infty[$ be two sequences such that $a_k<b_k$, $\lim_{k\to\infty}a_k/b_k=0$, and $\operatorname{sgn}(s)f(s)\leqslant 0$ for every $|s|\in[a_k,b_k]$. Let $s_k=(b_k/\kappa_0)^{\frac{p+1}{p}}$. Then,*

(a) $\max_{[-b_k,b_k]}F=\max_{[-a_k,a_k]}F\equiv F(\bar s_k)$ *with $\bar s_k\in[-a_k,a_k]$.*
(b) $\|\bar s_k w\|_E^{\frac{p+1}{p}}<s_k$ *for $k\in\mathbb{N}$ large enough.*

Proof. (a) It follows from the standard Mean Value Theorem and from the hypotheses that $\operatorname{sgn}(s)f(s)\leqslant 0$ for every $|s|\in[a_k,b_k]$.

(b) Since $\lim_{k\to\infty}a_k/b_k=0$, we may fix $k_0\in\mathbb{N}$ such that $a_k/b_k<\kappa_0^{-1}\|w\|_E^{-1}$ for $k>k_0$. Then, one has

$$\|\bar s_k w\|_E^{\frac{p+1}{p}}=|\bar s_k|^{\frac{p+1}{p}}\|w\|_E^{\frac{p+1}{p}}\leqslant a_k^{\frac{p+1}{p}}\|w\|_E^{\frac{p+1}{p}}<(b_k/\kappa_0)^{\frac{p+1}{p}}=s_k. \qquad \square$$

3. Proof of Theorem 1

Let $\{a_k\}$ and $\{b_k\}$ be as in the hypotheses. We recall from (6) that $\delta=\liminf_{s\to 0^+}\varphi(s)$.

Lemma 2. $\delta=0$.

Proof. By definition, $\delta\geqslant 0$. Suppose that $\delta>0$. By the first inequality of (H_0^1), there exist two positive numbers ℓ_0 and ϱ_0 such that

$$F(s)>-\ell_0|s|^{\frac{p+1}{p}}\quad\text{for every } s\in\,]-\varrho_0,\varrho_0[. \tag{9}$$

Furthermore, let $s_k,\bar s_k$ be as in Lemma 1 and let $\overline{w}_k=\bar s_k w\in E$, where w is defined in (7). By (H_0^3) and condition $\lim_{k\to\infty}\frac{\bar s_k}{b_k}=0$ ($|\bar s_k|\leqslant a_k$), there exists $k_0\in\mathbb{N}$ such that for $k>k_0$ we have

$$m(\Omega)\frac{F(\bar s_k)}{b_k^{\frac{p+1}{p}}}+\left(\frac{\delta}{2}\|w\|_E^{\frac{p+1}{p}}+m(\Omega)\ell_0\right)\left(\frac{|\bar s_k|}{b_k}\right)^{\frac{p+1}{p}}<\frac{\delta}{2\kappa_0^{\frac{p+1}{p}}}. \tag{10}$$

Let $v\in E$ be arbitrarily fixed with $\|v\|_E^{\frac{p+1}{p}}\leqslant s_k$. Thus, due to the embedding from (4), we have $\|v\|_\infty\leqslant b_k$. Due to Lemma 1(a), we obtain

$$F\big(v(x)\big)\leqslant\max_{[-b_k,b_k]}F=F(\bar s_k)\quad\text{for every } x\in\Omega.$$

Since $0\leqslant|\overline{w}_k(x)|\leqslant|\bar s_k|<\varrho_0$ for large $k\in\mathbb{N}$ and for all $x\in\Omega$, taking into account (9) and (10), it follows that

$$\sup_{\|v\|_E^{p+1}\leqslant s_k^p} \mathcal{F}(v)-\mathcal{F}(\overline{w}_k) = \sup_{\|v\|_E^{\frac{p+1}{p}}\leqslant s_k} \int_\Omega F(v)\,dx - \int_\Omega F(\overline{w}_k)\,dx$$
$$\leqslant m(\Omega)F(\bar{s}_k) + m(\Omega)\ell_0|\bar{s}_k|^{\frac{p+1}{p}}$$
$$< \frac{\delta}{2}\bigl(s_k - \|\overline{w}_k\|_E^{\frac{p+1}{p}}\bigr).$$

Since $\|\overline{w}_k\|_E^{\frac{p+1}{p}} < s_k$ (cf. Lemma 1(b)), and $s_k \to 0$ as $k \to \infty$, we obtain

$$\delta \leqslant \liminf_{k\to\infty} \varphi(s_k) \leqslant \liminf_{k\to\infty} \frac{\sup_{\|v\|_E^{p+1}\leqslant s_k^p} \mathcal{F}(v)-\mathcal{F}(\overline{w}_k)}{s_k - \|\overline{w}_k\|_E^{\frac{p+1}{p}}} \leqslant \frac{\delta}{2},$$

contradiction. This proves our claim. □

Lemma 3. 0 *is not a local minimum of* $I = \frac{p}{p+1}\Psi + \Phi$.

Proof. Let $\ell_0 > 0$ and $\varrho_0 > 0$ from the proof of Lemma 2, and $x_0 \in \Omega$ and $r, R > 0$ from the definition of the function w, see (7). Let $\mathcal{L}_0 > 0$ be such that

$$r^N\omega_N\mathcal{L}_0 - \frac{p}{p+1}\|w\|_E^{\frac{p+1}{p}} - \bigl(R^N - r^N\bigr)\omega_N\ell_0 > 0, \tag{11}$$

where ω_N is the volume of the N-dimensional unit ball. By the right-hand side of (H_0^1) we deduce the existence of a sequence $\{s_k^0\} \subset]-\varrho_0, \varrho_0[$ converging to zero such that

$$F\bigl(s_k^0\bigr) > \mathcal{L}_0\bigl|s_k^0\bigr|^{\frac{p+1}{p}}. \tag{12}$$

Let $w_k^0 = s_k^0 w \in E$. Due to (8), (9), (11) and (12), we have

$$\begin{aligned}
I\bigl(w_k^0\bigr) &= \frac{p}{p+1}\bigl\|w_k^0\bigr\|_E^{\frac{p+1}{p}} - \int_\Omega F\bigl(w_k^0\bigr) \\
&= \frac{p}{p+1}\|w\|_E^{\frac{p+1}{p}}\bigl|s_k^0\bigr|^{\frac{p+1}{p}} - \int_{B(x_0,r)} F\bigl(w_k^0\bigr) - \int_{B(x_0,R)\setminus B(x_0,r)} F\bigl(w_k^0\bigr) \\
&\leqslant \frac{p}{p+1}\|w\|_E^{\frac{p+1}{p}}\bigl|s_k^0\bigr|^{\frac{p+1}{p}} - F\bigl(s_k^0\bigr)m\bigl(B(x_0,r)\bigr) + \ell_0\bigl(m\bigl(B(0,R)\bigr) - m\bigl(B(0,r)\bigr)\bigr)\bigl|s_k^0\bigr|^{\frac{p+1}{p}} \\
&\leqslant \bigl|s_k^0\bigr|^{\frac{p+1}{p}}\left(\frac{p}{p+1}\|w\|_E^{\frac{p+1}{p}} - r^N\omega_N\mathcal{L}_0 + \bigl(R^N - r^N\bigr)\omega_N\ell_0\right) \\
&< 0 = I(0).
\end{aligned}$$

Since $\|w_k^0\|_E \to 0$ as $k \to \infty$, 0 is not a local minimum of I, as claimed. □

Proof of Theorem 1. Applying Theorem R(A), with $\lambda = \frac{p}{p+1}$ (see Lemma 2), we can exclude condition (A1) (see Lemma 3). Therefore there exists a sequence $\{u_k\} \subset E$ of pairwise distinct critical points of $I = \frac{p}{p+1}\Psi + \Phi$ such that

$$\lim_{k\to\infty}\|u_k\|_E = 0. \tag{13}$$

Thus, $\{(u_k, v_k)\} = \{(u_k, (-\Delta u_k)^{\frac{1}{p}})\} \subset E \times E$ is a sequence of distinct pairs of solutions to the system $(\tilde{S})$.

It remains to prove (2). First, due to (4) and (13), we have that $\lim_{k\to\infty}\|u_k\|_\infty = 0$. For every $k \in \mathbb{N}$ let $m_k \in [-\|u_k\|_\infty, \|u_k\|_\infty] =: J_k$ such that $|f(m_k)| = \max_{s\in J_k}|f(s)|$. Note that $\operatorname{diam} J_k \to 0$ as $k \to \infty$; thus $\lim_{k\to\infty} m_k = 0$ which implies that $\lim_{k\to\infty} f(m_k) = 0$. On the other hand, from the second equation of system $(\tilde{S})$ we have that

$$\|v_k\|_E^{\frac{p+1}{p}} = \int_\Omega |\Delta v_k|^{\frac{p+1}{p}}\,dx = \int_\Omega \left|f(u_k)\right|^{\frac{p+1}{p}}\,dx \leqslant \left|f(m_k)\right|^{\frac{p+1}{p}} m(\Omega),$$

which implies that $\lim_{k\to\infty}\|v_k\|_E = 0$. Using again (4) we have $\lim_{k\to\infty}\|v_k\|_\infty = 0$. □

4. Proof of Theorem 2

The proof of Theorem 2 is similar to that of Theorem 1. Let $\{a_k\}$ and $\{b_k\}$ be from Theorem 2 and $\gamma = \liminf_{s\to+\infty}\varphi(s)$ from (6).

Lemma 4. $\gamma = 0$.

Proof. It is clear that $\gamma \geqslant 0$. Suppose that $\gamma > 0$. Due to the left-hand side of (H^1_∞), one can find two positive numbers ℓ_∞ and ϱ_∞ such that

$$F(s) > -\ell_\infty|s|^{\frac{p+1}{p}} \quad \text{for every } |s| > \varrho_\infty. \tag{14}$$

Let s_k, $\bar{s}_k$ be as in Lemma 1. By the fact that $\lim_{k\to\infty} b_k = \infty$, hypothesis (H^3_∞) and condition $\lim_{k\to\infty}\frac{\bar{s}_k}{b_k} = 0$ $(-a_k \leqslant \bar{s}_k \leqslant a_k)$, there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$ we have

$$m(\Omega)\frac{\max_{[-\varrho_\infty,\varrho_\infty]}|F| + F(\bar{s}_k)}{b_k^{\frac{p+1}{p}}} + \left(\frac{\gamma}{2}\|w\|_E^{\frac{p+1}{p}} + m(\Omega)\ell_\infty\right)\left(\frac{|\bar{s}_k|}{b_k}\right)^{\frac{p+1}{p}} < \frac{\gamma}{2\kappa_0^{\frac{p+1}{p}}}. \tag{15}$$

Let $\overline{w}_k = \bar{s}_k w \in E$, where w is defined in (7). A similar estimation as in Lemma 2 gives throughout relations (14) and (15) that

$$\begin{aligned}
\sup_{\|v\|_E^{p+1}\leqslant s_k^p} \mathcal{F}(v) - \mathcal{F}(\overline{w}_k) &= \sup_{\|v\|_E^{\frac{p+1}{p}}\leqslant s_k} \int_\Omega F(v)\,dx - \int_\Omega F(\overline{w}_k)\,dx \\
&= \sup_{\|v\|_E^{\frac{p+1}{p}}\leqslant s_k} \int_\Omega F(v)\,dx - \int_{\{|w_k(x)|>\varrho_\infty\}} F(\overline{w}_k)\,dx \\
&\quad - \int_{\{|w_k(x)|\leqslant\varrho_\infty\}} F(\overline{w}_k)\,dx \\
&\leqslant m(\Omega)F(\bar{s}_k) + m(\Omega)\ell_\infty|\bar{s}_k|^{\frac{p+1}{p}} + m(\Omega)\max_{[-\varrho_\infty,\varrho_\infty]}|F| \\
&< \frac{\gamma}{2}\big(s_k - \|\overline{w}_k\|_E^{\frac{p+1}{p}}\big).
\end{aligned}$$

Since $s_k \to +\infty$,

$$\gamma \leqslant \liminf_{k\to\infty} \varphi(s_k) \leqslant \liminf_{k\to\infty} \frac{\sup_{\|v\|_E^{p+1} \leqslant s_k^p} \mathcal{F}(v) - \mathcal{F}(\overline{w}_k)}{s_k - \|\overline{w}_k\|_E^{\frac{p+1}{p}}} \leqslant \frac{\gamma}{2},$$

which contradicts $\gamma > 0$. $\square$

Lemma 5. *$I = \frac{p}{p+1}\Psi + \Phi$ is not bounded from below on E.*

Proof. Let ℓ_∞ and ϱ_∞ be from the proof of Lemma 4, and let $\mathcal{L}_\infty > 0$ be such that

$$r^N \omega_N \mathcal{L}_\infty - \frac{p}{p+1}\|w\|_E^{\frac{p+1}{p}} - \left(R^N - r^N\right)\omega_N \ell_\infty > 0, \tag{16}$$

where r and R are from the definition of the function w, see (7). By the second part of (H^1_∞) we deduce the existence of a sequence $\{s_k^\infty\} \subset \mathbb{R}$ with $\lim_{k\to\infty} |s_k^\infty| = \infty$ and

$$F\left(s_k^\infty\right) > \mathcal{L}_\infty \left|s_k^\infty\right|^{\frac{p+1}{p}}. \tag{17}$$

Let $w_k^\infty = s_k^\infty w \in E$. We clearly have that

$$I\left(w_k^\infty\right) = \frac{p}{p+1}\|w\|_E^{\frac{p+1}{p}} \left|s_k^\infty\right|^{\frac{p+1}{p}} - F\left(s_k^\infty\right)\omega_N r^N - \int_{B(x_0,R)\setminus B(x_0,r)} F\left(w_k^\infty\right).$$

For abbreviation, we choose the set $X = B(x_0, R) \setminus B(x_0, r)$. Then, on account of (14) we have

$$\begin{aligned}
\int_X F\left(w_k^\infty\right) &= \int_{X\cap\{|w_k^\infty(x)|>\varrho_\infty\}} F\left(w_k^\infty\right) + \int_{X\cap\{|w_k^\infty(x)|\leqslant\varrho_\infty\}} F\left(w_k^\infty\right) \\
&\geqslant -\ell_\infty \int_{X\cap\{|w_k^\infty(x)|>\varrho_\infty\}} \left|w_k^\infty\right|^{\frac{p+1}{p}} - \left(R^N - r^N\right)\omega_N \max_{[-\varrho_\infty,\varrho_\infty]} |F| \\
&\geqslant -\left(R^N - r^N\right)\omega_N \left(\ell_\infty \left|s_k^\infty\right|^{\frac{p+1}{p}} + \max_{[-\varrho_\infty,\varrho_\infty]} |F|\right).
\end{aligned}$$

Consequently, due to (17) and the above estimation, we have

$$I\left(w_k^\infty\right) \leqslant \left|s_k^\infty\right|^{\frac{p+1}{p}} \left(\frac{p}{p+1}\|w\|_E^{\frac{p+1}{p}} - r^N\omega_N\mathcal{L}_\infty + \left(R^N - r^N\right)\omega_N\ell_\infty\right) + \left(R^N - r^N\right)\omega_N \max_{[-\varrho_\infty,\varrho_\infty]} |F|.$$

Since $\lim_{k\to\infty} |s_k^\infty| = \infty$, due to (16), we have $\lim_{k\to\infty} I(w_k^\infty) = -\infty$; consequently, $\inf_E I = -\infty$. $\square$

Proof of Theorem 2. In Theorem R(B) we may choose $\lambda = \frac{p}{p+1}$ (see Lemma 4). On account of Lemma 5 the alternative (B1) can be excluded. Therefore, there exists a sequence $\{u_k\} \subset E$ of distinct critical points of $I = \frac{p}{p+1}\Psi + \Phi$ such that

$$\lim_{k\to\infty} \|u_k\|_E = \infty. \tag{18}$$

Thus, $\{(u_k, v_k)\} = \{(u_k, (-\Delta u_k)^{\frac{1}{p}})\} \subset E \times E$ is a sequence of distinct pairs of solutions to the system $(\tilde{S})$.

We now prove the rest of (3). Assume that for every $k \in \mathbb{N}$ we have $\|v_k\|_\infty \leqslant M$ for some $M > 0$. In particular, from the first equation of system $(\tilde{S})$ we obtain that

$$\|u_k\|_E^{\frac{p+1}{p}} = \int_\Omega |\Delta u_k|^{\frac{p+1}{p}}\,dx = \int_\Omega |v_k|^{p+1}\,dx \leqslant M^{p+1} m(\Omega),$$

which contradicts relation (18). Consequently, $\lim_{k\to\infty} \|v_k\|_\infty = \infty$. But, this fact and (4) give at once that $\lim_{k\to\infty} \|v_k\|_E = \infty$ as well. Assume finally that for every $k \in \mathbb{N}$ we have $\|u_k\|_\infty \leqslant M'$ for some $M' > 0$. The second equation of system $(\tilde{S})$ shows that

$$\|v_k\|_E^{\frac{p+1}{p}} = \int_\Omega |\Delta v_k|^{\frac{p+1}{p}}\,dx = \int_\Omega \left|f(u_k)\right|^{\frac{p+1}{p}}\,dx \leqslant m(\Omega) \max_{s\in[-M',M']} \left|f(s)\right|^{\frac{p+1}{p}},$$

which contradicts the fact that $\lim_{k\to\infty} \|v_k\|_E = \infty$. The proof is complete. □

5. Further problems related to system $(\tilde{S})$

1. We assume $(\tilde{1})$ holds and consider the perturbed system

$$\begin{cases} -\Delta u = v^p & \text{in } \Omega; \\ -\Delta v = f(u) + \varepsilon g(u) & \text{in } \Omega; \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\tilde{S}_\varepsilon)$$

   where $g : \mathbb{R} \to \mathbb{R}$ is any continuous function with $g(0) = 0$. We predict that for every $n \in \mathbb{N}$ there exists $\varepsilon_n > 0$ such that for every $\varepsilon \in [-\varepsilon_n, \varepsilon_n]$, system $(\tilde{S}_\varepsilon)$ has at least n distinct pairs of solutions whenever $f : \mathbb{R} \to \mathbb{R}$ verifies the set of assumptions from Theorem 1 or Theorem 2. This statement is not unexpected taking into account the recent papers of Anello and Cordaro [1] and Kristály [6] where a prescribed number of solutions were guaranteed for certain elliptic problems of *scalar* type whenever the parameter in the front of the perturbation is small enough. In both papers (i.e., [1] and [6]) the uniform Lipschitz truncation function $h_a : \mathbb{R} \to \mathbb{R}$ $(a > 0)$, $h_a(s) = \min(a, \max(s, 0))$ plays a key role, fulfilling as well the so-called *Markovian property* concerning the superposition operators: for any $u \in W_0^{1,r}(\Omega)$ one also has $h_a \circ u \in W_0^{1,r}(\Omega)$, where $r > 1$. Note however that a similar property is not available any longer replacing the space $W_0^{1,r}(\Omega)$ by a higher order Sobolev space $W^{2,r}(\Omega)$; in particular, the Markovian property is not valid for $E = W^{2,\frac{p+1}{p}}(\Omega) \cap W_0^{1,\frac{p+1}{p}}(\Omega)$.
2. Based on $(\tilde{1})$, the embedding $E \subset\subset C(\overline{\Omega})$ is essential in our investigations, see the proof of Lemmas 2 and 4. Is it possible to obtain similar conclusions as in Theorems 1 and 2 by omitting $(\tilde{1})$ and considering the whole region below the critical hyperbola?

Acknowledgments

The author would like to thank the organizers of the Summer School in Nonlinear PDEs for their kind invitation and Professor D.J. de Figueiredo for his nice lectures "On semilinear elliptic systems" presented during the same event at the Université Catholique de Louvain, Louvain-la-Neuve, Belgium, 26–30 May, 2008.

References

[1] G. Anello, G. Cordaro, Perturbation from Dirichlet problem involving oscillating nonlinearities, J. Differential Equations 234 (1) (2007) 80–90.
[2] D.G. de Figueiredo, P. Felmer, On superquadratic elliptic systems, Trans. Amer. Math. Soc. 343 (1) (1994) 99–116.
[3] D.G. de Figueiredo, J.M. do Ó, B. Ruf, Critical and subcritical elliptic systems in dimension two, Indiana Univ. Math. J. 53 (4) (2004) 1037–1054.
[4] D.G. de Figueiredo, B. Ruf, Elliptic systems with nonlinearities of arbitrary growth, Mediterr. J. Math. 1 (4) (2004) 417–431.
[5] J. Hulshof, E. Mitidieri, R. van der Vorst, Strongly indefinite systems with critical Sobolev exponents, Trans. Amer. Math. Soc. 350 (6) (1998) 2349–2365.
[6] A. Kristály, Detection of arbitrarily many solutions for perturbed elliptic problems involving oscillatory terms, J. Differential Equations 245 (12) (2008) 3849–3868.
[7] E. Mitidieri, A Rellich type identity and applications, Comm. Partial Differential Equations 18 (1–2) (1993) 125–151.
[8] B. Ricceri, A general variational principle and some of its applications, J. Comput. Appl. Math. 113 (2000) 401–410.
[9] B. Ricceri, Infinitely many solutions of the Neumann problem for elliptic equations involving the p-Laplacian, Bull. Lond. Math. Soc. 33 (3) (2001) 331–340.
[10] A. Salvatore, Multiple solutions for elliptic systems with nonlinearities of arbitrary growth, J. Differential Equations 244 (10) (2008) 2529–2544.
[11] R. van der Vorst, Variational identities and applications to differential systems, Arch. Ration. Mech. Anal. 116 (4) (1992) 375–398.