Variational Principles in Mathematical Physics, Geometry, and Economics

Qualitative Analysis of Nonlinear Equations and Unilateral Problems

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Preface

> For since the fabric of the universe is most perfect and the work of a most wise Creator, nothing at all takes place in the universe in which some rule of maximum or minimum does not appear.
>
> Leonhard Euler (1707–1783)

The roots of the calculus of variations go back to the 17th century. Indeed, Johann Bernoulli raised as a challenge the "Brachistochrone Problem" in 1696. The same year, when he heard of this problem, Sir Isaac Newton found that he could not sleep until he had solved it. Having done so, he published the solution anonymously. Bernoulli, however, knew at once that the author of the solution was Newton and, cf. [291], in a famous remark asserted that he "recognized the Lion by the print of its paw".

However, the modern calculus of variations appeared in the middle of the 19th century, as a basic tool in the qualitative analysis of models arising in physics. Indeed, *it was Riemann who aroused great interest in them [problems of the calculus of variations] by proving many interesting results in function theory by assuming Dirichlet's principle* (Charles B. Morrey Jr., [219]). The characterization of phenomena by means of variational principles has been a cornerstone in the transition from classical to contemporary physics. Since the middle part of the twentieth century, the use of variational principles has developed into a range of tools for the study of nonlinear partial differential equations and many problems arising in applications. Cf. Ioffe and Tikhomirov [143], *the term "variational principle" refers essentially to a group of re-*

sults showing that a lower semi-continuous, lower bounded function on a complete metric space possesses arbitrarily small perturbations such that the perturbed function will have an absolute (and even strict) minimum.

This monograph is an original attempt to develop the modern theory of the calculus of variations from the points of view of several disciplines. This theory is one of the twin pillars on which nonlinear functional analysis is built. The authors of this volume are fully aware of the limited achievements of this volume as compared with the task of understanding the force of variational principles in the description of many processes arising in various applications. Even though necessarily limited, the results in this book benefit from many years of work by the authors and from interdisciplinary exchanges between them and other researchers in this field.

One of the main objectives of this book is to let physicists, geometers, engineers, and economists know about some basic mathematical tools from which they might benefit. We would also like to help mathematicians learn what applied calculus of variations is about, so that they can focus their research on problems of real interest to physics, economics, engineering, as well as geometry or other fields of mathematics. We have tried to make the mathematical part accessible to the physicist and economist, and the physical part accessible to the mathematician, without sacrificing rigor in either case. The mathematical technicalities are kept to a minimum within the book, enabling the discussion to be understood by a broad audience. Each problem we develop in this book has its own difficulties. That is why we intend to develop some standard and appropriate methods that are useful and that can be extended to other problems. However, we do our best to restrict the prerequisites to the essential knowledge. We define as few concepts as possible and give only basic theorems that are useful for our topic. The authors use a first-principles approach, developing only the minimum background necessary to justify mathematical concepts and placing mathematical developments in context. The only prerequisite for this volume is a standard graduate course in partial differential equations, drawing especially from linear elliptic equations to elementary variational methods, with a special emphasis on the maximum principle (weak and strong variants). This volume may be used for self-study by advanced graduate students and as a valuable reference for researchers in pure and applied mathematics and related fields. Nevertheless, both the presentation style and the choice of the material make the present book accessible to all newcomers to this modern research field which lies at the interface between pure and applied mathematics.

Each chapter gives full details of the mathematical proofs and subtleties. The book also contains many exercises, some included to clarify simple points of exposition, others to introduce new ideas and techniques, and a few containing relatively deep mathematical results. Each chapter concludes with historical notes. Five appendices illustrate some basic mathematical tools applied in this book: elements of convex analysis, function spaces, category and genus, Clarke and Degiovanni gradients, and elements of set–valued analysis. These auxiliary chapters deal with some analytical methods used in this volume, but also include some complements. This unique presentation should ensure a volume of interest to mathematicians, engineers, economists, and physicists. Although the text is geared toward graduate students at a variety of levels, many of the book's applications will be of interest even to experts in the field.

We are very grateful to Diana Gillooly, Editor for Mathematics, for her efficient and enthusiastic help, as well as for numerous suggestions related to previous versions of this book. Our special thanks go also to Clare Dennison, Assistant Editor for Mathematics and Computer Science, and to the other members of the editorial technical staff of Cambridge University Press for the excellent quality of their work.

Our vision throughout this volume is closely inspired by the following prophetic words of Henri Poincaré on the role of partial differential equations in the development of other fields of mathematics and in applications: *A wide variety of physically significant problems arising in very different areas (such as electricity, hydrodynamics, heat, magnetism, optics, elasticity, etc...) have a family resemblance and should be treated by common methods.* (Henri Poincaré, [245]).

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PART I

Variational Principles in Mathematical Physics

1
Variational Principles

> A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator the smaller the fraction.
>
> Leo Tolstoy (1828–1910)

Variational principles are very powerful techniques at the interplay between nonlinear analysis, calculus of variations, and mathematical physics. They have been inspired and have important applications in modern research fields such as geometrical analysis, constructive quantum field theory, gauge theory, superconductivity, etc.

In this chapter we shortly recall the main variational principles which will be used in the sequel, as Ekeland and Borwein-Preiss variational principles, minimax- and minimization-type principles (mountain pass theorem, Ricceri-type multiplicity theorems, Brézis-Nirenberg minimization technique), the principle of symmetric criticality for non-smooth Szulkin-type functionals, as well as the Pohozaev's fibering method.

1.1 Minimization techniques and Ekeland variational principle

Many phenomena arising in applications (such as geodesics or minimal surfaces) can be understood in terms of the minimization of an energy functional over an appropriate class of objects. For the problems of mathematical physics, phase transitions, elastic instability, and diffraction of the light are among the phenomena that can be studied from this point of view.

A central problem in many nonlinear phenomena is if a bounded from below and lower semi-continuous functional f attains its infimum. A simple function when the above statement clearly fails is $f : \mathbb{R} \to \mathbb{R}$ defined by $f(s) = e^{-s}$. Nevertheless, further assumptions either on f or on its domain may give a satisfactory answer. In the sequel, we give two useful forms of the well-known Weierstrass theorem.

Theorem 1.1 [Minimization; compact case] *Let X be a compact topological space and $f : X \to \mathbb{R}\cup\{+\infty\}$ a lower semi-continuous functional. Then f is bounded from below and its infimum is attained on X.*

Proof The set X can be covered by the open family of sets $S_n := \{u \in X : f(u) > -n\}$, $n \in \mathbb{N}$. Since X is compact, there exists a finite number of sets $S_{n_0}, \ldots, S_{n_l}$ which also cover X. Consequently, $f(u) > -\max\{n_0, \ldots, n_l\}$ for all $u \in X$.

Let $s = \inf_X f > -\infty$. Arguing by contradiction, we assume that s is not achieved which means in particular that $X = \cup_{n=1}^{\infty}\{u \in X : f(u) > s+1/n\}$. Due to the compactness of X, there exists a number $n_0 \in \mathbb{N}$ such that $X = \cup_{n=1}^{n_0}\{u \in X : f(u) > s + 1/n\}$. In particular, $f(u) > s + 1/n_1$ for all $u \in X$ which is in contradiction with $s = \inf_X f > -\infty$. □

The following result is a very useful tool in the study of various partial differential equations where no compactness is assumed on the domain of the functional.

Theorem 1.2 [Minimization; noncompact case] *Let X be a reflexive Banach space, M be a weakly closed, bounded subset of X, and $f : M \to \mathbb{R}$ be a sequentially weak lower semi-continuous function. Then f is bounded from below and its infimum is attained on M.*

Proof We argue by contradiction, that is, we assume that f is not bounded from below on M. Then, for every $n \in \mathbb{N}$ there exists $u_n \in M$ such that $f(u_n) < -n$. Since M is bounded, the sequence $\{u_n\} \subset M$ is so. Due to the reflexivity of X, one may subtract a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ which weakly converges to an element $\tilde{x} \in X$. Since M is weakly closed, $\tilde{x} \in M$. Since $f : M \to \mathbb{R}$ is sequentially weak lower semi-continuous, we obtain that $f(\tilde{x}) \leq \liminf_{k\to\infty} f(u_{n_k}) = -\infty$, contradiction. Therefore, f is bounded from below.

Let $\{u_n\} \subset M$ be a minimizing sequence of f over M, that is, $\lim_{n\to\infty} f(u_n) = \inf_M f > -\infty$. As before, there is a subsequence

$\{u_{n_k}\}$ of $\{u_n\}$ which weakly converges to an element $\overline{x} \in M$. Due to the sequentially weak lower semi-continuity of f, we have that $f(\overline{x}) \leq \liminf_{k\to\infty} f(u_{n_k}) = \inf_M f$, which concludes the proof. $\square$

For any bounded from below, lower semi-continuous functional f, Ekeland's variational principle provides a minimizing sequence whose elements minimize an appropriate sequence of perturbations of f which converges locally uniformly to f. Roughly speaking, Ekeland's variational principle states that there exist points which are almost points of minima and where the "gradient" is small. In particular, it is not always possible to minimize a nonnegative continuous function on a complete metric space. Ekeland's variational principle is a very basic tool that is effective in numerous situations, which led to many new results and strengthened a series of known results in various fields of analysis, geometry, the Hamilton-Jacobi theory, extremal problems, the Ljusternik-Schnirelmann theory, etc.

Its precise statement is as follows.

Theorem 1.3 [Ekeland's variational principle] *Let (X, d) be a complete metric space and let $f : X \to \mathbb{R} \cup \{\infty\}$ be a lower semi-continuous, bounded from below functional with $D(f) = \{u \in X : f(u) < \infty\} \neq \emptyset$. Then for every $\varepsilon > 0$, $\lambda > 0$, and $u \in X$ such that*

$$f(u) \leq \inf_X f + \varepsilon$$

there exists an element $v \in X$ such that

a) $f(v) \leq f(u)$;
b) $d(v, u) \leq \frac{1}{\lambda}$;
c) $f(w) > f(v) - \varepsilon\lambda d(w, v)$ *for each* $w \in X \setminus \{v\}$.

Proof It is sufficient to prove our assertion for $\lambda = 1$. The general case is obtained by replacing d by an equivalent metric λd. We define the relation on X:

$$w \leq v \Longleftrightarrow f(w) + \varepsilon d(v, w) \leq f(v).$$

It is easy to see that this relation defines a partial ordering on X. We now construct inductively a sequence $\{u_n\} \subset X$ as follows: $u_0 = u$, and assuming that u_n has been defined, we set

$$S_n = \{w \in X : w \leq u_n\}$$

and choose $u_{n+1} \in S_n$ so that

$$f(u_{n+1}) \leq \inf_{S_n} f + \frac{1}{n+1}.$$

Since $u_{n+1} \leq u_n$ then $S_{n+1} \subset S_n$ and by the lower semi-continuity of f, S_n is closed. We now show that $\mathrm{diam} S_n \to 0$. Indeed, if $w \in S_{n+1}$, then $w \leq u_{n+1} \leq u_n$ and consequently

$$\varepsilon d(w, u_{n+1}) \leq f(u_{n+1}) - f(w) \leq \inf_{S_n} f + \frac{1}{n+1} - \inf_{S_n} f = \frac{1}{n+1}.$$

This estimate implies that

$$\mathrm{diam} S_{n+1} \leq \frac{2}{\varepsilon(n+1)}$$

and our claim follows. The fact that X is complete implies that $\bigcap_{n\geq 0} S_n = \{v\}$ for some $v \in X$. In particular, $v \in S_0$, that is, $v \leq u_0 = u$ and hence

$$f(v) \leq f(u) - \varepsilon d(u,v) \leq f(u)$$

and moreover

$$d(u,v) \leq \frac{1}{\varepsilon}(f(u) - f(v)) \leq \frac{1}{\varepsilon}(\inf_X f + \varepsilon - \inf_X f) = 1.$$

Now, let $w \neq v$. To complete the proof we must show that $w \leq v$ implies $w = v$. If $w \leq v$, then $w \leq u_n$ for each integer $n \geq 0$, that is $w \in \cap_{n\geq 0} S_n = \{v\}$. So, $w \not\leq v$, which is actually c). □

In $\mathbb{R}^N$ with the Euclidean metric, properties a) and c) in the statement of Ekeland's variational principle are completely intuitive as Figure 1.1 shows. Indeed, assuming that $\lambda = 1$, let us consider a cone lying below the graph of f, with slope $+1$, and vertex projecting onto u. We move up this cone until it first touches the graph of f at some point $(v, f(v))$. Then the point v satisfies both a) and c).

In the particular case $X = \mathbb{R}^N$ we can give the following simple alternative proof to Ekeland's variational principle, due to Hiriart-Urruty, [139]. Indeed, consider the perturbed functional

$$g(w) := f(w) + \varepsilon\lambda \|w - u\|, \qquad w \in \mathbb{R}^N.$$

Since f is lower semi-continuous and bounded from below, then g is lower semi-continuous and $\lim_{\|w\|\to\infty} g(w) = +\infty$. Therefore there exists $v \in \mathbb{R}^N$ minimizing g on $\mathbb{R}^N$ such that for all $w \in \mathbb{R}^N$

$$f(v) + \varepsilon\lambda \|v - u\| \leq f(w) + \varepsilon\lambda \|w - u\|. \tag{1.1}$$

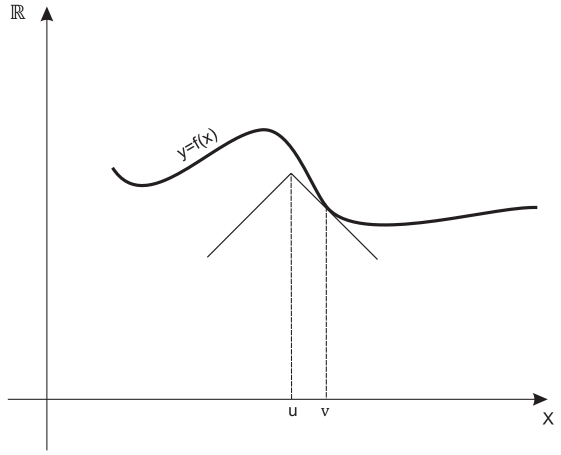

Fig. 1.1. Geometric illustration of Ekeland's variational principle.

By letting $w = u$ we find

$$f(v) + \varepsilon\lambda \|v - u\| \leq f(u)$$

and a) follows. Now, since $f(u) \leq \inf_{\mathbb{R}^N} f + \varepsilon$, we also deduce that $\|v - u\| \leq 1/\lambda$.

We infer from relation (1.1) that for any w,

$$f(v) \leq f(w) + \varepsilon\lambda \left[\|w - u\| - \|v - u\|\right] \leq f(w) + \varepsilon\lambda \|w - u\|,$$

which is the desired inequality c).

Taking $\lambda = \dfrac{1}{\sqrt{\varepsilon}}$ in the above theorem we obtain the following property.

Corollary 1.1 *Let (X, d) be a complete metric space and let $f : X \to \mathbb{R} \cup \{\infty\}$ be a lower semi-continuous, bounded from below and $D(f) = \{u \in X : f(u) < \infty\} \neq \emptyset$. Then for every $\varepsilon > 0$ and every $u \in X$ such that*

$$f(u) \leq \inf_X f + \varepsilon$$

there exists an element $u_\varepsilon \in X$ such that

a) $f(u_\varepsilon) \leq f(u)$;
b) $d(u_\varepsilon, u) \leq \sqrt{\varepsilon}$;
c) $f(w) > f(u_\varepsilon) - \sqrt{\varepsilon}\, d(w, u_\varepsilon)$ *for each* $w \in X \setminus \{u_\varepsilon\}$.

Let $(X, \|\cdot\|)$ be a real Banach space, $X^\star$ its topological dual endowed with its natural norm, denoted for simplicity also by $\|\cdot\|$. We denote by $\langle\cdot,\cdot\rangle$ the duality mapping between X and $X^\star$, that is, $\langle x^\star, u\rangle = x^\star(u)$ for every $x^\star \in X^\star, u \in X$. Theorem 1.3 readily implies the following property, which asserts the existence of *almost critical points.* In order words, Ekeland's variational principle can be viewed as a generalization of the Fermat theorem which establishes that interior extrema points of a smooth functional are, necessarily, critical points of this functional.

Corollary 1.2 *Let X be a Banach space and let $f : X \to \mathbb{R}$ be a lower semi-continuous functional which is bounded from below. Assume that f is Gâteaux differentiable at every point of X. Then for every $\varepsilon > 0$ there exists an element $u_\varepsilon \in X$ such that*

(i) $f(u_\varepsilon) \le \inf\limits_X f + \varepsilon$;
(ii) $\|f'(u_\varepsilon)\| \le \varepsilon$.

Letting $\varepsilon = 1/n$, $n \in \mathbb{N}$, Corollary 1.2 gives rise a minimizing sequence for the infimum of a given function which is bounded from below. Note however that such a sequence need not converge to any point. Indeed, let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(s) = e^{-s}$. Then, $\inf_{\mathbb{R}} f = 0$, and any minimizing sequence fulfilling (a) and (b) from Corollary 1.2 tends to $+\infty$. The following definition is dedicated to handle such situations.

Definition 1.1 (a) A function $f \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ (shortly, $(PS)_c$-condition) if every sequence $\{u_n\} \subset X$ such that $\lim_{n\to\infty} f(u_n) = c$ and $\lim_{n\to\infty} \|f'(u_n)\| = 0$, possesses a convergent subsequence.

(b) A function $f \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition (shortly, (PS)-condition) if it satisfies the Palais-Smale condition at every level $c \in \mathbb{R}$.

Combining this compactness condition with Corollary 1.2, we obtain the following result.

Theorem 1.4 *Let X be a Banach space and a function $f \in C^1(X, \mathbb{R})$ which is bounded from below. If f satisfies the $(PS)_c$-condition at level $c = \inf_X f$, then c is a critical value of f, that is, there exists a point $u_0 \in X$ such that $f(u_0) = c$ and u_0 is a critical point of f, that is, $f'(u_0) = 0$.*

1.2 Borwein-Preiss variational principle

The Borwein-Preiss variational principle [44] is an important tool in infinite dimensional nonsmooth analysis. This basic result is strongly related with Stegall's variational principle [278], *smooth bumps* on Banach spaces, Smulyan's test describing the relationship between Fréchet differentiability and the strong extremum, properties of continuous convex functions on separable Asplund spaces, variational characterizations of Banach spaces, the Bishop-Phelps theorem or Phelps' lemma [239]. The generalized version we present here is due to Loewen and Wang [195] and enables us to deduce the standard form of the Borwein-Preiss variational principle, as well as other related results.

Let X be a Banach space and assume that $\rho : X \to [0, \infty)$ is a continuous function satisfying

$$\rho(0) = 0 \quad \text{and} \quad \rho_M := \sup\{\|x\|;\ \rho(x) < 1\} < +\infty\,. \tag{1.2}$$

An example of function with these properties is $\rho(x) = \|x\|^p$ with $p > 0$.

Given the families of real numbers $\mu_n \in (0, 1)$ and vectors $e_n \in X$ ($n \geq 0$), we associate to ρ the *penalty function* ρ_∞ defined for all $x \in X$,

$$\rho_\infty(x) = \sum_{n=0}^{\infty} \rho_n(x - e_n), \quad \text{where } \rho_n(x) := \mu_n \rho((n+1)x)\,. \tag{1.3}$$

Definition 1.2 For the function $f : X \to \mathbb{R} \cup \{+\infty\}$, a point $x_0 \in X$ is a strong minimizer if $f(x_0) = \inf_X f$ and every minimizing sequence (z_n) of f satisfies $\|z_n - x_0\| \to 0$ as $n \to \infty$.

We observe that any strong minimizer of f is, in fact, a strict minimizer, that is $f(x) > f(x_0)$ for all $x \in X \setminus \{x_0\}$. The converse is true, as shown by $f(x) = x^2 e^x$, $x \in \mathbb{R}$, $x_0 = 0$.

The generalized version of the Borwein-Preiss variational principle due to Loewen and Wang is the following.

Theorem 1.5 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function. Assume that $x_0 \in X$ and $\varepsilon > 0$ satisfy*

$$f(x_0) < \varepsilon + \inf_X f\,.$$

Let $(\mu_n)_{n \geq 0}$ be a decreasing sequence in $(0, 1)$ such that the series $\sum_{n=0}^{\infty} \mu_n$ is convergent. Then for any continuous function ρ satisfying (1.2), there exists a sequence $(e_n)_{n \geq 0}$ in X converging to e such that

(i) $\rho(x_0 - e) < 1$;
(ii) $f(e) + \varepsilon\rho_\infty(e) \le f(x_0)$;
(iii) *e is a strong minimizer of $f + \varepsilon\rho_\infty$. In particular, e is a strict minimizer of $f + \varepsilon\rho_\infty$, that is,*

$$f(e) + \varepsilon\rho_\infty(e) < f(x) + \varepsilon\rho_\infty(x) \qquad \text{for all } x \in X \setminus \{e\}.$$

Proof Define the sequence $(f_n)_{n\ge 0}$ such that $f_0 = f$ and for any $n \ge 0$,

$$f_{n+1}(x) := f_n(x) + \varepsilon\rho_n(x - e_n)\,.$$

Then $f_n \le f_{n+1}$ and f_n is lower semi-continuous.

Set $e_0 = x_0$. We observe that for any $n \ge 0$,

$$\inf_X f_{n+1} \le f_{n+1}(e_n) = f_n(e_n)\,. \tag{1.4}$$

If this inequality is strict, then there exists $e_{n+1} \in X$ such that

$$f_{n+1}(e_{n+1}) \le \frac{\mu_{n+1}}{2}\, f_n(e_n) + \left(1 - \frac{\mu_{n+1}}{2}\right) \inf_X f_{n+1} \le f_n(e_n)\,. \tag{1.5}$$

If equality holds in relation (1.4) then (1.5) also holds, but for e_{n+1} replaced with e_n. Consequently, there exists a sequence $(e_n)_{n\ge 0}$ in X such that relation (1.5) holds true.

Set

$$D_n := \left\{x \in X;\ f_{n+1}(x) \le f_{n+1}(e_{n+1}) + \frac{\varepsilon\mu_n}{2}\right\}\,.$$

Then D_n is not empty, since $e_{n+1} \in D_n$. By the lower semi-continuity of functions f_n we also deduce that D_n is a closed set. Since $\mu_{n+1} \in (0,1)$, relation (1.5) implies

$$\begin{aligned} f_{n+1}(e_{n+1}) - \inf_X f_{n+1} &\le \frac{\mu_{n+1}}{2}\left[f_n(e_n) - \inf_X f_{n+1}\right] \\ &\le f_n(e_n) - \inf_X f_n\,. \end{aligned} \tag{1.6}$$

We also observe that

$$f_0(e_0) - \inf_X f_0 = f(x_0) - \inf_X f < \varepsilon\,.$$

Next, we prove that

$$\text{the sequence } (D_n)_{n\ge 0} \text{ is decreasing} \tag{1.7}$$

and

$$\operatorname{diam}(D_n) \to 0 \quad \text{as } n \to \infty. \tag{1.8}$$

In order to prove (1.7), assume that $x \in D_n$, $n \geq 1$. Since the sequence $(\mu_n)_{n\geq 0}$ is decreasing, relation (1.5) implies

$$f_n(x) \leq f_{n+1}(x) \leq f_{n+1}(e_{n+1}) + \frac{\varepsilon\mu_n}{2} \leq f_n(e_n) + \frac{\varepsilon\mu_{n-1}}{2}\,,$$

hence $x \in D_{n-1}$.

Since $f_n \geq f_{n-1}$, relations (1.5) and (1.6) imply

$$\begin{aligned} f_n(e_n) - \inf_X f_n \quad & \leq \frac{\mu_n}{2}\,[f_{n-1}(e_{n-1}) - \inf_X f_n] \\ & \leq \frac{\mu_n}{2}\,[f_{n-1}(e_{n-1}) - \inf_X f_{n-1}] < \frac{\varepsilon\mu_n}{2}\,. \end{aligned} \tag{1.9}$$

For any $x \in D_n$, combining relation (1.9) and the definitions of f_{n+1} and D_n we obtain

$$\begin{aligned} \varepsilon\mu_n\rho((n+1)(x-e_n)) \quad & \leq f_{n+1}(e_{n+1}) - f_n(x) + \frac{\varepsilon\mu_n}{2} \\ & \leq f_{n+1}(e_{n+1}) - \inf_X f_n + \frac{\varepsilon\mu_n}{2} \\ & \leq f_n(e_n) - \inf_X f_n + \frac{\varepsilon\mu_n}{2} < \varepsilon\mu_n\,. \end{aligned} \tag{1.10}$$

Therefore $\rho((n+1)(x-e_n)) < 1$. So, by (1.2),

$$(n+1)\|x-e_n\| \leq \rho_M\,,$$

which shows that $\operatorname{diam}(D_n) \leq 2\rho_M/(n+1)$. This implies (1.8).

Since D_n is a closed set for any $n \geq 1$, then (1.7) and (1.8) imply that $\cap_{n=1}^{\infty} D_n$ contains a single point, denoted by e. Then $e_n \to e$ as $n \to \infty$. Thus, using $\rho((n+1)(x-e_n)) < 1$ for all $n \geq 0$ and $x \in X$, we deduce that $\rho(x_0 - e) < 1$.

Since the sequence $(f_n(e_n))_{n\geq 0}$ is nonincreasing and $f_0(e_0) = f(x_0)$ it follows that, in order to prove (ii), it is enough to deduce that

$$f(e) + \varepsilon\rho_\infty(e) \leq f_n(e_n)\,. \tag{1.11}$$

For this purpose we define the nonempty closed sets

$$C_n := \{x \in X;\ f_{n+1}(x) \leq f_{n+1}(e_{n+1})\}\,.$$

Since $f_n \leq f_{n+1}$ and $f_n(e_n) \geq f_{n+1}(e_{n+1})$ for all n, it follows that the sequence $(C_n)_{n\geq 0}$ is nested and $C_n \subset D_n$ for all n. Therefore $\cap_{n=0}^{\infty} C_n = \{e\}$ and

$$f_m(e) \leq f_m(e_m) \leq f_n(e_n) \leq f(x_0) \qquad \text{provided that } m > n. \tag{1.12}$$

Taking $m \to \infty$ we obtain (1.11).

It remains to argue that e is a strong minimizer of $f_\varepsilon := f + \varepsilon\rho_\infty$. Since

$$f_\varepsilon(x) \le \inf_X f_\varepsilon + \frac{\varepsilon\mu_n}{2}\,,$$

relation (1.12) yields

$$f_{n+1}(x) \le f_\varepsilon(x) \le f_\varepsilon(e) + \frac{\varepsilon\mu_n}{2} \le f_{n+1}(e_{n+1}) + \frac{\varepsilon\mu_n}{2}\,.$$

Setting

$$A_n := \left\{x \in X;\ f_\varepsilon(x) \le \frac{\varepsilon\mu_n}{2} + \inf_X f\right\}\,,$$

the above relation shows that $A_n \subset D_n$. So, by (1.8), we deduce that $\operatorname{diam}(A_n)\to 0$ as $n\to\infty$ which shows that e is a strong minimizer of $f_\varepsilon := f + \varepsilon\rho_\infty$. $\square$

Assume that $p \ge 1$ and $\lambda > 0$. Taking

$$\rho(x) = \frac{\|x\|^p}{\lambda^p} \qquad \text{and} \qquad \mu_n = \frac{1}{2^{n+1}(n+1)}$$

we obtain the initial smooth version of the Borwein-Preiss variational principle. Roughly speaking, it asserts that the Lipschitz perturbations obtained in Ekeland's variational principle can be replaced by *superlinear perturbations* in a certain class of admissible functions.

Theorem 1.6 *Given $f : X\to\mathbb{R}\cup\{+\infty\}$ lower semi-continuous function, $x_0 \in X$, $\varepsilon > 0$, $\lambda > 0$, and $p \ge 1$, suppose*

$$f(x_0) < \varepsilon + \inf_X f\,.$$

Then there exists a sequence $(\mu_n)_{n\ge 0}$ with $\mu_n \ge 0$, $\sum_{n=0}^\infty \mu_n = 1$, and a point e in X, expressible as the limit of some sequence (e_n), such that for all $x \in X$,

$$f(x) + \frac{\varepsilon}{\lambda^p}\sum_{n=0}^{\infty}\mu_n\,\|x - e_n\|^p \ge f(e) + \frac{\varepsilon}{\lambda^p}\sum_{n=0}^{\infty}\mu_n\,\|e - e_n\|^p\,.$$

Moreover, $\|x_0 - e\| < \lambda$ and $f(e) \le \varepsilon + \inf_X f$.

We have seen in Corollary 1.2 of Ekeland's variational principle that any smooth bounded from below functional on a Banach space admits a sequence of “almost critical points”. The next consequence of Borwein-Preiss' variational principle asserts that, in the framework of Hilbert spaces, such a functional admits a sequence of *stable* “almost critical points”.

Corollary 1.3 *Let f be a real-valued C^2-functional that is bounded from below on a Hilbert space X. Assume that (u_n) is a minimizing sequence of f. Then there exists a minimizing sequence (v_n) of f such that the following properties hold true:*

(i) $\lim_{n\to\infty} \|u_n - v_n\| = 0$;
(ii) $\lim_{n\to\infty} \|f'(v_n)\| = 0$;
(iii) $\liminf_{n\to\infty} \langle f''(v_n)w, w\rangle \geq 0$ *for any* $w \in X$.

1.3 Minimax principles

In this section we are interested in some powerful techniques for finding solutions of some classes of stationary nonlinear boundary value problems. These solutions are viewed as critical points of a natural functional, often called the *energy* associated to the system. The critical points obtained in this section by means of topological techniques are generally *nonstable* critical points which are neither maxima nor minima of the energy functional.

1.3.1 Mountain pass type results

In many nonlinear problems we are interested in finding solutions as stationary points of some associated "energy" functionals. Often such a mapping is unbounded from above and below, so that it has no maximum or minimum. This forces us to look for saddle points, which are obtained by minimax arguments. In such a case one maximizes a functional f over a closed set A belonging to some family Γ of sets and then one minimizes with respect to the set A in the family. Thus we define

$$c = \inf_{A\in\Gamma} \sup_{u\in A} f(u) \tag{1.13}$$

and one tries to prove, under various hypotheses, that this number c is a critical value of f, hence there is a point u such that $f(u) = c$ and $f'(u) = 0$. Indeed, it seems intuitively obvious that c defined in (1.13) is a critical value of f. However, this is not true in general, as showed by the following example in the plane: let $f(x, y) = x^2 - (x-1)^3y^2$. Then $(0, 0)$ is the only critical point of f but c is not a critical value. Indeed, looking for sets A lying in a small neighborhood of the origin, then $c > 0$. This example shows that the heart of the matter is to find appropriate conditions on f and on the family Γ.

One of the most important minimax results is the so-called *mountain* *pass theorem* whose geometrical interpretation will be roughly described in the sequel. Denote by f the function which measures the altitude of a mountain terrain and assume that there are two points in the horizontal plane L_1 and L_2, representing the coordinates of two locations such that $f(L_1)$ and $f(L_2)$ are the deepest points of two separated valleys. Roughly speaking, our aim is to walk along an optimal path on the mountain from the point $(L_1, f(L_1))$ to $(L_2, f(L_2))$ spending the least amount of energy by passing the mountain ridge between the two valleys. Walking on a path $(\gamma, f(\gamma))$ from $(L_1, f(L_1))$ to $(L_2, f(L_2))$ such that the maximal altitude along γ is the smallest among all such continuous paths connecting $(L_1, f(L_1))$ and $(L_2, f(L_2))$, we reach a point L on γ passing the ridge of the mountain which is called a *mountain pass point*. As pointed out by Brezis and Browder [47], the mountain pass theorem "extends ideas already present in Poincaré and Birkhoff".

In the sequel, we give a first formulation of the mountain pass theorem.

Theorem 1.7 [Mountain pass theorem; positive altitude] *Let X be a Banach space, $f : X \to \mathbb{R}$ be a function of class C^1 such that*

$$\inf_{\|u-e_0\|=\rho} f(u) \geq \alpha > \max\{f(e_0), f(e_1)\}$$

for some $\alpha \in \mathbb{R}$ and $e_0 \neq e_1 \in X$ with $0 < \rho < \|e_0 - e_1\|$. If f satisfies the $(PS)_c$-condition at level

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = e_0,\ \gamma(1) = e_1\},$$

then c is a critical value of f with $c \geq \alpha$.

There are two different ways to prove Theorem 1.7; namely, via Ekeland's variational principle, or by using a sort of deformation lemma. We present its proof by means of the latter argument. We refer to Figure 1.3.1 for a geometric illustration of Theorem 1.7.

Proof of Theorem 1.7. We may assume that $e_0 = 0$ and let $e := e_1$. Since $f(u) \geq \alpha$ for every $u \in X$ with $\|u\| = \rho$ and $\rho < \|e\|$, the definition of the number c shows that $\alpha \leq c$.

It remains to prove that c is a critical value of f. Arguing by contradiction, assume $K_c = \emptyset$. Thus, on account of Remark D.1, we may choose $\mathcal{O} = \emptyset$ in Theorem D.1, see Appendix D, and $\bar{\varepsilon} > 0$ such that $\bar{\varepsilon} < \min\{\alpha - f(0), \alpha - f(e)\}$. Consequently, there exist $\varepsilon > 0$ and a

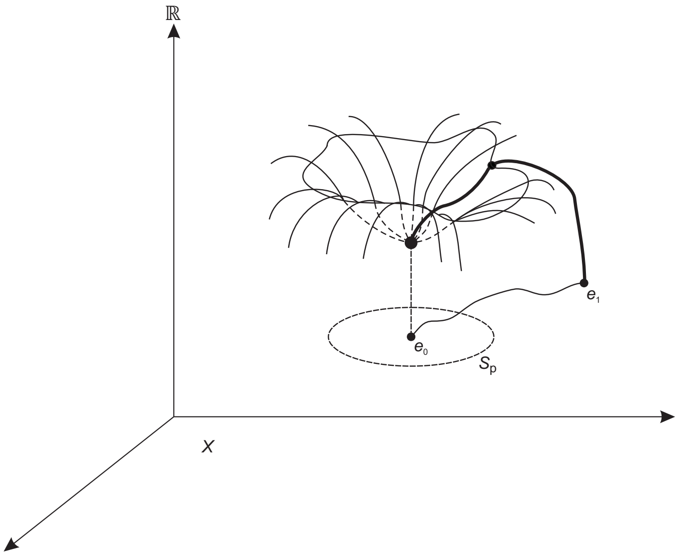

Fig. 1.2. Mountain pass landscape between "villages" e_0 and e_1.

continuous map $\eta : X \times [0,1] \to X$ verifying the properties (i)-(iv) from Theorem D.1.

From the definition of the number c, there exists $\gamma \in \Gamma$ such that

$$\max_{t\in[0,1]} f(\gamma(t)) \leq c + \varepsilon. \tag{1.14}$$

Let $\gamma_1 : [0,1] \to X$ defined by $\gamma_1(t) = \eta(\gamma(t), 1)$, $\forall t \in [0,1]$.

We prove that $\gamma_1 \in \Gamma$. The choice of $\overline{\varepsilon} > 0$ gives that $\max\{f(0), f(e)\} < \alpha - \overline{\varepsilon} \leq c - \overline{\varepsilon}$, thus $0, e \notin f^{-1}(]c - \overline{\varepsilon}, c + \overline{\varepsilon}[)$. Consequently, due to Theorem D.1 (iii), we have $\gamma_1(0) = \eta(\gamma(0), 1) = \eta(0,1) = 0$ and $\gamma_1(1) = \eta(\gamma(1), 1) = \eta(e, 1) = e$. Therefore, $\gamma_1 \in \Gamma$.

Note that (1.14) means that $\gamma(t) \in f^{c+\varepsilon} = \{u \in X : f(u) \leq c+\varepsilon\}$ for all $t \in [0,1]$. Then by Theorem D.1 (iv) and the definition of γ_1 we have

$$c \leq \max_{t\in[0,1]} f(\gamma_1(t)) = \max_{t\in[0,1]} f(\eta(\gamma(t), 1)) \leq c - \varepsilon,$$

contradiction.

Remark 1.1 Using Theorem D.1, we can provide an alternative proof to Theorem 1.4. Indeed, if we suppose that $K_c = \emptyset$, with $c = \inf_X f > -\infty$, one may deform continuously the level set $f^{c+\varepsilon}$ (for $\varepsilon > 0$ small enough) into a subset of $f^{c-\varepsilon}$, see Theorem D.1 (iv). But, $f^{c-\varepsilon} = \emptyset$, contradiction.

Remark 1.2 Note that the choice of $0 < \overline{\varepsilon} < \min\{\alpha - f(0), \alpha - f(e)\}$ with $\inf_{\|u\|=\rho} f(u) \geq \alpha$ is crucial in the proof of Theorem 1.7. Actually, it means that the ridge of the mountain between the two valleys has a positive altitude. However, a more involved proof allows us to handle the so-called "zero altitude" case. More precisely, the following result holds.

Theorem 1.8 [Mountain pass theorem; zero altitude] *Let X be a Banach space, $f : X \to \mathbb{R}$ be a function of class C^1 such that*

$$\inf_{\|u-e_0\|=\rho} f(u) \geq \max\{f(e_0), f(e_1)\}$$

for some $e_0 \neq e_1 \in X$ with $0 < \rho < \|e_0 - e_1\|$. If f satisfies the $(PS)_c$-condition at level

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = e_0,\ \gamma(1) = e_1\},$$

then c is a critical value of f with $c \geq \max\{f(e_0), f(e_1)\}$.

Remark 1.3 In both Theorems 1.7 and 1.8 the Palais-Smale condition may be replaced by a weaker one, called the *Cerami compactness condition.* More precisely, a function $f \in C^1(X, \mathbb{R})$ satisfies the *Cerami condition at level $c \in \mathbb{R}$* (shortly, $(C)_c$-condition) if every sequence $\{u_n\} \subset X$ such that $\lim_{n\to\infty} f(u_n) = c$ and $\lim_{n\to\infty}(1 + \|u_n\|)\|f'(u_n)\| = 0$, possesses a convergent subsequence.

1.3.2 Minimax results via Ljusternik-Schnirelmann category

A very useful method to find multiple critical points for a given functional is related to the notion of *Ljusternik-Schnirelmann category.* In the case of variational problems on finite dimensional manifolds, this notion is useful to find a lower bound for the number of critical points in terms of topological invariants. In this subsection we present a general form of this approach. We first present some basic properties of Finsler-Banach manifolds.

Let M be a Banach manifold of class C^1 and let TM be the usual tangent bundle, T_pM the tangent space at the point $p \in M$. A Finsler structure on the Banach manifold M can be introduced as follows.

Definition 1.1 A Finsler structure on TM is a continuous function $\|\cdot\| : TM \to [0, \infty)$ such that

(a) for each $p \in M$ the restriction $\|\cdot\|_p = \|\cdot\||_{T_pM}$ is a norm on T_pM.

(b) for each $p_0 \in M$ and $k > 1$, there is a neighborhood $U \subset M$ of p_0 such that

$$\frac{1}{k}\|\cdot\|_p \leq \|\cdot\|_{p_0} \leq k\|\cdot\|_p$$

for all $p \in U$.

The Banach manifold M of class C^1 endowed with a Finsler structure is called Banach-Finsler manifold M of class C^1.

Definition 1.2 Let M be a Banach-Finsler manifold of class C^1 and $\sigma : [a, b] \to M$ a C^1-path. The length of σ is defined by $l(\sigma) = \displaystyle\int_a^b \|\sigma'(t)\|dt$. If p and q are in the same component of M we define the distance from p to q by

$$\rho(p, q) = \inf\{l(\sigma) : \sigma \text{ is a } \ C^1 - \text{path from } p \text{ to } q\}.$$

Definition 1.3 A Banach-Finsler manifold M of class C^1 is said to be complete if the metric space (M, ρ) is complete.

Theorem 1.1 *If M is a Banach-Finsler manifold of class C^1 then the function $\rho : M \times M \to \mathbb{R}$ (from Definition 1.2) is a metric function for each component of M. Moreover, this metric is consistent with the topology of M.*

Theorem 1.2 *If M is a Banach-Finsler manifold of class C^1 and N is a C^1 submanifold of M, then $\|\cdot\| \,|_{T(N)}$ is a Finsler structure for N (called the Finsler structure induced from M). If M is complete and N is a closed C^1 submanifold of M, then N is a complete Banach-Finsler manifold in the induced Finsler structure.*

On a Banach-Finsler manifold M the cotangent bundle $TM^\star$ has a dual Finsler structure defined by

$$\|y\| = \sup\{\langle y, v\rangle : v \in T_pM, \|v\|_p = 1\},$$

where $y \in T_pM^\star$ and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $T_pM^\star$ and T_pM. Consequently, for a functional $f : M \to \mathbb{R}$ of class C^1, the application $p \mapsto \|f'(p)\|$ is well-defined and continuous. These facts allow to introduce the following definition.

Definition 1.3 Let M be a Banach-Finsler manifold of class C^1.

(a) A function $f \in C^1(M, \mathbb{R})$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ (shortly, $(PS)_c$-condition) if every sequence $\{u_n\} \subset M$ such that $\lim_{n\to\infty} f(u_n) = c$ and $\lim_{n\to\infty} \|f'(u_n)\| = 0$, possesses a convergent subsequence.

(b) A function $f \in C^1(M, \mathbb{R})$ satisfies the Palais-Smale condition (shortly, (PS)-condition) if it satisfies the Palais-Smale condition at every level $c \in \mathbb{R}$.

Let M be a Banach-Finsler manifold of class C^1. For $h \geq 1$, we introduce the set

$$\Gamma_h = \{A \subseteq M : \mathrm{cat}_M(A) \geq h,\ A \text{ compact}\}.$$

Here, $\mathrm{cat}_M(A)$ denotes the *Ljusternik-Schnirelmann category* of the set A relative to M, see Appendix C.

If $h \leq \mathrm{cat}_M(M) = \mathrm{cat}(M)$, then $\Gamma_h \neq \emptyset$. Moreover, if $\mathrm{cat}(M) = +\infty$, we may take any positive integer h in the above definition. If $f : M \to \mathbb{R}$ is a functional of class C^1, we denote by K the set of all critical points of f on M, that is, $K = \{u \in M : f'(u) = 0\}$ and $K_c = K \cap f^{-1}(c)$. Finally, for every $h \leq \mathrm{cat}(M)$ we define the number

$$c_h = \inf_{A\in\Gamma_h} \max_{u\in A} f(u).$$

Proposition 1.1 *Let $f : X \to \mathbb{R}$ be a functional of class C^1. Using the above notations, we have*

a) $c_1 = \inf_M f$.

b) $c_h \leq c_{h+1}$.

c) $c_h < \infty$ *for every* $h \leq \mathrm{cat}(M)$.

d) *If $c := c_h = c_{h+m-1}$ for some $h, m \geq 1$, and f satisfies the $(PS)_c$-condition, then $\mathrm{cat}_M(K_c) \geq m$. In particular, $K_c \neq \emptyset$.*

Proof Properties a)-c) are obvious. To prove d), we argue by contradiction. Thus, we assume that $\mathrm{cat}_M(K_c) \leq m - 1$. Since M is an ANR , there exists a neighborhood $\mathcal{O}$ of K_c such that $\mathrm{cat}_M(\overline{\mathcal{O}}) = \mathrm{cat}_M(K_c) \leq m - 1$. By using Theorem D.1, there exists a continuous map $\eta : X \times [0,1] \to X$ and $\varepsilon > 0$ such that $\eta(f^{c+\varepsilon} \setminus \mathcal{O}, 1) \subset f^{c-\varepsilon}$ and $\eta(u,0) = u$ for all $u \in X$. Let $A_1 \in \Gamma_{h+m-1}$ such that $\max_{u \in A_1} f(u) \leq c + \varepsilon$. Considering the set $A_2 = \overline{A_1 \setminus \mathcal{O}}$, we obtain that

$$\mathrm{cat}_M(A_2) \geq \mathrm{cat}_M(A_1) - \mathrm{cat}_M(\overline{\mathcal{O}}) \geq h + m - 1 - (m-1) = h.$$

Therefore, $A_2 \in \Gamma_h$. Moreover, $\max_{u \in A_2} f(u) \leq \max\limits_{u \in A_1} f(u) \leq c + \varepsilon$ and $A_2 \cap \mathcal{O} = \emptyset$. In conclusion, we have $A_2 \subset f^{c+\varepsilon} \setminus \mathcal{O}$. Due to Theorem D.1 (iv), $\eta(A_2, 1) \subset f^{c-\varepsilon}$. Moreover, $\mathrm{cat}_M(\eta(A_2,1)) \geq \mathrm{cat}_M(A_2) \geq h$, thus $\eta(A_2, 1) \in \Gamma_h$. But $c = \inf\limits_{A \in \Gamma_h} \max\limits_{u \in A} f(u) \leq \max f(\eta(A_2,1)) \leq c - \varepsilon$, contradiction. □

The main result of the present subsection is the following theorem.

Theorem 1.9 *Let M be a complete Banach-Finsler manifold of class C^1, and $f : M \to \mathbb{R}$ a functional of class C^1 which is bounded from below on M. If f satisfies the (PS)-condition then f has at least* $\mathrm{cat}(M)$ *critical points.*

Proof Since f is bounded from below, every c_h is finite, $h = 1, \ldots, \mathrm{cat}(M)$, see Proposition 1.1 c). To prove the statement, it is enough to show that

$$\mathrm{card}(K \cap f^{c_h}) \geq h, \quad h = 1, \ldots, \mathrm{cat}(M). \tag{1.15}$$

We proceed by induction. For $h = 1$, relation (1.15) is obvious since the global minimum belongs to K. Now, we assume that (1.15) holds for $h = 1, \ldots, k$. We will prove that (1.15) also holds for $k+1$. There are two cases:

I) $c_k \neq c_{k+1}$. Due to Proposition 1.1 d), we have that $K_{c_{k+1}} \neq \emptyset$. Consequently, each element of the set $K_{c_{k+1}}$ clearly differs from those of the set $K \cap f^{c_k}$. Therefore, $K \cap f^{c_{k+1}}$ contains at least $k+1$ points.

II) $c_k = c_{k+1} := c$. Let m be the least positive integer such that $c_m = c_{k+1}$. Due to Proposition 1.1 d), we have $\mathrm{cat}_M(K_c) \geq k + 1 - m + 1 = k - m + 2$. In particular, $\mathrm{card}(K_c) = \mathrm{card}(K_{c_{k+1}}) \geq k - m + 2$. Two distinguish again two subcases:

IIa) $m = 1$. The claim easily follows.

IIb) $m > 1$. Since $m \leq k$, then $\mathrm{card}(K \cap f^{c_{m-1}}) \geq m - 1$. Consequently,

$$\begin{aligned} \mathrm{card}(K \cap f^{c_{k+1}}) &\geq \mathrm{card}(K \cap f^{c_{m-1}}) + \mathrm{card}(K_{c_{k+1}}) \\ &\geq (m-1) + (k - m + 2) \\ &= k + 1, \end{aligned}$$

which complete the proof of (1.15), thus the theorem. □

Let X be an infinite dimensional, separable real Banach space and $S = \{u \in X : \|u\| = 1\}$ its unit sphere. Important questions appear when the study of some classes of nonlinear elliptic problems reduce to finding critical points of a given functional on S. Note that $\mathrm{cat}(S) = 1$; therefore, Theorem 1.9 does not give multiple critical points on S. However, exploiting the symmetric property of S, by means of the *Krasnoselski genus*, see Appendix C, a multiplicity result may be given, similar to Theorem 1.9. In order to state this result, we put ourselves to a general framework.

Let X be a real Banach space, and we denote the family of sets

$$\mathcal{A} = \{A \subset X \setminus \{0\} : A = -A,\ A \text{ is closed}\}.$$

Assume that $M = G^{-1}(0) \in \mathcal{A}$ is a submanifold of X, where $G : X \to \mathbb{R}$ is of class C^1 with $G'(u) \neq 0$ for every $u \in M$.

For any $h \leq \gamma_0(M) := \sup\{\gamma(K) : K \subseteq M, K \in \mathcal{A},\ K \text{ compact}\}$, we introduce the set

$$\mathcal{A}_h = \{A \subseteq M : A \in \mathcal{A}, \gamma(A) \geq h,\ A \text{ compact}\}.$$

Here, $\gamma(A)$ denotes the Krasnoselski genus of the set A, see Appendix C. Finally, if $f : X \to \mathbb{R}$ is an even function, we denote

$$\tilde{c}_h = \inf_{A \in \mathcal{A}_h} \max_{u \in A} f(u).$$

Proposition 1.2 *Let $f : X \to \mathbb{R}$ be an even functional of class C^1. Using the above notations, we have*

a) *$\tilde{c}_1 > -\infty$ whenever f is bounded from below on M.*

b) *$\tilde{c}_h \leq \tilde{c}_{h+1}$.*

c) *$\tilde{c}_h < \infty$ for every $h \leq \gamma_0(M)$.*

d) *If $c := \tilde{c}_h = \tilde{c}_{h+m-1}$ for some $h, m \geq 1$, and f satisfies the $(PS)_c$-condition on M, then $\gamma(K_c) \geq m$. In particular, $K_c \neq \emptyset$.*

The following result may be stated.

Theorem 1.10 *Let $M = G^{-1}(0) \subset X$ be a submanifold of a real Banach space, where $G : X \to \mathbb{R}$ is of class C^1 with $G'(u) \neq 0$ for every $u \in M$. Assume that $M \in \mathcal{A}$ and let $f : X \to \mathbb{R}$ be a functional of class C^1 which is bounded from below on M and even. If $\gamma(M) = +\infty$ and f satisfies the (PS)-condition on M then f has infinitely many critical points $\{u_k\} \subset M$ with $\lim_{k\to\infty} f(u_k) = \sup_M f$.*

1.4 Ricceri's variational results

The main part of the present section is dedicated to Ricceri's recent multiplicity results. First, several three critical points results with one/two parameter are stated. In the second subsection a general variational principle of Ricceri is presented. Finally, in the last subsection a new kind of multiplicity result of Ricceri is given which guarantees the existence of $k \geq 2$ distinct critical points for a fixed functional.

1.4.1 Three critical point results

It is a simple exercise to show that a function $f : \mathbb{R} \to \mathbb{R}$ of class C^1 having two local minima has necessarily a third critical point. However, once we are dealing with functions defined on a multi-dimensional space, the problem becomes much deeper. The well-known form of the three critical point theorem is due to Pucci-Serrin [246, 247] and it can be formulated as follows.

Theorem 1.11 *A function $f : X \to \mathbb{R}$ of class C^1 satisfies the (PS)-condition and it has two local minima. Then f has at least three distinct critical points.*

Proof Without loss of generality we may assume that 0 and $e \in X \setminus \{0\}$ are the two local minima of f and $f(e) \leq f(0) = 0$. We face two cases:

I) If there exist constants $\alpha, \rho > 0$ such that $\|e\| > \rho$ and $\inf_{\|u\|=\rho} f(u) \geq \alpha$, then the existence of a critical point of f at a minimax level c with $c \geq \alpha$ is guaranteed by Theorem 1.7. Consequently, this critical point certainly differs from 0 and e.

II) Assume now that such constants do not exist as in I). Since 0 is a local minima of f, we may choose $r < \|e\|$ such that $f(u) \geq 0$ for every $u \in X$ with $\|u\| \leq r$. We apply Ekeland's variational principle, see Theorem 1.3 with $\varepsilon = \dfrac{1}{n^2}$, $\lambda = n$ and f restricted to the closed ball $B[0, r] = \{u \in X : \|u\| \leq r\}$. Let us fix $0 < \rho < r$ small enough. Since $\inf_{\|u\|=\rho} f(u) = 0$, there exist $z_n \in X$ with $\|z_n\| = \rho$ and $u_n \in B[0, r]$ such that

$$0 \leq f(u_n) \leq f(z_n) \leq \frac{1}{n^2}, \quad \|u_n - z_n\| \leq \frac{1}{n}$$

and

$$f(w) - f(u_n) \geq -\frac{1}{n}\|w - u_n\|$$

for all $w \in B[0,r]$. Since $\rho < r$, for n enough large, we have $\|u_n\| < r$. Fix $v \in X$ arbitrarily and let $w = u_n + tv$. If $t > 0$ is small enough then $w \in B[0,r]$. Therefore, with this choice, the last inequality gives that $\|f'(u_n)\| \leq \frac{1}{n}$. Since $0 \leq f(u_n) \leq \frac{1}{n^2}$ and f satisfies the (PS)-condition, we may assume that $u_n \to u \in X$ which is a critical point of f. Since $|\|u_n\| - \rho| = |\|u_n\| - \|z_n\|| \leq \frac{1}{n}$, we actually have that $\|u\| = \rho$, thus $0 \neq u \neq e$. □

Let $f : X \times I \to X$ be a function such that $f(\cdot, \lambda)$ is of class C^1 for every $\lambda \in I \subset \mathbb{R}$. In view of Theorem 1.11, the main problem Ricceri dealt with is the *stability* of the three critical points of $f(\cdot, \lambda)$ with respect to the parameters $\lambda \in I$. Ricceri's main three critical point theorem is the following.

Theorem 1.12 *Let X be a separable and reflexive real Banach space, $I \subseteq \mathbb{R}$ be an interval, and $f : X \times I \to \mathbb{R}$ be a function satisfying the following conditions:*

(α_1) *for each $u \in X$, the function $f(u, \cdot)$ is continuous and concave;*

(β_1) *for each $\lambda \in I$, the function $f(\cdot, \lambda)$ is sequentially weakly lower semi-continuous and Gâteaux differentiable, and $\lim\limits_{\|u\| \to +\infty} f(u, \lambda) = +\infty$;*

(γ_1) *there exists a continuous concave function $h : I \to \mathbb{R}$ such that*

$$\sup_{\lambda \in I} \inf_{u \in X} (f(u, \lambda) + h(\lambda)) < \inf_{u \in X} \sup_{\lambda \in I} (f(u, \lambda) + h(\lambda)).$$

Then, there exist an open interval $J \subseteq I$ and a positive real number ρ, such that, for each $\lambda \in J$, the equation

$$f'_u(u, \lambda) = 0$$

has at least two solutions in X whose norms are less than ρ. If, in addition, the function f is (strongly) continuous in $X \times I$, and, for each $\lambda \in I$, the function $f(\cdot, \lambda)$ is of class C^1 and satisfies the (PS)-condition, then the above conclusion holds with "three" instead of "two".

The proof of Theorem 1.12 is quite involved combining deep arguments from nonlinear analysis as a topological minimax result of Saint Raymond [264], a general selector result of Kuratowski and Ryll-Nardzewski [182], [145], and the mountain pass theorem. Instead of its proof, we give a useful consequence of Theorem 1.12.

Corollary 1.4 *Let X be a separable and reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly* *lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on* $X^\star$, $\Psi : X \to \mathbb{R}$ *a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, and* $I \subseteq \mathbb{R}$ *an interval. Assume that*

$$\lim_{\|u\|\to+\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty$$

for all $\lambda \in I$, *and that there exists a continuous concave function* $h : I \to \mathbb{R}$ *such that*

$$\sup_{\lambda\in I}\inf_{u\in X}(\Phi(u) - \lambda\Psi(u) + h(\lambda)) < \inf_{u\in X}\sup_{\lambda\in I}(\Phi(u) - \lambda\Psi(u) + h(\lambda)).$$

Then, there exist an open interval $J \subseteq I$ *and a positive real number* ρ *such that, for each* $\lambda \in J$, *the equation*

$$\Phi'(u) - \lambda\Psi'(u) = 0$$

has at least three solutions in X *whose norms are less than* ρ.

Proof We may apply Theorem 1.12 to the function $f : X \times I \to \mathbb{R}$ defined by $f(u,\lambda) = \Phi(u) - \lambda\Psi(u)$ for each $(u,\lambda) \in X \times I$. In particular, the fact that Ψ' is compact implies that Ψ is sequentially weakly continuous. Moreover, our assumptions ensure that, for each $\lambda \in I$, the function $f(\cdot,\lambda)$ satisfies the (PS)-condition. □

In the sequel we point out a useful result concerning the *location* of parameters in Corollary 1.4. Actually, this location result is an improved version of Theorem 1.12 given by Bonanno [41].

Theorem 1.13 *Let* X *be a separable and reflexive real Banach space, and let* $\Phi, \Psi : X \to \mathbb{R}$ *be two continuously Gâteaux differentiable functionals. Assume that there exists* $u_0 \in X$ *such that* $\Phi(u_0) = \Psi(u_0) = 0$ *and* $\Phi(u) \geq 0$ *for every* $u \in X$ *and that there exists* $u_1 \in X$, $r > 0$ *such that*

(i) $r < \Phi(u_1)$;

(ii) $\displaystyle\sup_{\Phi(u)<r}\Psi(u) < r\frac{\Psi(u_1)}{\Phi(u_1)}$.

Further, put

$$\overline{a} = \frac{\zeta r}{r\dfrac{\Psi(u_1)}{\Phi(u_1)} - \displaystyle\sup_{\Phi(u)<r}\Psi(u)},$$

with $\zeta > 1$, and assume that the functional $\Phi - \lambda\Psi$ is sequentially weakly lower semi-continuous, satisfies the (PS)-condition and

(iii) $\lim\limits_{\|u\|\to+\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty$ *for every* $\lambda \in [0, \overline{a}]$.

Then, there exists an open interval $\Lambda \subset [0, \overline{a}]$ and a positive real number ρ such that, for each $\lambda \in \Lambda$, the equation

$$\Phi'(u) - \lambda\Psi'(u) = 0$$

admits at least three solutions in X whose norms are less than ρ.

Proof We prove the minimax inequality from Corollary 1.4 whit $I = [0, \overline{a}]$ and the function h suitable chosen. Fix $\zeta > 1$. Due to (ii), there exists $\sigma \in \mathbb{R}$ such that

$$\sup_{\Phi(u)<r} \Psi(u) + \frac{r\dfrac{\Psi(u_1)}{\Phi(u_1)} - \sup\limits_{\Phi(u)<r} \Psi(u)}{\zeta} < \sigma < r\frac{\Psi(u_1)}{\Phi(u_1)}. \tag{1.16}$$

Due to (i) and (1.16) we obtain that $0 < \sigma < \Psi(u_1)$; thus, introducing the notation $A := \sigma\frac{\Phi(u_1)}{\Psi(u_1)}$, we clearly have

$$\frac{A - \Phi(u_1)}{\sigma - \Psi(u_1)} = \frac{A}{\sigma}. \tag{1.17}$$

Let $\lambda \in \mathbb{R}$. Due to (1.17), one has either $\lambda \geq \frac{A-\Phi(u_1)}{\sigma-\Psi(u_1)}$ or $\lambda < \frac{A}{\sigma}$. When $\lambda \geq \frac{A-\Phi(u_1)}{\sigma-\Psi(u_1)}$ we have that

$$\inf_{u\in X}(\Phi(u) + \lambda(\sigma - \Psi(u))) \leq \Phi(u_1) + \lambda(\sigma - \Psi(u_1)) \leq A.$$

When $\lambda < \frac{A}{\sigma}$, we have

$$\inf_{u\in X}(\Phi(u) + \lambda(\sigma - \Psi(u))) \leq \Phi(u_0) + \lambda(\sigma - \Psi(u_0)) = \lambda\sigma < A.$$

Consequently,

$$\sup_{\lambda\in[0,\overline{a}]} \inf_{u\in X}(\Phi(u) + \lambda(\sigma - \Psi(u))) \leq \sup_{\lambda\in\mathbb{R}} \inf_{u\in X}(\Phi(u) + \lambda(\sigma - \Psi(u))) \leq A. \tag{1.18}$$

Set $k = [r\Psi(u_1)/\Phi(u_1) - \sup\limits_{\Phi(u)<r} \Psi(u)]/\zeta > 0$. On account of (1.16), one has $\sup\limits_{\Phi(u)<r} \Psi(u) < \sigma - k$. In particular, we have

$$\inf_{\Psi(u)>\sigma-k} \Phi(u) \geq r. \tag{1.19}$$

By (1.16) again, we have $r > \sigma \frac{\Phi(u_1)}{\Psi(u_1)} = A$. Consequently, from (1.19) one has

$$\inf_{\Psi(u)>\sigma-k} \Phi(u) > A.$$

Moreover, we also have

$$\overline{a}k = r > A.$$

After combining these facts, we obtain that

$$\inf_{u\in X} \sup_{\lambda\in[0,\overline{a}]} (\Phi(u) + \lambda(\sigma - \Psi(u))) = \inf_{u\in X} (\Phi(u) + \overline{a}\max\{0, \sigma - \Psi(u)\})$$

$$\geq \min\left\{ \inf_{\Psi(u)>\sigma-k} \Phi(u); \inf_{\Psi(u)\leq\sigma-k} \Phi(u) + \overline{a}k \right\} > A.$$

Combining this estimate with (1.18), the minimax inequality from Corollary 1.4 holds with the choices $I = [0, \overline{a}]$ and $h(\lambda) = \lambda\sigma$. The proof is complete. $\square$

We conclude the present subsection with a double eigenvalue problem. More precisely, we are interested in the minimal number of critical points of the functional $\Phi - \lambda\Psi$ (from Theorem 1.13) which is perturbed by an arbitrarily functional of class C^1. To state this result, we are going to apply a pure topological result due to Ricceri [262].

Theorem 1.14 *Let X be a reflexive real Banach space, $I \subseteq \mathbb{R}$ an interval, and $\varphi : X \times I \to \mathbb{R}$ a function such that $\varphi(u, \cdot)$ is concave in I for all $u \in X$, while $\varphi(\cdot, \lambda)$ is continuous, coercive and sequentially weakly lower semi-continuous in X for all $\lambda \in I$. Further, assume that*

$$\sup_I \inf_X \varphi < \inf_X \sup_I \varphi.$$

Then, for each $\eta > \sup_I \inf_X \varphi$, there exists a non-empty open set $A \subseteq I$ with the following property: for every $\lambda \in A$ and every sequentially weakly lower semi-continuous functional $\Upsilon : X \to \mathbb{R}$, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the functional $\varphi(\cdot, \lambda) + \mu\Upsilon(\cdot)$ has at least two local minima lying in the set $\{u \in X : \varphi(u, \lambda) < \eta\}$.

Now we prove another variant of Corollary 1.4 which appears in Ricceri [263].

Theorem 1.15 *Let X be a separable and reflexive real Banach space, $I \subseteq \mathbb{R}$ an interval, $\Phi : X \to \mathbb{R}$ a sequentially weakly lower semi-continuous functional of class C^1 whose derivative admits a continuous* *inverse on* $X^\star$ *and* $\Psi : X \to \mathbb{R}$ *a functional of class* C^1 *with compact derivative. Assume that*

$$\lim_{\|u\|\to+\infty} (\Phi(u) - \lambda\Psi(u)) = +\infty$$

for all $\lambda \in I$*, and that there exists* $\sigma \in \mathbb{R}$ *such that*

$$\sup_{\lambda\in I} \inf_{u\in X} (\Phi(u) + \lambda(\sigma - \Psi(u))) < \inf_{u\in X} \sup_{\lambda\in I} (\sigma - \Psi(u))). \tag{1.20}$$

Then, there exist a non-empty open set $A \subseteq I$ *and a positive real number* ρ *with the following property: for every* $\lambda \in A$ *and every functional* $\Upsilon : X \to \mathbb{R}$ *of class* C^1 *with compact derivative, there exists* $\delta > 0$ *such that, for each* $\mu \in [0, \delta]$*, the equation*

$$\Phi'(u) - \lambda\Psi'(u) + \mu\Upsilon'(u) = 0$$

has at least three solutions in X *whose norms are less than* ρ*.*

Proof Note that any functional on X of class C^1 with compact derivative is sequentially weakly continuous; in particular, it is bounded on each bounded subset of X, due to the reflexivity of X. Consequently, the function $\varphi : X \times I \to \mathbb{R}$ defined by

$$\varphi(u, \lambda) = \Phi(u) + \lambda(\sigma - \Psi(u))$$

satisfies all the hypotheses of Theorem 1.14. Fix $\eta > \sup_I \inf_X \varphi$ and consider a non-empty open set A with the property stated in Theorem 1.14. Fix also a compact interval $[a, b] \subset A$. We have

$$\bigcup_{\lambda\in[a,b]} \{u \in X : \varphi(u, \lambda) < \eta\}$$

$$\subseteq \{u \in X : \Phi(u) - a\Psi(u) < \eta - a\sigma\} \cup \{u \in X : \Phi(u) - b\Psi(u) < \eta - b\sigma\}.$$

Note that the set on the right-hand side is bounded, due to the coercivity assumption. Consequently, there is some $\tilde{\eta} > 0$ such that

$$\bigcup_{\lambda\in[a,b]} \{u \in X : \varphi(u, \lambda) < \eta\} \subseteq B_{\tilde{\eta}}, \tag{1.21}$$

where $B_{\tilde{\eta}} = \{u \in X : \|u\| < \tilde{\eta}\}$. Now, set

$$c^\star = \sup_{B_{\tilde{\eta}}} \Phi + \max\{|a|, |b|\} \sup_{B_{\tilde{\eta}}} |\Psi|$$

and fix $\rho > \tilde{\eta}$ such that

$$\bigcup_{\lambda\in[a,b]} \{u \in X : \Phi(u) - \lambda\Psi(u) \le c^\star + 2\} \subseteq B_\rho. \tag{1.22}$$

Now, let $\Upsilon : X \to \mathbb{R}$ be any functional of class C^1 with compact derivative. Let us choose a bounded function $g : \mathbb{R} \to \mathbb{R}$ of class C^1 such that $g(t) = t$ for all $t \in [-\sup_{B_\rho} |\Upsilon|, \sup_{B_\rho} |\Upsilon|]$. Put $\widetilde{\Psi}(u) = g(\Psi(u))$ for all $u \in X$. So, $\widetilde{\Psi}$ is a bounded functional on X of class C^1 such that

$$\widetilde{\Upsilon}(u) = \Upsilon(u) \text{ for all } x \in B_\rho. \tag{1.23}$$

For any set $Y \subset X$, we have

$$\widetilde{\Upsilon}'(Y) \subseteq g'(\Upsilon(Y))\Upsilon'(Y)$$

and hence it is clear that the derivative of $\widetilde{\Upsilon}$ is compact. Now, fix $\lambda \in [a, b]$. Then, taking (1.21) into account, there exists $\tilde{\delta} > 0$ such that, for each $\mu \in [0, \tilde{\delta}]$, the functional $\Phi - \lambda\Psi + \mu\widetilde{\Upsilon}$ has two local minima, say e_0, e_1 belonging to $B_{\tilde{\eta}}$. Further, set

$$\delta = \min\left\{\tilde{\delta}, \frac{1}{1 + \sup\limits_{\mathbb{R}} |g|}\right\}$$

and fix $\mu \in [0, \delta]$. Note that the functional $\Phi - \lambda\Psi + \mu\widetilde{\Upsilon}$ is coercive; moreover, an easy argument shows that this functional satisfies also the (PS)-condition. Consequently, if we denote by Γ the set of all continuous paths $\gamma : [0, 1] \to X$ such that $\gamma(0) = e_0$, $\gamma(1) = e_1$, and set

$$c_{\lambda,\mu} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} (\Phi(\gamma(t)) - \lambda\Psi(\gamma(t)) + \mu\widetilde{\Upsilon}(\gamma(t))),$$

by the mountain pass theorem for zero altitude (see Theorem 1.8), there exists $e \in X$ such that

$$\Phi'(e) - \lambda\Psi'(e) + \mu\widetilde{\Upsilon}'(e) = 0 \quad\text{and}\quad \Phi(e) - \lambda\Psi(e) + \mu\widetilde{\Upsilon}(e) = c_{\lambda,\mu}.$$

Now, choosing in particular $\gamma(t) = e_0 + t(e_1 - e_0)$, we observe that

$$\begin{aligned} c_{\lambda,\mu} &\le \max_{t\in[0,1]} (\Phi(e_0 + t(e_1 - e_0)) \\ &\quad -\lambda\Psi(e_0 + t(e_1 - e_0)) + \mu\widetilde{\Upsilon}(e_0 + t(e_1 - e_0))) \\ &\le \sup\nolimits_{B_{\tilde{\eta}}} \Phi + \max\{|a|, |b|\} \sup\nolimits_{B_{\tilde{\eta}}} |\Psi| + \delta \sup\nolimits_{\mathbb{R}} |g| \le c^\star + 1. \end{aligned}$$

Consequently, we have

$$\Phi(e) - \lambda\Psi(e) \le c^\star + 2.$$

By (1.22), it then follows that $e \in B_\rho$ and so, by (1.23), one has $\widetilde{\Upsilon}'(e_i) = \Upsilon'(e_i)$ for $i = 0, 1$ and $\widetilde{\Upsilon}'(e) = \Upsilon'(e)$. In conclusion, e_0, e_1 and e are three distinct solutions of the equation $\Phi'(u) - \lambda\Psi'(u) + \mu\Upsilon'(u) = 0$ lying in B_ρ. The proof is complete. □

1.4.2 A general variational principle

The general variational principle of Ricceri gives alternatives for the multiplicity of critical points of certain functions depending on a parameter. It can be stated in a following way, see Ricceri [256].

Theorem 1.16 *Let X be a reflexive real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semi-continuous, continuously Gâteaux differentiable functionals. Assume that Ψ is strongly continuous and coercive. For each $s > \inf_X \Psi$, set*

$$\varphi(s) := \inf_{u \in \Psi^s} \frac{\Phi(u) - \inf_{cl_w \Psi_s} \Phi}{s - \Psi(u)}, \tag{1.24}$$

where $\Psi_s := \{u \in X : \Psi(u) < s\}$ and $cl_w \Psi_s$ is the closure of Ψ_s in the weak topology of X. Furthermore, set

$$\delta := \liminf_{s \to (\inf_X \Psi)^+} \varphi(s), \qquad \gamma := \liminf_{s \to +\infty} \varphi(s). \tag{1.25}$$

Then, the following conclusions hold.

(A) *If $\delta < +\infty$ then, for every $\lambda > \delta$, either*

(A1) *there is a global minimum of Ψ which is a local minimum of $\Phi + \lambda\Psi$, or*

(A2) *there is a sequence $\{u_k\}$ of pairwise distinct critical points of $\Phi + \lambda\Psi$, with $\lim_{k\to\infty} \Psi(u_k) = \inf_X \Psi$, weakly converging to a global minimum of Ψ.*

(B) *If $\gamma < +\infty$ then, for every $\lambda > \gamma$, either*

(B1) *$\Phi + \lambda\Psi$ possesses a global minimum, or*

(B2) *there is a sequence $\{u_k\}$ of critical points of the functional $\Phi + \lambda\Psi$ such that $\lim_{k\to\infty} \Psi(u_k) = +\infty$.*

In order to prove Theorem 1.16, we recall without proof a topological result of Ricceri [256].

Theorem 1.17 *Let X be a topological space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially lower semi-continuous functions. Denote by I the set of all $s > \inf_X \Psi$ such that the set $\Psi_s := \{u \in X : \Psi(u) < s\}$ is contained in some sequentially compact subset of X. Assume that $I \neq \emptyset$. For each $s \in I$, denote by $\mathcal{F}_s$ the family of all sequentially compact subsets of X containing Ψ_s, and put*

$$\alpha(s) = \sup_{K \in \mathcal{F}_s} \inf_K \Phi.$$

Then, for each $s \in I$ and each λ satisfying

$$\lambda > \inf_{u \in \Psi_s} \frac{\Phi(u) - \alpha(s)}{s - \Psi(u)}$$

the restriction of the function $\Phi + \lambda\Psi$ to Ψ_s has a global minimum.

Proof of Theorem 1.16. Endowing the space X with the weak topology, it is easy to verify that $cl_w\Psi_s$ is the smallest sequentially weakly compact subset of X containing the set Ψ_s. Due to this observation, we also have that the function α and the interval I from Theorem 1.17 have the form $\alpha(s) = \inf_{u \in cl_w\Psi_s} \Phi(u)$ and $I =]\inf_X \Psi, \infty[$, respectively.

(A) We assume that $\delta < +\infty$; let $\lambda > \delta$. We may choose a sequence $\{s_k\} \subset I$ with the properties that $\lim_{k\to\infty} s_k = \inf_X \Psi$ and

$$\lambda > \inf_{u \in \Psi_{s_k}} \frac{\Phi(u) - \alpha(s_k)}{s_k - \Psi(u)}$$

for all $k \in \mathbb{N}$. Thanks to Theorem 1.17, for each $k \in \mathbb{N}$, the restriction of $\Phi + \lambda\Psi$ to Ψ_{s_k} has a global minimum in the weak topology; we denote it by u_k. Consequently, $u_k \in \Psi_{s_k}$ is a local minimum of $\Phi + \lambda\Psi$ in the strong topology too. Moreover, if $\overline{s} = \max_{k\in\mathbb{N}} s_k$, we clearly have that $\{u_k\} \subset \Psi_{\overline{s}}$. On the other hand, the set $\Psi_{\overline{s}}$ is contained in a sequentially compact subset of X. Therefore, the sequence $\{u_k\}$ admits a subsequence, denoted in the same way, which converges weakly to an element $u \in X$. We claim that u is a global minimum of Ψ. Indeed, since Ψ is sequentially weakly lower semi-continuous, we have that

$$\Psi(u) \leq \liminf_{k\to\infty} \Psi(u_k) \leq \lim_{k\to\infty} s_k = \inf_X \Psi.$$

Taking a subsequence if necessary, we have that

$$\lim_{k\to\infty} \Psi(u_k) = \Psi(u) = \inf_X \Psi.$$

Now, if $u = u_k$ for some $k \in \mathbb{N}$ then $(A1)$ holds; otherwise, we have $(A2)$.

(B) We assume that $\gamma < +\infty$; let $\lambda > \gamma$. We fix a sequence $\{s_k\} \subset I$ with the properties that $\lim_{k\to\infty} s_k = +\infty$ and

$$\lambda > \inf_{u\in\Psi_{s_k}} \frac{\Phi(u) - \alpha(s_k)}{s_k - \Psi(u)}$$

for all $k \in \mathbb{N}$. Due to Theorem 1.17, for each $k \in \mathbb{N}$, there exists $u_k \in \Psi_{s_k}$ such that

$$\Phi(u_k) + \lambda\Psi(u_k) = \min_{\Psi_{s_k}}(\Phi + \lambda\Psi). \tag{1.26}$$

We have basically two different cases.

When $\lim_{k\to\infty} \Psi(u_k) = \infty$, the proof is complete, since every set Ψ_{s_k} is open, thus u_k are different critical points of $\Phi + \lambda\Psi$, that is, $(B2)$ holds.

Now, we suppose that $\liminf_{k\to\infty} \Psi(u_k) < \infty$. Let us fix the number $r = \max\{\inf_X \Psi, \liminf_{k\to\infty} \Psi(u_k)\}$. For $k \in \mathbb{N}$ large enough, we clearly have that $\{u_k\} \subset \Psi_{r+1}$. In particular, since Ψ_{r+1} is contained in a sequentially compact subset of X, the sequence $\{u_k\}$ admits a subsequence, denoted in the same way, converging weakly to an element $u \in X$. We claim that u is a global minimum of Ψ. To see this, we fix $v \in X$. Since $\{s_k\}$ tends to $+\infty$, there exists $k_0 \in \mathbb{N}$ such that $\Psi(v) < s_{k_0}$, that is, $v \in \Psi_{s_{k_0}}$. Due to the sequentially weakly lower semi-continuity of $\Phi + \lambda\Psi$ (note that $\lambda \geq 0$), relation (1.26), we have that

$$\begin{aligned}\Phi(u) + \lambda\Psi(u) \quad &\leq \quad \liminf_{k\to\infty}(\Phi(u_k) + \lambda\Psi(u_k) \leq \Phi(u_{k_0}) + \lambda\Psi(u_{k_0})\\ &= \quad \min_{\Psi_{s_{k_0}}}(\Phi + \lambda\Psi) \leq \Phi(v) + \lambda\Psi(v).\end{aligned}$$

Consequently, $u \in X$ is a global minimum of $\Phi+\lambda\Psi$, that is, $(B1)$ holds. This completes the proof of Theorem 1.16.

The general variational principle of Ricceri (Theorem 1.16) implies immediately the following assertion: *if Φ and Ψ are two sequentially weakly lower semi-continuous functionals on a reflexive Banach space X and if Ψ is also continuous and coercive, then the functional $\Psi + \lambda\Phi$ has at least one local minimum for each $\lambda > 0$ small enough.*

A deep topological argument developed by Ricceri [260] shows a more precise conclusion. We present here this result without proof.

Theorem 1.18 *Let X be a separable and reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semi-continuous and continuously Gâteaux differentiable functionals, with Ψ coercive. Assume that the functional $\Psi+\lambda\Phi$ satisfies the (PS)-condition for every*

$\lambda > 0$ small enough and that the set of all global minima of Ψ has at least k connected components in the weak topology, with $k \geq 2$.

Then, for every $s > \inf_X \Psi$, there exists $\overline{\lambda} > 0$ such that for every $\lambda \in]0, \overline{\lambda}[$, the functional $\Psi + \lambda\Phi$ has at least $k+1$ critical points, k of which are lying in the set $\Psi_s = \{u \in X : \Psi(u) < s\}$.

Remark 1.4 The first k critical points of $\Psi + \lambda\Phi$ are actually local minima of the same functional which belong to the level set Ψ_s while the $(k+1)^{\text{th}}$ critical point is of mountain pass type where the assumption of the (PS)-condition is exploited. The mountain pass type point is located by means of the classification result of Ghoussoub-Preiss [128].

1.5 H^1 versus C^1 local minimizers

The result contained in this section establishes a surprising property of local minimizers, due to H. Brezis and L. Nirenberg [51].

Let Ω be a bounded open set in $\mathbb{R}^N$ with smooth boundary. Assume that $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function in u, uniformly in x such that for all $(x, u) \in \Omega \times \mathbb{R}$,

$$|f(x,u)| \leq C(1 + |u|^p)\,, \tag{1.27}$$

where $1 < p \leq (N+2)/(N-2)$. We point out that the exponent p can attain the critical Sobolev exponent $(N+2)/(N-2)$. Set $F(x,u) := \int_0^u f(x,s)\,ds$ and define on $H_0^1(\Omega)$ the *energy* functional

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(x,u)\,dx\,.$$

Definition 1.4 We say that $u_0 \in C_0^1(\overline{\Omega})$ is a local minimizer of E in the C^1 topology if there exists $r > 0$ such that

$$E(u_0) \leq E(u) \quad \text{for every } u \in C_0^1(\overline{\Omega}) \text{ with } \|u - u_0\|_{C^1} \leq r.$$

We say that $u_0 \in H_0^1(\overline{\Omega})$ is a local minimizer of E in the H_0^1 topology if there exists $\varepsilon_0 > 0$ such that

$$E(u_0) \leq E(u) \quad \text{for every } u \in H_0^1(\overline{\Omega}) \text{ with } \|u - u_0\|_{H^1} \leq \varepsilon_0.$$

The main result of this section is the following.

Theorem 1.19 *Any local minimizer of E in the C^1 topology is also a local minimizer of E in the H_0^1 topology.*

Proof We can assume without loss of generality that 0 is a local minimizer of E in the C^1 topology.

Subcritical case: $p < (N+2)/(N-2)$. Arguing by contradiction, we deduce that for any $\varepsilon > 0$, there exists $u_\varepsilon \in H_0^1(\Omega)$ such that

$$\|u_\varepsilon\|_{H^1} \leq \varepsilon \qquad \text{and} \qquad E(u_\varepsilon) < E(0)\,. \tag{1.28}$$

This enables us to apply a standard lower semi-continuity argument to deduce that $\min_{u\in\overline{B_\varepsilon}} E(u)$ is attained by some point u_ε, where

$$B_\varepsilon := \{u \in H_0^1(\Omega);\ \|u\|_{H^1} < \varepsilon\}\,.$$

We claim that

$$u_\varepsilon \to 0 \qquad \text{in } C^1(\overline{\Omega}).$$

Thus, by (1.28), we contradict the hypothesis that 0 is a local minimizer of E in the C^1 topology.

In order to prove our claim, let $\mu_\varepsilon \leq 0$ be a Lagrange multiplier of the corresponding Euler equation for u_ε, that is,

$$\langle E'(u_\varepsilon), v\rangle_{H^{-1},H_0^1} = \mu_\varepsilon\,(u_\varepsilon, v)_{H_0^1} \qquad \text{for all } v \in H_0^1(\Omega).$$

Therefore

$$\int_\Omega [\nabla u_\varepsilon \cdot \nabla v - f(x,u_\varepsilon)v]\;dx = \mu_\varepsilon \int_\Omega \nabla u_\varepsilon \cdot \nabla v\,dx \qquad \text{for all } v \in H_0^1(\Omega).$$

This can be rewritten as

$$-(1-\mu_\varepsilon)\Delta u_\varepsilon = f(x,u_\varepsilon) \qquad \text{in } \Omega. \tag{1.29}$$

Since $u_\varepsilon \to 0$ in $H^1(\Omega)$ it follows that, in order to deduce our claim, it is enough to apply the Arzelà–Ascoli theorem, after observing that

$$\|u_\varepsilon\|_{C^{1,\alpha}} \leq C\,.$$

This uniform estimate follows by a standard bootstrap argument. Indeed, assuming that $u_\varepsilon \in L^{q_1}(\Omega)$ for some $q_1 \geq 2N/(N-2)$, then $f(x,u_\varepsilon) \in L^r(\Omega)$, where $r = q_1/p$. Thus, by Schauder elliptic estimates, $u_\varepsilon \in W^{2,r}(\Omega) \subset L^{q_2}(\Omega)$, with

$$\frac{1}{q_2} = \frac{p}{q_1} - \frac{2}{N}\,,$$

provided that the right hand-side is positive. In the remaining case we have $u_\varepsilon \in L^q(\Omega)$ for any $q > 1$. Since $p < (N+2)/(N-2)$, we deduce that $q_2 > q_1$. Starting with $q_1 = 2N/(N-2)$ and iterating this process, we obtain an increasing divergent sequence $(q_m)_{m\geq 1}$. This shows that

$u_\varepsilon \in L^q(\Omega)$ for any $q > 1$. Returning now to (1.29) we deduce that $u_\varepsilon \in W^{2,r}(\Omega)$ for any $r > 1$. Thus, by Sobolev embeddings, $u_\varepsilon \in C^{1,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$. This concludes the proof in the subcritical case corresponding to $p < (N+2)/(N-2)$.

Critical case: $p = (N+2)/(N-2)$. We argue again by contradiction. Thus, for all $\varepsilon > 0$, there exists $u_\varepsilon \in H_0^1(\Omega)$ such that relation (1.28) holds. For any positive integer m we define the truncation map

$$T_m(t) = \begin{cases} -m & \text{if } t \le -m \\ t & \text{if } -m \le t \le m \\ m & \text{if } t \ge m\,. \end{cases}$$

For any integer $m \ge 1$, define the energy functional

$$E_m(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \int_\Omega F_m(x,u)\, dx\,,$$

where $f_m(x,t) := f(x, T_m(t))$ and $F_m(x,u) := \int_0^u f_m(x,t)\, dt$.

We first observe that for all $u \in H_0^1(\Omega)$, $E_m(u) \to E(u)$ as $m \to \infty$. Thus, for each $\varepsilon > 0$, there exists an integer $m(\varepsilon)$ such that $E_{m(\varepsilon)}(u_\varepsilon) < E(0)$. Let $v_\varepsilon \in H_0^1(\Omega)$ be such that

$$E_{m(\varepsilon)}(v_\varepsilon) = \min_{u \in \overline{B_\varepsilon}} E_\varepsilon(u)\,.$$

Therefore

$$E_{m(\varepsilon)}(v_\varepsilon) \le E_{m(\varepsilon)}(u_\varepsilon) < E(0)\,.$$

We claim that $v_\varepsilon \in C_0^1(\overline{\Omega})$ and that $v_\varepsilon \to 0$ in $C^1(\overline{\Omega})$. The, if $\varepsilon > 0$ is sufficiently small, we have

$$E(v_\varepsilon) = E_{m(\varepsilon)}(v_\varepsilon) < E(0)\,,$$

which contradicts our assumption that 0 is a local minimizer of E in the C^1 topology.

Returning to the proof of our claim, we observe that v_ε satisfies the Euler equation

$$-(1-\mu_\varepsilon)\Delta v_\varepsilon = f_m(x, v_\varepsilon)\,, \tag{1.30}$$

where

$$|f_m(x,u)| \le C\,(1 + |u|^p) \tag{1.31}$$

with $p = (N+2)/(N-2)$ and C not depending on m. Since $v_\varepsilon \to 0$ in $H_0^1(\Omega) \subset L^{2N/(N-2)}(\Omega)$, Theorem IV.9 in [46] implies that there exists

$h \in L^{2N/(N-2)}(\Omega)$ such that, up to a subsequence still denoted by (v_ε), we have $|v_\varepsilon| \le h$ in Ω. Thus, by relation (1.31),

$$|f_m(x, v_\varepsilon)| \le C\,(1 + a\,|v_\varepsilon|)$$

with $a = h^{4/(N-2)} \in L^{N/2}(\Omega)$. Using a bootstrap argument as in the proof of the subcritical case $p < (N+2)/(N-2)$, we deduce that (v_ε) is bounded in $L^q(\Omega)$, for any $1 < q < \infty$. Returning now to relation (1.30) and using (1.31) we obtain that (v_ε) is bounded in $C^{1,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$. Now, since $v_\varepsilon \to 0$ in $H_0^1(\Omega)$, the Arzelà–Ascoli theorem implies that $v_\varepsilon \to 0$ in $C^{1,\alpha}(\overline{\Omega})$. This concludes our proof. □

1.5.1 Application to sub and supersolutions

Consider the nonlinear elliptic problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.32}$$

where $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function in u, uniformly in x.

The method of sub and supersolutions is a very useful tool in nonlinear analysis which asserts that if $\underline{u}$, $\overline{u} \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy $\underline{u} \le \overline{u}$ and

$$\begin{cases} -\Delta \underline{u} - f(x,\underline{u}) \le 0 \le -\Delta \overline{u} - f(x,\overline{u}) & \text{in } \Omega \\ \underline{u} \le 0 \le \overline{u} & \text{on } \partial\Omega\,, \end{cases} \tag{1.33}$$

then problem (1.32) has at least a solution. The standard proof of this result relies on the maximum principle. By means of Theorem 1.19 we are able to give a *variational* proof to the following qualitative result which guarantees the existence of a local minimizer between a subsolution and a supersolution.

Theorem 1.20 *Assume that functions $\underline{u}$, $\overline{u} \in C^2(\Omega) \cap C^1(\overline{\Omega})$ with $\underline{u} \le \overline{u}$ satisfy (1.33). In addition, we suppose that neither $\underline{u}$ nor $\overline{u}$ is a solution of (1.32). Then there exists a solution u_0 of problem (1.32) such that $\underline{u} < u_0 < \overline{u}$ in Ω and u_0 is a local minimum of the associated energy functional E in $H_0^1(\Omega)$, where*

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(x,u)\,dx\,.$$

Proof Define the truncated continuous function

$$f_0(x,t) := \begin{cases} f(x,\underline{u}(x)) & \text{if } t < \underline{u}(x) \\ f(x,t) & \text{if } \underline{u}(x) \le t \le \overline{u}(x) \\ f(x,\overline{u}(x)) & \text{if } t > \overline{u}(x)\,. \end{cases}$$

Set $F_0(x,u) := \int_0^u f_0(x,t)\,dt$ and

$$E_0(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F_0(x,u)\,dx\,.$$

Then f_0 is bounded on $\overline{\Omega}\times\mathbb{R}$, hence E_0 is bounded from below on $H_0^1(\Omega)$. Thus, by semi-continuity arguments, $\inf_{u\in H_0^1(\Omega)} E_0(u)$ is achieved at some point u_0 and

$$-\Delta u_0 = f_0(x,u_0) \qquad \text{in } \Omega.$$

Bootstrap techniques (as developed in the proof of Theorem 1.19) imply that $u_0 \in W^{2,p}(\Omega)$ for any $p<\infty$. So, by Sobolev embeddings, $u_0 \in C^{1,\alpha}(\overline{\Omega})$ for all $\alpha\in(0,1)$.

Next, we prove that $\underline{u} < u_0 < \overline{u}$ in Ω. Indeed, multiplying by $(\underline{u}-u_0)^+$ the inequality

$$-\Delta(\underline{u}-u_0) \le f(x,\underline{u}) - f(x,u_0)$$

and integrating on Ω, we find $(\underline{u}-u_0)^+ = 0$. Thus, $\underline{u} \le u-0$ in Ω. In fact, the strong maximum principle implies that this inequality is strict. With a similar argument we deduce that $u_0 < \overline{u}$ in Ω. Using now $\underline{u} < u_0 < \overline{u}$ and applying again the strong maximum principle, we find $\varepsilon.0$ such that for all $x\in\Omega$,

$$\underline{u}(x) + \varepsilon\,\mathrm{dist}\,(x,\partial\Omega) \le u_0(x) \le \overline{u}(x) - \varepsilon\,\mathrm{dist}\,(x,\partial\Omega)\,.$$

We observe that the mapping $[\underline{u}(x),\overline{u}(x)] \ni x \longmapsto F_0(x,u) - F(x,u)$ is a function depending only on x. This shows that $E(u)-E_0(u)$ is constant, provided that $\|u-u_0\|_{C^1}$ is small enough. Since u_0 is a global minimum of E_0, we deduce that u_0 is a local minimum of E on $C_0^1(\underline{\Omega})$. Using now Theorem 1.19 we conclude that u_0 is also a local minimum of E in $H_0^1(\Omega)$. $\square$

1.6 Szulkin type functionals

Let X be a real Banach space and $X^\star$ its dual and we denote by $\langle\cdot,\cdot\rangle$ the duality pair between X and $X^\star$. Let $E: X\to\mathbb{R}$ be a functional of class C^1 and let $\zeta: X\to\mathbb{R}\cup\{+\infty\}$ be a proper (that is, $\not\equiv +\infty$), convex, lower semi-continuous function. Then, $I = E + \zeta$ is a *Szulkin-type functional*, see Szulkin [282]. An element $u \in X$ is called a *critical point of* $I = E + \zeta$ if

$$\langle E'(u), v - u\rangle + \zeta(v) - \zeta(u) \geq 0 \quad \text{for all } v \in X. \tag{1.34}$$

The number $I(u)$ is a *critical value* of I.

For $u \in D(\zeta) = \{u \in X : \zeta(u) < \infty\}$ we consider the set

$$\partial\zeta(u) = \{x^\star \in X^\star : \zeta(v) - \zeta(u) \geq \langle x^\star, v - u\rangle, \quad \forall\, v \in X\ \}.$$

The set $\partial\zeta(u)$ is called the *subdifferential* of ζ at u. Note that an equivalent formulation for (1.34) is

$$0 \in E'(u) + \partial\zeta(u) \quad \text{in} \quad X^\star. \tag{1.35}$$

Proposition 1.3 *Every local minimum point of $I = E + \zeta$ is a critical point of I in the sense of* (1.34).

Proof Let $u \in D(\zeta)$ be a local minimum point of $I = E + \zeta$. Due to the convexity of ζ, for every $t > 0$ small we have

$$0 \leq I((1-t)u + tv) - I(u) \leq E(u + t(v-u)) - E(u) + t(\zeta(v) - \zeta(u)).$$

Dividing by $t > 0$ and letting $t \to 0^+$, we obtain (1.34). □

Definition 1.5 The functional $I = E + \zeta$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$, (shortly, $(PSZ)_c$-condition) if every sequence $\{u_n\} \subset X$ such that $\lim_{n\to\infty} I(u_n) = c$ and

$$\langle E'(u_n), v - u_n\rangle_X + \zeta(v) - \zeta(u_n) \geq -\varepsilon_n\|v - u_n\| \quad \text{for all } v \in X,$$

where $\varepsilon_n \to 0$, possesses a convergent subsequence.

Remark 1.5 When $\zeta = 0$, $(PSZ)_c$-condition is equivalent to the standard $(PS)_c$-condition.

1.6.1 Minimax results of Szulkin type

Theorem 1.21 *Let X be a Banach space, $I = E + \zeta : X \to \mathbb{R} \cup \{+\infty\}$ a Szulkin-type functional and we assume that*

(i) $I(u) \geq \alpha$ for all $\|u\| = \rho$ with α, $\rho > 0$, and $I(0) = 0$;
(ii) there is $e \in X$ with $\|e\| > \rho$ and $I(e) \leq 0$.

If I satisfies the $(PSZ)_c$-condition for

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\},$$

then c is a critical value of I and $c \geq \alpha$.

Theorem 1.22 *Let X be a separable and reflexive Banach space, let $I_1 = E_1 + \zeta_1$ Szulkin-type functionals and $I_2 = E_2 : X \to \mathbb{R}$ a function of class C^1 and let $\Lambda \subseteq \mathbb{R}$ be an interval. Suppose that*

(i) E_1 is weakly sequentially lower semi-continuous and E_2 is weakly sequentially continuous;

(ii) for every $\lambda \in \Lambda$ the function $I_1 + \lambda I_2$ fulfils $(PSZ)_c$, $c \in \mathbb{R}$, with

$$\lim_{\|u\|\to+\infty} (I_1(u) + \lambda I_2(u)) = +\infty;$$

(iii) there exists a continuous concave function $h : \Lambda \to \mathbb{R}$ satisfying

$$\sup_{\lambda\in\Lambda} \inf_{u\in X} (I_1(u)+\lambda I_2(u)+h(\lambda)) < \inf_{u\in X} \sup_{\lambda\in\Lambda}(I_1(u)+\lambda I_2(u)+h(\lambda)).$$

Then there is an open interval $\Lambda_0 \subseteq \Lambda$, such that for each $\lambda \in \Lambda_0$ the function $I_1 + \lambda I_2$ has at least three critical points in X.

In the next, we give a variant of Ricceri variational principle for Szulkin type functionals which will be used in the study of the existence of scalar systems.

Suppose now that X and Y are real Banach spaces such that X is compactly embedded in Y. Let $E_1 \colon Y \to \mathbb{R}$ and $E_2 \colon X \to \mathbb{R}$ be C^1 functions, and let $\zeta_1 \colon X \to]-\infty, +\infty]$ be convex, proper, and lower semi-continuous. Define the maps $I_1 \colon X \to]-\infty, +\infty]$ and $I_2 \colon X \to \mathbb{R}$ as

$$I_1(u) = E_1(u) + \zeta_1(u), \quad I_2(u) = E_2(u), \quad \text{for all} \ \ x \in X.$$

Denote by $D(\zeta_1) := \{u \in X \mid \zeta_1(u) < +\infty\}$ and assume that

$$I_2^{-1}(]-\infty, \rho[) \cap D(\zeta_1) \neq \emptyset, \quad \text{for all } \rho > \inf_X I_2. \tag{1.36}$$

For every $\rho > \inf\limits_X I_2$ put

$$\varphi(\rho) := \inf_{u\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(u) - \inf_{v\in\overline{(I_2^{-1}(]-\infty,\rho[))}_w} \Phi(v)}{\rho - I_2(u)}, \tag{1.37}$$

where $\overline{(I_2^{-1}(]-\infty,\rho[))}_w$ is the weak closure of $I_2^{-1}(]-\infty,\rho[)$,

$$\gamma := \liminf_{\rho\to+\infty} \varphi(\rho), \tag{1.38}$$

$$\delta := \liminf_{\rho\to(\inf_X I_2)^+} \varphi(\rho). \tag{1.39}$$

Theorem 1.23 *Suppose that X is reflexive, I_2 is weakly sequentially lower semi-continuous and coercive, and* (1.36) *is fulfilled. Then the following assertions hold:*

(a) *For every $\rho > \inf\limits_X I_2$ and every $\lambda > \varphi(\rho)$ the function $I_1 + \lambda I_2$ has a critical point (more exactly: a local minimum) lying in $I_2^{-1}(]-\infty,\rho[) \cap D(\zeta_1)$.*

(b) *If $\gamma < +\infty$ then, for each $\lambda > \gamma$, either*

(b1) *$I_1 + \lambda I_2$ has a global minimum, or*

(b2) *there is a sequence $\{u_n\}$ of critical points (more exactly: local minima) of $I_1+\lambda I_2$ lying in $D(\zeta_1)$ and such that $\lim_{n\to\infty} I_2(u_n) = +\infty$.*

(c) *If $\delta < +\infty$ then, for every $\lambda > \delta$, either*

(c1) *$I_1 + \lambda I_2$ has a local minimum which is also a global minimum of I_2, or*

(c2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (more exactly: local minima) of $I_1 + \lambda I_2$ lying in $D(\zeta_1)$ and such that $\{u_n\}$ converges weakly to a global minimum of I_2 and $\lim_{n\to\infty} I_2(u_n) = \inf\limits_X I_2$.*

1.6.2 Principle of symmetric criticality

The principle of symmetric criticality plays a central role in many problems from the differential geometry, physics and in partial differential equations. First it was proved by Palais [232] for functionals of class C^1. In this subsection we recall this principle for functionals of class C^1 defined on Banach spaces and we state its version for Szulkin type functionals, based on the paper of Kobayashi-Otani [158].

Let X be a Banach space and let $X^\star$ its dual. The norms of X and $X^\star$ will be denoted by $\|\cdot\|$ and $\|\cdot\|_\star$, respectively. We shall denote by $\langle\cdot,\cdot,\rangle$ the duality pairing between X and $X^\star$.

Let G be a group, e its identity element, and let π a representation of

G over X, that is $\pi(g) \in L(X)$ for each $g \in G$ (where $L(X)$ denotes the set of the linear and bounded operator from X into X), and

a) $\pi(e)u = u$, $\forall\ u \in X$;
b) $\pi(g_1 g_2)u = \pi(g_1)(\pi(g_2 u))$, $\forall\ g_1, g_2 \in G$ and $u \in X$.

The representation $\pi_\star$ of G over $X^\star$ is naturally induced by π by the relation

$$\langle \pi_\star(g)v^\star, u\rangle = \langle v^\star, \pi(g^{-1})u\rangle, \forall\ g \in G, v^\star \text{ and } u \in X. \tag{1.40}$$

We often write gu or $gv^\star$ instead of $\pi(g)u$ or $\pi_\star(g)v^\star$, respectively.

A function $h : X \to \mathbb{R}$ (respectively, $h : X^\star \to \mathbb{R}$) is called G-invariant if $h(gu) = h(u)$ (respectively, $h(gu^\star) = h(u^\star)$) for every $u \in X$ (respectively, $u^\star \in X^\star$) and $g \in G$. A subset M of X is called G-invariant (respectively, $M^\star$ of $X^\star$) if

$$gM = \{gu : u \in M\} \subseteq M \quad \text{(respectively, } gM^\star \subseteq M^\star) \quad \forall\ g \in G.$$

The fixed point sets of the group action G on X and $X^\star$ (some authors call them G-symmetric points) are defined as

$$\Sigma = X^G = \{u \in X : gu = u\ \forall\ g \in G\},$$

$$\Sigma_\star = (X^\star)^G = \{v^\star \in X^\star : gv^\star = v^\star\ \forall\ g \in G\}.$$

Hence, by (1.40), we can see that $v^\star \in X^\star$ is symmetric if and only if $v^\star$ is a G-invariant functional. The sets Σ and $\Sigma_\star$ are closed linear subspaces of X and $X^\star$, respectively. So Σ and $\Sigma_\star$ are regarded as Banach spaces with their induced topologies. We introduce the notation

$$C^1_G(X) = \{f : X \to \mathbb{R} : f \text{ is } G-\text{invariant and of class } C^1\}.$$

We consider the following statement, called as *principle of symmetric criticality* (shortly, (PSC)):

- *If $f \in C^1_G(X)$ and $(f|_\Sigma)'(u) = 0$, then $f'(u) = 0$.*

Theorem 1.24 (PSC) *is valid if and only if $\Sigma_\star \cap \Sigma^\perp = \{0\}$, where $\Sigma^\perp = \{v^\star \in X^\star : \langle v^\star, u\rangle = 0,\ \forall\ u \in \Sigma\}$.*

Proof " $\Rightarrow$ " Suppose that $\Sigma_\star \cap \Sigma^\perp = \{0\}$ and let $u_0 \in \Sigma$ be a critical point of $f|_\Sigma$. We show that $f'(u_0) = 0$. Since $f(u_0) = f|_\Sigma(u_0)$ and $f(u_0 + v) = f|_\Sigma(u_0 + v)$ for all $v \in \Sigma$, we obtain $\langle f'(u_0), v\rangle =$

$\langle (f|_\Sigma)'(u_0), v\rangle_\Sigma$ for every $v \in \Sigma$, where $\langle\cdot,\cdot\rangle_\Sigma$ denotes the duality pairing between Σ and its dual $\Sigma^\star$. This implies that $f'(u_0) \in \Sigma^\perp$. On the other hand, from the G-invariance of f follows that

$$\begin{aligned}\langle f'(gu), v\rangle &= \lim_{t\to 0}\frac{f(gu+tv)-f(gu)}{t} = \lim_{t\to 0}\frac{f(u+tg^{-1}v)-f(u)}{t}\\ &= \langle f'(u), g^{-1}v\rangle = \langle gf'(u), v\rangle\end{aligned}$$

for all $g \in G$ and $u, v \in X$. This means that f' is G-equivariant, that is, $f'(gu) = gf'(u)$ for every $g \in G$ and $u \in X$. Since $u_0 \in \Sigma$, we obtain $gf'(u_0) = f'(u_0)$ for all $g \in G$, that is, $f'(u_0) \in \Sigma_\star$. Thus we conclude $f'(u_0) \in \Sigma_\star \cap \Sigma^\perp = \{0\}$. Therefore $f'(u_0) = 0$.

" $\Leftarrow$ " Suppose that there exists a non-zero element $v^\star \in \Sigma_\star \cap \Sigma^\perp$ and define $f_\star(\cdot)$ by $f_\star(u) = \langle v^\star, u\rangle$. It is clear that $f_\star \in C^1_G(X)$ and $(f_\star)'(\cdot) = v^\star \neq 0$, so $f_\star$ has no critical point in X. On the other hand $v^\star \in \Sigma^\perp$ implies $v^\star|_\Sigma = 0$, thus $(f_\star|_\Sigma)'(u) = 0$ for every $u \in \Sigma$ which contradicts (PSC). □

In the sequel, we are interested to find conditions in order to have $\Sigma_\star \cap \Sigma^\perp = \{0\}$, that is, (PSC) to be verified. There are two ways to achieve this purpose, the so-called "compact" as well as the "isometric" cases. We are dealing now with the first one whose original form has been given by Palais [232].

Theorem 1.25 *Let G be a compact topological group and the representation π of G over X is continuous, that is, $(g, u) \to gu$ is a continuous function $G \times X$ into X. Then* (PSC) *holds.*

In order to prove this theorem, we recall that for each $u \in X$, there exists a unique element $Au \in X$ such that

$$\langle v^\star, Au\rangle = \int_G \langle v^\star, gu\rangle dg, \quad \forall\, v^\star \in X^\star, \tag{1.41}$$

where dg is the normalized Haar measure on G. The mapping $A : X \to \Sigma$ is called the *averaging operator* on G.

Proof of Theorem 1.25. On account of Theorem 1.24, it is enough to verify the condition $\Sigma_\star \cap \Sigma^\perp = \{0\}$. Let $v^\star \in \Sigma_\star \cap \Sigma^\perp$ fixed and suppose that $v^\star \neq 0$. Let us define the hyperplane $H_{v^\star} = \{u \in X : \langle v^\star, u\rangle = 1\}$ which is a non-empty, closed, convex subset of X. For any $u \in H_{v^\star}$, since $v^\star \in \Sigma_\star$, we have

$$\langle v^\star, Au\rangle = \int_G \langle v^\star, gu\rangle dg = \int_G \langle g^{-1}v^\star, u\rangle dg = \int_G \langle v^\star, u\rangle dg =$$

$$= \langle v^\star, u\rangle \int_G dg = \langle v^\star, u\rangle = 1.$$

Note however that $v^\star \in \Sigma^\perp$, thus $\langle v^\star, Au\rangle = 0$ for any $u \in H_{v^\star}$, contradiction.

Another possibility for (PSC) to be valid is the following one which will be given without proof.

Theorem 1.26 *Assume that X is a reflexive and strictly convex Banach space and G acts isometrically on X, that is, $\|gu\| = \|u\|$ for all $g \in G$ and $u \in X$. Then* (PSC) *holds.*

We conclude this subsection with a non-smooth version of (PSC). On account of relation (1.35), we are entitled to consider the following form of the principle of symmetric criticality for Szulkin functionals (shortly, (PSCSZ)):

- *If $I = E+\zeta : X \to \mathbb{R}\cup\{+\infty\}$ is a G-invariant Szulkin type functional and $0 \in (E|_\Sigma)'(u) + \partial(\zeta|_\Sigma)(u)$ in $\Sigma^\star$ then $0 \in E'(u) + \partial\zeta(u)$ in $X^\star$.*

Although a slightly more general version is proven by Kobayashi-Otani [158], we recall the following form of the *principle of symmetric criticality for Szulkin functionals* which will be applied in the next chapters.

Theorem 1.27 [158, Theorem 3.16] *Let X be a reflexive Banach space and let $I = E + \zeta : X \to \mathbb{R} \cup \{+\infty\}$ be a Szulkin-type functional on X. If a compact group G acts linearly and continuously on X, and the functionals E and ζ are G-invariant, then* (PSCSZ) *holds.*

The proof of Theorem 1.27 is quite involved, it combines in an ingenious way various methods and notions from convex and functional analysis. The interested reader may consult the paper of Kobayashi-Otani [158].

1.7 Pohozaev's fibering method

The *fibering scheme* was introduced by S.I. Pohozaev in [243], [244]. Although there are various versions of it, we present the so-called *one-parametric fibering method*.

Let X be a real Banach space, $f : X \to \mathbb{R}$ be a functional such that f is of class C^1 on $X \setminus \{0\}$. By means of f we define the function $\tilde f : (\mathbb{R}\setminus\{0\}) \times X \to \mathbb{R}$ by

$$\tilde f(\lambda, v) = f(\lambda v), \quad (\lambda, v) \in (\mathbb{R}\setminus\{0\}) \times X. \tag{1.42}$$

Let $S = \{u \in X : \|u\| = 1\}$.

Definition 1.6 A point $(\lambda, v) \in (\mathbb{R} \setminus \{0\}) \times S$ is conditionally critical of the function $\tilde{f}$ if

$$-\tilde{f}'(\lambda, v) \in N_{(\mathbb{R}\setminus\{0\})\times S}(\lambda, v),$$

where $N_{(\mathbb{R}\setminus\{0\})\times S}(\lambda, v)$ is the normal cone to the set $(\mathbb{R} \setminus \{0\}) \times S$ at the point (λ, v).

We have the following result, proved by Pohozaev [244].

Theorem 1.28 *Let X be a real Banach space with differentiable norm on $X \setminus \{0\}$ and let $(\lambda, v) \in (\mathbb{R} \setminus \{0\}) \times S$ be a conditionally critical point of $\tilde{f} : (\mathbb{R} \setminus \{0\}) \times X \to \mathbb{R}$. Then $u = \lambda v \in X \setminus \{0\}$ is a critical point of $f : X \to \mathbb{R}$, that is, $f'(u) = 0$.*

Proof Due to the hypothesis, we have that

$$-\tilde{f}'(\lambda, v) \in N_{(\mathbb{R}\setminus\{0\})\times S}(\lambda, v).$$

On account of Proposition A.2, see Appendix A, we obtain

$$(-\tilde{f}'_\lambda(\lambda, v), -\tilde{f}'_v(\lambda, v)) \in N_{\mathbb{R}\setminus\{0\}}(\lambda) \times N_S(v).$$

Here, $\tilde{f}'_v$ denotes the differential of $\tilde{f}$ with respect to v in the whole space X, while $\tilde{f}'_\lambda$ is the derivative of $\tilde{f}$ with respect to the variable λ.

Note that $N_{\mathbb{R}\setminus\{0\}}(\lambda) = \{0\}$, thus

$$\tilde{f}'_\lambda(\lambda, v) = 0. \tag{1.43}$$

Moreover, since $T_S(v)$ is a linear space, the condition $-\tilde{f}'_v(\lambda, v) \in N_S(v)$ reduces to the fact that

$$\langle \tilde{f}'_v(\lambda, v), w\rangle = 0 \text{ for all } w \in T_S(v).$$

Since $T_S(v) = \operatorname{Ker}\| \cdot \|'(v)$, see Zeidler [296, Theorem 43.C], there are $\kappa, \mu \in \mathbb{R}$ such that $\kappa^2 + \mu^2 \neq 0$ and

$$\kappa \tilde{f}'_v(\lambda, v) = \mu \| \cdot \|'(v). \tag{1.44}$$

From (1.44) it follows that for every $v \in X$,

$$\kappa\langle \tilde{f}'_v(\lambda, v), v\rangle = \mu\langle \| \cdot \|'(v), v\rangle. \tag{1.45}$$

A simple calculation shows that

$$\tilde{f}'_v(\lambda, v) = \lambda f'(\lambda v). \tag{1.46}$$

Moreover, we have

$$\langle \tilde{f}'_v(\lambda, v), v\rangle = \lambda\langle f'(\lambda v), v\rangle = \lambda \tilde{f}'_\lambda(\lambda, v).$$

Consequently, from (1.45) and the above relation we obtain that

$$\lambda\kappa \tilde{f}'_\lambda(\lambda, v) = \mu\langle \|\cdot\|'(v), v\rangle = \mu.$$

From this equality and (1.43) we get $\mu = 0$. Then, we necessarily have $\kappa \neq 0$. Combining (1.44) with (1.46), we have that

$$\lambda f'(u) = \tilde{f}'_v(\lambda, v) = 0$$

for $u = \lambda v$. Since $\lambda \neq 0$, we obtain $f'(u) = 0$ which concludes the proof. □

1.8 Historical comments

Ekeland's variational principle [100] was established in 1974 and is the nonlinear version of the Bishop–Phelps theorem [239, 240], with its main feature of how to use the norm completeness and a partial ordering to obtain a point where a linear functional achieves its supremum on a closed bounded convex set. A major consequence of Ekeland's variational principle is that even if it is not always possible to minimize a nonnegative C^1 functional Φ on a Banach space; however, there is always a minimizing sequence $(u_n)_{n\geq 1}$ such that $\Phi'(u_n)\to 0$ as $n\to\infty$.

Sullivan [281] observed that Ekeland's variational principle characterizes complete metric spaces in the following sense.

Theorem 1.29 *Let (M, d) be a metric space. Then M is complete if and only if the following holds: For every application $\Phi : M\to(-\infty, \infty]$, $\Phi \not\equiv \infty$, which is bounded from below, and for every $\varepsilon > 0$, there exists $z_\varepsilon \in M$ such that*

(i) $\Phi(z_\varepsilon) \leq \inf_M \Phi + \varepsilon$;
(ii) $\Phi(x) > \Phi(z_\varepsilon) - \varepsilon\, d(x, z_\varepsilon)$, *for any $x \in M \setminus \{z_\varepsilon\}$.*

The mountain pass theorem was established by Ambrosetti and Rabinowitz in [7]. Their original proof relies on some deep deformation techniques developed by Palais and Smale [231], [234], who put the main ideas of the Morse theory into the framework of differential topology on infinite dimensional manifolds. In this way, Palais and Smale replaced the finite dimensionality assumption with an appropriate compactness hypothesis.

The main underlying ideas of the fibering methods are the following:

1. Let X, Y be two real Banach spaces and $A : X \to Y$ be a nonlinear operator. Fix an element $h \in Y$ and consider the equation

$$A(u) = h. \tag{1.47}$$

We also consider the wider Banach space $\tilde{X}, \tilde{Y}$ such that $X \subset \tilde{X}$ and $Y \subset \tilde{Y}$. Now let $\tilde{A} : \tilde{X} \to \tilde{Y}$ be a nonlinear operator such that $\tilde{A}|_X = A$. Instead of (1.47) we consider the extended equation

$$\tilde{A}(\tilde{u}) = \tilde{h}. \tag{1.48}$$

2. We equip $\tilde{X}$ with a *nonlinear structure* associated to the nonlinear operator A.

For given spaces X, Y and a nonlinear operator $A : X \to Y$, that is, for the triple (X, A, Y) we construct a triple (ξ, α, η), where $\xi : \tilde{X} \to X$ and $\eta : \tilde{Y} \to Y$ are fibrations and $\alpha = (\tilde{A}, A) : (\tilde{X}, X) \to (\tilde{Y}, Y)$ is a morphism of fibrations, that is, $\eta \circ \tilde{A} = A \circ \xi$. If we take $\tilde{X} = \mathbb{R}^k \times X$, then we obtain the k-parametric fibering methods. If $k = 1$ we get the *one-parametric fibering methods.* The basic idea in *one parametric fibering methods* is the representation of solutions for $A(u) = h$ in the form

$$u = tv, \tag{1.49}$$

where $\in \mathbb{R} \setminus \{0\}$ is a real parameter and $v \in X \setminus \{0\}$ satisfying the fibering constraint

$$H(t, v) = c. \tag{1.50}$$

The function H will be said *fibering functional.* In particular case, when $H(t, v) \equiv \|v\|$, the condition (1.50) becomes $\|v\| = 1$ and it is called *spherical fibering.*

2
Variational Inequalities

> Just as houses are made of stones, so is science made of facts; but a pile of stones is not a house and a collection of facts is not necessarily science.
>
> Henri Poincaré (1854–1912)

2.1 Introduction

The theory of variational inequalities appeared in the middle 1960's, in connection with the notion of subdifferential in the sense of convex analysis. All the inequality problems treated to the beginning 1980's were related to convex energy functionals and therefore strictly connected to monotonicity: for instance, only monotone (possibly multivalued) boundary conditions and stress-strain laws could be studied. Nonconvex inequality problems first appeared in [203] in the setting of global analysis and were related to the subdifferential introduced in [82] (see Marino [202] for a survey of the developments in this direction). A typical feature of nonconvex problems is that, while in the convex case the stationary variational inequalities give rise to minimization problems for the potential or for the energy, in the nonconvex case the problem of the stationarity of the potential emerges and therefore it becomes reasonable to expect results also in the line of critical point theory.

2.2 Variational inequalities on $\Omega = \omega \times \mathbb{R}^l$

2.2.1 Case $l \geq 2$

Let $\Omega = \omega \times \mathbb{R}^l$ be an unbounded strip (or, in other words, a strip-like domain), where $\omega \subset \mathbb{R}^m$ is open bounded, and $l \geq 2$, $m \geq 1$. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function, which satisfies the following condition:

(f1) there exist $c_1 > 0$ and $p \in (2, 2^\star)$ such that

$$|f(x,s)| \leq c_1(|s| + |s|^{p-1}), \quad \text{for every} \quad (x,s) \in \Omega \times \mathbb{R}.$$

Here, we denoted by $2^* = 2(m+l)(m+l-2)^{-1}$ the critical Sobolev exponent.

As usual, $H_0^1(\Omega)$ is the Sobolev space endowed with the inner product

$$\langle u, v \rangle_{H_0^1} = \int_\Omega \nabla u \nabla v dx$$

and norm $\|\cdot\|_{H_0^1} = \sqrt{\langle \cdot, \cdot \rangle_{H_0^1}}$, while the norm of $L^\alpha(\Omega)$ will be denoted by $\|\cdot\|_\alpha$. It is well-known that the embedding $H_0^1(\Omega) \hookrightarrow L^\alpha(\Omega)$, $\alpha \in [2, 2^*]$, is continuous, that is, there exists $k_\alpha > 0$ such that $\|u\|_\alpha \leq k_\alpha \|u\|_{H_0^1}$ for every $u \in H_0^1(\Omega)$.

Consider finally the closed convex cone

$$\mathcal{K} = \{u \in H_0^1(\Omega) \ : \ u \geq 0 \text{ a.e. in } \ \Omega\}.$$

The aim of this section is to study the following (eigenvalue) problem for variational inequality (denoted by (P)):

Find $(u, \lambda) \in \mathcal{K} \times (0, \infty)$ *such that*

$$\int_\Omega \nabla u(x)(\nabla v(x) - \nabla u(x))dx - \lambda \int_\Omega f(x, u(x))(v(x) - u(x))dx \geq 0, \quad \forall v \in \mathcal{K}.$$

We say that a function $h : \Omega \to \mathbb{R}$ is *axially symmetric,* if $h(x,y) = h(x, gy)$ for all $x \in \omega$, $y \in \mathbb{R}^l$ and $g \in O(l)$. In particular, we denote by $H_{0,s}^1(\Omega)$ the closed subspace of axially symmetric functions of $H_0^1(\Omega)$. Define $F : \Omega \times \mathbb{R} \to \mathbb{R}$ by $F(x,s) = \displaystyle\int_0^s f(x,t)dt$ and beside of **(f1)**, we make the following assumptions:

(f2) $\displaystyle\lim_{s \to 0} \frac{|f(x,s)|}{s} = 0$ uniformly for every $x \in \Omega$.

(f3) There exist $q \in]0,2[, \nu \in [2,2^\star], \alpha \in L^{\nu/(\nu-q)}(\Omega), \beta \in L^1(\Omega)$ such that

$$F(x,s) \le \alpha(x)|s|^q + \beta(x).$$

(f4) There exists $u_0 \in H^1_{0,s}(\Omega) \cap \mathcal{K}$ such that $\displaystyle\int_\Omega F(x,u_0(x))dx > 0$.

The main result of this section can be formulated as follows.

Theorem 2.1 *Let $f : \Omega : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies* **(f1)-(f4)** *and $F(\cdot, s)$ is axially symmetric for every $s \in \mathbb{R}$. Then there is an open interval $\Lambda_0 \subset (0,\infty)$ such that for every $\lambda \in \Lambda_0$ there are at least three distinct elements $u_i^\lambda \in \mathcal{K}$ $(i \in \{1,2,3\})$ which are axially symmetric, having the property that (u_i^λ, λ) are solutions of* (P) *for every $i \in \{1,2,3\}$.*

From now on, we assume that the hypotheses of Theorem 2.1 are fulfilled. Before to prove Theorem 2.1, some preliminary results will be given.

Remark 2.1 *For every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that*

(i) $|f(x,s)| \le \varepsilon|s| + c(\varepsilon)|s|^{p-1}$ *for every* $(x,s) \in \Omega \times \mathbb{R}$.

(ii) $|F(x,s)| \le \varepsilon s^2 + c(\varepsilon)|s|^p$ *for every* $(x,s) \in \Omega \times \mathbb{R}$.

Let us define $\mathcal{F} : H^1_0(\Omega) \to \mathbb{R}$ by $\mathcal{F}(u) = \displaystyle\int_\Omega F(x,u(x))dx$. From condition **(f1)** follows that $\mathcal{F}$ is of class C^1 and $\mathcal{F}'(u)(v) = \displaystyle\int_\Omega f(x,u(x))v(x)dx$.

We consider the indicator function of the set $\mathcal{K}$, $\zeta_{\mathcal{K}} : H^1_0(\Omega) \to]-\infty,\infty]$, i.e.,

$$\zeta_{\mathcal{K}}(u) = \begin{cases} 0, & \text{if } u \in \mathcal{K} \\ +\infty, & \text{if } u \notin \mathcal{K}, \end{cases}$$

which is clearly convex, proper and lower semi-continuous. Moreover, define for $\lambda > 0$ the function $I_\lambda : H^1_0(\Omega) \to]-\infty,\infty]$ by

$$I_\lambda(u) = \frac{1}{2}\|u\|^2_{H^1_0} - \lambda\mathcal{F}(u) + \zeta_{\mathcal{K}}(u). \tag{2.1}$$

It is easily seen that I_λ is a Szulkin-type functional. Furthermore, one has

Remark 2.2 *If $u \in H^1_0(\Omega)$ is a critical point of I_λ, then (u,λ) is a solution of* (P).

Let $G = id_{\mathbb{R}^m} \times O(l) \subset O(m+l)$. Define the action of G on $H_0^1(\Omega)$ by $gu(x) = u(g^{-1}x)$ for every $g \in G$, $u \in H_0^1(\Omega)$ and $x \in \Omega$. Since $\mathcal{K}$ is a G-invariant set, then $\zeta_{\mathcal{K}}$ is a G-invariant function. Since $F(\cdot, s)$ is axially symmetric for every $s \in \mathbb{R}$, then $\mathcal{F}$ is also a G-invariant function. The norm $\|\cdot\|_{H_0^1}$ is a G-invariant function as well. In conclusion, if we consider the set

$$\Sigma = H_{0,s}^1(\Omega) = \{u \in H_0^1(\Omega) : gu = u \text{ for every } g \in G\},$$

then, in view of Theorem 1.27, every critical point of $I_\lambda|_\Sigma$ becomes as well critical point of I_λ.
We will apply Theorem 1.22

$$X = \Sigma = H_{0,s}^1(\Omega),\;\; E_1 = \frac{1}{2}\|\cdot\|_\Sigma^2,\;\; \zeta_1 = \zeta_{\mathcal{K}}|_\Sigma,\;\; E_2 = -\mathcal{F}|_\Sigma,\;\; \Lambda = [0, \infty[.$$

As usual, $\|\cdot\|_\Sigma$, $\zeta_{\mathcal{K}}|_\Sigma$ and $\mathcal{F}|_\Sigma$ denote the restrictions of $\|\cdot\|_{H_0^1}$, $\zeta_{\mathcal{K}}$ and $\mathcal{F}$ to Σ, respectively. We will use also the notation $\langle\cdot,\cdot\rangle_\Sigma$ for the restriction of $\langle\cdot,\cdot\rangle_{H_0^1}$ to Σ and also we denote by $\langle\cdot,\cdot\rangle_\Sigma$ the duality mapping restricted to $\Sigma \times \Sigma^\star$.

Now, we are going to verify the hypotheses (i)-(iii) of Theorem 1.22.

Step 1. (Verification of (i)). The weakly sequentially lower semicontinuity of E_1 is standard. We prove that E_2 is weakly sequentially continuous.

Let $\{u_n\}$ be a sequence from Σ which converges weakly to some $u \in \Sigma$. In particular, $\{u_n\}$ is bounded in Σ and by virtue of Lemma 2.1, $F(x,s) = o(s^2)$ as $s \to 0$, and $F(x,s) = o(s^{2^*})$ as $s \to +\infty$, uniformly for every $x \in \Omega$. But, from [107, Lemma 4, p. 368] follows that $E_2(u_n) \to E_2(u)$ as $n \to \infty$, that is, E_2 is weakly sequentially continuous.

Step 2. (Verification of (ii)). Fix $\lambda \in \Lambda$. First, we will prove that $I_1 + \lambda I_2 \equiv E_1 + \zeta_1 + \lambda E_2$ is coercive. Indeed, due to **(f3)**, by Hölder's inequality we have for every $u \in \Sigma$ that

$$\begin{aligned} I_1(u) + \lambda I_2(u) &\geq \frac{1}{2}\|u\|_\Sigma^2 - \lambda \int_\Omega \alpha(x)|u(x)|^q dx - \lambda \int_\Omega \beta(x) dx \\ &\geq \frac{1}{2}\|u\|_\Sigma^2 - \lambda \|\alpha\|_{\nu/(\nu-q)} k_\nu^q \|u\|_\Sigma^q - \lambda \|\beta\|_1. \end{aligned}$$

Since $q < 2$, it is clear that $\|u\|_\Sigma \to +\infty$ implies $I_1(u) + \lambda I_2(u) \to +\infty$, as claimed.

Now, we will prove that $I_1+\lambda I_2$ verifies $(PSZ)_c$, $c\in\mathbb{R}$. Let $\{u_n\}\subset\Sigma$ be a sequence such that

$$I_1(u_n)+\lambda I_2(u_n)\to c \tag{2.2}$$

and for every $v\in\Sigma$ we have

$$\langle u_n, v-u_n\rangle_\Sigma+\lambda\langle E_2'(u_n), v-u_n\rangle_\Sigma+\zeta_1(v)-\zeta_1(u_n)\geq -\varepsilon_n\|v-u_n\|_\Sigma, \tag{2.3}$$

for a sequence $\{\varepsilon_n\}$ in $[0,+\infty[$ with $\varepsilon_n\to 0$. In particular, (2.2) shows that $\{u_n\}\subset\mathcal{K}$. Moreover, the coerciveness of the function $I_1+\lambda I_2$ implies that the sequence $\{u_n\}$ is bounded in $\Sigma\cap\mathcal{K}$. Therefore, there exists an element $u\in\mathcal{K}\cap\Sigma$ such that $\{u_n\}$ converges weakly to u in Σ. (Note that $\mathcal{K}$ is convex and closed, thus, weakly closed.) Moreover, since the embedding $\Sigma\hookrightarrow L^p(\Omega)$ is compact (see [107]), up to a subsequence, $\{u_n\}$ converges strongly to u in $L^p(\Omega)$. Choosing in particular $v=u$ in (2.3), we have

$$\|u_n-u\|_\Sigma^2\leq\lambda\langle E_2'(u_n), u-u_n\rangle_\Sigma+\langle u, u-u_n\rangle_\Sigma+\varepsilon_n\|u-u_n\|_\Sigma.$$

The last two terms tend to zero as $n\to\infty$. Thus, in order to prove $\|u_n-u\|_\Sigma\to 0$, it is enough to show that the first term in the right hand side tends to zero as well. From Remark 2.1 a), we obtain

$$\begin{aligned}\langle E_2'(u_n), u-u_n\rangle_\Sigma &= \int_\Omega f(x,u_n(x))(-u(x)+u_n(x))dx\\ &\leq \int_\Omega \left[\varepsilon|u_n(x)|+c(\varepsilon)|u_n(x)|^{p-1}\right]|u_n(x)-u(x)|dx\\ &\leq \varepsilon k_2^2\|u_n\|_\Sigma\|u_n-u\|_\Sigma+c(\varepsilon)\|u_n\|_p^{p-1}\|u_n-u\|_p.\end{aligned}$$

Due to the arbitrariness of $\varepsilon>0$, the last term tends to zero, therefore, $\|u_n-u\|_\Sigma\to 0$ as $n\to\infty$.

Step 3. (Verification of (iii)). Let us define the function $\gamma:[0,\infty[\to\mathbb{R}$ by

$$\gamma(s)=\sup\{-E_2(u):\ \|u\|_\Sigma^2\leq 2s\}.$$

Due to Remark 2.1 (ii), one has

$$\gamma(s)\leq 2\varepsilon k_2^2 s+2^p c(\varepsilon)k_p^p s^{\frac{p}{2}}.$$

On the other hand, we know that $\gamma(s)\geq 0$ for $s\geq 0$. Due to the arbitrariness of $\varepsilon>0$, we deduce

$$\lim_{s\to 0^+}\frac{\gamma(s)}{s}=0.$$

By **(f4)** it is clear that $u_0 \neq 0$ ($E_2(0) = 0$). Therefore it is possible to choose a number η such that

$$0 < \eta < -2E_2(u_0)\|u_0\|_\Sigma^{-2}.$$

From $\lim_{s\to 0^+} \gamma(s)/s = 0$ follows the existence of a number $s_0 \in]0, \|u_0\|_\Sigma^2/2[$ such that $\gamma(s_0) < \eta s_0$. Therefore,

$$\gamma(s_0) < -2E_2(u_0)\|u_0\|_\Sigma^{-2} s_0.$$

Choose $\rho_0 > 0$ such that

$$\gamma(s_0) < \rho_0 < -2E_2(u_0)\|u_0\|_\Sigma^{-2} s_0. \tag{2.4}$$

Due to the choice of t_0 and (2.4) we have

$$\rho_0 < -E_2(u_0). \tag{2.5}$$

Define $h : \Lambda = [0, +\infty[\to \mathbb{R}$ by $h(\lambda) = \rho_0\lambda$. We prove that the function h satisfies the inequality

$$\sup_{\lambda\in\Lambda} \inf_{u\in\Sigma} (I_1(u) + \lambda I_2(u) + \rho_0\lambda) < \inf_{u\in\Sigma} \sup_{\lambda\in\Lambda} (I_1(u) + \lambda I_2(u) + \rho_0\lambda).$$

Note, that in the previous inequality we can put $\Sigma \cap \mathcal{K}$ instead of Σ; indeed, if $u \in \Sigma \setminus \mathcal{K}$, then $I_1(u) = +\infty$.

The function

$$\Lambda \ni \lambda \mapsto \inf_{u\in\Sigma\cap\mathcal{K}} [\|u\|_\Sigma^2/2 + \lambda(\rho_0 + E_2(u))]$$

is upper semi-continuous on Λ. Relation (2.5) implies that

$$\lim_{\lambda\to+\infty} \inf_{u\in\Sigma\cap\mathcal{K}} [I_1(u) + \lambda I_2(u) + \rho_0\lambda] \le \lim_{\lambda\to+\infty} \left[\|u_0\|_\Sigma^2/2 + \lambda(\rho_0 + E_2(u_0))\right] = -\infty.$$

Thus we find an element $\overline{\lambda} \in \Lambda$ such that

$$\sup_{\lambda\in\Lambda} \inf_{u\in\Sigma\cap\mathcal{K}} (I_1(u) + \lambda I_2(u) + \rho_0\lambda) = \inf_{u\in\Sigma\cap\mathcal{K}} \left[\|u\|_\Sigma^2/2 + \overline{\lambda}(\rho_0 + E_2(u))\right]. \tag{2.6}$$

Since $\gamma(s_0) < \rho_0$, for all $u \in \Sigma$ such that $\|u\|_\Sigma^2 \le 2s_0$, we have $E_2(u) > -\rho_0$. Thus, we have

$$s_0 \le \inf\{\|u\|_\Sigma^2/2 : E_2(u) \le -\rho_0\} \le \inf\{\|u\|_\Sigma^2/2 : u \in \mathcal{K},\ E_2(u) \le -\rho_0\}. \tag{2.7}$$

On the other hand

$$\inf_{u\in\Sigma\cap\mathcal{K}} \sup_{\lambda\in\Lambda} (I_1(u) + \lambda I_2(u) + \rho_0\lambda) = \inf_{u\in\Sigma\cap\mathcal{K}} \left[\|u\|_\Sigma^2/2 + \sup_{\lambda\in\Lambda} (\lambda(\rho_0 + E_2(u)))\right]$$

$$= \inf_{u\in\Sigma\cap\mathcal{K}} \left\{ \|u\|_\Sigma^2/2 \quad : \; E_2(u) \le -\rho_0 \; \right\}.$$

Therefore, relation (2.7) can be written as

$$s_0 \le \inf_{u\in\Sigma\cap\mathcal{K}} \sup_{\lambda\in\Lambda} (I_1(u) + \lambda I_2(u) + \rho_0\lambda). \tag{2.8}$$

There are two distinct cases:

(A) If $0 \le \overline{\lambda} < s_0/\rho_0$, we have

$$\inf_{u\in\Sigma\cap\mathcal{K}} \left[\|u\|_\Sigma^2/2 + \overline{\lambda}(\rho_0 + E_2(u))\right] \le \overline{\lambda}(\rho_0 + E_2(0)) = \overline{\lambda}\rho_0 < s_0.$$

Combining this inequality with (2.6) and (2.8) we obtain the desired inequality.

(B) If $s_0/\rho_0 \le \overline{\lambda}$, from $\rho_0 < -E_2(u_0)$ and (2.4) follows

$$\begin{aligned}\inf_{u\in\Sigma\cap\mathcal{K}} \left[\|u\|_\Sigma^2/2 + \overline{\lambda}(\rho_0 + E_2(u))\right] &\le \|u_0\|_\Sigma^2/2 + \overline{\lambda}(\rho_0 + E_2(u_0)) \\ &\le \|u_0\|_\Sigma^2/2 + s_0(\rho_0 + E_2(u_0))/\rho_0 < s_0.\end{aligned}$$

Now, we will repeat the last part of (A), which concludes Step 3.

Proof of Theorem 2.1. Due to the above three steps, Theorem 1.22 implies the existence of an open interval $\Lambda_0 \subset [0, \infty[$, such that for each $\lambda \in \Lambda_0$, the function $I_\lambda|_\Sigma \equiv I_1 + \lambda I_2$ has at least three critical points in $\Sigma \cap \mathcal{K}$. It remains to apply Theorem 1.27 and Remark 2.2. □

2.2.2 *Case* $l = 1$

In this section we will continue our studies on the strip-like domains, but contrary to the previous section, we consider domains of the form $\Omega = \omega \times \mathbb{R}$, where $\omega \subset \mathbb{R}^m (m \ge 1)$ is a bounded open subset.

J.-L. Lions [98] observed that defining the closed convex cone

$$\begin{aligned}\mathcal{K} = \{u \quad \in \quad & H_0^1(\omega \times \mathbb{R}) : u \text{ is nonnegative,} \\ & y \mapsto u(x,y) \text{ is nonincreasing for } x \in \omega,\ y \ge 0, \text{ and} \\ & y \mapsto u(x,y) \text{ is nondecreasing for } x \in \omega,\ y \le 0\},\end{aligned} \tag{K}$$

the bounded subsets of $\mathcal{K}$ are relatively compact in $L^p(\omega \times \mathbb{R})$ whenever $p \in (2, 2^*)$. Note that $2^\star = \infty$, if $m = 1$.

The main goal of this section is to give a new approach to treat elliptic (eigenvalue) problems on domains of the type $\Omega = \omega \times \mathbb{R}$. The genesis of our method relies on the Szulkin-type functionals. Indeed, since the indicator function of a closed convex subset of a vector space (so, in particular $\mathcal{K}$ in $H_0^1(\omega \times \mathbb{R})$) is convex, lower semi-continuous and proper, this approach arises in a natural manner. In order to formulate our problem, we shall consider a continuous function $f : (\omega \times \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ such that

(F_1) $f(x,0) = 0$, and there exist $c_1 > 0$ and $p \in (2, 2^*)$ such that

$$|f(x,s)| \leq c_1(|s| + |s|^{p-1}), \ \forall\ (x,s) \in (\omega \times \mathbb{R}) \times \mathbb{R}.$$

Let $a \in L^1(\omega \times \mathbb{R}) \cap L^\infty(\omega \times \mathbb{R})$ with $a \geq 0$, $a \not\equiv 0$, and $q \in (1,2)$. For $\lambda > 0$, we denote by (P_λ) the following *variational inequality problem*: *Find* $u \in \mathcal{K}$ *such that*

$$\int_{\omega \times \mathbb{R}} \nabla u(x) \nabla (v(x) - u(x))dx + \int_{\omega \times \mathbb{R}} f(x, u(x))(-v(x) + u(x))dx$$

$$\geq \lambda \int_{\omega \times \mathbb{R}} a(x)|u(x)|^{q-2}u(x)(v(x) - u(x))dx, \quad \forall v \in \mathcal{K}.$$

For the sake of notation, we introduce $\Omega = \omega \times \mathbb{R}$.

Define $F : \Omega \times \mathbb{R} \to \mathbb{R}$ by $F(x,s) = \displaystyle\int_0^s f(x,t)dt$ and beside of (F_1) we make the following assumptions:

(F_2) $\displaystyle\lim_{s \to 0} \frac{f(x,s)}{s} = 0$ uniformly for every $x \in \Omega$;

(F_3) There exists $\nu > 2$ such that

$$\nu F(x,s) - sf(x,s) \leq 0, \ \forall (x,s) \in \Omega \times \mathbb{R};$$

(F_4) There exists $R > 0$ such that

$$\inf\{F(x,s) : (x, |s|) \in \omega \times [R, \infty)\} > 0.$$

Lemma 2.1 *If the functions* $f, F : \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfies* (F_1), (F_3) *and* (F_4) *then there exist* $c_2, c_3 > 0$ *such that*

$$F(x,s) \geq c_2|s|^\nu - c_3 s^2, \ \forall (x,s) \in \Omega \times \mathbb{R}.$$

Proof First, for arbitrary fixed $(x,u) \in \Omega \times \mathbb{R}$ we consider the function $g : (0, +\infty) \to \mathbb{R}$ defined by

$$g(s) = s^{-\nu} F(x, su).$$

Clearly, g is a function of class $\mathcal{C}^1$ and we have

$$g'(s) = -\nu s^{-\nu-1} F(x, su) + s^{-\nu} u f(x, su), \ s > 0.$$

Let $s > 1$ and by mean value theorem, there exist $\tau = \tau(x, u) \in (1, s)$ such that $g(s) - g(1) = g'(\tau)(s-1)$. Therefore, $g'(\tau) = -\nu\tau^{-\nu-1}F(x, \tau u) + \tau^{-\nu}uf(x, \tau u)$, thus

$$g(s) - g(1) = -\tau^{-\nu-1}[\nu F(x, \tau u) - \tau uf(x, \tau u)](s-1).$$

By (F_3) one has $g(s) \geq g(1)$, i.e. $F(x, su) \geq s^\nu F(x, u)$, for every $s \geq 1$. Let $c_R = \inf\{F(x, s) : (x, |s|) \in \omega \times [R, \infty)\}$, which is a strictly positive number, due to (F_4). Combining the above facts we derive

$$F(x, s) \geq \frac{c_R}{R^\nu}|s|^\nu, \ \forall (x, s) \in \Omega \times \mathbb{R} \text{ with } |s| \geq R. \tag{2.9}$$

On the other hand, by $((F_1))$ we have $|F(x, s)| \leq c_1(s^2 + |s|^p)$ for every $(x, s) \in \Omega \times \mathbb{R}$. In particular, we have

$$-F(x, s) \leq c_1(s^2 + |s|^p) \leq c_1(1 + R^{p-2} + R^{\nu-2})s^2 - c_1|s|^\nu$$

for every $(x, s) \in \Omega \times \mathbb{R}$ with $|s| \leq R$. Combining the above inequality with (2.9), the desired inequality yields if one chooses $c_2 = \min\{c_1, c_R/R^\nu\}$ and $c_3 = c_1(1 + R^{p-2} + R^{\nu-2})$. $\square$

Remark 2.3 In particular, from Lemma 2.1 we observe that $2 < \nu < p$.

To investigate the existence of solutions of (P_λ) we shall construct a functional $I_\lambda : H_0^1(\Omega) \to \mathbb{R}$ associated to (P_λ) which is defined by

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(x, u(x))dx - \frac{\lambda}{q}\int_\Omega a(x)|u|^q + \zeta_{\mathcal{K}}(u),$$

where $\zeta_{\mathcal{K}}$ is the indicator function of the set $\mathcal{K}$.

If we consider the function $\mathcal{F} : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{F}(u) = \int_\Omega F(x, u(x))dx,$$

then $\mathcal{F}$ is of class $\mathcal{C}^1$ and

$$\langle \mathcal{F}'(u), v\rangle_{H_0^1} = \int_\Omega f(x, u(x))v(x)dx, \quad \forall \ u, v \in H_0^1(\Omega).$$

By standard arguments we have that the functionals $A_1, A_2 : H_0^1(\Omega) \to \mathbb{R}$, defined by $A_1(u) = \|u\|^2_{H_0^1}$ and $A_2(u) = \int_\Omega a(x)|u|^q dx$ are of class $\mathcal{C}^1$ with derivatives

$$\langle A_1'(u), v\rangle_{H_0^1} = 2\langle u, v\rangle_{H_0^1}$$

and

$$\langle A_2'(u), v\rangle_{H_0^1} = q\int_\Omega a(x)|u|^{q-2}uvdx.$$

Therefore the function

$$E_\lambda(u) = \frac{1}{2}\|u\|^2_{H_0^1} - \frac{\lambda}{q}\int_\Omega a(x)|u|^q - \mathcal{F}(u)$$

on $H_0^1(\Omega)$ is of class $\mathcal{C}^1$. On the other hand, the indicator function of the set $\mathcal{K}$, i.e.

$$\zeta_{\mathcal{K}}(u) = \begin{cases} 0, & \text{if } u \in \mathcal{K}, \\ +\infty, & \text{if } u \notin \mathcal{K}, \end{cases}$$

is convex, proper, and lower semi-continuous. In conclusion, $I_\lambda = E_\lambda + \zeta_{\mathcal{K}}$ is a Szulkin-type functional.

Moreover, one easily obtain the following

Proposition 2.1 *Fix $\lambda > 0$ arbitrary. Every critical point $u \in H_0^1(\Omega)$ of $I_\lambda = E_\lambda + \zeta_{\mathcal{K}}$ (in the sense of Szulkin) is a solution of* (P_λ).

Proof Since $u \in H_0^1(\Omega)$ is a critical point of $I_\lambda = E_\lambda + \zeta_{\mathcal{K}}$, one has

$$\langle E'_\lambda(u), v - u\rangle_{H_0^1} + \zeta_{\mathcal{K}}(v) - \zeta_{\mathcal{K}}(u) \geq 0, \forall v \in H_0^1(\Omega).$$

We have immediately that u belongs to $\mathcal{K}$. Otherwise, we would have $\zeta_{\mathcal{K}}(u) = +\infty$ which led us to a contradiction, letting for instance $v = 0 \in \mathcal{K}$ in the above inequality. Now, we fix $v \in \mathcal{K}$ arbitrary and we obtain the desired inequality. $\square$

Remark 2.4 It is easy to see that $0 \in \mathcal{K}$ is a trivial solution of (P_λ) for every $\lambda \in \mathbb{R}$.

Proposition 2.2 *If the conditions $(F_1) - (F_3)$ hold, then $I_\lambda = E_\lambda + \zeta_{\mathcal{K}}$ satisfies $(PSZ)_c$-condition for every $c \in \mathbb{R}$ and $\lambda > 0$.*

Proof Let $\lambda > 0$ and $c \in \mathbb{R}$ be some fixed numbers and let $\{u_n\}$ be a sequence from $H_0^1(\Omega)$ such that

$$I_\lambda(u_n) = E_\lambda(u_n) + \zeta_{\mathcal{K}}(u_n) \to c; \tag{2.10}$$

$$\langle E'_\lambda(u_n), v - u_n\rangle_{H_0^1} + \zeta_{\mathcal{K}}(v) - \zeta_{\mathcal{K}}(u_n) \geq -\varepsilon_n\|v - u_n\|_{H_0^1}, \forall v \in H_0^1(\Omega), \tag{2.11}$$

for a sequence $\{\varepsilon_n\}$ in $[0, \infty)$ with $\varepsilon_n \to 0$. By (2.10) one concludes that the sequence $\{u_n\}$ belongs entirely to $\mathcal{K}$. Setting $v = 2u_n$ in (2.11), we obtain

$$\langle E'_\lambda(u_n), u_n\rangle_{H_0^1} \geq -\varepsilon_n\|u_n\|_{H_0^1}.$$

From the above inequality we derive

$$\|u_n\|^2_{H^1_0} - \lambda \int_\Omega a(x)|u_n|^q - \int_\Omega f(x,u_n(x))u_n(x)dx \geq -\varepsilon_n \|u_n\|_{H^1_0}. \quad (2.12)$$

By (2.10) one has for large $n \in \mathbb{N}$ that

$$c+1 \geq \frac{1}{2}\|u_n\|^2_{H^1_0} - \frac{\lambda}{q}\int_\Omega a(x)|u_n|^q - \int_\Omega F(x,u_n(x))dx \qquad (2.13)$$

Multiplying (2.12) by ν^{-1} and adding this one to (2.13), by Hölder's inequality we have for large $n \in \mathbb{N}$

$$\begin{aligned} c+1+\frac{1}{\nu}\|u_n\|_{H^1_0} &\geq (\frac{1}{2}-\frac{1}{\nu})\|u_n\|^2_{H^1_0} - \lambda(\frac{1}{q}-\frac{1}{\nu})\int_\Omega a(x)|u_n|^q \\ &\quad -\frac{1}{\nu}\int_\Omega [\nu F(x,u_n(x)) - u_n(x)f(x,u_n(x))]dx \\ &\overset{(F_3)}{\geq} (\frac{1}{2}-\frac{1}{\nu})\|u_n\|^2_{H^1_0} - \lambda(\frac{1}{q}-\frac{1}{\nu})\|a\|_{\nu/(\nu-q)}\|u_n\|^q_\nu \\ &\geq (\frac{1}{2}-\frac{1}{\nu})\|u_n\|^2_{H^1_0} - \lambda(\frac{1}{q}-\frac{1}{\nu})\|a\|_{\nu/(\nu-q)}k^q_\nu\|u_n\|^q_{H^1_0}. \end{aligned}$$

In the above inequalities we used the Remark 2.3 and the hypothesis $a \in L^1(\Omega) \cap L^\infty(\Omega)$ thus, in particular, $a \in L^{\nu/(\nu-q)}(\Omega)$. Since $q < 2 < \nu$, from the above estimation we derive that the sequence $\{u_n\}$ is bounded in $\mathcal{K}$. Therefore, $\{u_n\}$ is relatively compact in $L^p(\Omega)$, $p \in (2, 2^*)$. Up to a subsequence, we can suppose that

$$u_n \to u \text{ weakly in } H^1_0(\Omega); \qquad (2.14)$$

$$u_n \to u \text{ strongly in } L^\mu(\Omega),\ \mu \in (2, 2^*). \qquad (2.15)$$

Since $\mathcal{K}$ is (weakly) closed then $u \in \mathcal{K}$. Setting $v = u$ in (2.11), we have

$$\langle u_n, u-u_n\rangle_{H^1_0} + \int_\Omega f(x,u_n(x))(u_n(x)-u(x))dx$$

$$-\lambda\int_\Omega a(x)|u_n|^{q-2}u_n(u-u_n) \geq -\varepsilon_n\|u-u_n\|_{H^1_0}.$$

Therefore, in view of Remark 2.1 (i) we derive

$$\begin{aligned}
\|u-u_n\|^2_{H^1_0} &\le \langle u, u-u_n\rangle_{H^1_0} + \int_\Omega f(x,u_n(x))(u_n(x)-u(x))dx \\
&\quad -\lambda \int_\Omega a(x)|u_n|^{q-2}u_n(u-u_n) + \varepsilon_n\|u-u_n\|_{H^1_0} \\
&\le \langle u, u-u_n\rangle_{H^1_0} + \lambda\|a\|_{\nu/(\nu-q)}\|u_n\|^{q-1}_\nu\|u-u_n\|_\nu + \varepsilon_n\|u-u_n\|_{H^1_0} \\
&\quad +\varepsilon\|u_n\|_{H^1_0}\|u_n-u\|_{H^1_0} + c(\varepsilon)\|u_n\|^{p-1}_p\|u_n-u\|_p,
\end{aligned}$$

where $\varepsilon > 0$ is arbitrary small. Taking into account relations (2.14) and (2.15), the facts that $\nu, p \in (2, 2^*)$, the arbitrariness of $\varepsilon > 0$ and $\varepsilon_n \to 0^+$, one has that $\{u_n\}$ converges strongly to u in $H^1_0(\Omega)$. This completes the proof. $\square$

Proposition 2.3 *If the conditions $(F_1) - (F_4)$ are verified, then there exists a $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$ the function I_λ satisfies the Mountain Pass Geometry, i.e. the following assertions are true:*

(i) *there exist constants $\alpha_\lambda > 0$ and $\rho_\lambda > 0$ such that $I_\lambda(u) \ge \alpha_\lambda$ for all $\|u\|_{H^1_0} = \rho_\lambda$;*
(ii) *there exists $e_\lambda \in H^1_0(\Omega)$ with $\|e_\lambda\|_{H^1_0} > \rho_\lambda$ and $I_\lambda(e_\lambda) \le 0$.*

Proof (i). Due to Remark 2.1 (ii), for every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that $\mathcal{F}(u) \le \varepsilon\|u\|^2_{H^1_0} + c(\varepsilon)\|u\|^p_p$ for every $u \in H^1_0(\Omega)$. It suffices to restrict our attention to elements u which belong to $\mathcal{K}$; otherwise $I_\lambda(u)$ will be $+\infty$, i.e. (i) holds trivially. Fix $\varepsilon_0 \in (0, \frac{1}{2})$. One has

$$\begin{aligned}
I_\lambda(u) &\ge (\frac{1}{2} - \varepsilon_0)\|u\|^2_{H^1_0} - k_p^p c(\varepsilon_0)\|u\|^p_{H^1_0} - \frac{\lambda k_p^q}{q}\|a\|_{p/(p-q)}\|u\|^q_{H^1_0} \qquad (2.16)\\
&= (A - B\|u\|^{p-2}_{H^1_0} - \lambda C\|u\|^{q-2}_{H^1_0})\|u\|^2_{H^1_0},
\end{aligned}$$

where $A = (\frac{1}{2} - \varepsilon_0) > 0$, $B = k_p^p c(\varepsilon_0) > 0$ and $C = k_p^q\|a\|_{p/(p-q)}/q > 0$.

For every $\lambda > 0$, let us define a function $g_\lambda : (0,\infty) \to \mathbb{R}$ by

$$g_\lambda(s) = A - Bs^{p-2} - \lambda C s^{q-2}.$$

Clearly, $g'_\lambda(s_\lambda) = 0$ if and only if $s_\lambda = (\lambda\frac{2-q}{p-2}\frac{C}{B})^{\frac{1}{p-q}}$. Moreover, $g_\lambda(s_\lambda) = A - D\lambda^{\frac{p-2}{p-q}}$, where $D = D(p,q,B,C) > 0$. Choosing $\lambda_0 > 0$ such that $g_{\lambda_0}(s_{\lambda_0}) > 0$, one clearly has for every $\lambda \in (0, \lambda_0)$ that $g_\lambda(s_\lambda) > 0$. Therefore, for every $\lambda \in (0, \lambda_0)$, setting $\rho_\lambda = s_\lambda$ and $\alpha_\lambda = g_\lambda(s_\lambda)s^2_\lambda$, the assertion from (i) holds true.

ii). By Lemma 2.1 we have $\mathcal{F}(u) \geq c_2\|u\|_\nu^\nu - c_3\|u\|_2^2$ for every $u \in H_0^1(\Omega)$. Let us fix $u \in \mathcal{K}$. Then we have

$$I_\lambda(u) \leq (\frac{1}{2} + c_3 k_2^2)\|u\|_{H_0^1}^2 - c_2\|u\|_\nu^\nu + \frac{\lambda}{q}\|a\|_{\nu/(\nu-q)} k_\nu^q \|u\|_{H_0^1}^q. \tag{2.17}$$

Fix arbitrary $u_0 \in \mathcal{K} \setminus \{0\}$. Letting $u = su_0$ $(s > 0)$ in (2.17), we have that $I_\lambda(su_0) \to -\infty$ as $s \to +\infty$, since $\nu > 2 > q$. Thus, for every $\lambda \in (0, \lambda_0)$, it is possible to set $s = s_\lambda$ so large that for $e_\lambda = s_\lambda u_0$, we have $\|e_\lambda\|_{H_0^1} > \rho_\lambda$ and $I_\lambda(e_\lambda) \leq 0$. This ends the proof of the proposition. □

The main result of this section can be read as follows.

Theorem 2.2 *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function which satisfies (F_1)-(F_4). Then there exists $\lambda_0 > 0$ such that* (P$_\lambda$) *has at least two nontrivial, distinct solutions u_λ^1, $u_\lambda^2 \in \mathcal{K}$ whenever $\lambda \in (0, \lambda_0)$.*

Proof In the first step we prove the existence of the first nontrivial solution of (P$_\lambda$). By Proposition 2.2, the functional I_λ satisfies $(PSZ)_c$-condition for every $c \in \mathbb{R}$ and clearly $I_\lambda(0) = 0$ for every $\lambda > 0$. Let us fix $\lambda \in (0, \lambda_0)$, λ_0 being from Proposition 2.3. It follows that there are constants $\alpha_\lambda, \rho_\lambda > 0$ and $e_\lambda \in H_0^1(\Omega)$ such that I_λ fulfills the properties (i) and (ii) from Theorem 1.21. Therefore, the number $c_\lambda^1 = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \mathcal{J}_\lambda(\gamma(t))$, where $\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = e_\lambda\}$, is a critical value of I_λ with $c_\lambda^1 \geq \alpha_\lambda > 0$. It is clear that the critical point $u_\lambda^1 \in H_0^1(\Omega)$ which corresponds to c_λ^1 cannot be trivial since $I_\lambda(u_\lambda^1) = c_\lambda^1 > 0 = I_\lambda(0)$. It remains to apply Proposition 2.1 which concludes that u_λ^1 is actually an element of $\mathcal{K}$ and it is a solution of (P$_\lambda$).

In the next step we prove the existence of the second solution of the problem (P_λ). For this let us fix $\lambda \in (0, \lambda_0)$ arbitrary, λ_0 being from the first step. By Proposition 2.3, there exists $\rho_\lambda > 0$ such that

$$\inf_{\|u\|_{H_0^1} = \rho_\lambda} I_\lambda(u) > 0. \tag{2.18}$$

On the other hand, since $a \geq 0$, $a \not\equiv 0$, there exists $u_0 \in \mathcal{K}$ such that $\int_\Omega a(x)|u_0(x)|^q dx > 0$. Thus, for $t > 0$ small one has

$$I_\lambda(tu_0) \leq t^2(\frac{1}{2} + c_3 k_2^2)\|u_0\|_{H_0^1}^2 - c_2 t^\nu \|u_0\|_\nu^\nu - \frac{\lambda}{q} t^q \int_\Omega a(x)|u_0(x)|^q dx < 0.$$

For $r > 0$, let us denote by $B_r = \{u \in H_0^1(\Omega) : \|u\|_{H_0^1} \leq r\}$ and

$S_r = \{u \in H_0^1(\Omega) : \|u\|_{H_0^1} = r\}$. With these notations, relation (2.18) and the above inequality can be summarized as

$$c_\lambda^2 = \inf_{u \in B_{\rho_\lambda}} I_\lambda(u) < 0 < \inf_{u \in S_{\rho_\lambda}} I_\lambda(u). \tag{2.19}$$

We point out that c_λ^2 is finite, due to (2.16). Moreover, we will show that c_λ^2 is another critical point of I_λ. To this end, let $n \in \mathbb{N} \setminus \{0\}$ such that

$$\frac{1}{n} < \inf_{u \in S_{\rho_\lambda}} I_\lambda(u) - \inf_{u \in B_{\rho_\lambda}} I_\lambda(u). \tag{2.20}$$

By Ekeland's variational principle, applied to the lower semi-continuous functional $I_{\lambda|_{B_{\rho_\lambda}}}$, which is bounded below (see (2.19)), there is $u_{\lambda,n} \in B_{\rho_\lambda}$ such that

$$I_\lambda(u_{\lambda,n}) \leq \inf_{u \in B_{\rho_\lambda}} I_\lambda(u) + \frac{1}{n}; \tag{2.21}$$

$$I_\lambda(w) \geq I_\lambda(u_{\lambda,n}) - \frac{1}{n}\|w - u_{\lambda,n}\|_{H_0^1},\ \forall w \in B_{\rho_\lambda}. \tag{2.22}$$

By (2.20) and (2.21) we have that $I_\lambda(u_{\lambda,n}) < \inf_{u \in S_{\rho_\lambda}} I_\lambda(u)$; therefore $\|u_{\lambda,n}\|_{H_0^1} < \rho_\lambda$.

Fix an element $v \in H_0^1(\Omega)$. It is possible to choose $t > 0$ so small that $w = u_{\lambda,n} + t(v - u_{\lambda,n}) \in B_{\rho_\lambda}$. Putting this element into (2.22), using the convexity of $\zeta_\mathcal{K}$ and dividing by $t > 0$, one concludes

$$\frac{E_\lambda(u_{\lambda,n} + t(v - u_{\lambda,n})) - E_\lambda(u_{\lambda,n})}{t} + \zeta_\mathcal{K}(v) - \zeta_\mathcal{K}(u_{\lambda,n}) \geq -\frac{1}{n}\|v - u_{\lambda,n}\|_{H_0^1}.$$

Letting $t \to 0^+$, we derive

$$\langle E'_\lambda(u_{\lambda,n}), v - u_{\lambda,n}\rangle_{H_0^1(\Omega)} + \zeta_\mathcal{K}(v) - \zeta_\mathcal{K}(u_{\lambda,n}) \geq -\frac{1}{n}\|v - u_{\lambda,n}\|_{H_0^1}. \tag{2.23}$$

By (2.19) and (2.21) we obtain that

$$I_\lambda(u_{\lambda,n}) = E_\lambda(u_{\lambda,n}) + \zeta_\mathcal{K}(u_{\lambda,n}) \to c_\lambda^2 \tag{2.24}$$

as $n \to \infty$. Since v was arbitrary fixed in (2.23), the sequence $\{u_{\lambda,n}\}$ fulfills (2.10) and (2.11), respectively. Hence, it is possible to prove in a similar manner as in Proposition 2.2 that $\{u_{\lambda,n}\}$ contains a convergent subsequence; denote it again by $\{u_{\lambda,n}\}$ and its limit point by u_λ^2. It is clear that u_λ^2 belongs to B_{ρ_λ}. By the lower semi-continuity of $\zeta_\mathcal{K}$ we have $\zeta_\mathcal{K}(u_\lambda^2) \leq \liminf_{n\to\infty} \zeta_\mathcal{K}(u_{\lambda,n})$. Combining this inequality with $\lim_{n\to\infty}\langle E'_\lambda(u_{\lambda,n}), v - u_{\lambda,n}\rangle_{H_0^1} = \langle E'_\lambda(u_\lambda^2), v - u_\lambda^2\rangle_{H_0^1}$ and (2.23) we have

$$\langle E'_\lambda(u_\lambda^2), v - u_\lambda^2\rangle_{H_0^1} + \zeta_\mathcal{K}(v) - \zeta_\mathcal{K}(u_\lambda^2) \geq 0,\ \forall v \in H_0^1(\Omega),$$

i.e. u_λ^2 is a critical point of I_λ. Moreover,

$$c_\lambda^2 \stackrel{(2.19)}{=} \inf_{u \in B_{\rho_\lambda}} I_\lambda(u) \le I_\lambda(u_\lambda^2) \le \liminf_{n\to\infty} I_\lambda(u_{\lambda,n}) \stackrel{(2.24)}{=} c_\lambda^2,$$

i.e. $I_\lambda(u_\lambda^2) = c_\lambda^2$. Since $c_\lambda^2 < 0$, it follows that u_λ^2 is not trivial. We apply again Proposition 2.1, concluding that u_λ^2 is a solution of (P_λ) which differs from u_λ^1. This completes the proof of Theorem 2.2. □

2.3 Area–type variational inequalities

In this section we are interested to obtain existence and multiplicity results for hemivariational inequalities associated with energies which come from the relaxation of functionals

$$f(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx + \int_\Omega G(x,u)\, dx\,,$$

where $u \in W_0^{1,1}(\Omega;\mathbb{R}^N)$, Ω open in $\mathbb{R}^n$, $n \ge 2$. The first feature is that the functional f does not satisfy the Palais-Smale condition in the space of *functions of bounded variations*

$$BV(\Omega;\mathbb{R}^N) := \left\{ u \in L^1(\Omega);\ \frac{\partial u}{\partial x_i} \text{ is a measure, for all } i = 1,2,\ldots,n \right\},$$

which is the natural domain of f. Therefore we extend f to $L^{n/(n-1)}(\Omega;\mathbb{R}^N)$ with value $+\infty$ outside $BV(\Omega;\mathbb{R}^N)$. This larger space is better behaved for the compactness properties, but the nonsmoothness of the functional increases. The second feature is that the assumptions we impose on G imply the second term of f to be continuous on $L^{n/(n-1)}(\Omega;\mathbb{R}^N)$, but not locally Lipschitz. More precisely, the function $\{s \mapsto G(x,s)\}$ is supposed to be locally Lipschitz for a.e. $x \in \Omega$, but the growth conditions we impose do not ensure the corresponding property for the integral on $L^{n/(n-1)}(\Omega;\mathbb{R}^N)$.

After recalling the main tools we need in the sequel, we establish some general results for a class of lower semicontinuous functionals $f : L^p(\Omega;\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$. Next, we show that the area-type integrals fall into the class considered in this section. By the way, we also prove a relation between the convergence in the so-called intermediate topologies of $BV(\Omega;\mathbb{R}^N)$ and the convergence in $L^{n/(n-1)}(\Omega;\mathbb{R}^N)$. Finally, we obtain multiplicity results of Clark and Ambrosetti-Rabinowitz type.

We start with some notions and properties on nonsmooth analysis. We refer to Appendix D for basic definitions and related results.

Definition 2.1 Let $c \in \mathbb{R}$. We say that f satisfies condition $(epi)_c$, if there exists $\varepsilon > 0$ such that

$$\inf\{|d\mathcal{G}_f|(u,\lambda) : \; f(u) < \lambda, \; |\lambda - c| < \varepsilon\} > 0\,.$$

The next two results are useful in dealing with condition $(epi)_c$.

Proposition 2.4 *Let $(u,\lambda) \in \mathrm{epi}\,(f)$. Assume that there exist positive numbers ϱ, σ, δ, ε and a continuous map*

$$\mathcal{H} : \{w \in B_\delta(u) : f(w) < \lambda + \delta\} \times [0,\delta] \to X$$

satisfying

$$d(\mathcal{H}(w,t), w) \le \varrho t\,, \qquad f(\mathcal{H}(w,t)) \le \max\{f(w) - \sigma t, \lambda - \varepsilon\}$$

whenever $w \in B_\delta(u)$, $f(w) < \lambda + \delta$ and $t \in [0,\delta]$.

Under these assumptions,

$$|d\mathcal{G}_f|(u,\lambda) \ge \frac{\sigma}{\sqrt{\varrho^2 + \sigma^2}}\,.$$

If, moreover, X is a normed space, f is even, $u = 0$ and $\mathcal{H}(-w,t) = -\mathcal{H}(w,t)$, then

$$|d_{\mathbb{Z}_2}\mathcal{G}_f|(0,\lambda) \ge \frac{\sigma}{\sqrt{\varrho^2 + \sigma^2}}\,.$$

Proof Let $\delta' \in]0,\delta]$ be such that $\delta' + \sigma\delta' \le \varepsilon$ and let

$$\mathcal{K} : (B_{\delta'}(u,\lambda) \cap \mathrm{epi}\,(f)) \times [0,\delta'] \to \mathrm{epi}\,(f)$$

be defined by $\mathcal{K}\big((w,\mu),t\big) = \big(\mathcal{H}(w,t), \mu - \sigma t\big)$. If $(w,\mu) \in B_{\delta'}(u,\lambda) \cap \mathrm{epi}\,(f)$ and $t \in [0,\delta']$, we have

$$\lambda - \varepsilon \le \lambda - \delta' - \sigma\delta' < \mu - \sigma t\,, \qquad f(w) - \sigma t \le \mu - \sigma t\,,$$

hence

$$f(\mathcal{H}(w,t)) \le \max\{f(w) - \sigma t, \lambda - \varepsilon\} \le \mu - \sigma t\,.$$

Therefore $\mathcal{K}$ actually takes its values in $\mathrm{epi}\,(f)$. Furthermore, it is

$$d\big(\mathcal{K}\big((w,\mu),t\big),(w,\mu)\big) \le \sqrt{\varrho^2 + \sigma^2}\,t\,,$$

$$\mathcal{G}_f\big(\mathcal{K}\big((w,\mu),t\big)\big) = \mu - \sigma t = \mathcal{G}_f(w,\mu) - \sigma t\,.$$

Taking into account Definition D.6, the first assertion follows.

In the symmetric case, $\mathcal{K}$ automatically satisfies the further condition required in Definition D.14. $\square$

Corollary 2.1 *Let $(u, \lambda) \in \operatorname{epi}(f)$ with $f(u) < \lambda$. Assume that for every $\varrho > 0$ there exist $\delta > 0$ and a continuous map*

$$\mathcal{H} : \{w \in B_\delta(u) : f(w) < \lambda + \delta\} \times [0, \delta] \to X$$

satisfying

$$d(\mathcal{H}(w,t), w) \le \varrho t\,, \qquad f(\mathcal{H}(w,t)) \le f(w) + t\big(f(u) - f(w) + \varrho\big)$$

whenever $w \in B_\delta(u)$, $f(w) < \lambda + \delta$ and $t \in [0, \delta]$.

Then we have $|d\mathcal{G}_f|(u, \lambda) = 1$. If moreover X is a normed space, f is even, $u = 0$ and $\mathcal{H}(-w, t) = -\mathcal{H}(w, t)$, then $|d_{\mathbb{Z}_2}\mathcal{G}_f|(0, \lambda) = 1$.

Proof Let $\varepsilon > 0$ with $\lambda - 2\varepsilon > f(u)$, let $0 < \varrho < \lambda - f(u) - 2\varepsilon$ and let δ and $\mathcal{H}$ be as in the hypothesis. By reducing δ, we may also assume that

$$\delta \le 1\,, \qquad \delta\big(|\lambda - 2\varepsilon| + |f(u) + \varrho|\big) \le \varepsilon\,.$$

Now consider $w \in B_\delta(u)$ with $f(w) < \lambda + \delta$ and $t \in [0, \delta]$. If $f(w) \le \lambda - 2\varepsilon$, we have

$$\begin{aligned} f(w) + t(f(u) - f(w) + \varrho) &= (1-t)f(w) + t(f(u) + \varrho) \le \\ &\le (1-t)(\lambda - 2\varepsilon) + t(f(u) + \varrho) \le \\ &\le \lambda - 2\varepsilon + t|\lambda - 2\varepsilon| + t|f(u) + \varrho| \le \lambda - \varepsilon\,, \end{aligned}$$

while, if $f(w) > \lambda - 2\varepsilon$, we have

$$f(w) + t(f(u) - f(w) + \varrho) \le f(w) - (\lambda - f(u) - 2\varepsilon - \varrho)t\,.$$

In any case it follows

$$f(\mathcal{H}(w,t)) \le \max\{f(w) - (\lambda - f(u) - 2\varepsilon - \varrho)t, \lambda - \varepsilon\}\ .$$

From Proposition 2.4 we get

$$|d\mathcal{G}_f|(u, \lambda) \ge \frac{\lambda - f(u) - 2\varepsilon - \varrho}{\sqrt{\varrho^2 + (\lambda - f(u) - 2\varepsilon - \varrho)^2}}$$

and the first assertion follows by the arbitrariness of ϱ.

The same proof works also in the symmetric case. □

2.3.1 Statement of the problem

Let $n \ge 1$, $N \ge 1$, Ω be an open subset of $\mathbb{R}^n$ and $1 < p < \infty$. In the following, we denote by $\|\cdot\|_q$ the usual norm in L^q $(1 \le q \le \infty)$. We now define the functional setting we are interested in.

Let $\mathcal{E} : L^p(\Omega; \mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ be a functional such that:

$(\mathcal{E}_1)$ $\mathcal{E}$ is convex, lower semicontinuous and $0 \in \mathcal{D}(\mathcal{E})$, where

$$\mathcal{D}(\mathcal{E}) = \left\{u \in L^p(\Omega;\mathbb{R}^N) : \mathcal{E}(u) < +\infty\right\} ;$$

$(\mathcal{E}_2)$ there exists $\vartheta \in C_c(\mathbb{R}^N)$ with $0 \le \vartheta \le 1$ and $\vartheta(0) = 1$ such that

$(\mathcal{E}_2.1)$ $\quad \forall u \in \mathcal{D}(\mathcal{E}),\ \forall v \in \mathcal{D}(\mathcal{E}) \cap L^\infty(\Omega;\mathbb{R}^N),\ \forall c > 0 :$

$$\lim_{h\to\infty} \left[\sup_{\substack{\|z-u\|_p \le c \\ \mathcal{E}(z) \le c}} \mathcal{E}\left(\vartheta\left(\frac{z}{h}\right) v\right)\right] = \mathcal{E}(v) ;$$

$(\mathcal{E}_2.2)$ $\quad \forall u \in \mathcal{D}(\mathcal{E}) : \displaystyle\lim_{h\to\infty} \mathcal{E}\left(\vartheta\left(\frac{u}{h}\right) u\right) = \mathcal{E}(u) .$

Moreover, let $G : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a function such that

(G_1) $G(\cdot, s)$ is measurable for every $s \in \mathbb{R}^N$;

(G_2) for every $t > 0$ there exists $\alpha_t \in L^1(\Omega)$ such that

$$|G(x, s_1) - G(x, s_2)| \le \alpha_t(x)|s_1 - s_2|$$

for a.e. $x \in \Omega$ and every $s_1, s_2 \in \mathbb{R}^N$ with $|s_j| \le t$; for a.e. $x \in \Omega$ we set

$$G^\circ(x, s; \hat{s}) = \gamma^\circ(s; \hat{s}) , \qquad \partial_s G(x, s) = \partial\gamma(s) ,$$

where $\gamma(s) = G(x, s)$;

(G_3) there exist $a_0 \in L^1(\Omega)$ and $b_0 \in \mathbb{R}$ such that

$$G(x, s) \ge -a_0(x) - b_0|s|^p \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}^N ;$$

(G_4) there exist $a_1 \in L^1(\Omega)$ and $b_1 \in \mathbb{R}$ such that

$$G^\circ(x, s; -s) \le a_1(x) + b_1|s|^p \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}^N .$$

Because of $(\mathcal{E}_1)$ and (G_3), we can define a lower semicontinuous functional $f : L^p(\Omega;\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ by

$$f(u) = \mathcal{E}(u) + \int_\Omega G(x, u(x))\, dx .$$

Remark 2.1 *According to $(\mathcal{E}_1)$, the functional $\mathcal{E}$ is lower semicontinuous. Condition $(\mathcal{E}_2)$ ensures that $\mathcal{E}$ is continuous at least on some particular restrictions.*

Remark 2.2 *If $\{s \mapsto G(x,s)\}$ is of class C^1 for a.e $x \in \Omega$, the estimates in (G_2) and in (G_4) are respectively equivalent to*

$$|s| \le t \implies |D_s G(x,s)| \le \alpha_t(x)\,,$$

$$D_s G(x,s) \cdot s \ge -a_1(x) - b_1 |s|^p\,.$$

Because of (G_2), for a.e. $x \in \Omega$ and any $t > 0$ and $s \in \mathbb{R}^N$ with $|s| < t$ we have

$$\forall \hat{s} \in \mathbb{R}^N : \, |G^\circ(x,s;\hat{s})| \le \alpha_t(x)|\hat{s}|\,; \tag{2.25}$$

$$\forall s^* \in \partial_s G(x,s) : \, |s^*| \le \alpha_t(x)\,. \tag{2.26}$$

In the following, we set $\vartheta_h(s) = \vartheta(s/h)$, where ϑ is a function as in $(\mathcal{E}_2)$, and we fix $M > 0$ such that $\vartheta = 0$ outside $B_M(0)$. Therefore

$$\forall s \in \mathbb{R}^N : \, |s| \ge hM \implies \vartheta_h(s) = 0\,. \tag{2.27}$$

Our first result concerns the connection between the notions of generalized directional derivative and subdifferential in the functional space $L^p(\Omega;\mathbb{R}^N)$ and the more concrete setting of hemivariational inequalities, which also involves the notion of generalized directional derivative, but in $\mathbb{R}^N$.

If $u, v \in L^p(\Omega;\mathbb{R}^N)$, we can define $\int_\Omega G^\circ(x,u;v)\,dx$ as

$$\int_\Omega G^\circ(x,u;v)\,dx = +\infty \quad \text{whenever} \quad \int_\Omega [G^\circ(x,u;v)]^+\,dx = \int_\Omega [G^\circ(x,u;v)]^-\,dx = +\infty\,.$$

With this convention, $\left\{v \mapsto \int_\Omega G^\circ(x,u;v)\,dx\right\}$ is a convex functional from $L^p(\Omega;\mathbb{R}^N)$ into $\overline{\mathbb{R}}$.

Theorem 2.3 *Let $u \in \mathcal{D}(f)$. Then the following facts hold:*

(a) *for every $v \in \mathcal{D}(\mathcal{E})$ there exists a sequence (v_h) in $\mathcal{D}(\mathcal{E}) \cap L^\infty(\Omega;\mathbb{R}^N)$ satisfying $[G^\circ(x,u;v_h - u)]^+ \in L^1(\Omega)$, $\|v_h - v\|_p \to 0$ and $\mathcal{E}(v_h) \to \mathcal{E}(v)$;*

(b) *for every $v \in \mathcal{D}(\mathcal{E})$ we have*

$$f^\circ(u;v-u) \le \mathcal{E}(v) - \mathcal{E}(u) + \int_\Omega G^\circ(x,u;v-u)\,dx\,; \tag{2.28}$$

(c) *if $\partial f(u) \ne \emptyset$, we have $G^\circ(x,u;-u) \in L^1(\Omega)$ and*

$$\mathcal{E}(v) - \mathcal{E}(u) + \int_\Omega G^\circ(x,u;v-u)\,dx \ge \int_\Omega u^* \cdot (v-u)\,dx \tag{2.29}$$

for every $u^* \in \partial f(u)$ *and* $v \in \mathcal{D}(\mathcal{E})$ *(the dual space of* $L^p(\Omega;\mathbb{R}^N)$ *is identified with* $L^{p'}(\Omega;\mathbb{R}^N)$ *in the usual way);*

(*d*) *if* $N = 1$, *we have* $[G^\circ(x,u;v-u)]^+ \in L^1(\Omega)$ *for every* $v \in L^\infty(\Omega;\mathbb{R}^N)$.

Proof

(*a*) Given $\varepsilon > 0$, by $(\mathcal{E}_2.2)$ we have $\|\vartheta_h(v)v - v\|_p < \varepsilon$ and $|\mathcal{E}(\vartheta_h(v)v) - \mathcal{E}(v)| < \varepsilon$ for h large enough. Then, by $(\mathcal{E}_2.1)$ we get $\|\vartheta_k(u)\vartheta_h(v)v - v\|_p < \varepsilon$ and $|\mathcal{E}(\vartheta_k(u)\vartheta_h(v)v) - \mathcal{E}(v)| < \varepsilon$ for k large enough. Of course $\vartheta_k(u)\vartheta_h(v)v \in L^\infty(\Omega;\mathbb{R}^N)$ and by (2.25) we have

$$\begin{aligned} G^\circ(x,u;\vartheta_k(u)\vartheta_h(v)v-u) &\le \vartheta_k(u)\vartheta_h(v)G^\circ(x,u;v-u) + \\ &\quad +(1-\vartheta_k(u)\vartheta_h(v))G^\circ(x,u;-u) \le \\ &\le (h+k)M\alpha_{kM}(x) + [G^\circ(x,u;-u)]^+ . \end{aligned}$$

From (G_4) we infer that $[G^\circ(x,u;-u)]^+ \in L^1(\Omega)$ and assertion (*a*) follows.

(*b*) Without loss of generality, we may assume that $[G^\circ(x,u;v-u)]^+ \in L^1(\Omega)$. Suppose first that $v \in \mathcal{D}(\mathcal{E}) \cap L^\infty(\Omega;\mathbb{R}^N)$ and take $\varepsilon > 0$.

We claim that for every $z \in L^p(\Omega;\mathbb{R}^N)$, $t \in]0,1/2]$ and $h \ge 1$ with $hM > \|v\|_\infty$, we have

$$\frac{G(x,z+t(\vartheta_h(z)v-z)) - G(x,z)}{t} \le 2\left(\|v\|_\infty\,\alpha_{hM} + a_1 + b_1(|z|+|v|)^p\right) . \tag{2.30}$$

In fact, for a.e. $x \in \Omega$, by Lebourg's theorem (see Appendix D) there exist $\bar t \in]0,t[$ and $u^* \in \partial_s G(x, z+\bar t(\vartheta_h(z)v-z))$ such that

$$\begin{aligned} \frac{G(x,z+t(\vartheta_h(z)v-z)) - G(x,z)}{t} &= u^*\cdot(\vartheta_h(z)v-z) = \\ &= \frac{1}{1-\bar t}\left[\vartheta_h(z)u^*\cdot v - u^*\cdot(z+\bar t(\vartheta_h(z)v-z))\right] . \end{aligned}$$

By (2.26) and (2.27), it easily follows that

$$\frac{|\vartheta_h(z)u^*\cdot v|}{1-\bar t} \le 2\,\|v\|_\infty\,\alpha_{hM} .$$

On the other hand, from (G_4) we deduce that for a.e. $x \in \Omega$

$$\begin{aligned} \frac{u^*\cdot(z+\bar t(\vartheta_h(z)v-z))}{1-\bar t} &\ge -\frac{1}{1-\bar t}G^\circ(x,z+\bar t(\vartheta_h(z)v-z);-(z+\bar t(\vartheta_h(z)v-z)) \ge \\ &\ge -\frac{1}{1-\bar t}(a_1 + b_1|z+\bar t(\vartheta_h(z)v-z)|^p) \ge -2\,(a_1+b_1(|z|+|v|)^p) . \end{aligned}$$

Then (2.30) easily follows.

For a.e. $x \in \Omega$ we have

$$\begin{aligned} G^\circ(x,u;\vartheta_h(u)v-u) &\leq \vartheta_h(u)G^\circ(x,u;v-u)+(1-\vartheta_h(u))G^\circ(x,u;-u) \leq \\ &\leq [G^\circ(x,u;v-u)]^+ + [G^\circ(x,u;-u)]^+ . \end{aligned}$$

Furthermore, for a.e. $x \in \Omega$ and every $s \in \mathbb{R}^N$, (G_2) implies $G^\circ(x,s;\cdot)$ to be Lipschitz continuous, so in particular

$$\lim_h G^\circ(x,u;\vartheta_h(u)v-u) = G^\circ(x,u;v-u) \qquad \text{a.e. in } \Omega .$$

Then, given

$$\lambda > \int_\Omega G^\circ(x,u;v-u)\,dx ,$$

by Fatou's lemma there exists $\overline{h} \geq 1$ such that

$$\forall h \geq \overline{h} : \int_\Omega G^\circ(x,u;\vartheta_h(u)v-u)\,dx < \lambda \quad \text{and} \quad \|\vartheta_h(u)v-v\|_p < \varepsilon . \tag{2.31}$$

By the lower semicontinuity of $\mathcal{G}$, there exists $\overline{\delta} \in]0,1/2]$ such that for every $z \in B_{\overline{\delta}}(u)$ it is $\mathcal{G}(z) \geq \mathcal{G}(u) - \frac{1}{2}$. Then for every $(z,\mu) \in B_{\overline{\delta}}(u,f(u)) \cap \mathrm{epi}\,(f)$ it follows

$$\mathcal{E}(z) \leq \mu - \mathcal{G}(z) \leq \mu + \frac{1}{2} - \mathcal{G}(u) \leq f(u) + \overline{\delta} - \mathcal{G}(u) + \frac{1}{2} \leq \mathcal{E}(u) + 1 .$$

Let now $\sigma > 0$. By assumptions $(\mathcal{E}_1)$ and $(\mathcal{E}_2.1)$ there exist $h \geq \overline{h}$ and $\delta \leq \overline{\delta}$ such that

$$\|v\|_\infty < hM ,$$

$$\mathcal{E}(z) > \mathcal{E}(u) - \sigma , \quad \mathcal{E}(\vartheta_h(z)v) < \mathcal{E}(v) + \sigma , \quad \|(\vartheta_h(z)v - z) - (v-u)\|_p < \varepsilon ,$$

for any $z \in B_\delta(u)$ with $\mathcal{E}(z) \leq \mathcal{E}(u) + 1$.

Taking into account (D.1), (2.30) and (2.31), we deduce by Fatou's lemma that, possibly reducing δ, for any $t \in]0,\delta]$ and for any $z \in B_\delta(u)$ we have

$$\int_\Omega \frac{G(x, z + t(\vartheta_h(z)v - z)) - G(x,z)}{t}\,dx < \lambda .$$

Now let $\mathcal{V} : (B_\delta(u,f(u)) \cap \mathrm{epi}\,(f)) \times]0,\delta] \to B_\varepsilon(v-u)$ be defined setting

$$\mathcal{V}((z,\mu),t) = \vartheta_h(z)v - z .$$

Since $\mathcal{V}$ is evidently continuous and

$$\begin{aligned} f(z+t\mathcal{V}((z,\mu),t)) &= f(z+t(\vartheta_h(z)v-z)) \leq \\ &\leq \mathcal{E}(z)+t\left(\mathcal{E}(\vartheta_h(z)v)-\mathcal{E}(z)\right)+\mathcal{G}(z+t(\vartheta_h(z)v-z) \leq \\ &\leq \mathcal{E}(z)+(\mathcal{E}(v)-\mathcal{E}(u)+2\sigma)t+\mathcal{G}(z)+\lambda t = \\ &= f(z)+(\mathcal{E}(v)-\mathcal{E}(u)+\lambda+2\sigma)t\,, \end{aligned}$$

we have

$$f_\varepsilon^\circ(u;v-u) \leq \mathcal{E}(v)-\mathcal{E}(u)+\lambda+2\sigma\,.$$

By the arbitrariness of $\sigma > 0$ and $\lambda > \int_\Omega G^\circ(x,u;v-u)\,dx$, it follows

$$f_\varepsilon^\circ(u;v-u) \leq \mathcal{E}(v)-\mathcal{E}(u)+\int_\Omega G^\circ(x,u;v-u)\,dx\,.$$

Passing to the limit as $\varepsilon \to 0^+$, we get (2.28) when $v \in \mathcal{D}(\mathcal{E})\cap L^\infty(\Omega;\mathbb{R}^N)$.

Let us now treat the general case. If we set $v_h = \vartheta_h(v)v$, we have $v_h \in L^\infty(\Omega;\mathbb{R}^N)$. Arguing as before, it is easy to see that

$$G^\circ(x,u;v_h-u) \leq [G^\circ(x,u;v-u)]^+ + [G^\circ(x,u;-u)]^+\,,$$

so that

$$\limsup_h \int_\Omega G^\circ(x,u;v_h-u)\,dx \leq \int_\Omega G^\circ(x,u;v-u)\,dx\,.$$

On the other hand, by the previous step it holds

$$f^\circ(u;v_h-u) \leq \mathcal{E}(v_h)-\mathcal{E}(u)+\int_\Omega G^\circ(x,u;v_h-u)\,dx\,.$$

Passing to the lower limit as $h \to \infty$ and taking into account the lower semicontinuity of $f^\circ(u,\cdot)$ and $(\mathcal{E}_2.2)$, we get (2.28).
(c) We already know that $[G^\circ(x,u;-u)]^+ \in L^1(\Omega)$. If we choose $v=0$ in (2.28), we obtain

$$f^\circ(u;-u) \leq \mathcal{E}(0)-\mathcal{E}(u)+\int_\Omega G^\circ(x,u;-u)\,dx\,.$$

Since $\partial f(u) \neq \emptyset$, it is $f^\circ(u;-u) > -\infty$, hence

$$\int_\Omega [G^\circ(x,u;-u)]^-\,dx < +\infty\,.$$

Finally, if $u^* \in \partial f(u)$ we have by definition that

$$f^\circ(u;v-u) \geq \int_\Omega u^*\cdot(v-u)\,dx$$

and (2.29) follows from (2.28).
(d) From (2.25) it readily follows that $G^\circ(x,u;v-u)$ is summable where $|u(x)| \leq \|v\|_\infty$. On the other hand, where $|u(x)| > \|v\|_\infty$ we have

$$G^\circ(x,u;v-u) = \left(1 - \frac{v}{u}\right) G^\circ(x,u;-u)$$

and the assertion follows from (G_4). $\square$

Since f is only lower semicontinuous, we are interested in the verification of the condition $(epi)_c$. For this purpose, we consider an assumption (G_3') on G stronger than (G_3).

Theorem 2.4 *Assume that*

(G_3') *there exist $a \in L^1(\Omega)$ and $b \in \mathbb{R}$ such that*

$$|G(x,s)| \leq a(x) + b|s|^p \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}^N\,.$$

Then for every $(u,\lambda) \in \mathrm{epi}\,(f)$ with $\lambda > f(u)$ it is $|d\mathcal{G}_f|(u,\lambda) = 1$. Moreover, if $\mathcal{E}$ and $G(x,\cdot)$ are even, for every $\lambda > f(0)$ we have $|d_{\mathbb{Z}_2}\mathcal{G}_f|(0,\lambda) = 1$.

Proof Let $\varrho > 0$. Since

$$\forall \tau \in [0,1] : G^\circ(x,u;\tau u - u) = (1-\tau)G^\circ(x,u;-u) \leq [G^\circ(x,u;-u)]^+\,,$$

by $(\mathcal{E}_2.2)$ and (G_4) there exists $\overline{h} \geq 1$ such that

$$\|\vartheta_{\overline{h}}(u)u - u\|_p < \varrho\,, \qquad \mathcal{E}(\vartheta_{\overline{h}}(u)u) < \mathcal{E}(u) + \varrho\,,$$

$$\forall h \geq \overline{h} : \int_\Omega G^\circ(x,u;\vartheta_h(u)\vartheta_{\overline{h}}(u)u - u)\,dx < \varrho\,.$$

Set $v = \vartheta_{\overline{h}}(u)u$.

By $(\mathcal{E}_2.1)$ there exist $h \geq \overline{h}$ and $\delta \in]0,1]$ such that

$$\|\vartheta_h(z)v - z\|_p < \varrho\,, \qquad \mathcal{E}(\vartheta_h(z)v) < \mathcal{E}(u) + \varrho\,,$$

whenever $\|z-u\|_p < \delta$ and $\mathcal{E}(z) \leq \lambda + 1 - \mathcal{G}(u) + \varrho$.

By decreasing δ, from (G_3'), (2.30) and (D.1) we deduce that

$$|\mathcal{G}(z) - \mathcal{G}(u)| < \varrho\,, \qquad \int_\Omega \frac{G(x,z+t(\vartheta_h(z)v - z)) - G(x,z)}{t}\,dx < \varrho$$

whenever $\|z-u\|_p < \delta$ and $0 < t \leq \delta$.

Define a continuous map

$$\mathcal{H} : \{z \in B_\delta(u) : f(z) < \lambda + \delta\} \times [0,\delta] \to X$$

by $\mathcal{H}(z,t) = z + t(\vartheta_h(z)v - z)$. It is readily seen that $\|\mathcal{H}(z,t) - z\|_p \le \varrho t$.

If $z \in B_\delta(u)$, $f(z) < \lambda + \delta$ and $0 \le t \le \delta$, we have

$$\mathcal{E}(z) = f(z) - \mathcal{G}(z) < \lambda + \delta - \mathcal{G}(u) + \varrho \le \lambda + 1 - \mathcal{G}(u) + \varrho\,,$$

hence, taking into account the convexity of $\mathcal{E}$,

$$\mathcal{E}(z+t(\vartheta_h(z)v-z)) \le \mathcal{E}(z)+t(\mathcal{E}(\vartheta_h(z)v)-\mathcal{E}(z)) \le \mathcal{E}(z)+t(\mathcal{E}(u)-\mathcal{E}(z)+\varrho)\,.$$

Moreover, we also have

$$\mathcal{G}(z + t(\vartheta_h(z)v - z)) \le \mathcal{G}(z) + t\varrho \le \mathcal{G}(z) + t(\mathcal{G}(u) - \mathcal{G}(z) + 2\varrho)\,.$$

Therefore

$$f(z + t(\vartheta_h(z)v - z)) \le f(z) + t(f(u) - f(z) + 3\varrho)\,.$$

and the first assertion follows by Corollary 2.1.

Now assume that $\mathcal{E}$ and $G(x,\cdot)$ are even and that $u = 0$. Then, in the previous argument, we have $v = 0$, so that $\mathcal{H}(-z,t) = -\mathcal{H}(z,t)$ and the second assertion also follows. □

Now we want to provide a criterion which helps in the verification of the Palais-Smale condition. For this purpose, we consider further assumptions on $\mathcal{E}$, which ensure a suitable coerciveness, and a new condition (G_4') on G, stronger than (G_4), which is a kind of one-sided subcritical growth condition.

Theorem 2.5 *Let $c \in \mathbb{R}$. Assume that*

($\mathcal{E}_3$) *for every (u_h) bounded in $L^p(\Omega;\mathbb{R}^N)$ with $(\mathcal{E}(u_h))$ bounded, there exists a subsequence (u_{h_k}) and a function $u \in L^p(\Omega;\mathbb{R}^N)$ such that*

$$\lim_{k\to\infty} u_{h_k}(x) = u(x) \qquad \text{for a.e. } x \in \Omega\,;$$

($\mathcal{E}_4$) *if (u_h) is a sequence in $L^p(\Omega;\mathbb{R}^N)$ weakly convergent to $u \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(u_h)$ converges to $\mathcal{E}(u)$, then (u_h) converges to u strongly in $L^p(\Omega;\mathbb{R}^N)$;*

(G_4') *for every $\varepsilon > 0$ there exists $a_\varepsilon \in L^1(\Omega)$ such that*

$$G^\circ(x,s;-s) \le a_\varepsilon(x) + \varepsilon|s|^p \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}^N\,.$$

Then any $(PS)_c$-sequence (u_h) for f bounded in $L^p(\Omega;\mathbb{R}^N)$ admits a subsequence strongly convergent in $L^p(\Omega;\mathbb{R}^N)$.

Proof From (G_3) we deduce that $(\mathcal{G}(u_h))$ is bounded from below. Taking into account $(\mathcal{E}_1)$, it follows that $(\mathcal{E}(u_h))$ is bounded. By $(\mathcal{E}_3)$ there exists a subsequence, still denoted by (u_h), converging weakly in $L^p(\Omega;\mathbb{R}^N)$ and a.e. to some $u \in \mathcal{D}(\mathcal{E})$.

Given $\varepsilon > 0$, by $(\mathcal{E}_2.2)$ and (G_4) we may find $k_0 \geq 1$ such that

$$\mathcal{E}(\vartheta_{k_0}(u)u) < \mathcal{E}(u) + \varepsilon\,,$$

$$\int_\Omega (1 - \vartheta_{k_0}(u))G^\circ(x, u; -u)\, dx < \varepsilon\,.$$

Since $\vartheta_{k_0}(u)u \in \mathcal{D}(\mathcal{E}) \cap L^\infty(\Omega;\mathbb{R}^N)$, by $(\mathcal{E}_2.1)$ there exists $k_1 \geq k_0$ such that

$$\forall h \in \mathbb{N} : \quad \mathcal{E}(\vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u) < \mathcal{E}(u) + \varepsilon\,, \tag{2.32}$$

$$\int_\Omega (1 - \vartheta_{k_1}(u)\vartheta_{k_0}(u))G^\circ(x, u; -u)\, dx < \varepsilon\,.$$

It follows that $\vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u \in \mathcal{D}(\mathcal{E})$. Moreover, from (2.25) and (G'_4) we get

$$\begin{aligned}
&G^\circ(x, u_h; \vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u - u_h) \leq \\
&\leq \vartheta_{k_1}(u_h)G^\circ(x, u_h; \vartheta_{k_0}(u)u - u_h) + (1 - \vartheta_{k_1}(u_h))G^\circ(x, u_h; -u_h) \leq \\
&\qquad\qquad \leq \alpha_{k_1 M}(x)(k_0 M + k_1 M) + a_\varepsilon(x) + \varepsilon |u_h|^p\,.
\end{aligned}$$

From (D.2) and Fatou's Lemma we deduce that

$$\begin{aligned}
\limsup_{h\to\infty} &\int_\Omega \left[G^\circ(x, u_h; \vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u - u_h) - \varepsilon |u_h|^p\right]\, dx \leq \\
&\leq \int_\Omega \left[G^\circ(x, u; \vartheta_{k_1}(u)\vartheta_{k_0}(u)u - u) - \varepsilon |u|^p\right]\, dx \leq \\
&\qquad\qquad \leq \int_\Omega (1 - \vartheta_{k_1}(u)\vartheta_{k_0}(u))G^\circ(x, u; -u)\, dx < \varepsilon\,,
\end{aligned}$$

hence

$$\limsup_{h\to\infty} \int_\Omega G^\circ(x, u_h; \vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u - u_h)\, dx < \varepsilon \sup_h \|u_h\|_p^p + \varepsilon\,. \tag{2.33}$$

Since (u_h) is a $(PS)_c$-sequence, by Theorem D.2 there exists $u_h^* \in \partial f(u_h)$ with $\|u_h^*\|_{p'} \leq |df|(u_h)$, so that $\lim\limits_{h\to\infty} \|u_h^*\|_{p'} = 0$. Applying (c) of Theorem 2.3, we get

$$\mathcal{E}(\vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u) \geq \mathcal{E}(u_h) - \int_\Omega G^\circ(x, u_h; \vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u - u_h)\, dx + \\ + \int_\Omega u_h^* \cdot (\vartheta_{k_1}(u_h)\vartheta_{k_0}(u)u - u_h)\, dx\,.$$

Taking into account (2.32), (2.33) and passing to the upper limit, we obtain

$$\limsup_{h\to\infty} \mathcal{E}(u_h) \leq \mathcal{E}(u) + 2\varepsilon + \varepsilon \sup_h \|u_h\|_p^p\,.$$

By the arbitrariness of $\varepsilon > 0$, we finally have

$$\limsup_{h\to\infty} \mathcal{E}(u_h) \leq \mathcal{E}(u)$$

and the strong convergence of (u_h) to u follows from $(\mathcal{E}_4)$. $\square$

2.3.2 Area type functionals

Let $n \geq 2$, $N \geq 1$, Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary and let

$$\Psi : \mathbb{R}^{nN} \to \mathbb{R}$$

be a convex function satisfying

$$(\Psi) \qquad \begin{cases} \Psi(0) = 0,\ \Psi(\xi) > 0 \text{ for any } \xi \neq 0 \text{ and} \\ \text{there exists } c > 0 \text{ such that } \Psi(\xi) \leq c|\xi| \text{ for any } \xi \in \mathbb{R}^{nN}. \end{cases}$$

We want to study the functional $\mathcal{E} : L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathcal{E}(u) = \begin{cases} \displaystyle\int_\Omega \Psi(Du^a)\, dx + \int_\Omega \Psi^\infty\left(\frac{Du^s}{|Du^s|}\right) d|Du^s|(x) + & \\ \displaystyle + \int_{\partial\Omega} \Psi^\infty(u \otimes \nu)\, d\mathcal{H}^{n-1}(x) & \text{if } u \in BV(\Omega;\mathbb{R}^N), \\ +\infty & \text{if } u \in L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^N) \setminus BV(\Omega;\mathbb{R}^N), \end{cases}$$

where $Du = Du^a\, dx + Du^s$ is the Lebesgue decomposition of Du, $|Du^s|$ is the total variation of Du^s, $Du^s/|Du^s|$ is the Radon-Nikodym derivative of Du^s with respect to $|Du^s|$, Ψ^∞ is the recession functional associated with Ψ, ν is the outer normal to Ω and the trace of u on $\partial\Omega$ is still denoted by u (see e.g. [?, ?]).

Theorem 2.6 *The functional $\mathcal{E}$ satisfies conditions $(\mathcal{E}_1)$, $(\mathcal{E}_2)$, $(\mathcal{E}_3)$ and $(\mathcal{E}_4)$.*

The section will be devoted to the proof of this result. We begin establishing some technical lemmas. For notions concerning the space BV, such as those of $\tilde{u}$, S_u, u^+ and u^-, we refer the reader to [**?**, **?**].

In $BV(\Omega;\mathbb{R}^N)$ we will consider the norm

$$\|u\|_{BV} = \int_\Omega |Du^a|\,dx + |Du^s|(\Omega) + \int_{\partial\Omega} |u|\,d\mathcal{H}^{n-1}(x)\,,$$

which is equivalent to the standard norm of $BV(\Omega;\mathbb{R}^N)$.

Lemma 2.2 *For every $u \in BV(\Omega;\mathbb{R}^N)$ and every $\varepsilon > 0$ there exists $v \in C_c^\infty(\Omega;\mathbb{R}^N)$ such that*

$$\|v-u\|_{\frac{n}{n-1}} < \varepsilon\,, \quad \left|\int_\Omega |Dv|\,dx - \|u\|_{BV}\right| < \varepsilon\,, \quad |\mathcal{E}(v)-\mathcal{E}(u)| < \varepsilon\,, \quad \|v\|_\infty \leq \mathrm{esssup}_\Omega|\mathrm{u}|\,.$$

Proof Let $\delta > 0$, let $R > 0$ with $\overline{\Omega} \subseteq B_R(0)$ and let

$$\vartheta_h(x) = 1 - \min\left\{\max\left\{\frac{h+1}{h}[1 - h\,d(x,\mathbb{R}^n\setminus\Omega)],0\right\},1\right\}\,.$$

Define $\hat{u} \in BV(B_R(0);\mathbb{R}^N)$ by

$$\hat{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega\,,\\ 0 & \text{if } x \in B_R(0)\setminus\Omega\,.\end{cases}$$

According to [**?**, Lemma 7.4 and formula (7.2)], if h is sufficiently large, we have that $\vartheta_h u \in BV(\Omega;\mathbb{R}^N)$, $\|\vartheta_h u - u\|_{\frac{n}{n-1}} < \delta$ and

$$\begin{aligned}\int_\Omega \sqrt{1+|D(\vartheta_h u)^a|^2}\,d\mathcal{L}^n + |D(\vartheta_h u)^s|(\Omega) <\\ < \int_\Omega \sqrt{1+|Du^a|^2}\,d\mathcal{L}^n + |Du^s|(\Omega) + \int_{\partial\Omega}|u|\,d\mathcal{H}^{n-1} + \delta =\\ = \int_{B_R(0)} \sqrt{1+|D\hat{u}^a|^2}\,d\mathcal{L}^n + |D\hat{u}^s|(B_R(0)) + \delta\,.\end{aligned}$$

Moreover, $\vartheta_h u$ has compact support in Ω and $\mathrm{esssup}_\Omega|\vartheta_\mathrm{h}\mathrm{u}| \leq \mathrm{esssup}_\Omega|\mathrm{u}|$.

If we regularize $\vartheta_h u$ by convolution, we easily get $v \in C_c^\infty(\Omega;\mathbb{R}^N)$ with

$$\|v\|_\infty \leq \mathrm{esssup}_\Omega|\mathrm{u}|\,, \qquad \|\mathrm{v}-\mathrm{u}\|_{\frac{\mathrm{n}}{\mathrm{n}-1}} < \delta$$

and

$$\int_\Omega \sqrt{1+|Dv|^2}\, d\mathcal{L}^n < \int_{B_R(0)} \sqrt{1+|D\hat{u}^a|^2}\, d\mathcal{L}^n + |D\hat{u}^s|(B_R(0)) + \delta\,.$$

Since

$$\|u\|_{BV} = \int_{B_R(0)} |D\hat{u}^a|\, dx + |D\hat{u}^s|(B_R(0))\,,$$

$$\mathcal{E}(u) = \int_{B_R(0)} \Psi(D\hat{u}^a)\, dx + \int_{B_R(0)} \Psi^\infty\left(\frac{D\hat{u}^s}{|D\hat{u}^s|}\right) d|D\hat{u}^s|\,,$$

by the results of [**?**] the assertion follows (see also [**?**, Fact 3.1]). $\square$

Lemma 2.3 *The following facts hold:*

- (a) $\Psi : \mathbb{R}^{nN} \to \mathbb{R}$ *is Lipschitz continuous of some constant* $\mathrm{Lip}(\Psi) > 0$*;*
- (b) *for any* $\xi \in \mathbb{R}^{nN}$ *and* $s \in [0,1]$ *we have* $\Psi(s\xi) \le s\Psi(\xi)$*;*
- (c) *for every* $\sigma > 0$ *there exists* $d_\sigma > 0$ *such that*

$$\forall \xi \in \mathbb{R}^{nN} : \quad \Psi(\xi) \ge d_\sigma(|\xi| - \sigma)\,;$$

- (d) $\mathcal{E} : BV(\Omega;\mathbb{R}^N) \to \mathbb{R}$ *is Lipschitz continuous of constant* $\mathrm{Lip}(\Psi)$*;*
- (e) *if* σ *and* d_σ *are as in* (c)*, we have*

$$\forall u \in BV(\Omega;\mathbb{R}^N) : \quad \mathcal{E}(u) \ge d_\sigma\Big(\|u\|_{BV} - \sigma\mathcal{L}^n(\Omega)\Big)\,.$$

Proof Properties (a) and (b) easily follow from the convexity of Ψ and assumption (Ψ).

To prove (c), assume by contradiction that $\sigma > 0$ and (ξ_h) is a sequence with $\Psi(\xi_h) < \frac{1}{h}(|\xi_h| - \sigma)$. If $|\xi_h| \to +\infty$, we have eventually

$$\Psi\left(\frac{\xi_h}{|\xi_h|}\right) \le \frac{\Psi(\xi_h)}{|\xi_h|} < \frac{1}{h}\left(1 - \frac{\sigma}{|\xi_h|}\right)\,.$$

Up to a subsequence, $(\xi_h/|\xi_h|)$ is convergent to some $\eta \ne 0$ with $\Psi(\eta) \le 0$, which is impossible. Since $|\xi_h|$ is bounded, up to a subsequence we have $\xi_h \to \xi$ with $|\xi| \ge \sigma$ and $\Psi(\xi) \le 0$, which is again impossible.

Finally, (d) easily follows from (a) and the definition of $\|\cdot\|_{BV}$, while (e) follows from (c) (see e.g. [**?**, Lemma 4.1]). $\square$

Let now $\vartheta \in C^1_c(\mathbb{R}^N)$ with $0 \le \vartheta \le 1$, $\|\nabla\vartheta\|_\infty \le 2$, $\vartheta(s) = 1$ for $|s| \le 1$ and $\vartheta(s) = 0$ for $|s| \ge 2$. Define $\vartheta_h : \mathbb{R}^N \to \mathbb{R}$ and $T_h, R_h : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\vartheta_h(s) = \vartheta\left(\frac{s}{h}\right), \quad T_h(s) = \vartheta_h(s)s\,, \quad R_h(s) = (1 - \vartheta_h(s))s\,.$$

Lemma 2.4 *There exists a constant $c_\Psi > 0$ such that*

$$\mathcal{E}\left(\vartheta\left(\frac{u}{h}\right)v\right) \le \mathcal{E}(v) + \frac{c_\Psi}{h}\|v\|_\infty\|u\|_{BV}\,,$$

$$\mathcal{E}(T_h \circ u) \le \mathcal{E}(u) + c_\Psi\Bigg[|Du|(\{x \in \Omega \setminus S_u : |\tilde{u}(x)| > h\}) +$$

$$+ \int_{\{x\in S_u:|u^+(x)|>h \text{ or } |u^-(x)|>h\}} |u^+ - u^-|\, d\mathcal{H}^{n-1}(x) + \int_{\{x\in\partial\Omega:|u(x)|>h\}} |u|\, d\mathcal{H}^{n-1}(x)\Bigg]\,,$$

$$\mathcal{E}(T_h \circ w) + \mathcal{E}(R_h \circ w) \le \mathcal{E}(w) + c_\Psi \int_{\{x\in\Omega:h<|w(x)|<2h\}} |Dw|\, dx$$

whenever $h \ge 1$, $u \in BV(\Omega;\mathbb{R}^N)$, $v \in BV(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$ and $w \in C^\infty_c(\Omega;\mathbb{R}^N)$.

Proof Suppose first that $u, v \in C^\infty_c(\Omega;\mathbb{R}^N)$. Then, since

$$D\left[\vartheta\left(\frac{u}{h}\right)v\right] = \vartheta\left(\frac{u}{h}\right)Dv + \frac{1}{h}\, v \otimes \left[D\vartheta\left(\frac{u}{h}\right)Du\right]\,,$$

by (Ψ) and Lemma 2.3 it follows that

$$\mathcal{E}\left(\vartheta\left(\frac{u}{h}\right)v\right) \le \mathcal{E}(v) + \mathrm{Lip}(\Psi)\frac{\|D\vartheta\|_\infty}{h}\|v\|_\infty \int_\Omega |Du|\, dx\,. \tag{2.34}$$

In the general case, let us consider two sequences (u_k), (v_k) in $C^\infty_c(\Omega;\mathbb{R}^N)$ converging to u, v in $L^1(\Omega;\mathbb{R}^N)$ with $\int_\Omega |Du_k|\, dx \to \|u\|_{BV}$, $\mathcal{E}(v_k) \to \mathcal{E}(v)$ and $\|v_k\|_\infty \le \|v\|_\infty$. Passing to the lower limit in (2.34), we obtain the first inequality in the assertion.

To prove the second inequality, we first observe that by Lemma 2.3 we have

$$\mathcal{E}(T_h \circ u) \le \mathcal{E}(u) + \mathrm{Lip}(\Psi)\|R_h \circ u\|_{BV}\,. \tag{2.35}$$

In order to estimate the last term in (2.35), we apply the chain rule of

[?, ?]. Since $R_h(s)=0$ if $|s|\le h$ and $\|DR_h\|_\infty \le k_\vartheta$ for some $k_\vartheta>0$, we have

$$\int_\Omega |D(R_h(u))^a|\,dx \le \int_{\Omega\setminus S_u} |DR_h(\tilde u)||Du^a|\,dx \le k_\vartheta \int_{\{x\in\Omega\setminus S_u:|\tilde u(x)|>h\}} |Du^a|\,dx\,,$$

$$\Big|D(R_h(u))^s\Big|(\Omega) \le \int_{\Omega\setminus S_u} |DR_h(\tilde u)|\,d|Du^s|(x) + \int_{S_u} |R_h(u^+)-R_h(u^-)|\,d\mathcal{H}^{n-1}(x) \le$$
$$\le k_\vartheta\Big(|Du^s|\,(\{x\in\Omega\setminus S_u : |\tilde u(x)|>h\}) + \int_{\{x\in S_u:|u^+(x)|>h \text{ or } |u^-(x)|>h\}} |u^+-u^-|\,d\mathcal{H}^{m-1}(x)\Big)$$

and

$$\int_{\partial\Omega} |R_h(u)|\,d\mathcal{H}^{n-1}(x) \le k_\vartheta \int_{\{x\in\partial\Omega:|u(x)|>h\}} |u|\,d\mathcal{H}^{n-1}(x)\,.$$

Combining these three estimates, we get

$$\|R_h\circ u\|_{BV} \le k_\vartheta\Bigg(\int_{\{x\in\Omega\setminus S_u:|\tilde u(x)|>h\}} |Du^a|\,dx + |Du^s|(\{x\in\Omega\setminus S_u : |\tilde u(x)|>h\}) + \tag{2.36}$$
$$+\int_{\{x\in S_u:|u^+(x)|>h \text{ or } |u^-(x)|>h\}} |u^+-u^-|\,d\mathcal{H}^{m-1}(x) + \int_{\{x\in\partial\Omega:|u(x)|>h\}} |u|\,d\mathcal{H}^{n-1}(x)\Bigg)\,.$$

Then the second inequality follows from (2.35) and (2.36).

Again, since Ψ is Lipschitz continuous, we have

$$\begin{aligned}\left|\int_\Omega \Psi(D(T_h\circ w))\,dx - \int_\Omega \Psi(\vartheta_h(w)Dw)\,dx\right| &\le \frac{\mathrm{Lip}(\Psi)}{h}\int_\Omega \left|D\vartheta\left(\frac{w}{h}\right)Dw\right||w|\,dx \le \\ &\le 2\,\mathrm{Lip}(\Psi)\|\nabla\vartheta\|_\infty \int_{\{h<|w|<2h\}} |Dw|\,dx\,.\end{aligned}$$

In a similar way, it is also

$$\left|\int_\Omega \Psi(D(R_h\circ w))\,dx - \int_\Omega \Psi((1-\vartheta_h(w))Dw)\,dx\right| \le 2\,\mathrm{Lip}(\Psi)\|\nabla\vartheta\|_\infty \int_{\{h<|w|<2h\}} |Dw|\,dx\,.$$

Hence, combining the last two estimates and taking into account (b) of Lemma 2.3, we get

$$\begin{aligned}&\int_\Omega \Psi\big(D(T_h\circ w)\big)\,dx + \int_\Omega \Psi\big(D(R_h\circ w)\big)\,dx \le \\ &\qquad\le \int_\Omega \Psi(Dw)\,dx + 4\,\mathrm{Lip}(\Psi)\|\nabla\vartheta\|_\infty \int_{\{h<|w|<2h\}} |Dw|\,dx\end{aligned}$$

and the proof is complete. □

Lemma 2.5 *Let (u_h) be a sequence in $C_c^\infty(\Omega;\mathbb{R}^N)$ and assume that (u_h) is bounded in $BV(\Omega;\mathbb{R}^N)$.*

Then for every $\varepsilon > 0$ and every $\overline{k} \in \mathbb{N}$ there exists $k \geq \overline{k}$ such that

$$\liminf_{h\to\infty} \int\limits_{\{k<|u_h|<2k\}} |Du_h|\,dx < \varepsilon\,.$$

Proof Let $m \geq 1$ be such that

$$\sup_h \int_\Omega |Du_h|\,dx \leq \frac{m\varepsilon}{2}$$

and let $i_0 \in \mathbb{N}$ with $2^{i_0} \geq \overline{k}$. Then, since

$$\sum_{i=i_0}^{i_0+m-1} \int_{\{2^i<|u_h|<2^{i+1}\}} |Du_h|\,dx \leq \int_\Omega |Du_h|\,dx \leq \frac{m\varepsilon}{2}\,,$$

there exists i_h between i_0 and i_0+m-1 such that

$$\int_{\{2^{i_h}<|u_h|<2^{i_h+1}\}} |Du_h|\,dx \leq \frac{\varepsilon}{2}\,.$$

Passing to a subsequence (i_{h_j}), we can suppose $i_{h_j} \equiv i \geq i_0$, and setting $k = 2^i$ we get

$$\forall j \in \mathbb{N}: \int\limits_{\{k<|u_{h_j}|<2k\}} |Du_{h_j}|\,dx \leq \frac{\varepsilon}{2}\,.$$

Then the assertion follows. □

Lemma 2.6 *Let (u_h) be a sequence in $C_c^\infty(\Omega;\mathbb{R}^N)$ and let $u \in BV(\Omega;\mathbb{R}^N)$ with $\|u_h - u\|_1 \to 0$ and $\mathcal{E}(u_h) \to \mathcal{E}(u)$.*

Then for every $\varepsilon > 0$ and every $\overline{k} \in \mathbb{N}$ there exists $k \geq \overline{k}$ such that

$$\liminf_{h\to\infty} \|R_k \circ u_h\|_{BV} < \varepsilon\,.$$

Proof Given $\varepsilon > 0$, let $d > 0$ be such that

$$\forall \xi \in \mathbb{R}^{nN}: \quad \Psi(\xi) \geq d\left(|\xi| - \frac{\varepsilon}{3\mathcal{L}^n(\Omega)}\right)\,,$$

according to Lemma 2.3. Let also $c_\Psi > 0$ be as in Lemma 2.4. By (2.36) and Lemma 2.5, there exists $k \geq \overline{k}$ such that

$$\|R_k \circ u\|_{BV} < \frac{d\varepsilon}{3\mathrm{Lip}(\Psi)}\,,$$

$$\liminf_{h\to\infty} \int\limits_{\{k<|u_h|<2k\}} |Du_h|\,dx < \frac{d\varepsilon}{3c_\Psi}\,.$$

From Lemma 2.4 we deduce that

$$\begin{aligned}
\mathcal{E}(T_k\circ u) + \liminf_{h\to\infty}\mathcal{E}(R_k\circ u_h) &\le \liminf_{h\to\infty}\mathcal{E}(T_k\circ u_h) + \liminf_{h\to\infty}\mathcal{E}(R_k\circ u_h) \le \\
&\le \liminf_{h\to\infty}\Big(\mathcal{E}(T_k\circ u_h) + \mathcal{E}(R_k\circ u_h)\Big) \le \\
&\le \mathcal{E}(u) + c_\Psi \liminf_{h\to\infty} \int\limits_{\{k<|u_h|<2k\}} |Du_h|\,dx < \\
&< \mathcal{E}(u) + \frac{d\varepsilon}{3} \le \mathcal{E}(T_k\circ u) + \mathrm{Lip}(\Psi)\|R_k\circ u\|_{BV} + \frac{d\varepsilon}{3} < \\
&< \mathcal{E}(T_k\circ u) + \frac{2}{3}d\varepsilon\,,
\end{aligned}$$

whence

$$\liminf_{h\to\infty}\mathcal{E}(R_k\circ u_h) < \frac{2}{3}d\varepsilon\,.$$

On the other hand, by Lemma 2.3 we have

$$\mathcal{E}(R_k\circ u_h) \ge d\left(\|R_k\circ u_h\|_{BV} - \frac{\varepsilon}{3}\right)$$

and the assertion follows. □

Now we can prove the main auxiliary result we need for the proof of Theorem 2.6. It is a property of the space BV which could be interesting also in itself.

Theorem 2.7 *Let (u_h) be a sequence in $BV(\Omega;\mathbb{R}^N)$ and let $u\in BV(\Omega;\mathbb{R}^N)$ with $\|u_h-u\|_1\to 0$ and $\mathcal{E}(u_h)\to\mathcal{E}(u)$.*

Then (u_h) is strongly convergent to u in $L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^N)$.

Proof By Lemma 2.2 we may find $v_h\in C_c^\infty(\Omega;\mathbb{R}^N)$ with

$$\|v_h-u_h\|_1 < \frac{1}{h}\,,\quad \|v_h-u_h\|_{\frac{n}{n-1}} < \frac{1}{h}\,,\quad |\mathcal{E}(v_h)-\mathcal{E}(u_h)| < \frac{1}{h}\,.$$

Therefore it is sufficient to treat the case in which $u_h\in C_c^\infty(\Omega;\mathbb{R}^N)$.

By contradiction, up to a subsequence we may assume that there exists $\varepsilon>0$ such that $\|u_h-u\|_{\frac{n}{n-1}}\ge\varepsilon$. Let $\tilde{c}$ be a constant such that $\|w\|_{\frac{n}{n-1}}\le\tilde{c}\|w\|_{BV}$ for any $w\in BV(\Omega;\mathbb{R}^N)$ (see [**?**, Theorem 1.28]). According to Lemma 2.6, let $k\in\mathbb{N}$ be such that

$$\|R_k\circ u\|_{\frac{n}{n-1}} < \frac{\varepsilon}{2}\,,\quad \liminf_{h\to\infty}\|R_k\circ u_h\|_{\frac{n}{n-1}} \le \tilde{c}\liminf_{h\to\infty}\|R_k\circ u_h\|_{BV} < \frac{\varepsilon}{2}\,.$$

Then we have

$$\|u_h - u\|_{\frac{n}{n-1}} \leq \|R_k \circ u_h\|_{\frac{n}{n-1}} + \|T_k \circ u_h - T_k \circ u\|_{\frac{n}{n-1}} + \|R_k \circ u\|_{\frac{n}{n-1}} . \quad (2.37)$$

Since $T_k \circ u_h \to T_k \circ u$ in $L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^N)$ as $h \to \infty$, passing to the lower limit in (2.37) we get

$$\liminf_{h\to\infty} \|u_h - u\|_{\frac{n}{n-1}} < \varepsilon ,$$

whence a contradiction. □

Proof of Theorem 2.6. It is well known that $\mathcal{E}$ satisfies condition $(\mathcal{E}_1)$. Conditions $(\mathcal{E}_2)$ are an immediate consequence of Lemma 2.4. From (e) of Lemma 2.3 and Rellich's Theorem (see [**?**, Theorem 1.19]) it follows that $\mathcal{E}$ satisfies condition $(\mathcal{E}_3)$. To prove $(\mathcal{E}_4)$, let (u_h) be a sequence in $L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^N)$ weakly convergent to $u \in BV(\Omega; \mathbb{R}^N)$ such that $\mathcal{E}(u_h)$ converges to $\mathcal{E}(u)$. Again by (e) of Lemma 2.3 and Rellich's Theorem we deduce that (u_h) is strongly convergent to u in $L^1(\Omega; \mathbb{R}^N)$. Then the assertion follows from Theorem 2.7.

2.3.3 A result of Clark type

Let $n \geq 2$ and Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, let $\Psi : \mathbb{R}^{nN} \to \mathbb{R}$ be an even convex function satisfying (Ψ) and let $G : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a function satisfying (G_1), (G_2), (G'_3), (G'_4) with $p = \frac{n}{n-1}$ and the following conditions:

$$\begin{cases} \text{there exist } \tilde{a} \in L^1(\Omega) \text{ and } \tilde{b} \in L^n(\Omega) \text{ such that} \\ G(x,s) \geq -\tilde{a}(x) - \tilde{b}(x)|s| \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}^N ; \end{cases} \tag{2.38}$$

$$\lim_{|s|\to\infty} \frac{G(x,s)}{|s|} = +\infty \qquad \text{for a.e. } x \in \Omega ; \tag{2.39}$$

$$\{s \longmapsto G(x,s)\} \text{ is even for a.e. } x \in \Omega . \tag{2.40}$$

Finally, define $\mathcal{E}$ as in Section 4. The main result of this section is:

Theorem 2.8 *For every $k \in \mathbb{N}$ there exists Λ_k such that for any $\lambda \geq \Lambda_k$ the problem*

$$\begin{cases} u \in BV(\Omega; \mathbb{R}^N) \\ \mathcal{E}(v) - \mathcal{E}(u) + \displaystyle\int_\Omega G^\circ(x,u;v-u)\,dx \geq \lambda \int_\Omega \frac{u}{\sqrt{1+|u|^2}} \cdot (v-u)\,dx \quad \forall v \in BV(\Omega; \mathbb{R}^N) \end{cases}$$

admits at least k pairs $(u, -u)$ of distinct solutions.

For the proof we need the following

Lemma 2.7 *Let (u_h) be a bounded sequence in $L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^N)$, which is convergent a.e. to u, and let (ϱ_h) be a positively divergent sequence of real numbers.*

Then we have

$$\lim_h \int_\Omega \frac{G(x, \varrho_h u_h)}{\varrho_h}\, dx = +\infty \quad \text{if } u \neq 0\,,$$
$$\liminf_h \int_\Omega \frac{G(x, \varrho_h u_h)}{\varrho_h}\, dx \geq 0 \quad \text{if } u = 0\,.$$

Proof If $u = 0$, the assertion follows directly from (2.38). If $u \neq 0$, we have

$$\int_\Omega \frac{G(x, \varrho_h u_h)}{\varrho_h}\, dx \geq \int_{\{u\neq 0\}} \frac{G(x, \varrho_h u_h)}{\varrho_h}\, dx - \frac{1}{\varrho_h} \int_{\{u=0\}} \tilde{a}\, dx - \int_{\{u=0\}} \tilde{b}|u_h|\, dx\,.$$

From (2.38), (2.39) and Fatou's Lemma, we deduce that

$$\lim_h \int_{\{u\neq 0\}} \frac{G(x, \varrho_h u_h)}{\varrho_h}\, dx = +\infty\,,$$

whence the assertion. □

Proof of Theorem 2.8. First of all, set

$$\widetilde{G}(x,s) = G(x,s) - \lambda\left(\sqrt{1+|s|^2} - 1\right).$$

It is easy to see that also $\widetilde{G}$ satisfies (G_1), (G_2), (G_3'), (G_4'), (2.38), (2.39), (2.40) and that

$$\widetilde{G}^\circ(x, s; \hat{s}) = G^\circ(x, s; \hat{s}) - \lambda \frac{s}{\sqrt{1+|s|^2}} \cdot \hat{s}\,.$$

Now define a lower semicontinuous functional $f : L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ by

$$f(u) = \mathcal{E}(u) + \int_\Omega \widetilde{G}(x,u)\, dx\,.$$

Then f is even by (2.40) and satisfies condition $(epi)_c$ by Theorem 2.4. We claim that

$$\lim_{\|u\|_{\frac{n}{n-1}} \to \infty} f(u) = +\infty\,. \tag{2.41}$$

To prove it, let (u_h) be a sequence in $BV(\Omega;\mathbb{R}^N)$ with $\|u_h\|_{\frac{n}{n-1}} = 1$ and let $\varrho_h \to +\infty$. By (e) of Lemma 2.3 there exist $\tilde{c} > 0$ and $\tilde{d} > 0$ such that

$$\forall u \in BV(\Omega;\mathbb{R}^N): \quad \mathcal{E}(u) \geq \tilde{d}\Big(\|u\|_{BV} - \tilde{c}\mathcal{L}^n(\Omega)\Big)\,.$$

If $\|u_h\|_{BV} \to +\infty$, it readily follows from (2.38) that $f(\varrho_h u_h) \to +\infty$. Otherwise, up to a subsequence, u_h is convergent a.e. and the assertion follows from the previous Lemma and the inequality

$$f(\varrho_h u_h) \geq \varrho_h \left[\tilde{d}\left(\|u_h\|_{BV} - \frac{\tilde{c}}{\varrho_h}\mathcal{L}^n(\Omega)\right) + \int_\Omega \frac{\widetilde{G}(x,\varrho_h u_h)}{\varrho_h}\,dx\right].$$

Since f is bounded below on bounded subsets of $L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^N)$, it follows from (2.41) that f is bounded below on all $L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^N)$; furthermore, it also turns out from (2.41) that any $(PS)_c$ sequence is bounded, hence f satisfies $(PS)_c$ by Theorem 2.5.

Finally, let $k \geq 1$, let $w_1, \ldots, w_k$ be linearly independent elements of $BV(\Omega;\mathbb{R}^N)$ and let $\psi : S^{k-1} \to L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^N)$ be the odd continuous map defined by

$$\psi(\xi) = \sum_{j=1}^{k} \xi_j w_j\,.$$

Because of (G_3'), it is easily seen that

$$\sup\left\{\mathcal{E}(u) + \int_\Omega G(x,u)\,dx : u \in \psi(S^{k-1})\right\} < +\infty$$

and

$$\inf\left\{\int_\Omega \big(\sqrt{1+|u|^2} - 1\big)\,dx : u \in \psi(S^{k-1})\right\} > 0\,.$$

Therefore there exists $\Lambda_k > 0$ such that $\sup\limits_{\xi \in S^{k-1}} f(\psi(\xi)) < 0$ whenever $\lambda \geq \Lambda_k$.

Next, we recall the following result which is due to Clark. We refer to Theorem 2.5 in [84] for a nonsmooth version and complete proof.

Theorem 2.9 *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ an even lower semicontinuous function. Assume that*

(a) *f is bounded from below;*

(b) *for every $c < f(0)$, the function f satisfies $(PS)_c$ and $(epi)_c$;*

(c) *there exist $k \geq 1$ and an odd continuous map $\psi : S^{k-1} \to X$ such that*

$$\sup\left\{f(\psi(x)) : x \in S^{k-1}\right\} < f(0)\,,$$

where S^{k-1} denotes the unit sphere in $\mathbb{R}^k$.

Then f admits at least k pairs $(u_1, -u_1), \ldots, (u_k, -u_k)$ of critical points with $f(u_j) < f(0)$.

Applying now Theorem 2.9, it follows that f admits at least k pairs $(u_k, -u_k)$ of critical points. Therefore, by Theorem D.2, for any u_k it is possible to apply Theorem 2.3 (with $\widetilde{G}$ instead of G), which completes the proof.

2.3.4 A superlinear potential

Let $n \geq 2$ and Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, let $\Psi : \mathbb{R}^{nN} \to \mathbb{R}$ be an even convex function satisfying (Ψ) and let $G : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a function satisfying (G_1), (G_2), (G'_3), (G'_4), (2.40) with $p = \frac{n}{n-1}$ and the following condition:

$$\begin{cases} \text{there exist } q > 1 \text{ and } R > 0 \text{ such that} \\ G^\circ(x, s; s) \leq qG(x, s) < 0 \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}^N \text{ with } |s| \geq R\,. \end{cases} \tag{2.42}$$

Define $\mathcal{E}$ as in section 4 and an even lower semicontinuous functional $f : L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ by

$$f(u) = \mathcal{E}(u) + \int_\Omega G(x, u)\, dx\,.$$

Theorem 2.10 *There exists a sequence (u_h) of solutions of the problem*

$$\begin{cases} u \in BV(\Omega; \mathbb{R}^N) \\ \mathcal{E}(v) - \mathcal{E}(u) + \displaystyle\int_\Omega G^\circ(x, u; v - u)\, dx \geq 0 \quad \forall v \in BV(\Omega; \mathbb{R}^N) \end{cases}$$

with $f(u_h) \to +\infty$.

Proof According to (2.25), we have

$$|s| < R \quad \Longrightarrow \quad |G^\circ(x, s; s)| \leq \alpha_R(x)|s|\,.$$

Combining this fact with (2.42) and (G_3'), we deduce that there exists $a_0 \in L^1(\Omega)$ such that

$$G^\circ(x,s;s) \leq qG(x,s) + a_0(x) \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}^N . \tag{2.43}$$

Moreover, from (2.42) and Lebourg's mean value theorem it follows that for every $s \in \mathbb{R}^N$ with $|s| = 1$ the function $\{t \to t^{-q}G(x,ts)\}$ is nonincreasing on $[R,+\infty[$. Taking into account (G_3') and possibly substituting a_0 with another function in $L^1(\Omega)$, we deduce that

$$G(x,s) \leq a_0(x) - b_0(x)|s|^q \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}^N , \tag{2.44}$$

where

$$b_0(x) = \inf_{|s|=1} (-R^{-q}G(x,Rs)) > 0 \qquad \text{for a.e. } x \in \Omega .$$

Finally, since $\{\hat{s} \to G^\circ(x,s;\hat{s})\}$ is a convex function vanishing at the origin, we have $G^\circ(x,s;s) \geq -G^\circ(x,s;-s)$. Combining (2.43) with (G_4'), we deduce that for every $\varepsilon > 0$ there exists $\tilde{a}_\varepsilon \in L^1(\Omega)$ such that

$$G(x,s) \geq -\tilde{a}_\varepsilon(x) - \varepsilon|s|^{\frac{n}{n-1}} \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}^N . \tag{2.45}$$

By Theorem 2.4 we have that f satisfies $(epi)_c$ for any $c \in \mathbb{R}$ and that $|d_{\mathbb{Z}_2}\mathcal{G}_f|(0,\lambda) = 1$ for any $\lambda > f(0)$.

We also recall that, since Ψ is Lipschitz continuous, there exists $M \in \mathbb{R}$ such that

$$(q+1)\Psi(\xi) - \Psi(2\xi) \geq \frac{q-1}{2}\Psi(\xi) - M , \tag{2.46}$$

$$(q+1)\Psi^\infty(\xi) - \Psi^\infty(2\xi) \geq \frac{q-1}{2}\Psi^\infty(\xi) \tag{2.47}$$

(see also [**?**]).

We claim that f satisfies the condition $(PS)_c$ for every $c \in \mathbb{R}$. Let (u_h) be a $(PS)_c$-sequence for f. By Theorem D.2 there exists a sequence (u_h^*) in $L^n(\Omega;\mathbb{R}^N)$ with $u_h^* \in \partial f(u_h)$ and $\|u_h^*\|_n \to 0$. According to Theorem 2.3 and (2.43), we have

$$\begin{aligned}
\mathcal{E}(2u_h) \quad &\geq \quad \mathcal{E}(u_h) - \int_\Omega G^\circ(x,u_h;u_h)\,dx + \int_\Omega u_h^* \cdot u_h\,dx \geq \\
&\geq \quad \mathcal{E}(u_h) - q\int_\Omega G(x,u_h)\,dx + \int_\Omega u_h^* \cdot u_h\,dx - \int_\Omega a_0(x)\,dx .
\end{aligned}$$

By the definition of f, it follows

$$qf(u_h) + \|u_h^*\|_n \|u_h\|_{\frac{n}{n-1}} + \int_\Omega a_0(x)\,dx \geq (q+1)\mathcal{E}(u_h) - \mathcal{E}(2u_h)\,.$$

Finally, applying (2.46) and (2.47) we get

$$qf(u_h) + \|u_h^*\|_n \|u_h\|_{\frac{n}{n-1}} + \int_\Omega a_0(x)\,dx \geq \frac{q-1}{2}\mathcal{E}(u_h) - M\mathcal{L}^n(\Omega)\,.$$

By (e) of Lemma 2.3 we deduce that (u_h) is bounded in $BV(\Omega;\mathbb{R}^N)$, hence in $L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^N)$. Applying Theorem 2.5 we get that (u_h) admits a strongly convergent subsequence and $(PS)_c$ follows.

By [**?**, Lemma 3.8], there exist a strictly increasing sequence (W_h) of finite-dimensional subspaces of $BV(\Omega;\mathbb{R}^N)\cap L^\infty(\Omega;\mathbb{R}^N)$ and a strictly decreasing sequence (Z_h) of closed subspaces of $L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^N)$ such that $L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^N) = W_h \oplus Z_h$ and $\bigcap\limits_{h=0}^{\infty} Z_h = \{0\}$. By (e) of Lemma 2.3 there exists $\varrho > 0$ such that

$$\forall u \in L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^N): \qquad \|u\|_{\frac{n}{n-1}} = \varrho \quad \Longrightarrow \quad \mathcal{E}(u) \geq 1\,.$$

We claim that

$$\lim_h \Big(\inf\{f(u) : u \in Z_h, \|u\|_{\frac{n}{n-1}} = \varrho\}\Big) > f(0)\,.$$

Actually, assume by contradiction that (u_h) is a sequence with $u_h \in Z_h$, $\|u_h\|_{\frac{n}{n-1}} = \varrho$ and

$$\limsup_h f(u_h) \leq f(0)\,.$$

Taking into account (G_3') and Lemma 2.3, we deduce that $(\mathcal{E}(u_h))$ is bounded, so that (u_h) is bounded in $BV(\Omega;\mathbb{R}^N)$. Therefore, up to a subsequence, (u_h) is convergent a.e. to 0. From (2.45) it follows that

$$\liminf_h \int_\Omega \Big(G(x,u_h) + \varepsilon |u_h|^{\frac{n}{n-1}}\Big)\,dx \geq \int_\Omega G(x,0)\,dx\,,$$

hence

$$\liminf_h \int_\Omega G(x,u_h)\,dx \geq \int_\Omega G(x,0)\,dx$$

by the boundedness of (u_h) in $L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^N)$ and the arbitrariness of ε. Therefore

$$\limsup_h \mathcal{E}(u_h) \leq \mathcal{E}(0) = 0$$

which contradicts the choice of ϱ.

Now, fix $\overline{h}$ with

$$\inf\{f(u) : u \in Z_{\overline{h}}, \|u\|_{\frac{n}{n-1}} = \varrho\}\Big) > f(0)$$

and set $Z = Z_{\overline{h}}$ and $V_h = W_{\overline{h}+h}$. Then Z satisfies assumption (a) of Theorem D.3 for some $\alpha > f(0)$.

Finally, since V_h is finite-dimensional,

$$\|u\|_G := \left(\int_\Omega b_0|u|^q dx\right)^{\frac{1}{q}}$$

is a norm on V_h equivalent to the norm of $BV(\Omega;\mathbb{R}^N)$. Then, combining (2.44) with (d) of Lemma 2.3, we see that also assumption (b) of Theorem D.3 is satisfied.

Therefore there exists a sequence (u_h) of critical points for f with $f(u_h) \to +\infty$ and, by Theorems D.2 and 2.3, the result follows. $\square$

2.4 Historical notes and comments

The space of functions of bounded variations is very useful in the calculus of variations (geometric measure theory, fracture mechanics and image processing), as well as in the study of shock waves for nonlinear hyperbolic conservation laws.

3
Nonlinear Eigenvalue Problems

> All truths are easy to understand once they are discovered; the point is to discover them.
>
> Galileo Galilei (1564–1642)

The study of nonlinear eigenvalue problems for quasilinear operators on unbounded domains involving the p-Laplacian is motivated by various applications, for instance, in Fluid Mechanics, in mathematical models of the torsional creep, in nonlinear field equations from quantum mechanics. For instance, in fluid mechanics, the shear stress $\vec{\tau}$ and the velocity gradient $\nabla_p u$ of certain fluids obey a relation of the form $\vec{\tau}(x) = a(x)\nabla_p u(x)$, where $\nabla_p u = |\nabla u|^{p-2}\nabla u$ and $p > 1$ is an arbitrary real number. The case $p = 2$ (respectively $p < 2$, $p > 2$) corresponds to a Newtonian (respectively pseudo-plastic, dilatant) fluid. Then the resulting equations of motion involve $\operatorname{div}(a\nabla_p u)$, which reduces to $a\Delta_p u = a \operatorname{div}(\nabla_p u)$, provided that a is a constant. The p-Laplace operator appears in the study of flows through porous media ($p = 3/2$, see Showalter-Walkington [275]) or glacial sliding ($p \in (1, 4/3]$, see Pélissier-Reynaud [236]). We also refer to Aronsson-Janfalk [13] for the mathematical treatment of the Hele-Shaw flow of "power-law fluids". The concept of Hele-Shaw flow corresponds to a flow between two closely-spaced parallel plates, where the gap between the plates is small compared to the dimension of the plates. Quasilinear problems with a variable coefficient also appear in mathematical models of torsional creeps (elastic for $p = 2$, plastic as $p \to \infty$, see Bhattacharya-DiBenedetto-Manfredi [38] and Kawohl [155]). This study is based on the observation that a prismatic material rod subject to a torsional moment, at sufficiently high temperature and for an extended period of time, exhibits a permanent deformation, called *creep*. The corresponding equations are derived under the assumptions that the components of strain and stress are linked by a power law referred to as the *creep-law* see Kachanov [151, Chapters IV, VIII], [152], and Findley-Lai-Onaran [120]). A nonlinear field equation in quantum mechanics involving the p-Laplacian, for $p = 6$, has been proposed in Benci-Fortunato-Pisani [34].

In this chapter we are concerned to study the following quasilinear eigenvalue problem

$$(P_{\lambda,\mu}) \qquad \begin{cases} -\mathrm{div}\,(a(x)|\nabla u|^{p-2}\nabla u) = \lambda\alpha(x,u) + \beta(x,u) & \text{in } \Omega, \\ a(x)|\nabla u|^{p-2}\nabla u \cdot \mathsf{n} + b(x)|u|^{p-2}u = \mu h(x,u) & \text{on } \Gamma\,, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an unbounded domain with (possible noncompact) smooth boundary Γ, n denotes the unit outward normal on Γ, $a \in L^\infty(\Omega), b : \Gamma \to \mathbb{R}, \alpha, \beta : \Omega \times \mathbb{R} \to \mathbb{R}$ and $h : \Gamma \times \mathbb{R} \to \mathbb{R}$ are functions which satisfy some growth conditions and $\lambda, \mu \in \mathbb{R}$ are parameters. Our aim is to study the existence and multiplicity of the solutions of problem $(P_{\lambda,\mu})$ for different values of the parameters $\lambda, \mu \in \mathbb{R}$ and for various functions α, β and h. Eigenvalue problems involving the p-Laplacian have been the subject of much recent interest, see for example the papers of Allegretto-Huang [4], Anane [11], Drábek [92], Drábek-Pohozaev [94], Drábek-Simader [95], García-Montefusco-Peral [122], García-Peral [123].

In the first section we describe the weighted Sobolev spaces, where we define the energy functional associated to the problem $(P_{\lambda,\mu})$. In the next section our attention is focused to prove existence and multiplicity result for problem $(P_{\lambda,\mu})$, when $\mu = 1$, using the Mountain Pass Theorem, and Ljusternik-Schnirelmann theory. In the following section we study the problem $(P_{\lambda,\mu})$, when $\lambda = \mu$. Using the Mountain Pass type result proved by Motreanu, we prove the problem has a nontrivial solutions. In the last parts of this chapter we prove that the problem $(P_{\lambda,\mu})$ has two nontrivial weak solutions if the parameters λ and μ belongs to some interval.

3.1 Weighted Sobolev spaces

Let $\Omega \subset \mathbb{R}^N$ be an unbounded domain with smooth boundary Γ. We assume that p, q and m are real numbers satisfying

$$1 < p < N, \quad p < q < p^* = \frac{Np}{N-p}, \ (p^* = +\infty, \text{ if } p \geq N), \tag{3.1}$$

and

$$q \leq m < \frac{p(N-1)}{N-p} \text{ if } p < N \ (q \leq m < +\infty, \text{ when } p \geq N). \tag{3.2}$$

Let $C_\delta^\infty(\Omega)$ be the space of $C_0^\infty(\mathbb{R}^N)$-functions restricted on Ω.

We define the weighted Sobolev space E as the completion of $C_\delta^\infty(\Omega)$ in the norm

$$\|u\|_E = \left(\int\limits_\Omega \left(|\nabla u(x)|^p + \frac{1}{(1+|x|)^p} |u(x)|^p \right) dx \right)^{1/p}.$$

Denote by $L^p(\Omega;\, w_1)$ and $L^m(\Gamma;\, w_2)$ the weighted Lebesgue spaces with weight functions

$$w_i(x) = (1+|x|)^{\beta_i}, \quad (i = 1, 2), \tag{3.3}$$

and the norms defined by

$$\|u\|_{p,w_1}^p = \int_\Omega w_1 |u(x)|^p \, dx, \quad \|u\|_{q,w_2}^q = \int_\Gamma w_2 |u(x)|^q \, dx$$

where

$$-N < \beta_1 < -p, \ \text{ if } \ p < N \ (\beta_1 < -p, \ \text{ when } \ p \geq N), \tag{3.4}$$

$$-N < \beta_2 < q\frac{N-p}{p} - N + 1, \ \text{ if } \ p < N \, (-N < \beta_2 < 0, \ \text{ when } \ p \geq N). \tag{3.5}$$

We have the following weighted Hardy-type inequality.

Lemma 3.1 *Let $1 < p < N$. Then, there exist positive constants C_1 and C_2, such that for every $u \in E$ it holds*

$$\int_\Omega \frac{1}{(1+|x|)^p} |u|^p \, dx \leq C_1 \int_\Omega |\nabla u|^p \, dx + C_2 \int_\Gamma \frac{|\mathsf{n} \cdot x|}{(1+|x|)^p} |u|^p \, d\Gamma \,. \tag{3.6}$$

Proof Using the divergence theorem we obtain for $u \in C_\delta^\infty(\Omega)$

$$\int_\Omega x\cdot\nabla\left(\frac{1}{(1+|x|)^p}|u|^p\right)dx = \int_\Gamma (\mathsf{n}\cdot x)\frac{1}{(1+|x|)^p}|u|^p\,d\Gamma - N\int_\Omega \frac{1}{(1+|x|)^p}|u|^p\,dx\,.$$

This implies

$$\begin{aligned} N\int_\Omega \frac{1}{(1+|x|)^p}|u|^p\,dx \quad &\le \quad \int_\Gamma \frac{|\mathsf{n}\cdot x|}{(1+|x|)^p}|u|^p\,d\Gamma + p\int_\Omega \frac{1}{(1+|x|)^p}|u|^p\,dx \\ &\quad +p\int_\Omega \frac{1}{(1+|x|)^{p-1}}|u|^{p-1}|\nabla u|\,dx\,. \end{aligned}$$

Using Hölder's and Young's inequalities, the last term can be estimated by

$$\begin{aligned} &p\left(\int_\Omega \frac{1}{(1+|x|)^p}|u|^p\,dx\right)^{(p-1)/p}\left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p} \\ &\le \quad \varepsilon(p-1)\int_\Omega \frac{1}{(1+|x|)^p}|u|^p\,dx + \varepsilon^{1-p}\int_\Omega |\nabla u|^p\,dx\,, \end{aligned}$$

where $\varepsilon > 0$ is an arbitrary real number. It follows that

$$(N-\varepsilon(p-1)-p)\int_\Omega \frac{1}{(1+|x|)^p}|u|^p\,dx \le \varepsilon^{1-p}\int_\Omega |\nabla u|^p\,dx + \int_\Gamma \frac{|\mathsf{n}\cdot x|}{(1+|x|)^p}|u|^p\,d\Gamma\,,$$

and for ε small enough, the desired inequality follows by standard density arguments. □

In this chapter we shall use the following embedding result, see Pflüger [237] and [238].

Theorem 3.1 *If*

$$p \le r \le \frac{pN}{N-p} \quad \textit{and} \quad -N < \beta_1 \le r\,\frac{N-p}{p} - N\,, \tag{3.7}$$

then the embedding $E \hookrightarrow L^r(\Omega; w_1)$ is continuous. If the upper bounds for r in (3.7) *are strict, then the embedding is compact. If*

$$p \le q \le \frac{p(N-1)}{N-p} \quad \textit{and} \quad -N < \beta_2 \le q\,\frac{N-p}{p} - N + 1\,, \tag{3.8}$$

then the trace operator $E \hookrightarrow L^q(\Gamma; w_2)$ is continuous. If the upper bounds for q in (3.8) *are strict, then the trace operator is compact.*

As a corollary of Lemma 3.1 and Theorem 3.1 we obtain:

Lemma 3.2 *Let b satisfying $\dfrac{c}{(1+|x|)^{p-1}} \le b(x) \le \dfrac{C}{(1+|x|)^{p-1}}$ for some constants $0 < c \le C$. Then*

$$\|u\|_b^p = \int_\Omega a(x)|\nabla u|^p \, dx + \int_\Gamma b(x)|u|^p \, d\Gamma$$

defines an equivalent norm on E.

Proof The inequality $\|u\|_E \le C_1\|u\|_b$ follows directly from Lemma 3.1, while Theorem 3.1 (for $p = q$ and $\beta_2 = -(p-1)$) implies

$$\begin{aligned} \|u\|_b^p &\le \|a\|_{L^\infty} \int_\Omega |\nabla u|^p dx + C \int_\Gamma |u|^p (1+|x|)^{-(p-1)} d\Gamma \\ &\le \|a\|_{L^\infty} \int_\Omega |\nabla u|^p dx + C_2 \|u\|_E^p, \end{aligned}$$

which shows the desired equivalence. □

We assume that $\mathrm{h} : \Gamma \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that satisfies the following conditions:
$(\mathcal{H})$ $|\mathrm{h}(x,s)| \le \mathrm{h}_0(x) + \mathrm{h}_1(x)|s|^{m-1}$; $p \le m < p \cdot \dfrac{N-1}{N-p}$, where $\mathrm{h}_i : \Gamma \to \mathbb{R}$ are nonnegative, measurable functions such that

$$0 \le \mathrm{h}_i(x) \le C_h w_2 \ \text{a.e.}, \quad \mathrm{h}_0 \in L^{m/(m-1)}(\Gamma;\, w_2^{1/(1-m)}),$$

where $-N < \beta_2 < m \cdot \dfrac{N-p}{p} - N + 1$ and w_2 is defined as in (3.3).

Set $\mathrm{H}(x,s) = \displaystyle\int_0^s \mathrm{h}(x,t)dt$. We denote by $N_{\mathrm{h}}, N_{\mathrm{H}}$ the corresponding Nemytskii operators.

Lemma 3.3 *The operators*

$$N_{\mathrm{h}} : L^m(\Gamma;\, w_2) \to L^{m/(m-1)}(\Gamma;\, w_2^{1/(1-m)}), \ \ N_{\mathrm{H}} : L^m(\Gamma;\, w_2) \to L^1(\Gamma)$$

are bounded and continuous.

Proof Let $m' = m/(m-1)$ and $u \in L^m(\Gamma; w_2)$. Then, by $(\mathcal{H})$,

$$\int_\Gamma |N_{\mathrm{h}}(u)|^{m'} \cdot w_2^{1/(1-m)}\, d\Gamma \leq$$

$$\leq 2^{m'-1}\left(\int_\Gamma \mathrm{h}_0^{m'} \cdot w_2^{1/(1-m)}\, d\Gamma + \int_\Gamma h_1^{m'} |u|^m \cdot w_2^{1/(1-m)}\, d\Gamma\right) \leq$$

$$\leq 2^{m'-1}\left(C + C_h \cdot \int_\Gamma |u|^m \cdot w_2\, d\Gamma\right),$$

which shows that N_{h} is bounded. In a similar way we obtain

$$\int_\Gamma |N_{\mathrm{H}}(u)|\, d\Gamma \leq \int_\Gamma \mathrm{h}_0 |u|\, d\Gamma + \int_\Gamma \mathrm{h}_1 |u|^m\, d\Gamma \leq$$

$$\leq \left(\int_\Gamma \mathrm{h}_0^{m'} \cdot w_2^{1/(1-m)}\, d\Gamma\right)^{1/m'} \cdot \left(\int_\Gamma |u|^m \cdot w_2\, d\Gamma\right)^{1/m} + C_g \cdot \int_\Gamma |u|^m \cdot w_2\, d\Gamma,$$

which imply the boundedness of N_{H}.

From the usual properties of Nemytskii operators we deduce the continuity of N_{h} and N_{H}. □

3.2 Eigenvalue problems

We suppose that p, q and m are real numbers satisfying the conditions (3.1) and (3.2). In this section we consider the particular case for $(P_{\lambda,\mu})$, when $\alpha(x,u) = f(x)|u|^{p-2}u$, $\beta(x,u) = g(x)|u|^{q-2}u$ and $\mu = 1$. In this case problem $(P_{\lambda,\mu})$ becomes

$$(P_\lambda) \begin{cases} -\mathrm{div}\,(a(x)|\nabla u|^{p-2}\nabla u) = \lambda f(x)|u|^{p-2}u + g(x)|u|^{q-2}u & \text{in } \Omega, \\ a(x)|\nabla u|^{p-2}\nabla u \cdot \mathsf{n} + b(x)|u|^{p-2}u = h(x,u) & \text{on } \Gamma, \end{cases}$$

where n denotes the unit outward normal on Γ, $0 < a_0 \leq a$, where $a \in L^\infty(\Omega)$ and $b : \Gamma \to \mathbb{R}$ is a continuous function satisfying

$$\frac{c}{(1+|x|)^{p-1}} \leq b(x) \leq \frac{C}{(1+|x|)^{p-1}},$$

for some constants $0 < c \leq C$.

We assume that f and g are nontrivial measurable functions satisfying

$$0 \leq f(x) \leq C(1+|x|)^{\alpha_1} \quad \text{and} \quad 0 \leq g(x) \leq C(1+|x|)^{\alpha_2}, \quad \text{for a.e. } x \in \Omega,$$

where the numbers α_1, α_2 satisfy the conditions

$$-N < \alpha_1 < -p, \quad \text{if } \ p < N(\alpha_1 < -p, \quad \text{when } \ p \geq N), \tag{3.9}$$

and

$$-N < \alpha_2 < q\frac{N-p}{p} - N + 1, \quad \text{if } \ p < N \,(-N < \alpha_2 < 0, \quad \text{when } \ p \geq N). \tag{3.10}$$

The mapping $h : \Gamma \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which fulfills the assumption

$(H1)$ $\quad |h(x,s)| \leq h_0(x) + h_1(x)|s|^{m-1}$,

where $h_i : \Gamma \to \mathbb{R}$ $(i = 0, 1)$ are measurable functions satisfying

$$h_0 \in L^{m/(m-1)}(\Gamma;\, w_3^{1/(1-m)}) \quad \text{and} \quad 0 \leq h_i \leq C_h w_3 \quad \text{a.e. on } \Gamma,$$

and $w_3 = (1 + |x|)^{\alpha_3}$, where α_3 satisfies the conditions

$$-N < \alpha_3 < m\frac{N-p}{p} - N + 1, \quad \text{if } \ p < N \quad (-N < \alpha_3 < 0 \ \text{ when } \ p \geq N). \tag{3.11}$$

We also assume:

$(H2)$ $\quad \displaystyle\lim_{s \to 0} \frac{h(x,s)}{b(x)|s|^{p-1}} = 0$ uniformly in x.

$(H3)$ $\quad$ There exists $\mu \in (p, q]$ such that

$$\mu H(x,t) \leq t h(x,t) \text{ for a.e. } x \in \Gamma \text{ and every } t \in \mathbb{R}\,.$$

$(H4)$ $\quad$ There exists a nonempty open set $U \subset \Gamma$ with $H(x,t) > 0$ for $(x,t) \in U \times (0, \infty)$, where $H(x,t) = \displaystyle\int_0^t h(x,s)\,ds$.

By a weak solution of problem (P_λ) we mean a function $u \in E$ such that for any $v \in E$ it holds

$$\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla v \ dx + \int_\Gamma b(x)|u|^{p-2}uv \ dS$$

$$= \lambda \int_\Omega f(x)|u|^{p-2}uvdx + \int_\Omega g(x)|u|^{q-2}uvdx + \int_\Gamma h(x,u)vdS.$$

The energy functional corresponding to (P_λ) is defined as $\mathcal{E}_\lambda : E \to \mathbb{R}$

$$\begin{aligned} \mathcal{E}_\lambda(u) \quad &= \quad \frac{1}{p}\int_\Omega a(x)|\nabla u|^p \, dx + \frac{1}{p}\int_\Gamma b(x)|u|^p \, dS \\ &- \quad \frac{\lambda}{p}\int_\Omega f(x)|u|^p \, dx - \int_\Gamma H(x,u)\, dS - \frac{1}{q}\int_\Omega g(x)|u|^q \, dx. \end{aligned}$$

By Lemma 3.2 we have $\|\cdot\|_b \simeq \|\cdot\|_E$. We may write

$$\mathcal{E}_\lambda(u) = \frac{1}{p}\|u\|_b^p - \frac{\lambda}{p}\int_\Omega f(x)|u|^p\,dx - \int_\Gamma H(x,u)\,dS - \frac{1}{q}\int_\Omega g(x)|u|^q\,dx.$$

Since $p < q < p^*$, $-N < \alpha_1 < -p$ and $-N < \alpha_2 < q\frac{N-p}{p} - N$, we can apply Theorem 3.1 and obtain that the embedding $E \hookrightarrow L^p(\Omega;\, w_1)$, $E \hookrightarrow L^q(\Omega;\, w_2)$ and $E \hookrightarrow L^q(\Gamma; w_w)$ are compact. So, the functional $\mathcal{E}_\lambda$ is well defined.

Lemma 3.4 *Under the assumptions* (*H*1)-(*H*4)*, the functional* $\mathcal{E}_\lambda$ *is Fréchet differentiable on* E *and satisfies the Palais-Smale condition.*

Proof Denote $I(u) = \frac{1}{p}\|u\|_b^p$, $K_H(u) = \displaystyle\int_\Gamma H(x,u)\,dS$, $K_\Psi(u) = \displaystyle\int_\Omega \Psi(x,u)\,dx$ and $K_\Phi(u) = \displaystyle\int_\Omega \Phi(x,u)\,dx$, where $\Phi(x,u) = \frac{1}{p}f(x)|u|^p$ and $\Psi(x,u) = \frac{1}{q}g(x)|u|^q$.

Then the directional derivative of $\mathcal{E}'_\lambda$ in the direction $v \in E$ is

$$\langle \mathcal{E}'_\lambda(u), v\rangle = \langle I'(u), v\rangle - \lambda\langle K'_\Phi(u), v\rangle - \langle K'_\Psi(u), v\rangle - \langle K'_H(u), v\rangle,$$

where

$$\langle I'(u), v\rangle = \int_\Omega a(x)|\nabla u|^{p-2}\nabla u\nabla v\,dx + \int_\Gamma b(x)|u|^{p-2}uv\,dS,$$

$$\langle K'_H(u), v\rangle = \int_\Gamma h(x,u)v\,dS,$$

$$\langle K'_\Psi(u), v\rangle = \int_\Omega g(x)|u|^{q-2}uv\,dx,$$

$$\langle K'_\Phi(u), v\rangle = \int_\Omega f(x)|u|^{p-2}uv\,dx.$$

Clearly, $I' : E \to E^\star$ is continuous. The operator $K_H^{'}$ is a composition of the operators

$$K_H^{'} : E \to L^m(\Gamma;\, w_3) \xrightarrow{N_h} L^{m/(m-1)}(\Gamma;\, w_3^{1/(1-m)}) \xrightarrow{l} E^\star$$

where $\langle l(u), v\rangle = \displaystyle\int_\Gamma uv\,dS$. Since

$$\int_\Gamma |uv|\,dS \le \left(\int_\Gamma |u|^{m'} w_3^{1/(1-m)}\,dS\right)^{1/m'} \cdot \left(\int_\Gamma |v|^m w_3\,dS\right)^{1/m},$$

then by Theorem 3.1 l is continuous. As a composition of continuous operators, $K_H^{'}$ is continuous, too. Moreover, by our assumptions on w_3, the trace operator $E \hookrightarrow L^m(\Gamma;\, w_3)$ is compact and therefore, $K_H^{'}$ is also compact.

Set $\varphi(u) = f(x)\,|u|^{p-2}u$. Lemma 3.3 implies that that the Nemytskii operator corresponding to any function which satisfies $(H1)$ is bounded and continuous. Hence N_h and N_φ are bounded and continuous. Note that

$$K_\Phi^{'} : E \subset L^p(\Omega;\, w_1) \stackrel{N_\varphi}{\to} L^{p/(p-1)}(\Omega;\, w_1^{1/(1-p)}) \stackrel{\eta}{\to} E^\star,$$

where $\langle \eta(u), v\rangle = \displaystyle\int_\Omega uv\, dx$. Since

$$\int_\Omega |uv|\, dx \le \left(\int_\Omega |u|^{p/(p-1)} w_1^{1/(1-p)}\, dx\right)^{(p-1)/p} \cdot \left(\int_\Omega |v|^p w_1\, dx\right)^{1/p},$$

it follows that η is continuous. But $K_\Phi^{'}$ is the composition of three continuous operators and by the assumptions on w_1, the embedding $E \hookrightarrow L^p(\Omega;\, w_1)$ is compact. This implies that $K_\Phi^{'}$ is compact. In a similar way we obtain that $K_\Psi^{'}$ is compact and this implies the continuous Fréchet differentiability of $\mathcal{E}_\lambda$.

Now, let $u_n \in E$ be a Palais-Smale sequence, that is,

$$|\mathcal{E}_\lambda(u_n)| \le C \text{ for all } n \tag{3.12}$$

and

$$\|\mathcal{E}_\lambda'(u_n)\|_{E^\star} \to 0 \text{ as } n \to \infty. \tag{3.13}$$

We first prove that $\{u_n\}$ is bounded in E. Remark that (3.13) implies that

$$|\langle \mathcal{E}_\lambda'(u_n), u_n\rangle| \le \mu \cdot \|u_n\|_b \text{ for } n \text{ large enough.}$$

Then by (3.12) we obtain

$$C + \|u_n\|_b \ge \mathcal{E}_\lambda(u_n) - \frac{1}{\mu}\langle \mathcal{E}_\lambda'(u_n), u_n\rangle. \tag{3.14}$$

But

$$\begin{aligned}\langle \mathcal{E}_\lambda'(u_n), u_n\rangle \;&=\; \int_\Omega a(x)|\nabla u_n|^p\, dx + \int_\Gamma b(x)|u_n|^p\, dS\\ &-\; \lambda\int_\Omega f(x)|u_n|^p\, dx - \int_\Omega g(x)|u_n|^q\, dx - \int_\Gamma h(x,u_n)u_n\, dS.\end{aligned}$$

We have

$$\mathcal{E}_\lambda(u_n) - \frac{1}{\mu}\langle \mathcal{E}'_\lambda(u_n), u_n\rangle = \left(\frac{1}{p} - \frac{1}{\mu}\right)\left(\|u_n\|_b^p - \lambda \int_\Omega f(x)|u|^p dx\right)$$

$$-\left(\int_\Gamma H(x,u_n)\, dS - \frac{1}{\mu}\int_\Gamma h(x,u_n)u_n\, dS\right) - \left(\frac{1}{q} - \frac{1}{\mu}\right)\int_\Omega g(x)|u_n|^q\, dx).$$

By $(H3)$ we deduce that

$$\int_\Gamma H(x,u_n)\, dS \le \frac{1}{\mu}\int_\Gamma h(x,u_n)u_n\, dS. \tag{3.15}$$

Therefore

$$\mathcal{E}_\lambda(u_n) - \frac{1}{\mu}\langle \mathcal{E}'_\lambda(u_n), u_n\rangle \ge \left(\frac{1}{p} - \frac{1}{\mu}\right) C_0\|u_n\|_b^p. \tag{3.16}$$

Relations (3.14) and (3.16) yield

$$C + \|u_n\|_b \ge \left(\frac{1}{p} - \frac{1}{\mu}\right) C_0\|u_n\|_b^p.$$

This shows that $\{u_n\}$ is bounded in E.

To prove that $\{u_n\}$ contains a Cauchy sequence we use the following well-known inequalities:

$$|\xi - \zeta|^p \le C(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta), \quad \text{for } p \ge 2, \tag{3.17}$$

$$|\xi - \zeta|^2 \le C(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta)(|\xi| + |\zeta|)^{2-p}, \quad \text{for } 1 < p < 2, \tag{3.18}$$

for every $\xi, \zeta \in \mathbb{R}^N$.

In the case $p \ge 2$ we obtain:

$$\begin{aligned}
\|u_n - u_k\|_b^p &= \int_\Omega a(x)|\nabla u_n - \nabla u_k|^p\, dx + \int_\Gamma b(x)|u_n - u_k|^p\, dS \\
&\le C(\langle I'(u_n), u_n - u_k\rangle - \langle I'(u_k), u_n - u_k\rangle) \\
&= C(\langle \mathcal{E}'_\lambda(u_n), u_n - u_k\rangle - \langle \mathcal{E}'_\lambda(u_k), u_n - u_k\rangle \\
&\quad + \lambda\langle K'_\Phi(u_n), u_n - u_k\rangle - \lambda\langle K'_\Phi(u_k), u_n - u_k\rangle \\
&\quad + \langle K'_H(u_n), u_n - u_k\rangle - \langle K'_H(u_k), u_n - u_k\rangle \\
&\quad + \langle K'_\Psi(u_n), u_n - u_k\rangle - \langle K'_\Psi(u_k), u_n - u_k\rangle) \\
&\le C(\|\mathcal{E}'_\lambda(u_n)\|_{E^\star} + \|\mathcal{E}'_\lambda(u_k)\|_{E^\star} + |\lambda|\, \|K'_\Phi(u_n) - K'_\Phi(u_k)\|_{E^\star} \\
&\quad + \|K'_H(u_n) - K'_H(u_k)\|_{E^\star} + \|K'_\Psi(u_n) - K'_\Psi(u_k)\|_{E^\star})\|u_n - u_k\|_b.
\end{aligned}$$

Since $\mathcal{E}'_\lambda(u_n) \to 0$ and K'_Φ, K'_Ψ, K'_H are compact, we conclude (passing eventually to a subsequence) that $\{u_n\}$ converges in E.

If $1 < p < 2$, then we use the estimate

$$\|u_n - u_k\|_b^2 \leq C' |\langle I'(u_n), u_n - u_k\rangle - \langle I'(u_k), u_n - u_k\rangle| (\|u_n\|_b^{2-p} + \|u_k\|_b^{2-p}). \tag{3.19}$$

Since $\|u_n\|_b$ is bounded, the same arguments lead to a convergent subsequence. In order to prove the estimate (3.19) we recall the following result: for all $s \in (0, \infty)$ there is a constant $C_s > 0$ such that

$$(x + y)^s \leq C_s (x^s + y^s) \qquad \text{for any } x, y \in (0, \infty). \tag{3.20}$$

Then we obtain

$$\begin{aligned} \|u_n - u_k\|_b^2 &= \left(\int_\Omega a(x) |\nabla u_n - \nabla u_k|^p \, dx + \int_\Gamma b(x) |u_n - u_k|^p \, dS \right)^{\frac{2}{p}} \\ &\leq C_p \left[\left(\int_\Omega a(x) |\nabla u_n - \nabla u_k|^p \, dx \right)^{\frac{2}{p}} + \left(\int_\Gamma b(x) |u_n - u_k|^p \, dS \right)^{\frac{2}{p}} \right]. \end{aligned} \tag{3.21}$$

Using (3.18), (3.20) and the Hölder inequality we find

$$
\begin{aligned}
&\int_\Omega a(x)|\nabla u_n - \nabla u_k|^p\,dx = \int_\Omega a(x)(|\nabla u_n - \nabla u_k|^2)^{\frac{p}{2}}\,dx \\
\leq\;& C\int_\Omega a(x)\left((|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\right)^{\frac{p}{2}} \\
\times\;& (|\nabla u_n| + |\nabla u_k|)^{\frac{p(2-p)}{2}}\,dx \\
=\;& C\int_\Omega \left(a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\right)^{\frac{p}{2}} \\
\times\;& (a(x)(|\nabla u_n| + |\nabla u_k|)^p)^{\frac{2-p}{2}}\,dx \\
\leq\;& C\left(\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\,dx\right)^{\frac{p}{2}} \\
\times\;& \left(\int_\Omega a(x)(|\nabla u_n| + |\nabla u_k|)^p\,dx\right)^{\frac{2-p}{2}} \\
\leq\;& \tilde{C}_p\left(\int_\Omega a(x)|\nabla u_n|^p\,dx + \int_\Omega a(x)|\nabla u_k|^p\,dx\right)^{\frac{2-p}{2}} \\
& \left(\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\,dx\right)^{\frac{p}{2}} \\
\leq\;& \overline{C}_p\left[\left(\int_\Omega a(x)|\nabla u_n|^p\,dx\right)^{\frac{2-p}{2}} + \left(\int_\Omega a(x)|\nabla u_k|^p\,dx\right)^{\frac{2-p}{2}}\right] \\
\times\;& \left(\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\,dx\right)^{\frac{p}{2}} \\
\leq\;& \overline{C}_p\left[\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\,dx\right]^{\frac{p}{2}} \\
\times\;& \left(\|u_n\|_b^{\frac{(2-p)p}{2}} + \|u_k\|_b^{\frac{(2-p)p}{2}}\right).
\end{aligned}
$$

Using the last inequality and (3.20) we have the estimate

$$
\begin{aligned}
&\left(\int_\Omega a(x)|\nabla u_n - \nabla u_k|^p\,dx\right)^{\frac{2}{p}} \qquad (3.22)\\
&\leq C_p'\left(\int_\Omega a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_k|^{p-2}\nabla u_k)(\nabla u_n - \nabla u_k)\,dx\right) \\
&\cdot(\|u_n\|_b^{2-p} + \|u_k\|_b^{2-p}).
\end{aligned}
$$

In a similar way we can obtain the estimate

$$\left(\int_\Gamma b(x)|u_n-u_k|^p\,dS\right)^{\frac{2}{p}} \le C_p'\left(\int_\Gamma b(x)(|u_n|^{p-2}u_n-|u_k|^{p-2}u_k)(u_n-u_k)\,dx\right) \tag{3.23}$$

$$\cdot(\|u_n\|_b^{2-p}+\|u_k\|_b^{2-p}).$$

It is now easy to observe that the inequalities (3.21), (3.22) and (3.23) imply the estimate (3.19). The proof of Lemma 3.4 is complete. □

Define

$$\tilde{\lambda} := \inf_{u\in E;\ u\neq 0}\left(\frac{\displaystyle\int_\Omega a(x)|\nabla u|^p\,dx+\int_\Gamma b(x)|u|^p\,dS}{\displaystyle\int_\Omega f(x)|u|^p\,dx}\right).$$

We have the following existence result.

Theorem 3.2 *Assume that the conditions* $(H1)$-$(H4)$ *hold. Then, for every* $\lambda<\tilde{\lambda}$, *problem* (P_λ) *has a nontrivial weak solution.*

Our hypothesis $\lambda<\tilde{\lambda}$ implies the existence of some $C_0>0$ such that, for every $v\in E$

$$\|v\|_b^p-\lambda\int_\Omega f(x)|v|^p dx \ge C_0\|v\|_b^p. \tag{3.24}$$

Proof of Theorem 3.2. We have to verify the geometric assumptions of the mountain pass theorem, see Theorem 1.7. First we show that there exist positive constants R and c_0 such that

$$\mathcal{E}_\lambda(u)\ge c_0, \qquad \text{for any } u\in E \text{ with } \|u\|=R. \tag{3.25}$$

By Theorem 3.1 there exists $A>0$ such that

$$\|u\|_{q,w_2}^q \le A\|u\|_b^q \quad \text{for all } u\in E.$$

This inequality together with (3.24) imply that

$$\begin{aligned}\mathcal{E}_\lambda(u) &= \frac{1}{p}\left(\|u\|_b^p-\lambda\|u\|_{p,w_1}^p\right)-\frac{1}{q}\int_\Omega g(x)|u|^q\,dx-\int_\Gamma H(x,u)\,dS\\ &\ge \frac{C_0}{p}\|u\|_b^p-\frac{A}{q}\|u\|_b^q-\int_\Gamma H(x,u)\,dS.\end{aligned}$$

By $(H1)$ and $(H2)$ we deduce that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\frac{1}{q}|g(x)||u|^q \leq \varepsilon b(x)|u|^p + C_\varepsilon w_3(x)|u|^m.$$

Consequently,

$$\begin{aligned}
\mathcal{E}_\lambda(u) &\geq \frac{C_0}{p}\|u\|_b^p - \frac{A}{q}\|u\|_b^q - \int_\Gamma (\varepsilon b(x)|u|^p + C_\varepsilon w_3(x)|u|^m)\, dS \\
&\geq \frac{C_0}{p}\|u\|_b^p - \frac{A}{q}\|u\|_b^q - \varepsilon c_1\|u\|_b^p - C_\varepsilon C_2\|u\|_b^m.
\end{aligned}$$

For $\varepsilon > 0$ and $R > 0$ small enough, we deduce that for every $u \in E$ with $\|u\|_b = R$ we have $\mathcal{E}_\lambda(u) \geq c_0 > 0$, which yields (3.25).

In what follows we verify the second geometric assumption of Theorem 1.7, namely

$$\exists v \in E \text{ with } \|v\| > R \text{ such that } \mathcal{E}_\lambda(v) < c_0. \tag{3.26}$$

Choose $\psi \in C_\delta^\infty(\Omega)$, $\psi \geq 0$, such that $\emptyset \neq \mathrm{supp}\psi \cap \Gamma \subset U$. From

$$\frac{1}{q}g(x)u^q \geq c_3u^\mu - c_4 \quad \text{on} \quad (u,s) \in U \times (0,\infty)$$

and $(H1)$ we claim that

$$\mathcal{E}_\lambda(t\psi) = \frac{t^p}{p}\left(\|\psi\|_b^p - \lambda\|\psi\|_{p,w_1}^p\right) - \frac{1}{q}\int_\Omega g(x)|t\psi|^q\, dx - \int_\Gamma H(x,t\psi)\, dS$$

$$\leq \frac{t^p}{p}\left(\|\psi\|_b^p - \lambda\|\psi\|_{p,w_1}^p\right) - c_3t^\mu\int_U \psi^\mu\, dS + c_4|U| - \frac{t^q}{q}\int_\Omega w_2\psi^q\, dx.$$

Since $q \geq \mu > p$, we obtain $\mathcal{E}_\lambda(t\psi) \to -\infty$ as $t \to \infty$. It follows that if $t > 0$ is large enough, $\mathcal{E}_\lambda(t\psi) < 0$, so $v = t\psi$ satisfies (3.26).

By Theorem 1.7 it follows that problem (P_λ) has a nontrivial weak solution.$\square$

3.3 Superlinear case

We assume throughout this section that p, q, r and α_1 are real numbers satisfying

$$1 < p < N, \quad p < q < r < p^\star = \frac{pN}{N-p}, \quad -N < \alpha_1 < q\frac{N-p}{p} - N. \tag{3.27}$$

Also, we consider $a \in L^\infty(\Omega)$ and $b \in L^\infty(\Gamma)$ such that

$$a(x) \geq a_0 > 0, \quad \text{for a.e. } \ x \in \Omega, \tag{3.28}$$

and

$$\frac{c}{(1+|x|)^{p-1}} \leq b(x) \leq \frac{C}{(1+|x|)^{p-1}}, \qquad \text{for a.e. } x \in \Gamma, \ \text{where } c, C > 0. \tag{3.29}$$

We assume that $h : \Gamma \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, which satisfies the following conditions:
(h1) $h(\cdot, 0) = 0, \ h(x,s) + h(x,-s) \geq 0$, for a.e. $x \in \Gamma$ and for any $s \in \mathbb{R}$;
(h2) $h(x,s) \leq h_0(x) + h_1(x)|s|^{m-1}, \ \ p \leq m < p\frac{N-1}{N-p}$, where $h_i : \Gamma \to \mathbb{R}, i = 1, 2$ are nonnegative, measurable functions such that

$$0 \leq h_i(x) \leq C_h w_2, \ \text{ a.e. } \ g_0 \in L^{m/(m-1)}(\Gamma; w_2^{1/(m-1)}),$$

where $-N < \alpha_2 < m \cdot \dfrac{N-p}{p} - N + 1$ and $w_2 = (1+|x|)^{\alpha_2}$.

Set $H(x,s) = \displaystyle\int_0^s h(x,t)dt$. We denote by H_h, N_H the corresponding Nemytskii operators. From Lemma 3.3 and condition (h2) it follows that the operators H_h, N_H are bounded and continuous.

In this section we consider the following double eigenvalue problem:

$$(\mathrm{P}'_{\lambda,\mu}) \begin{cases} -\mathrm{div}\,(a(x)|\nabla u|^{p-2}\nabla u) + h(x)|u|^{r-2}u = \lambda(1+|x|)^{\alpha_1}|u|^{q-2}u \text{ in } \Omega \subset \mathbb{R}^N, \\ a(x)|\nabla u|^{p-2}\nabla u \cdot \mathsf{n} + b(x) \cdot |u|^{p-2}u = \mu g(x,u) \quad \text{on } \Gamma\,, \\ u \geq 0, \ u \not\equiv 0 \quad \text{ in } \ \Omega\,. \end{cases}$$

Let $\chi : \Omega \to \mathbb{R}$ be a positive and continuous function satisfying

$$\int_\Omega \frac{w_1^{r/(r-q)}}{\chi^{q/(r-q)}}\,dx < \infty. \tag{3.30}$$

Define the Banach space

$$X = \{u \in E : \int_\Omega \chi(x)|u|^r\,dx < \infty\}$$

endowed with the norm

$$\|u\|_X^p = \|u\|_b^p + \left(\int_\Omega \chi(x)|u(x)|^r\,dx\right)^{p/r}.$$

The energy functional $\Phi : X \to \mathbb{R}$ corresponding to problem $(P_{\lambda,\mu})$ is given by

$$\begin{aligned}\Phi(u) &= \frac{1}{p}\int_\Omega a(x)|\nabla u|^p\,dx + \frac{1}{p}\int_\Gamma b(x)|u|^p\,d\Gamma - \frac{\lambda}{q}\int_\Omega w_1|u|^q\,dx + \\ &+ \frac{1}{r}\int_\Omega \chi(x)|u|^r\,dx - \mu\int_\Gamma H(x,u)\,d\Gamma.\end{aligned}$$

Theorem 3.1 implies that Φ is well defined. The solutions of problem $(P_{\lambda,\mu})$ will be found as critical points of Φ. Therefore, a function $u \in X$ is a solution of problem $(\mathrm{P}_{\lambda,\mu})$ provided that for any $v \in X$ it holds

$$\int_\Omega a|\nabla u|^{p-2}\nabla u\cdot\nabla v+\int_\Gamma b|u|^{p-2}uv = \lambda\int_\Omega w_1|u|^{p-2}uv-\int_\Omega \chi|u|^{p-2}uv+\mu\int_\Gamma hv\,.$$

We have the following result.

Theorem 3.3 *Assume hypotheses (3.27), (3.28), (3.29), (3.30), (h1) and (h2) hold. Then there exist real numbers μ_*, μ^* and $\lambda^* > 0$ such that problem $(P'_{\lambda,\mu})$ has no nontrivial solution, provided that $\mu_* < \mu < \mu^*$ and $0 < \lambda < \lambda^*$.*

Proof Suppose that u is a solution in X of $(P'_{\lambda,\mu})$. Then u satisfies

$$\begin{aligned}\int_\Omega a(x)|\nabla u|^p\,dx &+ \int_\Gamma b(x)|u|^p\,d\Gamma - \mu\int_\Gamma h(x,u)u\,d\Gamma + \\ &+ \int_\Omega \chi(x)|u|^r\,dx = \lambda\int_\Omega w_1|u|^q\,dx.\end{aligned}\tag{3.31}$$

It follows from Young's inequality that

$$\lambda\int_\Omega w_1|u|^q\,dx = \int_\Omega \frac{\lambda w_1}{\chi^{q/r}}\cdot\chi^{q/r}|u|^q\,dx \le \frac{r-q}{r}\lambda^{r/(r-q)}\int_\Omega \frac{w_1^{r/(r-q)}}{\chi^{q/(r-q)}}\,dx+\frac{q}{r}\int_\Omega \chi|u|^r\,dx.$$

This combined with (3.31) gives

$$\begin{aligned}\|u\|_b^p &- \mu\int_\Gamma h(x,u)u\,d\Gamma \le \frac{r-q}{r}\lambda^{r/(r-q)}\int_\Omega \frac{w_1^{r/(r-q)}}{\chi^{q/(r-q)}}\,dx + \\ &+ \frac{q-r}{r}\int_\Omega \chi|u|^r\,dx \le \frac{r-q}{r}\lambda^{r/(r-q)}\int_\Omega \frac{w_1^{r/(r-q)}}{\chi^{q/(r-q)}}\,dx.\end{aligned}\tag{3.32}$$

Set

$$A = \{u \in X : \int\limits_{\Gamma} h(x,u)u\,d\Gamma < 0\}, \qquad B = \{u \in X : \int\limits_{\Gamma} h(x,u)u\,d\Gamma > 0\},$$

$$\mu_* = \sup_{u\in A} \frac{\|u\|_b^p}{\int\limits_{\Gamma} h(x,u)u\,d\Gamma}, \qquad \mu^* = \inf_{u\in B} \frac{\|u\|_b^p}{\int\limits_{\Gamma} h(x,u)u\,d\Gamma}. \tag{3.33}$$

We introduce the convention that, if $A = \emptyset$, then $\mu_* = -\infty$ and if $B = \emptyset$, then $\mu^* = +\infty$.

We show that, if we take $\mu_\star < \mu < \mu^\star$, then there exists $C_0 > 0$ such that

$$C_0\|u\|_b^p \le \|u\|_b^p - \mu \int\limits_{\Gamma} h(x,u)u\,d\Gamma \text{ for all } u \in X. \tag{3.34}$$

If $\mu < \mu^*$, then there exists a constant $C_1 \in (0,1)$ such that

$$\mu \le (1-C_1)\mu^* \le (1-C_1)\frac{\|u\|_b^p}{\int\limits_{\Gamma} h(x,u)u\,d\Gamma} \text{ for all } u \in B,$$

which implies

$$\|u\|_b^p - \mu \int\limits_{\Gamma} h(x,u)u\,d\Gamma \ge C_1\|u\|_b^p \quad \text{for all } u \in B. \tag{3.35}$$

If $\mu_* < \mu$, then there exists a constant $C_2 \in (0,1)$ such that

$$(1-C_2)\frac{\|u\|_b^p}{\int\limits_{\Gamma} h(x,u)u\,d\Gamma} \le (1-C_2)\mu_* \le \mu \text{ for all } u \in A,$$

which yields

$$\|u\|_b^p - \mu \int\limits_{\Gamma} h(x,u)u\,d\Gamma \ge C_2\|u\|_b^p \quad \text{for all } u \in A. \tag{3.36}$$

From (3.35) and (3.36) we conclude that

$$\|u\|_b^p - \mu \int\limits_{\Gamma} h(x,u)u\,d\Gamma \ge \min\{C_1, C_2\}\|u\|_b^p \quad \text{for all } u \in X$$

and taking $C_0 = \min\{C_1, C_2\}$ we obtain (3.34).

By (3.31), (3.34) and Theorem 3.1 we have

$$C_0\overline{C}\left(\int_\Omega w_1|u|^q\,dx\right)^{p/q} \le C_0\|u\|_b^p \le \lambda\int_\Omega w_1|u|^q\,dx, \tag{3.37}$$

for some constant $\overline{C} > 0$. This inequality implies

$$(\overline{C}\lambda^{-1}C_0)^{q/(q-p)} \le \int_\Omega w_1|u|^q\,dx,$$

which combined with (3.37) yields

$$C_0\overline{C}(\overline{C}\lambda^{-1}C_0)^{p/(q-p)} \le C_0\|u\|_b^p.$$

Combining this with (3.32) and (3.34) we obtain

$$C_0\overline{C}(\overline{C}\lambda^{-1}C_0)^{p/(q-p)} \le \frac{r-q}{r}\lambda^{r/(r-q)}\int_\Omega \frac{w_1^{r/(r-q)}}{\chi^{q/(r-q)}}\,dx.$$

By taking

$$\lambda^* = \left((C_0\overline{C})^{q/(q-p)}\frac{r}{r-q}\left(\int_\Omega \frac{w_1^{r/(r-q)}}{\chi^{q/(r-q)}}\,dx\right)^{-1}\right)^{\frac{(r-q)(q-p)}{q(r-p)}},$$

the result follows. □

Set

$$U = \{u \in X : \int_\Gamma H(x,u)\,d\Gamma < 0\}, \qquad V = \{u \in X : \int_\Gamma H(x,u)\,d\Gamma > 0\},$$

$$\mu_- = \sup_{u\in U}\frac{\|u\|_b^p}{p\int_\Gamma H(x,u)\,d\Gamma}, \qquad \mu^+ = \inf_{u\in V}\frac{\|u\|_b^p}{p\int_\Gamma H(x,u)\,d\Gamma}. \tag{3.38}$$

If $U = \emptyset$ (resp. $V = \emptyset$), then we set $\mu_- = -\infty$ (resp. $\mu^+ = +\infty$). Proceeding in the same manner as we did for proving (3.34), we can show that if we take $\mu_- < \mu < \mu^+$, then there exists $c > 0$ such that

$$\frac{1}{p}\|u\|_b^p - \mu\int_\Gamma H(x,u)\,d\Gamma \ge c\|u\|_b^p \qquad \text{for all } u \in X. \tag{3.39}$$

In what follows we employ the following elementary inequality

$$k|u|^\beta - \chi|u|^\gamma \le C_{\beta,\gamma} k \left(\frac{k}{\chi}\right)^{\frac{\beta}{\gamma-\beta}}, \quad \forall u \in \mathbb{R}, \forall \chi, k \in (0,\infty),\ \forall 0<\beta<\gamma\,. \tag{3.40}$$

Proposition 3.1 *If $\mu_- < \mu < \mu^+$, then the functional Φ is coercive.*

Proof By virtue of (3.40) we have

$$\begin{aligned}\int_\Omega \left(\frac{\lambda}{q}|u|^q w_1 - \frac{\chi}{2r}|u|^r\right) dx &\le C_{r,q}\int_\Omega \lambda w_1 \left(\frac{\lambda w_1}{\chi}\right)^{q/(r-q)} dx \\ &= C_{r,q}\lambda^{r/(r-q)}\int_\Omega \frac{w_1^{r/(r-q)}}{h^{q/(r-q)}}\, dx.\end{aligned}$$

Using (3.39) it follows that

$$\begin{aligned}\Phi(u) &= \frac{1}{p}\|u\|_b^p - \mu\int_\Gamma H(x,u)\, d\Gamma - \int_\Omega \left(\frac{\lambda}{q}|u|^q w_1 - \frac{\chi}{2r}|u|^r\right) dx \\ &+ \frac{1}{2r}\int_\Omega \chi|u|^r\, dx \ge c\|u\|_b^p + \frac{1}{2r}\int_\Omega \chi|u|^r\, dx - C_1\end{aligned}$$

and the coercivity of Φ follows. $\square$

Proposition 3.2 *Suppose $\mu_- < \mu < \mu^+$ and let $\{u_n\}$ be a sequence in X such that $\Phi(u_n)$ is bounded. Then there exists a subsequence of $\{u_n\}$, relabeled again by $\{u_n\}$, such that $u_n \rightharpoonup u_0$ in X and*

$$\Phi(u_0) \le \liminf_{n\to\infty} \Phi(u_n).$$

Proof Since Φ is coercive in X we see that the boundedness of $\{\Phi(u_n)\}$ implies that $\|u_n\|_b$ and $\int_\Omega \chi|u_n|^r\, dx$ are bounded. From Theorem 3.1 we know that the embedding $E \hookrightarrow L^q(\Omega; w_1)$ is compact and using the fact that $\{u_n\}$ is bounded in E we may assume that $u_n \rightharpoonup u_0$ in E and $u_n \to u_0$ in $L^q(\Omega; w_1)$.

Set $F(x,u) = \frac{\lambda}{q}|u|^q w_1 - \frac{1}{r}h|u|^r$ and $f(x,u) = F_u(x,u)$. A simple computation yields

$$f_u(x,u) = (q-1)\lambda|u|^{q-2}w_1-(r-1)\chi|u|^{r-2} \le C_{r,q}\lambda w_1 \left(\frac{\lambda w_1}{\chi}\right)^{(q-2)/(r-q)}, \tag{3.41}$$

where the last inequality follows from (3.40). We obtain

$$\begin{aligned}
\Phi(u_0) \quad &- \quad \Phi(u_n) = \frac{1}{p}\int_\Omega a(x)|\nabla u_0|^p\,dx + \frac{1}{p}\int_\Gamma b(x)|u_0|^p\,d\Gamma \\
&- \quad \frac{1}{p}\int_\Omega a(x)|\nabla u_n|^p\,dx - \frac{1}{p}\int_\Gamma b(x)|u_n|^p\,d\Gamma - \mu\int_\Gamma G(x,u_0)\,d\Gamma \\
&+ \quad \mu\int_\Gamma H(x,u_n)\,d\Gamma + \int_\Omega (F(x,u_n)-F(x,u_0))\,dx = \|u_0\|_b^p - \|u_n\|_b^p \\
&+ \quad \mu\left(\int_\Gamma H(x,u_n)\,d\Gamma - \int_\Gamma H(x,u_0)\,d\Gamma\right) \\
&+ \quad \int_\Omega \left(\int_0^1\int_0^s f_u(x,u_0+t(u_n-u_0))\,dt\,ds\right)(u_n-u_0)^2\,dx \\
&\le \quad \|u_0\|_b^p - \|u_n\|_b^p + \mu\left(\int_\Gamma H(x,u_n)\,d\Gamma - \int_\Gamma H(x,u_0)\,d\Gamma\right) \\
&+ \quad C_2\int_\Omega (u_n-u_0)^2\frac{w_1^{(r-2)/(r-q)}}{\chi^{(q-2)/(r-q)}}\,dx,
\end{aligned}$$

where $C_2 = \frac{1}{2}C_{r,q}\lambda^{(r-2)/(r-q)}$.

We show that the last integral tends to 0 as $n\to\infty$. Indeed, applying Hölder's inequality we obtain

$$\int_\Omega (u_n-u_0)^2\frac{w_1^{(r-2)/(r-q)}}{\chi^{(q-2)/(r-q)}}\,dx \le \left(\int_\Omega \frac{w_1^{r/(r-q)}}{\chi^{q/(r-q)}}\,dx\right)^{(q-2)/q}\cdot\left(\int_\Omega w_1|u_n-u_0|^q\,dx\right)^{2/q}.$$

Since $u_n\to u_0$ in $L^q(\Omega;\, w_1)$ we have

$$\lim_{n\to\infty}\int_\Omega (u_n-u_0)^2\frac{w_1^{(r-2)/(r-q)}}{\chi^{(q-2)/(r-q)}}\,dx = 0. \tag{3.42}$$

The compactness of the trace operator $E\to L^m(\Gamma;\, w_2)$ and the continuity of the Nemytskii operator $N_H : L^m(\Gamma;\, w_2) \to L^1(\Gamma)$ imply

$N_H(u_n) \to N_H(u_0)$ in $L^1(\Gamma)$ i.e. $\int\limits_\Gamma |N_H(u_n) - N_H(u_0)|\, d\Gamma \to 0$ as $n \to \infty$. It follows that

$$\lim_{n\to\infty} \int\limits_\Gamma H(x, u_n)\, d\Gamma = \int\limits_\Gamma H(x, u_0)\, d\Gamma. \tag{3.43}$$

Since the norm in E is lower semicontinuous with respect to the weak topology our conclusion follows from (3.42) and (3.43). □

Proposition 3.3 *If $\mu_* < \mu < \mu^*$ and u is a solution of problem $(P'_{\lambda,\mu})$, then*

$$C_0\|u\|_b^p + \frac{r-q}{r} \int\limits_\Omega \chi|u|^r\, dx \le \frac{r-q}{r} \lambda^{r/(r-q)} \int\limits_\Omega \frac{w_1^{r/(r-q)}}{\chi^{q/(r-q)}}\, dx$$

and

$$\|u\|_b \ge K\lambda^{-1/(q-p)},$$

where $K > 0$ is a constant independent of u.

Proof If u is a solution of $(P'_{\lambda,\mu})$, then

$$\begin{aligned} \|u\|_b^p - \mu \int\limits_\Gamma h(x,u)u\, d\Gamma \quad &+ \quad \int\limits_\Omega \chi|u|^r\, dx = \lambda \int\limits_\Omega w_1|u|^q\, dx \le \\ &\le \quad \frac{r-q}{r}\lambda^{r/(r-q)} \int\limits_\Omega \frac{w_1^{r/(r-q)}}{\chi^{q/(r-q)}}\, dx + \frac{q}{r} \int\limits_\Omega \chi|u|^r\, dx. \end{aligned}$$

Using (3.34) we obtain the first part of the assertion.

From Theorem 3.1 we have that there exists $C_q > 0$ such that

$$\|u\|^q_{L^q(\Omega;\, w_1)} \le C_q\|u\|_b^q, \text{ for all } u \in E.$$

The above inequality and (3.34) imply

$$\|u\|_b \ge C_0^{1/(q-p)} C_q^{-1/(q-p)} \lambda^{-1/(q-p)}.$$

By taking $K = C_0^{1/(q-p)} C_q^{-1/(q-p)}$ the second assertion follows. □

Theorem 3.4 *Assume that hypotheses (3.27), (3.28), (3.29), (3.30), (h1) and (h2) hold. Set $\underline{\mu} = \max\{\mu_*, \mu_-\}$, $\overline{\mu} = \min\{\mu^*, \mu^+\}$ and $J = (\underline{\mu}, \overline{\mu})$. Then there exists $\lambda_0 > 0$ such that the following statements hold:*

(i) problem $(P'_{\lambda,\mu})$ admits a nontrivial solution, for any $\lambda \ge \lambda_0$ and every $\mu \in J$;

(ii) problem $(P'_{\lambda,\mu})$ *does not have any nontrivial solution, provided that* $0<\lambda<\lambda_0$ *and* $\mu\in J$.

Proof According to Propositions 3.1 and 3.2, Φ is coercive and lower semicontinuous. Therefore, there exists $\tilde u\in X$ such that $\Phi(\tilde u)=\inf_X \Phi(u)$. To ensure that $\tilde u\not\equiv 0$, we shall prove that $\inf_X \Phi<0$. Set

$$\tilde\lambda := \inf\Big\{\frac{q}{p}\|u\|_b^p - q\mu\int_\Gamma H(x,u)\,d\Gamma + \frac{q}{r}\int_\Omega \chi|u|^r\,dx : u\in X, \int_\Omega w_1|u|^q\,dx=1\Big\}.$$

First we check that $\tilde\lambda>0$. For this aim we consider the constrained minimization problem

$$M := \inf\Big\{\int_\Omega a(x)|\nabla u|^p\,dx + \int_\Gamma b(x)|u|^p\,d\Gamma : u\in E, \int_\Omega w_1|u|^q\,dx=1\Big\}.$$

Clearly, $M>0$. Since X is embedded in E, we have

$$\int_\Omega a(x)|\nabla u|^p\,dx + \int_\Gamma b(x)|u|^p\,d\Gamma \ge M$$

for all $u\in X$ with $\int_\Omega w_1|u|^q\,dx=1$. Now, applying the Hölder inequality we find

$$1=\int_\Omega w_1|u|^q\,dx = \int_\Omega \frac{w_1}{\chi^{q/r}}h^{q/r}|u|^q\,dx \le \left(\int_\Omega \frac{w_1^{r/(r-q)}}{\chi^{q/(r-q)}}\,dx\right)^{(r-q)/r}\cdot\left(\int_\Omega \chi|u|^r\,dx\right)^{q/r}. \tag{3.44}$$

Relation (3.39) implies

$$\frac{q}{p}\|u\|_b^p - q\,\mu\int_\Gamma H(x,u)\,d\Gamma \ge q\,c\|u\|_b^p.$$

By virtue of (3.44) we have

$$\frac{q}{p}\|u\|_b^p - q\,\mu\int_\Gamma H(x,u)\,d\Gamma + \frac{q}{r}\int_\Omega \chi|u|^r\,dx \ge qc\|u\|_b^p + \frac{q}{r}\int_\Omega \chi|u|^r\,dx \ge$$

$$\ge qcM + \frac{q}{r}\left(\int_\Omega \frac{w_1^{r/(r-q)}}{\chi^{q/(r-q)}}\,dx\right)^{-(r-q)/q}$$

for all $u \in X$ with $\int_\Omega w_1|u|^q\,dx = 1$. Then,

$$\tilde{\lambda} \geq qcM + \frac{q}{r}\left(\int_\Omega \frac{w_1^{r/(r-q)}}{\chi^{q/(r-q)}}\,dx\right)^{-(r-q)/q}$$

and our claim follows.

Let $\lambda > \tilde{\lambda}$. Then, there exists a function $u \in X$ with $\int_\Omega w_1|u|^q\,dx = 1$ such that

$$\lambda > \frac{q}{p}\|u\|_b^p - q\mu\int_\Gamma H(x,u)\,d\Gamma + \frac{q}{r}\int_\Omega \chi|u|^r\,dx.$$

This can be rewritten as

$$\Phi(u) = \frac{1}{p}\|u\|_b^p - \mu\int_\Gamma H(x,u)\,d\Gamma + \frac{1}{r}\int_\Omega \chi|u|^r\,dx - \frac{\lambda}{q}\int_\Omega w_1|u|^q\,dx < 0$$

and consequently $\inf_{u\in X}\Phi(u) < 0$. By Propositions 3.1 and 3.2 it follows that problem $(P'_{\lambda,\mu})$ has a solution.

We set

$$\lambda_0 = \inf\{\lambda > 0 : (P'_{\lambda,\mu}) \text{ admits a solution}\}.$$

Suppose $\lambda_0 = 0$. Then by taking $\lambda_1 \in (0,\lambda^*)$ (where λ^* is given by Theorem 3.3), we have that there exists $\bar{\lambda}$ such that problem $(P'_{\bar{\lambda},\mu})$ admits a solution. But this is a contradiction, according to Theorem 3.3. Consequently, $\lambda_0 > 0$.

We now show that for each $\lambda > \lambda_0$ problem $(P'_{\lambda,\mu})$ admits a solution. Indeed, for every $\lambda > \lambda_0$ there exists $\rho \in (\lambda_0,\lambda)$ such that problem $(P'_{\rho,\mu})$ has a solution u_ρ which is a subsolution of problem $(P'_{\lambda,\mu})$. We consider the variational problem

$$\inf\{\Phi(u) : u \in X \text{ and } u \geq u_\rho\}.$$

By Propositions 3.1 and 3.2 this problem admits a solution $\bar{u}$. This minimizer $\bar{u}$ is a solution of problem $(P'_{\lambda,\mu})$. Since the hypothesis $h(x,s) + h(x,-s) \geq 0$ for a.e. $x \in \Gamma$ and for all $s \in \mathbb{R}$ implies that $H(x,|\bar{u}|) \geq H(x,\bar{u})$ (that is, $\Phi(|\bar{u}|) \leq \Phi(\bar{u})$), we may assume that $\bar{u} \geq 0$ on Ω. It remains to show that problem $(P'_{\lambda_0,\mu})$ has also a solution. Let $\lambda_n \to \lambda_0$ and $\lambda_n > \lambda_0$ for each n. Problem $(P'_{\lambda_n,\mu})$ has a solution u_n for each n. By Proposition 3.3 the sequence $\{u_n\}$ is bounded in X. Therefore we may assume that $u_n \rightharpoonup u_0$ in X and $u_n \to u_0$ in $L^q(\Omega; w_1)$. We have that u_0 is a solution of $(P'_{\lambda_0,\mu})$. Since u_n and u_0 are solutions of $(P'_{\lambda_n,\mu})$ and $(P'_{\lambda_0,\mu})$, respectively, we have

$$\begin{aligned}
\int_\Omega & a(x)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_0|^{p-2}\nabla u_0)(\nabla u_n - \nabla u_0)\,dx \\
&+ \int_\Gamma b(x)(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0)(u_n - u_0)\,d\Gamma \\
&+ \int_\Omega \chi(|u_n|^{r-2}u_n - |u_0|^{r-2}u_0)(u_n - u_0)\,dx \\
&= \lambda_n \int_\Omega w_1(|u_n|^{q-2}u_n - |u_0|^{q-2}u_0)(u_n - u_0)\,dx \\
&+ (\lambda_n - \lambda_0)\int_\Omega w_1|u_0|^{q-2}u_0(u_n - u_0)\,dx \\
&+ \mu\int_\Gamma (h(x,u_n) - h(x,u_0))(u_n - u_0)\,d\Gamma = J_{1,n} + J_{2,n} + J_{3,n},
\end{aligned}$$

where

$$\begin{aligned}
J_{1,n} &= \lambda_n \int_\Omega w_1(|u_n|^{q-2}u_n - |u_0|^{q-2}u_0)(u_n - u_0)\,dx, \\
J_{2,n} &= (\lambda_n - \lambda_0)\int_\Omega w_1|u_0|^{q-2}u_0(u_n - u_0)\,dx, \\
J_{3,n} &= \mu\int_\Gamma (h(x,u_n) - h(x,u_0))(u_n - u_0)\,d\Gamma.
\end{aligned}$$

We have

$$|J_{1,n}| \le \sup_{n\ge 1}\lambda_n \left(\int_\Omega w_1|u_n|^{q-1}|u_n - u_0|\,dx + \int_\Omega w_1|u_0|^{q-1}|u_n - u_0|\,dx\right)$$

and it follows from Hölder's inequality that

$$|J_{1,n}| \le \sup_{n\ge 1}\lambda_n \Big[\left(\int_\Omega w_1|u_n|^q\,dx\right)^{(q-1)/q} \cdot \left(\int_\Omega w_1|u_n - u_0|^q\,dx\right)^{1/q} +$$

$$+\left(\int\limits_{\Omega} w_1|u_0|^q\,dx\right)^{(q-1)/q}\cdot\left(\int\limits_{\Omega} w_1|u_n-u_0|^q\,dx\right)^{1/q}\Bigg].$$

We easily observe that $J_{1,n}\to 0$ as $n\to\infty$.

From the estimate

$$|J_{2,n}|\le|\lambda_n-\lambda_0|\left(\int\limits_{\Omega} w_1|u_0|^q\,dx\right)^{(q-1)/q}\cdot\left(\int\limits_{\Omega} w_1|u_n-u_0|^q\,dx\right)^{1/q}$$

we obtain that $J_{2,n}\to 0$ as $n\to\infty$.

Using the compactness of the trace operator $E\to L^m(\Gamma;\,w_2)$, the continuity of Nemytskii operator $N_h: L^m(\Gamma;\,w_2)\to L^{m/(m-1)}(\Gamma;\,w_2^{1/(1-m)})$ and the estimate

$$\int\limits_{\Gamma}|h(x,u_n)-h(x,u_0)|\cdot|u_n-u_0|\,d\Gamma\le$$

$$\le\left(\int\limits_{\Gamma}|h(x,u_n)-h(x,u_0)|^{m/(m-1)}w_2^{1/(1-m)}\,d\Gamma\right)^{(m-1)/m}\cdot\left(\int\limits_{\Gamma} w_2|u_n-u_0|^m\,d\Gamma\right)^{1/m}$$

we see that $J_{3,n}\to 0$ as $n\to\infty$.
So, we have proved that

$$\lim_{n\to\infty}\Big(\int\limits_{\Omega} a(x)(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_0|^{p-2}\nabla u_0)(\nabla u_n-\nabla u_0)\,dx+$$
$$\int\limits_{\Gamma} b(x)(|u_n|^{p-2}u_n-|u_0|^{p-2}u_0)(u_n-u_0)\,d\Gamma\Big)=0.$$

Applying the inequality

$$|\xi-\zeta|^p\le C(|\xi|^{p-2}\xi-|\zeta|^{p-2}\zeta)(\xi-\zeta),\qquad\forall\xi,\zeta\in\mathbb{R}^N$$

we find

$$\|u_n-u_0\|_b^p=\int\limits_{\Omega} a(x)|\nabla u_n-\nabla u_0|^p\,dx+\int\limits_{\Gamma} b(x)|u_n-u_0|^p\,dx\le$$
$$\le C\Big(\int\limits_{\Omega} a(x)(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_0|^{p-2}\nabla u_0)(\nabla u_n-\nabla u_0)\,dx+$$
$$+\int\limits_{\Gamma} b(x)(|u_n|^{p-2}u_n-|u_0|^{p-2}u_0)(u_n-u_0)\,d\Gamma\Big)\to 0\ \text{ as } n\to\infty,$$

which shows that $\|u_n\|_b \to \|u_0\|_b$. By Proposition 3.3 we have $u_0 \not\equiv 0$. This concludes our proof. □

3.4 Sublinear case

In this section we suppose that the conditions (3.27), (3.28) and (3.29) are fulfilled. For $\lambda, \mu > 0$, we consider the following double eigenvalue problem:

$$(P''_{\lambda,\mu}) \quad \begin{cases} -\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u) = \lambda f(x, u(x)) \text{ in } \Omega, \\ a(x)|\nabla u|^{p-2}\nabla u \cdot \mathsf{n} + b(x)|u|^{p-2}u = \mu g(x, u(x)) \text{ on } \Gamma, \\ u \not\equiv 0 \text{ in } \Omega. \end{cases}$$

We consider the following assumptions:

(F1) Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $f(\cdot, 0) = 0$ and

$$|f(x, s)| \le f_0(x) + f_1(x)|s|^{r-1},$$

where $p < r < \dfrac{pN}{N-p}$, and f_0, f_1 are measurable functions which satisfy

$$0 < f_0(x) \le C_f w_1(x), \text{ and } 0 \le f_1(x) \le C_f w_1(x) \quad \text{a.e. } x \in \Omega,$$

$$f_0 \in L^{\frac{r}{r-1}}(\Omega; w_1^{\frac{1}{1-r}});$$

(F2)

$$\lim_{s\to 0} \frac{f(x, s)}{f_0(x)|s|^{p-1}} = 0, \quad \text{uniformly} \quad \text{in } x \in \Omega;$$

(F3) $\displaystyle\limsup_{s\to+\infty} \frac{1}{f_0(x)|s^p|} F(x, s) \le 0$ uniformly for all $x \in \Omega$, $\displaystyle\max_{|s|\le M} F(\cdot, s) \in L^1(\Omega)$ for all $M > 0$, where F denotes the primitive function of f with respect to the second variable, i.e. $F(x, u) = \displaystyle\int_0^u f(x, s)ds$;

(F4) there exist $x_0 \in \Omega$, $R_0 > 0$ and $s_0 \in \mathbb{R}$ such that $\displaystyle\min_{|x-x_0|<R_0} F(x, s_0) > 0$.

(G1) Let $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ be a a Carathéodory function such that $g(\cdot, 0) = 0$ and

$$|g(x,s)| \leq g_0(x) + g_1(x)|s|^{m-1},$$

where $p \leq m < p \cdot \dfrac{N-1}{N-p}$, and g_0, g_1 are measurable functions satisfying

$$0 < g_0(x) \leq C_g w_2(x) \text{ and } 0 \leq g_1(x) \leq C_g w_2(x), \quad \text{a.e. } x \in \Gamma,$$

$$g_0 \in L^{\frac{q}{q-1}}(\Gamma; w_2^{\frac{1}{1-q}});$$

(G2)

$$\lim_{s \to 0} \frac{g(x,s)}{g_0(x)|s|^{p-1}} = 0, \quad \text{uniformly} \quad \text{in } x \in \Gamma;$$

(G3) $\limsup_{s \to +\infty} \dfrac{1}{g_0(x)|s^p|} G(x,s) < +\infty$ uniformly for all $x \in \Gamma$, $\max_{|s| \leq M} G(\cdot, s) \in L^1(\Gamma)$ for all $M > 0$, where G is the primitive function of g with respect to the second variable, i.e. $G(x,u) = \displaystyle\int_0^u g(x,s)ds$.

The functions w_1 and w_2 are defined as in the previous section, that is,

$$w_i = (1 + |x|)^{\alpha_i}, \ i = 1, 2.$$

The first multiplicity result for problem $(P''_{\lambda,\mu})$ is the following:

Theorem 3.5 *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function satisfying the conditions* (F1)-(F4)*. Then, there exists a non-degenerate compact interval $[a,b] \subset]0, +\infty[$ with the following properties:*

1° *there exists a number $\sigma_0 > 0$ such that for every $\lambda \in [a,b]$ and for every function $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfying the conditions* (G1)-(G2)*, there exists $\mu_0 > 0$ such that for each $\mu \in]0, \mu_0[$, the functional $\mathcal{E}_{\lambda,\mu}$ has at least two critical points with norms less than σ_0;*

2° *there exists a number $\sigma_1 > 0$ such that for every $\lambda \in [a,b]$ and for every function $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfying the conditions* (G1)-(G3)*, there exists $\mu_1 > 0$ such that for each $\mu \in]0, \mu_1[$, the functional $\mathcal{E}_{\lambda,\mu}$ has at least three critical points with norms less than σ_1.*

To prove Theorem 3.5 we use Theorem 1.15. Before we prove this result, we need some auxiliary results.

Lemma 3.5 *We assume that the conditions* (F1) *and* (F2) *are satisfied. Then, the functional* J_F *is sequentially weakly continuous.*

Proof First we observe that, from assumption (F1) and (F2), for every $\varepsilon > 0$, there exists C_ε such that

$$|F(x,u)| \leq \varepsilon f_0(x)|u(x)|^p + C_\varepsilon(f_0(x)+f_1(x))|u(x)|^r. \tag{3.45}$$

Now we prove our assertion, arguing by contradiction: Let $\{u_n\}$ be a sequence in E weakly convergent to $u \in E$, and let $d > 0$ be such that

$$|J_F(u_n) - J_F(u)| \geq d \text{ for all } n \in \mathbb{N}.$$

Without loss of generality, we can assume that there is a positive constant M such that

$$\|u\|_b \leq M, \|u_n\|_b \leq M, \|u_n - u\|_b \leq M, \quad \text{for all} \quad n \in \mathbb{N}.$$

Because the embedding $E \hookrightarrow L^r(\Omega; w_1)$ is compact, follows $\|u_n - u\|_{r,w_1} \to 0$. Using (3.45), Theorem 3.1 and the Hölder inequality we have

$$\begin{aligned}
&|J_F(u_n) - J_F(u)| \leq \int_\Omega |F(x,u_n(x)) - F(x,u(x))| dx \\
\leq\ & \varepsilon\hat{c}\int_\Omega f_0(x)|u_n(x)-u(x)|(|u_n(x)|^{p-1}+|u(x)|^{p-1})dx \\
&+\hat{c}C_\varepsilon\int_\Omega (f_0(x)+f_1(x))|u_n(x)-u(x)|(|u_n(x)|^{r-1}+|u(x)|^{r-1})dx \\
\leq\ & \varepsilon\hat{c}C_f\int_\Omega w_1(x)|u_n(x)-u(x)|(|u_n(x)|^{p-1}+|u(x)|^{p-1})dx \\
&+2\hat{c}C_\varepsilon C_f\int_\Omega w_1(x)|u_n(x)-u(x)|(|u_n(x)|^{r-1}+|u(x)|^{r-1})dx \\
\leq\ & \varepsilon\hat{c}C_f\|u_n-u\|_{p,w_1}(\|u_n\|_{p,w_1}^{\frac{p}{p'}}+\|u\|_{p,w_1}^{\frac{p}{p'}}) \\
&+2\hat{c}C_\varepsilon C_f\|u_n-u\|_{r,w_1}(\|u_n\|_{r,w_1}^{\frac{r}{r'}}+\|u\|_{r,w_1}^{\frac{r}{r'}}),
\end{aligned}$$

where $\hat{c} > 0$ is a constant, $\dfrac{1}{p}+\dfrac{1}{p'} = 1$ and $\frac{1}{r}+\dfrac{1}{r'} = 1$. By the embedding results from Theorem 3.1 it follows that

$$d \leq |J_F(u_n) - J_F(u)| \leq 2\varepsilon\hat{c}C_fC_{p,w_1}^p M^p + 4\hat{c}C_\varepsilon C_f C_{r,w_1}^{\frac{r}{r'}} M^{\frac{r}{r'}}\|u_n-u\|_{r,w_1}.$$

Therefore, if $\varepsilon > 0$ is sufficiently small and $n \in \mathbb{N}$ is large enough, we have

$$d \leq |J_F(u_n) - J_F(u)| < d,$$

which is a contradiction. □

Remark 3.1 *Note that a similar result holds for the functional J_G, i.e. if the conditions (G1), (G2) are fulfilled, then the functional J_G is sequentially weakly continuous.*

Lemma 3.6 *There exists $u_0 \in E$ such that $J_F(u_0) > 0$.*

Proof Let $R_0 > 0$ and $s_0 \in \mathbb{R}$ from (F4) and fix $\varepsilon \in (0, \frac{R_0}{2})$. We consider the function $u_\varepsilon \in \mathcal{C}_0^\infty(\Omega)$ such that

$$u_\varepsilon(x) = \begin{cases} 0, & \text{if } |x - x_0| \geq R_0 \\ s_0, & \text{if } |x - x_0| \leq R_0 - \varepsilon \end{cases}$$

and $\|u_\varepsilon\|_\infty \leq |s_0|$. Then, we have $u_\varepsilon \in E$.

We denote by $Vol(B_r(x_0))$ the volume of the ball

$$B_r(x_0) = \{x \in \mathbb{R}^N : |x - x_0| \leq r\}.$$

We have

$$\begin{aligned} J_F(u_\varepsilon) &= \int_\Omega F(x, u_\varepsilon)dx = \int_{B_{R_0-\varepsilon}(x_0)\cap\Omega} F(x, u_\varepsilon(x))dx + \\ &+ \int_{(\mathbb{R}^N\setminus B_{R_0}(x_0))\cap\Omega} F(x, u_\varepsilon(x))dx + \int_{(B_{R_0}(x_0)\setminus B_{R_0-\varepsilon}(x_0))\cap\Omega} F(x, u_\varepsilon(x))dx \geq \\ &\geq A_0 Vol(B_{\frac{R_0}{2}}(x_0) \cap \Omega) - B_0[Vol(B_{R_0}(x_0) \cap \Omega) - Vol(B_{R_0-\varepsilon}(x_0) \cap \Omega)], \end{aligned}$$

where $A_0 = \min_{|x-x_0|\leq R_0} F(x, s_0) > 0$ and $B_0 = \max_{\frac{R_0}{2}\leq|x-x_0|\leq R_0} \max_{|s|\leq|s_0|} F(x, s)$.
If ε goes to 0, then $[Vol(B_{R_0}(x_0)\cap\Omega) - Vol(B_{R_0-\varepsilon}(x_0)\cap\Omega)] \to 0$ as well, so we can choose an $\varepsilon = \varepsilon_0$, such that $J_F(u_{\varepsilon_0}) > 0$. We take $u_0 = u_{\varepsilon_0}$ and the proof is complete. □

Lemma 3.7 *Suppose that the conditions* (F1) *and* (F3) *are satisfied. Then, for every $\lambda \geq 0$ the functional*

$$u \to \frac{\|u\|_b^p}{p} - \lambda J_F(u)$$

is coercive on E.

Proof If $\lambda = 0$, then statement is trivial. Let $\lambda > 0$ and $a \in \left]0, \frac{1}{\lambda p C_{p,w_1}^p}\right[$.

There exists a positive function $k_a \in L^1(\Omega; w_1)$ such that

$$F(x,s) \le a f_0(x)|s|^p + k_a(x) w_1(x), \quad \text{for all} \quad (x,s) \in \Omega \times \mathbb{R}.$$

Then, it follows that

$$\frac{\|u\|_b^p}{p} - \lambda J_F(u) \ge \frac{\|u\|_b^p}{p} - \lambda a \int_\Omega w_1(x)|u(x)|^p dx - \lambda \int_\Omega k_a(x) w_1(x) dx \ge$$

$$\ge \|u\|_b^p \left(\frac{1}{p} - \lambda a C_{p,w_1}^p \right) - \lambda \|k_a\|_{1,w_1},$$

which converges to ∞ as $\|u\|_b \to \infty$. $\square$

Proof of Theorem 3.5: We define the function $h :]0, +\infty[\to \mathbb{R}$ by

$$h(t) = \sup \left\{ J_F(u) \ : \ \frac{\|u\|_b^p}{p} \le t \right\}, \quad \text{for all} \quad t > 0.$$

By using (3.45) it follows that

$$0 \le h(t) \le \varepsilon p C_f C_{p,w_1}^p t + 2 p^{\frac{r}{p}} C_\varepsilon C_f C_{r,w_1}^r t^{\frac{r}{p}} \quad \text{for all } t > 0;$$

since $p < r$, this implies

$$\lim_{t \to 0^+} \frac{h(t)}{t} = 0.$$

By (F4) it is clear that $u_0 \ne 0$ (since $J_F(0) = 0$). Therefore, due to the convergence relation above and Lemma 3.6 it is possible to choose a real number t_0 such that $0 < t_0 < \dfrac{\|u_0\|_b^p}{p}$ and

$$\frac{h(t_0)}{t_0} < \left[\frac{\|u_0\|_b^p}{p} \right]^{-1} \cdot J_F(u_0).$$

We choose $\rho_0 > 0$ such that

$$h(t_0) < \rho_0 < \left[\frac{\|u_0\|_b^p}{p} \right]^{-1} \cdot J_F(u_0) t_0. \tag{3.46}$$

In particular, we get $\rho_0 < J_F(u_0)$.

Now, we are going to apply Theorem 1.15 to the space E, the interval $\Lambda =]0, +\infty[$ and the function $\Psi : E \times \Lambda \to \mathbb{R}$ defined by

$$\Psi(u, \lambda) = \frac{\|u\|_b^p}{p} + \lambda (\rho_0 - J_F(u)) \text{ for all } (u, \lambda) \in E \times \Lambda$$

and $\Phi : E \to \mathbb{R}$ by

$$\Phi(u) = -J_G(u) \text{ for all } u \in E.$$

Let us fix $\lambda \in]0, +\infty[$: from Lemma 3.7 it follows that the functional $\Psi(\cdot, \lambda)$ is coercive; moreover, $\Psi(\cdot, \lambda)$ is the sum of $u \mapsto \dfrac{\|u\|_b^p}{p}$, which is sequentially weakly l.s.c., and of $u \mapsto \lambda(\rho_0 - J_F(u))$, which is sequentially weakly continuous (see Lemma 3.5).

We prove now that Ψ complies with the minimax inequality (1.20) from Theorem 1.15. The function

$$\lambda \mapsto \inf_{u \in E} \Psi(u, \lambda)$$

is upper semicontinuous on Λ. Since

$$\inf_{u \in E} \Psi(u, \lambda) \leq \Psi(u_0, \lambda) = \frac{\|u_0\|_b^p}{p} + \lambda(\rho_0 - J_F(u_0))$$

and $\rho_0 < J_F(u_0)$, it follows that

$$\lim_{\lambda \to +\infty} \inf_{u \in E} \Psi(u, \lambda) = -\infty.$$

Thus we can find $\overline{\lambda} \in \Lambda$ such that

$$\beta_1 = \sup_{\lambda \in \Lambda} \inf_{u \in E} \Psi(u, \lambda) = \inf_{u \in E} \Psi(u, \overline{\lambda}).$$

In order to prove that $\beta_1 < t_0$, we distinguish two cases:
I. If $0 \leq \overline{\lambda} < \dfrac{t_0}{\rho_0}$, we have

$$\beta_1 \leq \Psi(0, \overline{\lambda}) = \overline{\lambda}\rho_0 < t_0.$$

II. If $\overline{\lambda} \geq \dfrac{t_0}{\rho_0}$, then we use $\rho_0 < J_F(u_0)$ and the inequality (3.46) to get

$$\beta_1 \leq \Psi(u_0, \overline{\lambda}) \leq \frac{\|u_0\|_b^p}{p} + \frac{t_0}{\rho_0}(\rho_0 - J_F(u_0)) < t_0.$$

Let us focus next on the right hand side of the inequality (1.20) of Theorem 1.15. Clearly,

$$\beta_2 = \inf_{u \in E} \sup_{\lambda \in \Lambda} \Psi(u, \lambda) = \inf \left\{ \frac{\|u\|_b^p}{p} : J_F(u) \geq \rho_0 \right\}.$$

On the other hand, by applying again (3.46) we easily get

$$t_0 \leq \inf \left\{ \frac{\|u\|_b^p}{p} \ : \ J_F(u) \geq \rho_0 \right\}.$$

We conclude that

$$\beta_1 < t_0 \leq \beta_2,$$

i.e., condition (1.20) from Theorem 1.15 holds .

Thus, we can apply Theorem 1.15. Fix $\delta > \beta_1$, and for every $\lambda \in \Lambda$ denote

$$S_\lambda = \{u \in E \ : \ \Psi(u, \lambda) < \delta\}\,.$$

There exists a non-empty open set $\Lambda_0 \subset]0, +\infty[$ with the following property: for every $\lambda \in \Lambda_0$ and every sequentially weakly l.s.c. $\Phi : E \to \mathbb{R}$, there exists $\mu_0 > 0$, such that for each $\mu \in]0, \mu_0[$, the functional

$$u \to \Psi(u, \lambda) + \mu\Phi(u)$$

has at least two local minima lying in the set S_λ. Let $[a, b] \subset \Lambda_0$ be a non–degenerate compact interval.

We prove now the two assertions of our theorem:
1° Let $\lambda \in [a, b]$ be a real number, and let $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfying the conditions (G1)-(G2), and let $\Phi = -J_G$. Then, by Remark 3.1, Φ is sequentially weakly continuous. From what stated above, there exists $\mu_0 > 0$ such that for all $\mu \in]0, \mu_0[$ the functional $\mathcal{E}_{\lambda,\mu}$ admits at least two local minima $u^1_{\lambda,\mu},\ u^2_{\lambda,\mu} \in S_\lambda$, therefore these are critical points of $\mathcal{E}_{\lambda,\mu}$.

Observe that

$$S := \bigcup_{\lambda\in[a,b]} S_\lambda \subseteq S_a \cup S_b.$$

Since $\Psi(\cdot, \lambda)$ is coercive for all $\lambda \geq 0$, the latter sets are bounded, hence S is bounded as well. Chosen $\sigma_0 > \sup\limits_{u\in S} \|u\|_b$, we get

$$\|u^1_{\lambda,\mu}\|_b, \|u^2_{\lambda,\mu}\|_b < \sigma_0.$$

2° Let $\lambda \in [a, b]$ be a real number, and let $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfying the conditions (G1)-(G3): as above, there exists $\mu_0 > 0$ such that for all $\mu \in]0, \mu_0[$ the functional $\mathcal{E}_{\lambda,\mu}$ has at least two local minima $u^1_{\lambda,\mu},\ u^2_{\lambda,\mu} \in E$ with norms less than σ_0. To prove the existence of a third critical point for $\mathcal{E}_{\lambda,\mu}$, we are going to apply Theorem 1.8. For this, it is enough to prove that the functional $\mathcal{E}_{\lambda,\mu}$ satisfies the (PS) condition for $\mu > 0$ small enough. Since (G3) holds, arguing as in Lemma 3.7, it is easy to prove that there exists $\mu_1 \in]0, \mu_0[$ such that $\mathcal{E}_{\lambda,\mu}$ is coercive in E for all $\mu \in [0, \mu_1]$. Let $\{u_n\}$ be a sequence such that $\{\mathcal{E}_{\lambda,\mu}(u_n)\}$ is bounded and $\mathcal{E}'_{\lambda,\mu}(u_n) \to 0$ holds. The coercivity of $\mathcal{E}_{\lambda,\mu}$ implies that $\{u_n\}$ is bounded E. Because E is a reflexive Banach space we can find a subsequence, which we still denote by $\{u_n\}$, weakly convergent to a point $u_\star \in E$. We denote $I(u) = \frac{1}{p}\|u\|_b^p$. Then the directional derivative of $\mathcal{E}_{\lambda,\mu}$ in the direction $h \in E$ is

$$\langle \mathcal{E}'_{\lambda,\mu}(u), h\rangle = \langle I'(u), h\rangle - \lambda\langle J'_F(u), h\rangle - \mu\langle J'_G(u), h\rangle,$$

where

$$\langle I'(u), h\rangle = \int_\Omega a(x)|\nabla u|^{p-2}\nabla u(x)\nabla h(x)dx + \int_\Gamma b(x)|u(x)|^{p-2}u(x)h(x)d\Gamma,$$

$$\langle J'_F(u), h\rangle = \int_\Omega f(x,u(x))h(x)dx \quad \text{and} \quad \langle J'_G(u), h\rangle = \int_\Gamma g(x,u(x))h(x)d\Gamma.$$

To show that $u_n \to u_\star$ strongly in E we use the inequalities (3.17) and (3.18).

In the case $p \geq 2$ we use (3.17) and we obtain:

$$\begin{aligned}
\|u_n - u_\star\|_b^p &= \int_\Omega a(x)|\nabla u_n(x) - \nabla u_\star(x)|^p dx + \int_\Gamma b(x)|u_n(x) - u_\star(x)|^p d\Gamma \\
&\leq C\Big(\langle I'(u_n), u_n - u_\star\rangle - \langle I'(u_\star), u_n - u_\star\rangle\Big) \\
&= C\Big(\langle \mathcal{E}'_{\lambda,\mu}(u_n), u_n - u_\star\rangle - \langle \mathcal{E}'_{\lambda,\mu}(u_\star), u_n - u_\star\rangle + \langle \lambda J'_F(u_n) \\
&\quad +\mu J'_G(u_n), u_n - u_\star\rangle - \langle \lambda J'_F(u_\star) + \mu J'_G(u_\star), u_n - u_\star\rangle\Big) \\
&\leq C\Big(\|\mathcal{E}'_{\lambda,\mu}(u_n)\|_{E'} + \lambda\|J'_F(u_n) - J'_F(u_\star)\|_{E'} \\
&\quad +\mu\|J'_G(u_n) - J'_G(u_\star)\|_{E'}\Big)\|u_n - u_\star\|_b - C\langle \mathcal{E}'_{\lambda,\mu}(u_\star), u_n - u_\star\rangle\,.
\end{aligned}$$

Since $\mathcal{E}'_{\lambda,\mu}(u_n) \to 0$ and J'_F, J'_G are compact, it follows that $u_n \to u_\star$ converges strongly in E.

In the case $1 < p < 2$, we use (3.18) and Hölder's inequality to obtain the estimate

$$\|u_n - u_\star\|_b^p \leq \hat{C}\Big|\langle I'(u_n), u_n - u_\star\rangle - \langle I'(u_\star), u_n - u_\star\rangle\Big|\Big(\|u_n\|_b^p + \|u_\star\|_b^p\Big)^{(2-p)/p},$$

where $\hat{C} > 0$ is a positive constant depending on p and C.

Thus, the condition (PS) is fulfilled for all $\mu \in [0, \mu_1]$. This concludes our proof.

Corollary 3.1 *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function satisfying conditions* (F1)-(F4). *Then, there exists a non-degenerate compact interval $[a, b] \subset]0, +\infty[$ with the following properties:*

I. *there exists a number $\sigma_0 > 0$ such that for every $\lambda \in [a,b]$ and for every function $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfying conditions* (G1)-(G2), *there exists $\mu_0 > 0$ such that for each $\mu \in]0, \mu_0[$, problem $(P''_{\lambda,\mu})$ has at least one non–trivial solution in E with norm less than σ_0;*

II. *for every $\lambda \in [a,b]$ and for every function $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfying conditions* (G1)-(G3), *there exists $\mu_1 > 0$ such that for each $\mu \in]0, \mu_1[$, problem $(P''_{\lambda,\mu})$ has at least two non–trivial solutions in E.*

In the last part of this section we consider for $\lambda > 0$ and $\mu \in \mathbb{R}$ the following double eigenvalue problem:

$$(P'''_{\lambda,\mu}) \qquad \begin{cases} -\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u) = \lambda f(x,u) \text{ in } \Omega, \\ a(x)|\nabla u|^{p-2}\nabla u \cdot \mathsf{n} + b(x)|u|^{p-2}u = \lambda\mu g(x,u) \text{ on } \Gamma \\ u \neq 0 \text{ in } \Omega, \end{cases}$$

where we use the same notations as in the case of problem $(P''_{\lambda,\mu})$.

The energy functional $\mathcal{F}_{\lambda,\mu} : E \to \mathbb{R}$ corresponding to $(P'''_{\lambda,\mu})$ is given by:

$$\mathcal{F}_{\lambda,\mu}(u) = \frac{1}{p}\int_\Omega a(x)|\nabla u(x)|^p dx + \frac{1}{p}\int_\Gamma b(x)|u(x)|^p d\Gamma - \lambda J_\mu(u),$$

where $J_\mu(u) = J_F(u) + \mu J_G(u)$.

Lemma 3.8 *Suppose that the conditions $(F1), (F3), (G1)$ and $(G3)$ are satisfied. Then, for every $\lambda \geq 0$ and $\mu \in \mathbb{R}$ the functional $\mathcal{F}_{\lambda,\mu}$ is coercive on E and satisfies the (PS) condition.*

Proof If $\lambda = 0$, then the statement is trivial.

Let $\lambda > 0$, $\mu \in \mathbb{R}$ and $a, b > 0$ such that $\lambda a C_f C^p_{p,w_1} + \lambda|\mu| b C_g C^p_{p,w_2} < \frac{1}{p}$. There exist positive functions $k_a \in L^1(\Omega; w_1)$, $k_b \in L^1(\Gamma; w_2)$ such that

$$F(x,s) \leq a f_0(x)|s|^p + k_a(x)w_1(x), \quad \text{for all} \quad (x,s) \in \Omega \times \mathbb{R},$$

$$G(x,s) \leq b g_0(x)|s|^p + k_b(x)w_2(x), \quad \text{for all} \quad (x,s) \in \Gamma \times \mathbb{R}.$$

Then, it follows that

$$\begin{aligned}
\mathcal{F}_{\lambda,\mu}(u) &= \frac{\|u\|_b^p}{p} - \lambda J_\mu(u) = \frac{\|u\|_b^p}{p} - \lambda\Big(\int_\Omega F(x,u(x))dx + \mu\int_\Gamma G(x,u(x))d\Gamma\Big) \geq \\
&\geq \frac{\|u\|_b^p}{p} - \lambda a C_f \int_\Omega w_1(x)|u(x)|^p dx - \lambda \int_\Omega k_a(x) w_1(x) dx - \\
&- \lambda|\mu| b C_g \int_\Gamma w_2(x)|u(x)|^p d\Gamma - \lambda|\mu| \int_\Gamma k_b(x) w_2(x) d\Gamma \geq \\
&\geq \|u\|_b^p \left(\frac{1}{p} - \lambda a C_f C_{p,w_1}^p - \lambda|\mu| b C_g C_{p,w_2}^p\right) - \lambda\|k_a\|_{1,w_1} - \lambda|\mu|\|k_b\|_{1,w_2}
\end{aligned}$$

which goes to ∞ as $\|u\|_b \to \infty$. We omit to prove that $\mathcal{F}_{\lambda,\mu}$ satisfies the (PS) condition, since it is likewise to the proof given in Lemma 3.4. □

Theorem 3.6 *We suppose that the functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfy the conditions $(F1) - (F4)$ and $(G1) - (G3)$ respectively. Then, there exists $\lambda_0 > 0$ such that to every $\lambda \in]\lambda_0, +\infty[$ it corresponds a nonempty open interval $I_\lambda \subset \mathbb{R}$ such that for every $\mu \in I_\lambda$ problem $(P'''_{\lambda,\mu})$ has at least two distinct, nontrivial weak solutions $u_{\lambda,\mu}$ and $v_{\lambda,\mu}$, with the property*

$$\mathcal{F}_{\lambda,\mu}(u_\lambda, \mu) < 0 < \mathcal{F}_{\lambda,\mu}(v_\lambda, \mu).$$

Lemma 3.9 *For $\lambda > \lambda_0$ and $|\mu| \in]0, \mu_\lambda^*]$ we have*

$$\inf_{u\in E} \mathcal{F}_{\lambda,\mu}(u) < 0.$$

Proof. It is sufficient to prove, that for $\lambda > \lambda_0$ and $|\mu| \in]0, \mu_\lambda^*]$ we have $\mathcal{F}_{\lambda,\mu}(u_0) < 0$. Indeed,

$$\begin{aligned}
\mathcal{F}_{\lambda,\mu}(u_0) &= I(u) - \lambda J_f(u_0) - \lambda\mu J_G(u_0) \leq \\
&\leq \lambda_0 J_F(u_0) - \lambda J_F(u_0) + \lambda|\mu| m = \\
&= (\lambda_0 - \lambda) J_F(u_0) + \lambda|\mu| m = \\
&= (\lambda_0 - \lambda)\frac{\lambda(1+c)\mu_\lambda^*}{\lambda - \lambda_0} + \lambda|\mu| m = \\
&= -(1+m)\lambda\mu_\lambda^* + \lambda|\mu| m = \\
&= -\lambda\mu_\lambda^* - m\lambda(|\mu| - \mu_\lambda^*) < 0
\end{aligned}$$

for all $\lambda > \lambda_0$ and $|\mu| \in]0, \mu_\lambda^*]$. □

Lemma 3.10 *For every $\lambda > \lambda_0$ and $\mu \in]0, \mu_\lambda^*]$, the functional $\mathcal{F}_{\lambda,\mu}$ satisfies the mountain pass geometry.*

Proof. The assumptions (F1),(F2), (G1) and (G2) imply for every $\varepsilon > 0$ the existence of $c_1(\varepsilon)$, $c_2(\varepsilon) > 0$ such that

$$|F(x,u(x))| \le \varepsilon f_0(x)|u(x)|^p + c_1(\varepsilon)(f_0(x)+f_1(x))|u(x)|^r, \tag{3.47}$$

$$|G(x,u(x))| \le \varepsilon g_0(x)|u(x)|^p + c_2(\varepsilon)(g_0(x)+g_1(x))|u(x)|^m, \tag{3.48}$$

where $r \in]p,p^*[$ and $m \in \left[p, p\frac{N-1}{N-p}\right]$. Using again (F1) and (G1), we get

$$|F(x,u(x))| \le \varepsilon w_1(x)C_f|u(x)|^p + 2c_1(\varepsilon)C_f w_1(x))|u(x)|^r, \tag{3.49}$$

$$|G(x,u(x))| \le \varepsilon w_2(x)C_g|u(x)|^p + 2c_2(\varepsilon)C_g w_2(x)|u(x)|^m. \tag{3.50}$$

Fix $\lambda > \lambda_0$ and $\mu \in]0,\mu^*_\lambda[$, then by using the inequalities from (3.49) and (3.50) we have for every $u \in E$

$$\begin{aligned}
\mathcal{F}_{\lambda,\mu}(u) &= \frac{1}{p}\|u\|_b^p - \lambda J_\mu(u) \ge \\
&\ge \frac{1}{p}\|u\|_b^p - \lambda\int_\Omega |F(x,u(x))|dx - \lambda|\mu|\int_\Gamma |G(x,u(x))|d\Gamma \ge \\
&\ge \frac{1}{p}\|u\|_b^p - \lambda\varepsilon C_f\int_\Omega w_1(x)|u(x)|^p dx - 2\lambda c_1(\varepsilon)C_f\int_\Omega w_1(x)|u(x)|^r dx \\
&\quad -\lambda|\mu|\varepsilon C_g\int_\Gamma w_2(x)|u(x)|^p d\Gamma - 2\lambda|\mu|c_2(\varepsilon)C_g\int_\Gamma w_2(x)|u(x)|^m d\Gamma = \\
&= \frac{1}{p}\|u\|_b^p - \lambda\varepsilon C_f\|u\|_{p,w_1}^p - 2\lambda c_1(\varepsilon)C_f\|u\|_{r,w_1}^r - \\
&\quad -\lambda|\mu|\varepsilon C_g\|u\|_{p,w_2}^p - 2\lambda|\mu|c_2(\varepsilon)C_g\|u\|_{m,w_2}^m \ge \\
&\ge \left(\frac{1}{p} - \lambda\varepsilon C_f C_{p,w_1}^p - \lambda|\mu|\varepsilon C_g C_{p,w_2}^p\right)\|u\|_b^p - \\
&- \quad 2\lambda c_1(\varepsilon)C_f C_{r,w_1}^r\|u\|_b^r - 2\lambda|\mu|c_2(\varepsilon)C_g C_{m,w_2}^m\|u\|_b^m.
\end{aligned}$$

Using the notations

$$A = \left(\frac{1}{p} - \lambda\varepsilon C_f C_{p,w_1}^p - \lambda|\mu|\varepsilon C_g C_{p,w_2}^p\right),$$

$$B = 2\lambda c_1(\varepsilon)C_f C_{r,w_1}^r, \quad C = 2\lambda|\mu|c_2(\varepsilon)C_g C_{m,w_2}^m,$$

we get

$$\mathcal{F}_{\lambda,\mu}(u) \ge (A - B\|u\|_b^{r-p} - C\|u\|_b^{m-p})\|u\|_b^p.$$

We choose $\varepsilon \in \left]0, \frac{1}{2p}\frac{1}{\lambda(C_f C_{p,w_1}^p + |\mu|C_g C_{p,w_2}^p)}\right[$, then $A > 0$. Let $l : \mathbb{R}_+ \to \mathbb{R}$ be defined by $l(t) = A - Bt^{r-p} - Ct^{m-p}$. We can see, that $l(0) = A > 0$, so there exists $\varepsilon^* > 0$ such that $l(t) > 0$ for every $t \in$

$]0, \varepsilon^*[$. Then, for every $u \in E$, with $\|u\| = \varepsilon^{**} < \min\{\varepsilon^*, \|u_0\|\}$, we have $\mathcal{F}_{\lambda,\mu}(u) \geq \eta(\lambda, \mu, \varepsilon^*) > 0$. From Lemma 3.9 we obtain $\mathcal{F}_{\lambda,\mu}(u_0) < 0$.

Therefore, the functional $\mathcal{F}_{\lambda,\mu}$ satisfies the hypotheses of Theorem 1.7. □

Proof of Theorem 3.6: Fix $\lambda > \lambda_0$ and $\mu \in]0, \mu^*_\lambda[= I_\lambda$. From Lemma 3.8 we have that the functional $\mathcal{F}_{\lambda,\mu}$ is coercive and satisfies the (PS) condition. Then, there exists an element $u_{\lambda,\mu} \in E$ such that $\mathcal{E}_{\lambda,\mu}(u_{\lambda,\mu}) = \inf_{v \in E} \mathcal{F}_{\lambda,\mu}(v)$ (see [249]). By using Lemma 3.9 we obtain $\mathcal{F}_{\lambda,\mu}(u_{\lambda,\mu}) < 0$. On the other hand, by Lemma 3.10 and Theorem 1.8, it follows that there exists an element $v_{\lambda,\mu} \in E$ such that $\mathcal{F}'_{\lambda,\mu}(v_{\lambda,\mu}) = 0$ and $\mathcal{F}_{\lambda,\mu}(v_{\lambda,\mu}) \geq \eta(\lambda, \mu, \varepsilon^*) > 0$. □

3.5 Comments and further perspectives

A. Comments. The problems considered in this chapter extend in a quasilinear framework the *stationary Schrödinger equation*

$$-\Delta u + V(x)u = au \qquad \text{in } \Omega\,.$$

This equation describes the motion of a particle interacting with a force field $F(x) = -\nabla V(x)$, where the function $V : \Omega \rightarrow \mathbb{R}$ is called a *potential*. The associated energy functional is

$$E(u) = E_1(u) + E_2(u)\,,$$

where

$$E_1(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx \qquad \text{and} \qquad E_2(u) = \frac{1}{2}\int_\Omega \left[V(x)u^2 - au^2\right]\,dx\,.$$

Physically, $E_1(u)$ is called the *kinetic energy* of the particle u, $E_2(u)$ is its *potential energy*, while $E(u)$ is the *total energy* of u.

In some situations (see, e.g., Proposition 3.1), the kinetic energy dominates the potential energy. Any inequality in which the kinetic energy dominates some kind of integral of u (but not depending on ∇u) is called an *uncertainty principle*. As stated in [189], "the historical reason for this strange appellation is that such an inequality implies that one cannot make the potential energy very negative without also making the kinetic energy large".

In this chapter we have been interested in the qualitative analysis of weak solutions to the quasilinear mixed boundary value problem

$$(P_{\lambda,\mu}) \qquad \begin{cases} -\mathrm{div}\,(a(x)|\nabla u|^{p-2}\nabla u) = \lambda\alpha(x,u) + \beta(x,u) & \text{in } \Omega, \\ a(x)|\nabla u|^{p-2}\nabla u\cdot \mathsf{n} + b(x)|u|^{p-2}u = \mu h(x,u) & \text{on } \Gamma\,, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an unbounded domain with (possible noncompact) smooth boundary Γ. In our treatment a central role is played by the compact embedding result Theorem 3.1, due to Pflüger [237], [238]. In our results, the functions α, β and h satisfies subcritical growth condition, which implies that the energy functional associated to the problem $(P_{\lambda,\mu})$ satisfies the (PS) condition. The (PS) condition play a key place in the mountain pass theorem and Ricceri type results used in the proof of our results. The results of this chapter are based on the papers Montefusco and Radulescu [218], Cârstea and Rădulescu [70], Lisei, Varga, and Horváth [192], and Mezei and Varga [210].

B. Further perspectives. In the last years the fibering method introduced and developed by Pohozaev [241], [242], [243] can be applied with success in the study of existence and multiplicity of solutions for bifurcation equations, homogenous Dirichlet or Neumann problems which contain p-Laplacian (see [244]). El Hamidi in the paper [102] use this method to study mixed homogeneous problems. Mitidieri and Bozhkov in [212] use an early modified version of fibering method and prove the existence and multiplicity results for quasilinear system, which contains p,q-Laplacians. In [142] the authors study positive solutions for indefinite inhomogeneous Neumann elliptic problems using this methods. Using the concentration compactness principle [191] combined with fibering method we propose the following problem:

$$(P^\star_{\lambda,\mu}) \ \begin{cases} -\mathrm{div}\,(a(x)|\nabla u|^{p-2}\nabla u) = \lambda K(x)|u|^r + k(x)|u|^{p-2}u & \text{in } \Omega, \\ a(x)|\nabla u|^{p-2}\nabla u\cdot \mathsf{n} + b(x)|u|^{p-2}u = \mu L(x)|u|^q & \text{on } \Gamma\,, \end{cases} \quad ,$$

where $r = p^\star = \frac{pN}{N-p}, q < \overline{p}^\star = \frac{p(N-1)}{N-p}$ or $r < p^\star = \frac{pN}{N-p}, q = \overline{p}^\star = \frac{p(N-1)}{N-p}$ or $r = p^\star = \frac{pN}{N-p}, q = \overline{p}^\star = \frac{p(N-1)}{N-p}$ or $p^\star = \frac{pN}{N-p}, \overline{p}^\star = \frac{p(N-1)}{N-p}$ represent the critical Sobolev exponent for the embedding $E \hookrightarrow L^r(\Omega; w_1)$ and $E \hookrightarrow L^q(\Gamma; w_2)$.

The problem $(P^\star_{\lambda,\mu})$ is similar with Yamabe problem on manifold with boundary, see the papers of Escobar [104], [105] and [106].

4

Elliptic Systems of Gradient Type

Rigor is to the mathematician
what morality is to men.

André Weil (1906–1998)

4.1 Introduction

In this chapter we are concerned to study the existence and multiplicity results for eigenvalue problems for elliptic systems of gradient type. On of the main results mathematical development was initiated by Esteban [107], Grossinho and Tersian [132], Fan and Zhao [110]. In this chapter we present some new results in this direction using results of Bartolo, Benci and Fortunato [24], and some recent results of Ricceri [255], [259], [257], [261] and Tsar'kov [285].

4.2 Formulation of the problems

Let $\Omega \subseteq \mathbb{R}^N$ a stripe-like domain, i.e. $\Omega = \omega \times \mathbb{R}^l$, $\omega \subset \mathbb{R}^m$ is bounded with smooth boundary and $m \geq 1, l \geq 2, 1 < p, q < N = m + l$. Let $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^1. We consider the following problem:

$$(\mathrm{S}^{\lambda}_{p,q}) \qquad \begin{cases} -\triangle_p u = \lambda F_u(x, u, v) & \text{in } \Omega, \\ -\triangle_q v = \lambda F_v(x, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

We suppose that the function $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the following condition:

(F1) $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $(s,t) \mapsto F(x,s,t)$ is of class C^1 and $F(x,0,0) = 0$ for every $x \in \Omega$.

(F2) There exist $c_1 > 0$ and $r \in (p, p^*)$, $s \in (q, q^*)$ such that

$$|F_u(x,u,v)| \le c_1(|u|^{p-1} + |v|^{(p-1)q/p} + |u|^{r-1}), \tag{4.1}$$

$$|F_v(x,u,v)| \le c_1(|v|^{q-1} + |u|^{(q-1)p/q} + |v|^{s-1}) \tag{4.2}$$

for every $x \in \Omega$ and $(u,v) \in \mathbb{R}^2$.

We consider the Sobolev space $W_0^{1,\alpha}(\Omega)(\alpha \in \{p,q\})$ endowed with the norm

$$\|u\|_{1,\alpha} = (\int_\Omega |\nabla u|^\alpha)^{1/\alpha},$$

with $\alpha \in \{p,q\}$.

We say that $(u,v) \in W_0^{1,p} \times W_0^{1,q}$ is a weak solutions of the problem $(\mathrm{S}_{p,q,\Omega})$ if

$$\begin{cases} \displaystyle\int_\Omega |\nabla u|^{p-2}\nabla u \nabla w_1 dx - \lambda \int_\Omega F'_u(x;u(x),v(x))w_1(x))dx = 0 \\ \displaystyle\int_\Omega |\nabla v|^{q-2}\nabla v \nabla w_2 dx - \lambda \int_\Omega F'_v(x;u(x),v(x))w_2(x)dx = 0, \end{cases}$$

for every $(w_1, w_2) \in W_0^{1,p} \times W_0^{1,q}$.

The product space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ is endowed with the norm $\|(u,v)\|_{1,p,q} = \|u\|_{1,p} + \|v\|_{1,q}$. We define the function $\mathcal{F} : W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \to \mathbb{R}$ by

$$\mathcal{F}(u,v) = \int_\Omega F(x;u,v)dx$$

for $u \in W_0^{1,p}(\Omega)$, $v \in W_0^{1,q}(\Omega)$. The energy functional associated to problem $(S^\lambda_{p,q})$ is given by

$$E(u,v) = \frac{1}{p}||u||^p_{1,p} + \frac{1}{q}||v||^q_{1,q} - \int_\Omega F(x;u,v)dx$$

and taking into account the conditions (**F1**) and (**F2**) follows that E is of class C^1 and its critical points are weak solutions of the problem $(\mathrm{S}^\lambda_{p,q})$.

The embedding $W_0^{1,\alpha}(\Omega) \hookrightarrow L^\beta(\Omega)$ for $\beta \in [\alpha, \alpha^*]$ $(\alpha \in \{p,q\})$ are continuous, and we denote by $c_{\beta,\alpha} > 0$ be the embedding constant, i.e. $\|u\|_\beta \le c_{\beta,\alpha}\|u\|_{1,\alpha}$ for all $u \in W_0^{1,\alpha}(\Omega)$. Since the embedding $W_0^{1,\alpha}(\Omega) \hookrightarrow$

$L^\beta(\Omega)$ for $\beta \in [\alpha, \alpha^*]$ $(\alpha \in \{p,q\})$ are not compact, we introduce the action of $G = id^m \times O(N-m)$ on $W_0^{1,\alpha}(\Omega)$ as

$$gu(x,y) = u(x, g_0^{-1}y)$$

for all $(x,y) \in \omega \times \mathbb{R}^{N-m}$, $g = id^m \times g_0 \in G$ and $u \in W_0^{1,\alpha}(\Omega)$, $\alpha \in \{p,q\}$. Moreover, the action G on $W_0^{1,\alpha}(\Omega)$ is isometric, that is $\|gu\|_{1,\alpha} = \|u\|_{1,\alpha}$ for all $g \in G$, $u \in W_0^{1,\alpha}(\Omega)$, $\alpha \in \{p,q\}$. Let us denote by

$$W_{0,G}^{1,\alpha}(\Omega) = \{u \in W_0^{1,\alpha}(\Omega) : gu = u \text{ for all } g \in G\}, \quad \alpha \in \{p,q\},$$

which is exactly the closed subspace of axially symmetric functions of $W_0^{1,\alpha}(\Omega)$. The embeddings $W_{0,G}^{1,\alpha}(\Omega) \hookrightarrow L^\beta(\Omega)$, $\alpha < \beta < \alpha^*$, $\alpha \in \{p,q\}$, are compact see **Appendix Sobolev space**. The fixed points set of the action G on $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ is $W_{0,G}^{1,p,q}(\Omega) = W_{0,G}^{1,p}(\Omega) \times W_{0,G}^{1,q}(\Omega)$. To be possible to use the Principle of symmetric criticality, i.e. Theorem ... , we introduce an invariance condition for the function F, i.e. we suppose that:

*(**FI**)* $F(gx,u,v) = F(x,u,v)$ for every $x \in \Omega, g \in G$ and $(u,v) \in \mathbb{R}^2$.

Therefore to study the weak solutions of the problem $(S_{p,q}^\lambda)$ it is sufficient to study the existence of critical points of the function $E_G = E|_{W_{0,G}^{1,p,q}(\Omega)}$. To manipulate easier the problem $(\mathrm{S}_{p,q}^\lambda)$ we introduce the following notations:

$$\langle A_p u, w_1\rangle_p = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla w_1 dx$$

and

$$\langle A_q v, w_2\rangle_q = \int_\Omega |\nabla v|^{p-2}\nabla v \nabla w_1 dx,$$

where $\langle\cdot,\cdot\rangle_p$ ($\langle\cdot,\cdot\rangle_q$) means the duality between $W_0^{1,p}(\Omega)$ and $(W_0^{1,p}(\Omega))^\star$ $(W_0^{1,q}(\Omega)$ and $(W_0^{1,q}(\Omega))^\star)$ respectively. Also we denote in the same way the restriction of the norms $\|\cdot\|_{1,p}, \|\cdot\|_{1,q}$ and $\langle\cdot,\cdot\rangle_p, \langle\cdot,\cdot\rangle_q$ restricted to $W_{0,G}^{1,p,q}(\Omega)$.

With the above notations we have $A_p : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^\star$ and $A_q : W_0^{1,q}(\Omega) \to (W_0^{1,q}(\Omega))^\star$ are the duality mappings induced by the the functions $t \in [0,+\infty[\mapsto t^{p-1} \in [0,+\infty[$, and $t \in [0,+\infty[\mapsto t^{q-1} \in [0,+\infty[$, respectively.

This chapter is divided in three sections. In the first section we study the case when the function F is superlinear at infinity, the second section contain the case when F is sublinear, while the section three contains a special class of system.

4.3 Systems with superlinear potential

In this section we prove an existence result for problem ($(\mathrm{S}^{\lambda}_{p,q})$ in the case when $\lambda = 1$ and the function F is independent of x, i.e. $F : \mathbb{R}^2 \to \mathbb{R}$ and satisfies the conditions (**F1**), (**F2**). In this case we have the following problem:

$$(\mathrm{S}_{p,q}) \qquad \begin{cases} -\triangle_p u = F_u(u,v) & \text{in } \Omega, \\ -\triangle_q v = F_v(u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

This problem consists in finding $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ such that

$$\begin{cases} \langle A_p(u), w_1 \rangle - \displaystyle\int_\Omega F'_u(u(x),v(x))w_1(x))dx = 0 \\ \\ \langle A_q(v), w_2 \rangle - \displaystyle\int_\Omega F'_v(u(x),v(x))w_2(x)dx = 0, \end{cases}$$

for every $(w_1, w_2) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

In this case the energy functional associated to the problem $(\mathrm{S}_{p,q})$ is given by

$$E(u,v) = \frac{1}{p}||u||^p_{1,p} + \frac{1}{q}||v||^q_{1,q} - \mathcal{F}(u,v), \tag{4.3}$$

where $\mathcal{F}(u,v) = \displaystyle\int_\Omega F(u(x),v(x))dx$. Its derivative in $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ is given by

$$E'(u,v)(w_1,w_2) = \langle A_p(u), w_1\rangle_p + \langle A_q(u), w_2\rangle_q - \mathcal{F}'(u,v)(w_1,w_2), \tag{4.4}$$

for every $(w_1, w_2) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, where

$$\mathcal{F}'(u,v)(w_1,w_2) = \int_\Omega [F_u(u,v)w_1 + F_v(u,v)w_2]dx. \tag{4.5}$$

As in introduction we denote by $E_G = E|_{W^{1,p,q}_{0,G}(\Omega)}$ and $\mathcal{F}_G = \mathcal{F}|_{W^{1,p,q}_{0,G}(\Omega)}$. In the following we study the only the existence of critical points of the functional E_G.

In the next we suppose that the function $F : \mathbb{R}^2 \to \mathbb{R}$ beside of

conditions (**F1**), (**F2**) satisfies the following assumptions:
(**F3**) there exist $c_2 > 0$ and $\mu, \nu \geq 1$ such that

$$-c_2(|u|^\mu + |v|^\nu) \geq F(u,v) - \frac{1}{p}F'_u(u,v)u - \frac{1}{q}F'_v(u,v)v, \tag{4.6}$$

for all $(u,v) \in \mathbb{R}^2$.

(**F4**)
$$\lim_{u,v\to 0} \frac{F'_u(u,v)}{|u|^{p-1}} = \lim_{u,v\to 0} \frac{F'_v(u,v)}{|v|^{q-1}} = 0,$$

The main result of this section can be formulated as follows:

Theorem 4.1 *We suppose that the function $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the conditions* (**F1**)-(**F4**) *with $ps = qr$ and*

$$\mu > \max\{p, N(r-p)/p\} \text{ and } \nu > \max\{q, N(s-q)/q\}. \tag{4.7}$$

Then, (S) admits at least one nonzero solution.

We will use the following inequality:

Proposition 4.1 *For all $t > 1$ and $(u,v) \in \mathbb{R}^2$ we have*

$$F(t^{\frac{1}{p}}u, t^{\frac{1}{q}}v) \geq tF(u,v) + c_2\left[\frac{p}{\mu-p}(t^{\frac{\mu}{p}} - t)|u|^\mu + \frac{q}{\nu-q}(t^{\frac{\nu}{q}} - t)|v|^\nu\right]. \tag{4.8}$$

Proof We fix an element $(u,v) \in \mathbb{R}^2$ arbitrary. We define the function $g :]0,\infty[\to \mathbb{R}$ by

$$g(t) = t^{-1}F(t^{\frac{1}{p}}u, t^{\frac{1}{q}}v) - c_2\frac{p}{\mu-p}t^{\frac{\mu}{p}-1}|u|^\mu - c_2\frac{q}{\nu-q}t^{\frac{\nu}{q}-1}|v|^\nu.$$

Since g is of class C^1, due to the Mean Value Theorem, for a fixed $t > 1$ there exists $\tau = \tau(t,u,v) \in]1,t[$ such that

$$g(t) - g(1) = g'(\tau)(t-1).$$

We have

$$\partial_t F(t^{\frac{1}{p}}u, t^{\frac{1}{q}}v) = \frac{1}{p}F'_u(t^{\frac{1}{p}}u, t^{\frac{1}{q}}v)t^{\frac{1}{p}-1}u + \frac{1}{q}F'_v(t^{\frac{1}{p}}u, t^{\frac{1}{q}}v)t^{\frac{1}{q}-1}v.$$

Hence,

$$\begin{aligned} g'(t) \quad &= \quad -t^{-2}F(t^{\frac{1}{p}}u, t^{\frac{1}{q}}v) + t^{-1}\left[\frac{1}{p}F'_u(t^{\frac{1}{p}}u, t^{\frac{1}{q}}v)t^{\frac{1}{p}-1}u \right. \\ &\quad \left. + \frac{1}{q}F'_v(t^{\frac{1}{p}}u, t^{\frac{1}{q}}v)t^{\frac{1}{q}-1}v\right] - c_2\left[t^{\frac{\mu}{p}-2}|u|^\mu + t^{\frac{\nu}{q}-2}|v|^\nu\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} g(t)-g(1) &= -\tau^{-2}[F(\tau^{\frac{1}{p}}u,\tau^{\frac{1}{q}}v)-\frac{1}{p}F'_u(\tau^{\frac{1}{p}}u,\tau^{\frac{1}{q}}v)\tau^{\frac{1}{p}}u \\ &- \frac{1}{q}F'_v(\tau^{\frac{1}{p}}u,\tau^{\frac{1}{q}}v)\tau^{\frac{1}{q}}v+c_2(|\tau^{\frac{1}{p}}u|^{\mu}+|\tau^{\frac{1}{q}}v|^{\nu})](t-1). \end{aligned}$$

Due to (4.6), we have $g(t)\geq g(1)$. Thus,

$$F(t^{\frac{1}{p}}u,t^{\frac{1}{q}}v)\geq tF(u,v)+c_2\left[\frac{p}{\mu-p}(t^{\frac{\mu}{p}}-t)|u|^{\mu}+\frac{q}{\nu-q}(t^{\frac{\nu}{q}}-t)|v|^{\nu}\right]$$

for all $t>1$ and $(u,v)\in\mathbb{R}^2$. □

Proposition 4.2 *If the assumptions of Theorem 4.1 are satisfied, then the functional E_G satisfies the Cerami $(C)_c$ condition for all $c>0$.*

Proof The norm of the Banach space $W^{1,p,q}_{0,G}(\Omega)$ is $||(u,v)||_{1,p.q}=||u||_{1,p}+||v||_{1,q}$. We denote the norm on the dual space $(W^{1,p,q}_{0,G}(\Omega))^{\star}$ by $||\cdot||_{\star}$. Now, let $(u_n,v_n)\in W^{1,p,q}_{0,G}(\Omega)$ be such that

$$E_G(u_n,v_n)\to c>0, \tag{4.9}$$

and

$$(1+||(u_n,v_n)||_{1,p,q})||E'_G(u_n,v_n)||_{\star}\to 0, \tag{4.10}$$

as $n\to+\infty$. Moreover, we have

$$\left\langle E'_G(u_n,v_n)(\frac{1}{p}u_n,\frac{1}{q}v_n)\right\rangle\geq-(1+\|(u_n,v_n)\|\|E'_G(u_n,v_n)\|_{\star}. \tag{4.11}$$

Using (4.9), (4.10) and (4.6) one has for n large enough that

$$\begin{aligned} c+1 &\geq E_G(u_n,v_n)-E'_G(u_n,v_n)(\frac{1}{p}u_n,\frac{1}{q}v_n) \\ &= -\mathcal{F}_G(u_n,v_n)+\mathcal{F}'_G(u_n,v_n)(\frac{1}{p}u_n,\frac{1}{q}v_n) \\ &= -\int_\Omega[F(u_n,v_n)-\frac{1}{p}F'_u(u_n,v_n)u_n \\ &\quad -\frac{1}{q}F'_v(u_n,v_n)v_n]dx\geq c_2\int_\Omega[|u_n|^{\mu}+|v_n|^{\nu}]dx. \end{aligned}$$

From the previous inequality we obtain that

$$\{(u_n,v_n)\}\text{ is bounded in }L^{\mu}(\Omega)\times L^{\nu}(\Omega). \tag{4.12}$$

By (**F4**) we have that for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that, if $|u|^{p-1} + |v|^{(p-1)q/p} < \delta(\varepsilon)$, then

$$|F_u'(u,v)| < \varepsilon \left(|u|^{p-1} + |v|^{(p-1)q/p}\right).$$

If $|u|^{p-1} + |v|^{(p-1)q/p} \geq \delta(\varepsilon)$, by using (4.1), we have

$$\begin{aligned} |F_u'(u,v)| &\leq c_1[(|u|^{p-1} + |v|^{(p-1)q/p})^{\frac{r-1}{p-1}}(\delta(\varepsilon))^{\frac{p-r}{p-1}} + |u|^{r-1}] \\ &\leq c(\varepsilon)(|u|^{r-1} + |v|^{(r-1)q/p}). \end{aligned}$$

Combining the above relations, we have that for all $\varepsilon > 0$ there exists $c_1(\varepsilon) > 0$ such that

$$|F_u'(u,v)| < \varepsilon \left(|u|^{p-1} + |v|^{(p-1)q/p}\right) + c_1(\varepsilon)\left(|u|^{r-1} + |v|^{(r-1)q/p}\right) \tag{4.13}$$

for all $(u,v) \in \mathbb{R}^2$. A similar calculation shows that for all $\varepsilon > 0$ there exists $c_2(\varepsilon) > 0$ such that

$$|F_v'(u,v)| < \varepsilon \left(|v|^{q-1} + |u|^{(q-1)p/q}\right) + c_2(\varepsilon)\left(|v|^{s-1} + |u|^{(s-1)p/q}\right) \tag{4.14}$$

for all $(u,v) \in \mathbb{R}^2$.

Using (4.13), (4.14), and keeping in mind that $F(0,0) = 0$, for all $\varepsilon > 0$ there exists $c(\varepsilon) = c(c_1(\varepsilon), c_2(\varepsilon)) > 0$ such that

$$\begin{aligned} F(u,v) &\leq \varepsilon(|u|^p + |v|^{(p-1)q/p}|u| + |v|^q + |u|^{(q-1)p/q}|v|) \\ &+ c(\varepsilon)(|u|^r + |v|^{(r-1)q/p}|u| + |v|^s + |u|^{(s-1)p/q}|v|) \end{aligned} \tag{4.15}$$

for all $(u,v) \in \mathbb{R}^2$.

After integration we use the relation $ps = qr$, as well as Young's inequality and Hölder's inequality to obtain

$$\begin{aligned} \mathcal{F}_G(u_n,v_n) &\leq \varepsilon\left[(2 + \frac{1}{p} - \frac{1}{q})\|u_n\|_p^p + (2 + \frac{1}{q} - \frac{1}{p})\|v_n\|_q^q\right] \\ &+ c(\varepsilon)\left[(2 + \frac{1}{r} - \frac{1}{s})\|u_n\|_r^r + (2 + \frac{1}{s} - \frac{1}{r})\|v_n\|_s^s\right]. \end{aligned}$$

Therefore, one has

$$\left[\frac{1}{p} - \varepsilon(2 + \frac{1}{p} - \frac{1}{q})c_{p,p}^p\right]\|u_n\|_{1,p}^p + \left[\frac{1}{q} - \varepsilon(2 + \frac{1}{q} - \frac{1}{p})c_{q,q}^q\right]\|v_n\|_{1,q}^q \leq$$

$$\leq E_G(u_n,v_n) + c(\varepsilon)\left[(2 + \frac{1}{r} - \frac{1}{s})\|u_n\|_r^r + (2 + \frac{1}{s} - \frac{1}{r})\|v_n\|_s^s\right].$$

Choosing

$$0 < \varepsilon < \frac{1}{3} \min \left\{ \frac{1}{pc_{p,p}^p}, \frac{1}{qc_{q,q}^q} \right\}, \tag{4.16}$$

we find $c_3(\varepsilon), c_4(\varepsilon) > 0$ such that

$$c_3(\varepsilon)(\|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q) \leq c + 1 + c_4(\varepsilon)(\|u_n\|_r^r + \|v_n\|_s^s) \tag{4.17}$$

for n large enough. Now, we will examine the behaviour of the sequences $\{\|u_n\|_r^r\}$ and $\{\|v_n\|_s^s\}$, respectively.

To this end, observe that $\mu \leq r$ and $\nu \leq s$. From relations (4.1), (4.2) and from the fact that for all $\beta \in]0, \infty[$ there exists a constant $c(\beta) > 0$ such that

$$(x + y)^\beta \leq c(\beta)(x^\beta + y^\beta), \quad \text{for all} \quad x, y \in [0, \infty[,$$

we have

$$|F(u, v) - F(w, y)| \leq \tag{4.18}$$

$$\begin{aligned} \leq \quad & c_3[|u - w|(|u|^{p-1} + |w|^{p-1} + |v|^{(p-1)q/p} + |y|^{(p-1)q/p} + |u|^{r-1} + |w|^{r-1}) \\ + \quad & |v - y|(|v|^{q-1} + |y|^{q-1} + |u|^{(q-1)p/q} + |w|^{(q-1)p/q} + |v|^{s-1} + |y|^{s-1})], \end{aligned}$$

where $c_3 = c_3(c_1, p, q, r, s) > 0$.

Keeping in mind the relation $ps = qr$, taking $w = y = 0$ and $u = ut^{\frac{1}{p}}$, $v = vt^{\frac{1}{q}}$ $(t > 1)$ in (4.18), from (4.8) follow the required relations.

We distinguish two cases:

I) $\mu = r$: From (4.12) we have that $\{\|u_n\|_r^r\}$ is bounded.

II) $\mu \in] \max \{p, N(r - p)/p\}, r[$: We have the interpolation inequality

$$\|u\|_r \leq \|u\|_\mu^{1-\delta} \|u\|_{p^*}^\delta \quad \text{for all } u \in L^\mu(\Omega) \cap L^{p^*}(\Omega)$$

with $\delta = \frac{p^*}{r} \cdot \frac{r-\mu}{p^*-\mu}$. From (4.12) and the fact that the embedding $W_{0,G}^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, is continuous, we have that there exists $c_5 > 0$ such that $\|u_n\|_r^r \leq c_5 \|u_n\|_{1,p}^{\delta r}$, with $\delta r < p$.

Taking into consideration the analogous relations for the sequence $\{\|v_n\|_s^s\}$, we conclude from (4.17) that the sequences $\{\|u_n\|_{1,p}\}$ and $\{\|v_n\|_{1,q}\}$ are bounded. Because the embeddings $W_{0,G}^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, $W_{0,G}^{1,q}(\Omega) \hookrightarrow L^s(\Omega)$ are compact, therefore up to a subsequence, we have

$$(u_n, v_n) \to (u, v) \text{ weakly in} W_{0,G}^{1,p,q}(\Omega), \tag{4.19}$$

$$u_n \to u \text{ strongly in } L^r(\Omega), \tag{4.20}$$

$$v_n \to v \text{ strongly in } L^s(\Omega). \tag{4.21}$$

Moreover, we have

$$\begin{aligned} E'_G(u_n,v_n)(u-u_n,v-v_n) &= \langle A_p(u_n),u-u_n\rangle_p + \langle A_q(v_n),v-v_n\rangle_q \\ &- \mathcal{F}'_G(u_n,v_n)(u-u_n,v-v_n) \end{aligned}$$

and

$$\begin{aligned} E'_G(u,v)(u_n-u,v_n-v) &= \langle A_p(u),u_n-u\rangle_p + \langle A_q(v),v_n-v\rangle_q \\ &- \mathcal{F}'_G(u,v)(u_n-u,v_n-v). \end{aligned}$$

By adding the above two relations, we obtain

$$J_n := \langle A_p(u)-A_p(u_n),u-u_n\rangle_p+\langle A_q(v)-A_q(v_n),v-v_n\rangle_q = J_n^1-J_n^2-J_n^3, \tag{4.22}$$

where

$$J_n^1 = \mathcal{F}'_G(u_n,v_n)(u_n-u,v_n-v)+\mathcal{F}'_G(u,v)(u-u_n,v-v_n),$$

$$J_n^2 = E'_G(u_n,v_n)(u-u_n,v-v_n) \text{ and } J_n^3 = E'_G(u,v)(u_n-u,v_n-v).$$

In the sequel, we will estimate J_n^i ($i \in \{1,2,3\}$). Using (4.13), (4.14) and $ps = qr$, one has

$$\begin{aligned} J_n^1 &= \int_\Omega [F'(u_n(x),v_n(x))(u_n(x)-u(x),v_n(x)-v(x)) \\ &\quad +F'(u(x),v(x))(u(x)-u_n(x),v(x)-v_n(x))]dx \\ &\le \int_\Omega [|F'_u(u_n(x),v_n(x))(u_n(x)-u(x))| \\ &\quad +|F'_v(u_n(x),v_n(x))(v_n(x)-v(x))| + |F'_u(u(x),v(x))(u(x)-u_n(x))| \\ &\quad +|F'_v(u(x),v(x))(v(x)-v_n(x))|]dx \\ &\le \varepsilon\Big[(\|u_n\|_p^{p-1}+\|u\|_p^{p-1}+\|v_n\|_q^{(p-1)q/p}+\|v\|_q^{(p-1)q/p})\|u-u_n\|_p \\ &\quad +(\|v_n\|_q^{q-1}+\|v\|_q^{q-1}+\|u_n\|_p^{(q-1)p/q}+\|u\|_p^{(q-1)p/q})\|v-v_n\|_q\Big] \\ &\quad +c_1(\varepsilon)\left(\|u_n\|_r^{r-1}+\|u\|_r^{r-1}+\|v_n\|_s^{(r-1)s/r}+\|v\|_s^{(r-1)s/r}\right)\|u-u_n\|_r \\ &\quad +c_2(\varepsilon)\left(\|v_n\|_s^{s-1}+\|v\|_s^{s-1}+\|u_n\|_r^{(s-1)r/s}+\|u\|_r^{(s-1)r/s}\right)\|v-v_n\|_s. \end{aligned}$$

The sequences $\{u_n\}$ and $\{v_n\}$ are bounded in $W^{1,p}_{0,G}(\Omega)(\hookrightarrow L^p(\Omega)\cap L^r(\Omega))$ and $W^{1,q}_{0,G}(\Omega)(\hookrightarrow L^q(\Omega)\cap L^s(\Omega))$, respectively, then by using the relations (4.20) and (4.21), one has

$$\limsup_{n\to\infty} J_n^1 \le 0, \tag{4.23}$$

since $\varepsilon > 0$ was arbitrary chosen.

The convergence in (4.10) implies $||E'_G(u_n, v_n)||_\star \to 0$, therefore $J_n^2 \to 0$. Taking into account that $(u_n, v_n) \rightharpoonup W_{0,G}^{1,p,q}(\Omega)$ weakly, it follows that $J_n^3 \to 0$. Therefore, if $n \to \infty$, then

$$\langle A_p(u) - A_p(u_n), u - u_n\rangle_p + \langle A_q(v) - A_q(v_n), v - v_n\rangle_q \to 0.$$

On the other hand, from the inequality

$$|t-s|^\alpha \leq \begin{cases} (|t|^{\alpha-2}t - |s|^{\alpha-2}s)(t-s), & \text{if } \alpha \geq 2, \\ ((|t|^{\alpha-2}t - |s|^{\alpha-2}s)(t-s))^{\frac{\alpha}{2}}(|t|^\alpha + |s|^\alpha)^{\frac{2-\alpha}{2}}, & \text{if } 1 < \alpha < 2, \end{cases}$$

for all $t, s \in \mathbb{R}^N$, we obtain that

$$\lim_{n\to\infty} \int_\Omega (|\nabla u_n - \nabla u|^p + |\nabla v_n - \nabla v|^q) = 0,$$

that is, the sequences $\{u_n\}$ and $\{v_n\}$ are strongly convergent in $W_{0,G}^{1,p}(\Omega)$ and $W_{0,G}^{1,q}(\Omega)$, respectively.

□

To prove Theorem 4.1 we use the following formulation of the Mountain Pass Theorem, see [24] or [174].

Proof of Theorem 4.1 If we combine the Principle of Symmetric criticality with Theorem **Cerami Mountain Pass** the assertion follows. From Proposition 4.2 it follows that the functional E_G satisfies the Cerami $(C)_c$ condition for every $c > 0$. We need to verify the conditions (i) and (ii) from Theorem **Cerami Mountain pass**. Using (4.15), we obtain

$$\begin{aligned} E_G(u,v) \quad &= \quad \frac{1}{p}\|u\|_{1,p}^p + \frac{1}{q}\|v\|_{1,q}^q - \int_\Omega F(u,v)dx \geq \\ &\left[\frac{1}{p} - \varepsilon(2 + \frac{1}{p} - \frac{1}{q})c_{p,p}^p\right]\|u\|_{1,p}^p + \left[\frac{1}{q} - \varepsilon(2 + \frac{1}{q} - \frac{1}{p})c_{q,q}^q\right]\|v\|_{1,q}^q \\ &-c(\varepsilon)\left[(2 + \frac{1}{r} - \frac{1}{s})\|u\|_r^r + (2 + \frac{1}{s} - \frac{1}{r})\|v\|_s^s\right]. \end{aligned}$$

Choosing ε as in (4.16), we can fix $c_5(\varepsilon), c_6(\varepsilon) > 0$ such that

$$E_G(u,v) \geq c_5(\varepsilon)(||u||_{1,p}^p + ||v||_{1,q}^q) - c_6(\varepsilon)(||u||_{1,p}^r + ||v||_{1,q}^s).$$

Since the function $t \mapsto (x^t + y^t)^{\frac{1}{t}}$, $t > 0$ is non-increasing $(x, y \geq 0)$, using again $ps = qr$, we have

$$\|u\|_{1,p}^r + \|v\|_{1,q}^s \leq \left[\|u\|_{1,p}^p + \|v\|_{1,q}^q\right]^{\frac{r}{p}(=\frac{s}{q})}.$$

Therefore,

$$E_G(u,v) \geq \left[c_5(\varepsilon) - c_6(\varepsilon)(\|u\|_{1,p}^p + \|v\|_{1,q}^q)^{\frac{r}{p}-1}\right](\|u\|_{1,p}^p + \|v\|_{1,q}^q).$$

Let $0 < \rho < 1$ and denote

$$B_\rho = \{(u,v) \in W_{0,G}^{1,p,q}(\Omega) : \|(u,v)\|_{1,p,q} = \rho\}.$$

Then, we have $(\rho/2)^{\max\{p,q\}} \leq \|u\|_{1,p}^p + \|v\|_{1,q}^q \leq \rho$ for all $(u,v) \in B_\rho$. Choosing ρ small enough, there exists $\eta > 0$ such that $E_G(u,v) \geq \eta$ for all $(u,v) \in B_\rho$, due to the fact that $r > p$. This proves (i) from Theorem **Cerami Mountain Pass**.

To prove (ii) from Theorem **Cerami Mountain Pass**, fix $u^0 \in W_{0,G}^{1,p}(\Omega)$ and $v^0 \in W_{0,G}^{1,q}(\Omega)$ such that $\|u^0\|_{1,p} = \|v^0\|_{1,q} = 1$. Then, for every $t > 1$ we have

$$\begin{aligned} E_G(t^{\frac{1}{p}}u^0, t^{\frac{1}{q}}v^0) &= (\frac{1}{p} + \frac{1}{q})t - \int_\Omega F(t^{\frac{1}{p}}u^0, t^{\frac{1}{q}}v^0)dx \\ &\leq \left(\frac{1}{p} + \frac{1}{q} - \int_\Omega F(u^0,v^0)dx\right)t \\ &\quad -c_2\left[\frac{p}{\mu-p}(t^{\frac{\mu}{p}} - t)\|u^0\|_\mu^\mu + \frac{q}{\nu-q}(t^{\frac{\nu}{q}} - t)\|v^0\|_\nu^\nu\right]. \end{aligned}$$

Therefore, $E_G(t^{\frac{1}{p}}u^0, t^{\frac{1}{q}}v^0) \to -\infty$ as $t \to \infty$ (recall that $\mu > p$ and $\nu > q$). Choosing $t = t_0$ large enough and denoting by $e_p = t_0^{\frac{1}{p}}u_p^0$ and $e_q = t_0^{\frac{1}{q}}v_q^0$, we are led to (ii). This completes the proof. $\square$

Example 4.1 Let $p = 3/2$, $q = 9/4$, $\Omega =]a,b[\times\mathbb{R}^2$ $(a < b)$ and

$$F(u,v) = u^2 + |v|^{7/2} + 1/4\{|u|^{5/2} + |v|^{5/2}\}.$$

The conditions (**F1**) an (**F4**) can be verified easily. Choosing $r = 5/2$, $s = 15/4$ and $\mu = 5/2$, $\nu = 7/2$, the assumptions (**F2**) and (**F3**) hold too. Therefore we can apply Theorem 4.1 and obtain at least one nonzero solution for problem $(\mathrm{S}'_{3/2,9/4,]a,b[\times\mathbb{R}^2})$.

4.4 Systems with sublinear potential

In the previous section we studied problem $(\mathrm{S}_{p,q})$, assuming that the function F satisfies the condition (**F3**), which is an Amrosetti-Rabinowitz type condition and which asserts that the energy functional E satisfies the Palais-Smale or Cerami compactness condition. This condition implies, in particular, some sort of *superlinearity* of F. In this section we will treat the case when F is *sub-(p,q)-linear*. We suppose that the conditions (**F1**)-(**F2**) are fulfilled as introduction. Also, we impose that the function $F:\Omega\times\mathbb{R}^2\to\mathbb{R}$ satisfies the following conditions:

(F'3) $\lim\limits_{u,v\to 0}\dfrac{F_u(x,u,v)}{|u|^{p-1}}=\lim\limits_{u,v\to 0}\dfrac{F_v(x,u,v)}{|v|^{q-1}}=0$ uniformly for every $x\in\Omega$.

(F'4) There exist $p_1\in(0,p)$, $q_1\in(0,q)$, $\mu\in[p,p^*]$, $\nu\in[q,q^*]$ and $a\in L^{\mu/(\mu-p_1)}(\Omega)$, $b\in L^{\nu/(\nu-q_1)}(\Omega)$, $c\in L^1(\Omega)$ such that

$$F(x,u,v)\leq a(x)|u|^{p_1}+b(x)|v|^{q_1}+c(x)$$

for every $x\in\Omega$ and $(u,v)\in\mathbb{R}^2$.

(F'5) There exists $(u_0,v_0)\in W_G^{p,q}$ such that

$$\int_\Omega F(x,u_0(x),v_0(x))dx>0.$$

In this section we suppose that the function F satisfies the conditions (**F1**)-(**F2**) and **(F'3)-(F'5)**. This section is dedicated to study the problem $(\mathrm{S}^\lambda_{p,q})$ which is equivalent with, *find* $(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ *such that*

$$\begin{cases}\langle A_p(u),w_1\rangle_p-\lambda\displaystyle\int_\Omega F_u'(x;u(x),v(x))w_1(x))dx=0\\[2ex] \langle A_q(v),w_2\rangle_q-\lambda\displaystyle\int_\Omega F_v'(x;u(x),v(x))w_2(x)dx=0,\end{cases}$$

for every $(w_1,w_2)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$, *where* $\lambda\in\mathbb{R}$ *is a parameter.*

Using the Principle of Symmetric Criticality it is enough to study the critical points of the functional $E_{\lambda,G}=E_\lambda|_{W^{1,p,q}_{0,G}(\Omega)}$. The functional $E_{\lambda,G}:W^{1,p,q}_{0,G}(\Omega)\to\mathbb{R}$ is given by

$$E_{\lambda,G}(u,v)=\frac{1}{p}||u||^p_{1,p}+\frac{1}{q}||v||^q_{1,q}-\lambda\mathcal{F}_G(u,v), \tag{4.24}$$

where $\mathcal{F}_G(u,v)=\displaystyle\int_\Omega F(x;u(x),v(x))dx$ and its critical points will be the weak solutions of the problem $(\mathrm{S}^\lambda_{p,q})$.

The main result of this section is the following:

Theorem 4.2 *Let $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be a function which satisfies the conditions* **(F1)**-**(F2)** *and* **(F'3)**-**(F'5)** *and $ps = qr$. Then, there exist an open interval $\Lambda \subset (0, \infty)$ and $\sigma > 0$ such that for all $\lambda \in \Lambda$ the system (S_λ) has at least two distinct, nontrivial weak solutions, denoted by $(u^i_\lambda, v^i_\lambda)$, $i \in \{1, 2\}$ such that $\|u^i_\lambda\|_{1,p} < \sigma$, $\|v^i_\lambda\|_{1,q} < \sigma$, $i \in \{1, 2\}$.*

Further in this section, we suppose that all assumptions of Theorem 4.2 are fulfilled. Before we prove Theorem 4.2, the main result of this section, we need some auxiliary results.

Lemma 4.1 *For every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that*
i) $|F_u(x, u, v)| \leq \varepsilon(|u|^{p-1} + |v|^{(p-1)q/p}) + c(\varepsilon)(|u|^{r-1} + |v|^{(r-1)q/p})$;
ii) $|F_v(x, u, v)| \leq \varepsilon(|v|^{q-1} + |u|^{(q-1)p/q}) + c(\varepsilon)(|v|^{s-1} + |u|^{(s-1)p/q})$;
iii) $|F(x, u, v)| \leq \varepsilon(|u|^p + |v|^{(p-1)q/p}|u| + |v|^q + |u|^{(q-1)p/q}|v|)$
$+c(\varepsilon)(|u|^r + |v|^{(r-1)q/p}|u| + |v|^s + |u|^{(s-1)p/q}|v|)$
for every $x \in \Omega$ and $(u, v) \in \mathbb{R}^2$.

Proof *i)* Let $\varepsilon > 0$ be arbitrary. Let us prove the first inequality, the second one being similar. From the first limit of **(F'3)** we have in particular that

$$\lim_{u,v\to 0} \frac{F_u(x, u, v)}{|u|^{p-1} + |v|^{(p-1)q/p}} = 0.$$

Therefore, there exists $\delta(\varepsilon) > 0$ such that, if $|u|^{p-1} + |v|^{(p-1)q/p} < \delta(\varepsilon)$, then $|F_u(x, u, v)| \leq \varepsilon(|u|^{p-1} + |v|^{(p-1)q/p})$. If $|u|^{p-1} + |v|^{(p-1)q/p} \geq \delta(\varepsilon)$, then (4.1) implies that

$$\begin{aligned} |F_u(x, u, v)| \quad &\leq \quad c_1[(|u|^{p-1} + |v|^{(p-1)q/p})^{(r-1)/(p-1)}\delta(\varepsilon)^{(p-r)/(p-1)} + |u|^{r-1}] \\ &+ \quad c(\varepsilon)(|u|^{r-1} + |v|^{(r-1)q/p}). \end{aligned}$$

Combining the above inequalities, we obtain the desired relation. The inequality *iii)* follows from the Mean Value Theorem, $i), ii)$ and $F(x, 0, 0) = 0$. $\square$

Lemma 4.2 *$\mathcal{F}_G$ is a sequentially weakly continuous function on $W^{1,p,q}_{0,G}(\Omega)$.*

Proof Suppose the contrary, i.e., let $\{(u_n, v_n)\} \subset W^{1,p,q}_{0,G}(\Omega)$ be a sequence which converges weakly to $(u, v) \in W^{1,p,q}_{0,G}(\Omega)$ and $\mathcal{F}_G(u_n, v_n) \nrightarrow \mathcal{F}_G(u, v)$. Therefore, there exists $\varepsilon_0 > 0$ and a subsequence of $\{(u_n, v_n)\}$ (denoted again by $\{(u_n, v_n)\}$) such that

$$0 < \varepsilon_0 \leq |\mathcal{F}_G(u_n, v_n) - \mathcal{F}_G(u, v)| \quad \text{for every } n \in \mathbb{N}.$$

For some $0 < \theta_n < 1$ we have

$$0 < \varepsilon_0 \leq |\mathcal{F}'(u_n + \theta_n(u - u_n), v_n + \theta_n(v - v_n))(u_n - u, v_n - v)| \quad (4.25)$$

for every $n \in \mathbb{N}$. Let us denote by $w_n = u_n + \theta_n(u - u_n)$ and $y_n = v_n + \theta_n(v - v_n)$. Since the embeddings $W^{1,p}_{0,G} \hookrightarrow L^r(\Omega)$ and $W^{1,q}_{0,G} \hookrightarrow L^s(\Omega)$ are compact, up to a subsequence, $\{(u_n, v_n)\}$ converges strongly to (u, v) in $L^r(\Omega) \times L^s(\Omega)$. By (4.5), Lemma 4.1, Hölder's inequality and $ps = qr$ one has

$$\begin{aligned}
& |\mathcal{F}'(w_n, y_n)(u_n - u, v_n - v)| \\
\leq\ & \int_\Omega [|F_u(x, w_n, y_n)||u_n - u| + |F_v(x, w_n, y_n)||v_n - v|]dx \\
\leq\ & \varepsilon \int_\Omega [(|w_n|^{p-1} + |y_n|^{(p-1)q/p})|u_n - u| + (|y_n|^{q-1} + |w_n|^{(q-1)p/q})|v_n - v|]dx \\
& + c(\varepsilon) \int_\Omega [(|w_n|^{r-1} + |y_n|^{(r-1)q/p})|u_n - u| + (|y_n|^{s-1} \\
& + |w_n|^{(s-1)p/q})|v_n - v|]dx \\
\leq\ & \varepsilon[(\|w_n\|_p^{p-1} + \|y_n\|_q^{(p-1)q/p})\|u_n - u\|_p + (\|y_n\|_q^{q-1} + \\
& \|w_n\|_p^{(q-1)p/q})\|v_n - v\|_q] + c(\varepsilon)[(\|w_n\|_r^{r-1} + \|y_n\|_s^{(r-1)q/p})\|u_n - u\|_r \\
& + (\|y_n\|_s^{s-1} + \|w_n\|_r^{(s-1)p/q})\|v_n - v\|_s].
\end{aligned}$$

Since $\{w_n\}$ and $\{y_n\}$ are bounded in $W^{1,p}_{0,G} \hookrightarrow L^p(\Omega) \cap L^r(\Omega)$ and $W^{1,q}(\Omega) \hookrightarrow L^q(\Omega) \cap L^s(\Omega)$, respectively, while $u_n \to u$ and $v_n \to v$ strongly in $L^r(\Omega)$ and $L^s(\Omega)$, respectively, choosing $\varepsilon > 0$ arbitrarily small, we obtain that $\mathcal{F}'(w_n, y_n)(u_n - u, v_n - v) \to 0$, as $n \to \infty$. But this contradicts (4.25). $\square$

Lemma 4.3 *Let $\lambda \geq 0$ be fixed and let $\{(u_n, v_n)\}$ be a bounded sequence in $W^{1,p,q}_{0,G}(\Omega)$ such that*

$$\|E'_{\lambda,G}(u_n, v_n)\|_\star \to 0$$

as $n \to \infty$. Then $\{(u_n, v_n)\}$ contains a strongly convergent subsequence in $W^{1,p,q}_{0,G}(\Omega)$.

Proof Because $W^{1,p,q}_{0,G}(\Omega)$ is a reflexive Banach space and $\{(u_n, v_n)\}$ is a bounded sequence, we can assume that

$$(u_n, v_n) \to (u, v) \text{ weakly in } W^{1,p,q}_{0,G}; \quad (4.26)$$

$$(u_n, v_n) \to (u, v) \text{ strongly in } L^r(\Omega) \times L^s(\Omega). \quad (4.27)$$

On the other hand, we have

$$E'_{\lambda,G}(u_n,v_n)(u-u_n,v-v_n)=\int_\Omega |\nabla u_n|^{p-2}\nabla u_n(\nabla u-\nabla u_n)$$

$$+\int_\Omega |\nabla v_n|^{q-2}\nabla v_n(\nabla v-\nabla v_n)-\lambda\mathcal{F}'_G(u_n,v_n)(u-u_n,v-v_n)$$

and

$$E'_{\lambda,G}(u,v)(u_n-u,v_n-v)=\int_\Omega |\nabla u|^{p-2}\nabla u(\nabla u_n-\nabla u)$$

$$+\int_\Omega |\nabla v|^{q-2}\nabla v(\nabla v_n-\nabla v)-\lambda\mathcal{F}'_G(u,v)(u_n-u,v_n-v).$$

Adding these two relations, one has

$$\begin{aligned}
a_n \quad &\overset{\text{not.}}{=} \quad \int_\Omega (|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u)(\nabla u_n-\nabla u)\\
&\qquad +\int_\Omega (|\nabla v_n|^{q-2}\nabla v_n-|\nabla v|^{q-2}\nabla v)(\nabla v_n-\nabla v)\\
&= \quad -E'_{\lambda,G}(u_n,v_n)(u-u_n,v-v_n)-E'_{\lambda,G}(u,v)(u_n-u,v_n-v)\\
&\qquad -\lambda\mathcal{F}'_G(u_n,v_n)(u-u_n,v-v_n)-\lambda\mathcal{F}'_G(u,v)(u_n-u,v_n-v).
\end{aligned}$$

Using (4.26) and (4.27), similar estimations as in Lemma 4.2 show that the last two terms tends to 0 as $n\to\infty$. Due to (4.26), the second terms tends to 0, while the inequality

$$|E'_{\lambda,G}(u_n,v_n)(u-u_n,v-v_n)|\le \|E'_{\lambda,G}(u_n,v_n)\|_\star\|(u-u_n,v-v_n)\|_{1,p,q}$$

and the assumption implies that the first term tends to 0 too. Thus,

$$\lim_{n\to\infty} a_n=0. \tag{4.28}$$

From the well-known inequality

$$|t-s|^\alpha\le\begin{cases}(|t|^{\alpha-2}t-|s|^{\alpha-2}s)(t-s), & \text{if } \alpha\ge 2,\\ ((|t|^{\alpha-2}t-|s|^{\alpha-2}s)(t-s))^{\alpha/2}(|t|^\alpha+|s|^\alpha)^{(2-\alpha)/2}, & \text{if } 1<\alpha<2,\end{cases}$$

for all $t,s\in\mathbb{R}^N$, and (4.28), we conclude that

$$\lim_{n\to\infty}\int_\Omega (|\nabla u_n-\nabla u|^p+|\nabla v_n-\nabla v|^q)=0,$$

hence, the sequence $\{(u_n,v_n)\}$ converges strongly to (u,v) in $W_G^{p,q}$. □

□

Theorem 4.3 *[255, Theorem 3] Let $(Z, \|\cdot\|)$ be a separable and reflexive real Banach space, let $I \subseteq \mathbb{R}$ be an interval, and let $g : Z \times I \to \mathbb{R}$ be a continuous function satisfying the following conditions:*

i) for every $z \in Z$, the function $g(z, \cdot)$ is concave;

ii) for every $\lambda \in I$, the function $g(\cdot, \lambda)$ is sequentially weakly lower semicontinuous and continuously Gâteaux differentiable, and satisfies the Palais-Smale condition, as well as

$$\lim_{\|z\|\to+\infty} g(z,\lambda) = +\infty;$$

iii) there exists a continuous concave function $h : I \to \mathbb{R}$ such that

$$\sup_{\lambda\in I}\inf_{z\in Z}(g(z,\lambda)+h(\lambda)) < \inf_{z\in Z}\sup_{\lambda\in I}(g(z,\lambda)+h(\lambda)).$$

Then, there exist an open interval $\Lambda \subseteq I$ and a number $\sigma > 0$ such that for each $\lambda \in \Lambda$, the function $g(\cdot, \lambda)$ has at least three critical points in X having the norms less than σ.

Proof of Theorem 4.2. We will show that the assumptions of Theorem 4.3 are fulfilled for: $Z = W^{1,p,q}_{0,G}(\Omega)$, $I = [0, \infty[$ and $g = E_{\lambda,G}$.

We fix $\lambda \geq 0$. It is clear that $W^{1,p,q}_{0,G}(\Omega) \ni (u, v) \mapsto \frac{1}{p}\|u\|^p_{1,p} + \frac{1}{q}\|v\|^q_{1,q}$ is sequentially weakly lower semicontinuous. Thus, from Lemma 4.2 it follows that $\mathcal{E}_\lambda(\cdot, \cdot)$ is also sequentially weakly lower semicontinuous.

First we prove that

$$\lim_{\|(u,v)\|\to\infty} E_{\lambda,G}(u, v) = +\infty. \tag{4.29}$$

Indeed, from **(F'4)**) and Hölder's inequality, one has

$$\begin{aligned} E_{\lambda,G}(u,v) &\geq \frac{1}{p}\|u\|^p_{1,p} + \frac{1}{q}\|v\|^q_{1,q} - \lambda\int_\Omega [a(x)|u|^{p_1} + b(x)|v|^{q_1} + c(x)]dx \\ &\geq \frac{1}{p}\|u\|^p_{1,p} + \frac{1}{q}\|v\|^q_{1,q} - \lambda[\|a\|_{\mu/(\mu-p_1)}\|u\|^{p_1}_\mu \\ &\quad +\|b\|_{\nu/(\nu-q_1)}\|v\|^{q_1}_\nu + \|c\|_1]. \end{aligned}$$

Since $W^{1,p}_{0,G}(\Omega) \hookrightarrow L^\mu(\Omega)$ and $W^{1,q}_{0,G}(\Omega) \hookrightarrow L^\nu(\Omega)$ are continuous, $p_1 < p$ and $q_1 < q$, then relation (4.29) yields immediately.

We prove that $E_{\lambda,G}(\cdot, \cdot)$ satisfies the Palais-Smale condition: Let $\{(u_n, v_n)\}$ be a sequence in $W^{1,p,q}_{0,G}(\Omega)$ such that $\sup_{n\in\mathbb{N}} |E_{\lambda,G}(u_n, v_n)| < +\infty$ and
$\lim_{n\to\infty} \|E'_{\lambda,G}(u_n, v_n)\|_\star = 0$. According to (4.29), $\{(u_n, v_n)\}$ must be bounded in $W^{1,p,q}_{0,G}(\Omega)$. The conclusion follows now by Lemma 4.3.

Now, we deal with *iii*): Let us define the function $f : (0, \infty) \to \mathbb{R}$ by

$$f(t) = \sup\{\mathcal{F}_G(u, v) : \frac{1}{p}\|u\|_{1,p}^p + \frac{1}{q}\|v\|_{1,q}^q \leq t\}.$$

After integration in the inequality from Lemma 4.1 *iii*), then by using the Young inequality, the fact that the embedding $W_{0,G}^{1,p}(\Omega) \hookrightarrow L^\mu(\Omega)$ and $W_{0,G}^{1,q}(\Omega) \hookrightarrow L^\nu(\Omega)$ are compact for $\mu \in]p, p^\star[$ and $\nu \in]q, q^\star[$ and the relation $ps = qr$, we have that for an arbitrary $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that

$$\mathcal{F}_G(u, v) \leq \varepsilon(\|u\|_{1,p}^p + \|v\|_{1,q}^q) + c(\varepsilon)(\|u\|_{1,p}^r + \|v\|_{1,q}^s) \tag{4.30}$$

for every $(u, v) \in W_{0,G}^{1,p,q}(\Omega)$. Since the function $x \mapsto (a^x + b^x)^{1/x}$, $x > 0$ is non-increasing ($a, b \geq 0$), and by using again $ps = qr$, one has that

$$\|u\|_{1,p}^r + \|v\|_{1,q}^s \leq \left[\|u\|_{1,p}^p + \|v\|_{1,q}^q\right]^{r/p}. \tag{4.31}$$

Therefore,

$$f(t) \leq \varepsilon \max\{p, q\}t + c(\varepsilon)(\max\{p, q\}t)^{\frac{r}{p}}, \ t > 0.$$

On the other hand, it is clear that $f(t) \geq 0$, $t > 0$. Taking into account the arbitrariness of $\varepsilon > 0$ and the fact that $r > p$, we conclude that

$$\lim_{t \to 0^+} \frac{f(t)}{t} = 0. \tag{4.32}$$

By **(F'5)** it is clear that $(u_0, v_0) \neq (0, 0)$ (note that $\mathcal{F}_G(0, 0) = 0$). Therefore, it is possible to choose a number η such that

$$0 < \eta < \mathcal{F}_G(u_0, v_0) \left[\frac{1}{p}\|u_0\|_{1,p}^p + \frac{1}{q}\|v_0\|_{1,q}^q\right]^{-1}.$$

Due to (4.32), there exists $t_0 \in \left(0, \frac{1}{p}\|u_0\|_{1,p}^p + \frac{1}{q}\|v_0\|_{1,q}^q\right)$ such that $f(t_0) < \eta t_0$. Thus, $f(t_0) < \mathcal{F}_G(u_0, v_0)t_0 \left[\frac{1}{p}\|u_0\|_{1,p}^p + \frac{1}{q}\|v_0\|_{1,q}^q\right]^{-1}$. Let $\rho_0 > 0$ be such that

$$f(t_0) < \rho_0 < \mathcal{F}_G(u_0, v_0)t_0 \left[\frac{1}{p}\|u_0\|_{1,p}^p + \frac{1}{q}\|v_0\|_{1,q}^q\right]^{-1}. \tag{4.33}$$

Define $h : I = [0, \infty) \to \mathbb{R}$ by $h(\lambda) = \rho_0\lambda$. We prove that h fulfils the inequality *iii*) from Theorem 4.3.

Due to the choice of t_0 and (4.33), one has

$$\rho_0 < \mathcal{F}_G(u_0, v_0). \tag{4.34}$$

The function

$$I \ni \lambda \mapsto \inf_{(u,v)\in W^{1,p,q}_{0,G}(\Omega)} \left[\frac{1}{p}\|u\|^p_{1,p} + \frac{1}{q}\|v\|^q_{1,q} + \lambda(\rho_0 - \mathcal{F}_G(u,v))\right]$$

is clearly upper semicontinuous on I. Thanks to (4.34), we have

$$\begin{aligned}\lim_{\lambda\to\infty} \inf_{(u,v)\in W^{1,p,q}_{0,G}(\Omega)} (E_{\lambda,G}(u,v) + \rho_0\lambda) \quad \le \quad & \lim_{\lambda\to\infty} \left[\frac{1}{p}\|u_0\|^p_{1,p} + \frac{1}{q}\|v_0\|^q_{1,q}\right. \\ & \left. +\lambda(\rho_0 - \mathcal{F}_G(u_0,v_0))\right] = -\infty.\end{aligned}$$

Thus we find an element $\overline{\lambda} \in I$ such that

$$\begin{aligned}&\sup_{\lambda\in I} \inf_{(u,v)\in W^{1,p,q}_{0,G}(\Omega)} (E_{\lambda,G}(u,v) + \rho_0\lambda) \qquad (4.35)\\ = & \inf_{(u,v)\in W^{1,p,q}_{0,G}(\Omega)} \left[\frac{1}{p}\|u\|^p_{1,p} + \frac{1}{q}\|v\|^q_{1,q} + \overline{\lambda}(\rho_0 - \mathcal{F}_G(u,v))\right].\end{aligned}$$

Since $f(t_0) < \rho_0$, for all $(u,v) \in W^{1,p,q}_{0,G}(\Omega)$ such that $\frac{1}{p}\|u\|^p_{1,p} + \frac{1}{q}\|v\|^q_{1,q} \le t_0$, we have $\mathcal{F}(u,v) < \rho_0$. Thus,

$$t_0 \le \inf\left\{\frac{1}{p}\|u\|^p_{1,p} + \frac{1}{q}\|v\|^q_{1,q} : \mathcal{F}_G(u,v) \ge \rho_0\right\}. \qquad (4.36)$$

On the other hand,

$$\begin{aligned}\inf_{(u,v)\in W^{1,p,q}_{0,G}(\Omega)} \sup_{\lambda\in I}(E_{\lambda,G}(u,v)+\rho_0\lambda) = \inf_{(u,v)\in W^{1,p,q}_{0,G}(\Omega)} \left[\frac{1}{p}\|u\|^p_{1,p} + \frac{1}{q}\|v\|^q_{1,q}\right.\\ \left.+\sup_{\lambda\in I}(\lambda(\rho_0 - \mathcal{F}_G(u,v)))\right] = \inf\left\{\frac{1}{p}\|u\|^p_{1,p} + \frac{1}{q}\|v\|^q_{1,q} : \mathcal{F}_G(u,v) \ge \rho_0\right\}.\end{aligned}$$

Thus, (4.36) is equivalent to

$$t_0 \le \inf_{(u,v)\in W^{1,p,q}_{0,G}(\Omega)} \sup_{\lambda\in I}(E_{\lambda,G}(u,v) + \rho_0\lambda). \qquad (4.37)$$

There are two distinct cases:

I) If $0 \le \overline{\lambda} < t_0/\rho_0$, we have

$$\begin{aligned}&\inf_{(u,v)\in W^{1,p,q}_{0,G}(\Omega)} \left[\frac{1}{p}\|u\|^p_{1,p} + \frac{1}{q}\|v\|^q_{1,q} + \overline{\lambda}(\rho_0 - \mathcal{F}_G(u,v))\right]\\ &\qquad \le E_{\overline{\lambda},G}(0,0) + \rho_0\overline{\lambda} = \overline{\lambda}\rho_0 < t_0.\end{aligned}$$

Combining the above inequality with (4.35) and (4.37), the desired relation from Theorem 4.3 *iii)* yields immediately.

II) If $t_0/\rho_0 \leq \overline{\lambda}$, from (4.34) and (4.33) we obtain

$$\begin{aligned}
&\inf_{(u,v)\in W^{1,p,q}_{0,G}(\Omega)} \left[\frac{1}{p}\|u\|^p_{1,p} + \frac{1}{q}\|v\|^q_{1,q} + \overline{\lambda}(\rho_0 - \mathcal{F}_G(u,v))\right] \\
&\leq \frac{1}{p}\|u_0\|^p_{1,p} + \frac{1}{q}\|v_0\|^q_{1,q} + \overline{\lambda}(\rho_0 - \mathcal{F}_G(u_0,v_0)) \\
&\leq \frac{1}{p}\|u_0\|^p_{1,p} + \frac{1}{q}\|v_0\|^q_{1,q} + \frac{t_0}{\rho_0}(\rho_0 - \mathcal{F}_G(u_0,v_0)) < t_0.
\end{aligned}$$

The conclusion holds similarly as in the first case.

Thus, the hypotheses of Theorem 4.3 are fulfilled. This implies the existence of an open interval $\Lambda \subset [0,\infty)$ and $\sigma > 0$ such that for all $\lambda \in \Lambda$ the function $E_{\lambda,G}(\cdot,\cdot)$ has at least three distinct critical points in $W^{1,p,q}_{0,G}(\Omega)$ (denote them by $(u^i_\lambda, v^i_\lambda)$, $i \in \{1,2,3\}$) and $\|(u^i_\lambda, v^i_\lambda)\| < \sigma$. Therefore $\|u^i_\lambda\|_{1,p} < \sigma$, $\|v^i_\lambda\|_{1,q} < \sigma$, $i \in \{1,2,3\}$.

Example 4.2 Let $\Omega = \omega \times \mathbb{R}^2$, where ω is a bounded open interval in $\mathbb{R}$. Let $\gamma : \Omega \to \mathbb{R}$ be a continuous, non-negative, not identically zero, axially symmetric function with compact support in Ω. Then, there exist an open interval $\Lambda \subset (0,\infty)$ and a number $\sigma > 0$ such that for every $\lambda \in \Lambda$, the system

$$\begin{cases}
-\Delta_{3/2}u = 5/2\lambda\gamma(x)|u|^{1/2}u\cos(|u|^{5/2} + |v|^3) & \text{in} \quad \Omega \\
-\Delta_{9/4}v = 3\lambda\gamma(x)|v|v\cos(|u|^{5/2} + |v|^3) & \text{in} \quad \Omega \\
u = v = 0 & \text{on} \quad \partial\Omega,
\end{cases}$$

has at least two distinct, nontrivial weak solutions with the properties from Theorem **??**.

Indeed, let us choose $F(x,u,v) = \gamma(x)\sin(|u|^{5/2} + |v|^3)$, $r = 11/4$, $s = 33/8$. **(F1)** and **(F'3)** hold immediately. For **(F'4)** we choose $a = b = 0$, $c = \gamma$. Since γ is an axially symmetric function, suppγ will be an $id \times \mathbf{O}(2)$-invariant set, i.e., if $(x,y) \in \text{supp}\gamma$, then $(x,gy) \in \text{supp}\gamma$ for every $g \in \mathbf{O}(2)$. Therefore, it is possible to fix an element $u_0 \in W^{1,3/2}_{0,id\times\mathbf{O}(2)}(\Omega)$ such that $u_0(x) = (\pi/2)^{2/5}$ for every $x \in \text{supp}\gamma$. Choosing $v_0 = 0$, one has that

$$\int_\Omega F(x,u_0(x),v_0(x))dx = \int_{\text{supp}\gamma} \gamma(x)\sin|u_0(x)|^{5/2}dx = \int_{\text{supp}\gamma} \gamma(x)dx > 0.$$

The conclusion follows from Theorem **??**. $\square$

4.5 Shift solutions for gradient systems

In the above two section we was supposed that the function $F : \Omega\times\mathbb{R}^2 \to \mathbb{R}$ is *superlinear* in $(0,0)$. If F is not superlinear in $(0,0)$ then the problem $(S^\lambda_{p,q})$ cannot be handled with minimax type results. Using recent ideas of B. Ricceri [257] it is possible to prove a multiplicity result obtaining shift solutions for problem $(S^\lambda_{p,q})$. Roughly speaking, under certain assumptions on the nonlinear term F we are able to show that the existence of a real parameter $\lambda > 0$, and of a pair $(u_0, v_0) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ such that the problem

$$((S^\lambda_{(u_0,v_0)}) : \begin{cases} -\Delta_p(\|u\|_{1,p}^{p'(q-1)}u) = \lambda F_u(x, u+u_0, v+v_0) \\ \\ -\Delta_q(\|u\|_{1,q}^{q'(p-1)}u) = \lambda F_v(x, u+u_0, v+v_0), \end{cases}$$

In this section we suppose that the function F has the following form $F(x,u,v) = b(x)G(u,v)$, where $G : \mathbb{R}^2 \to \mathbb{R}$ is of class C^1 and $b : \Omega \to \mathbb{R}$ is such that they fulfil the following conditions:
(**G**) $G(0,0) = 0$ and there exist real numbers $k > 0$, $p_1 \in]0, p-1[$, and $q_1 \in]0, q-1[$ with the following properties:

(i) For every $u, v \in \mathbb{R}$ we have $|G'_u(u,v)| \le k|u|^{p_1}$;
(ii) For every $u, v \in \mathbb{R}$ we have $|G'_v(u,v)| \le k|v|^{q_1}$.

(**b**) $b\colon \Omega \to [0, +\infty[$ belongs to $L^1(\Omega)\cap L^\infty(\Omega)$ and is not identically zero and $G = id^m \times O(N-m)$ invariant.

The problem $((S^\lambda_{(u_0,v_0)})$ can be reformulated in the following way: *For $(u_0, v_0) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ and $\lambda > 0$ we denote by $(S^\lambda_{(u_0,v_0)})$ the problem of finding $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ such that for every $(w_1, w_2) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ we have:*

$$\begin{cases} ||u-u_0||_{1,p}^{pq-p}\langle A_p(u-u_0)), w_1\rangle_p - \lambda \displaystyle\int_\Omega b(x)G'_u(u(x), v(x))w_1(x))dx = 0 \\ \\ ||v-v_0||_{1,q}^{pq-q}\langle A_q(v-v_0)), w_2\rangle_q - \lambda \displaystyle\int_\Omega b(x)G'_v(u(x), v(x))w_2(x))dx = 0. \end{cases}$$

In this case, the energy functional $E_{(u_0,v_0),\lambda} : W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega) \to R$ associated to the problem $(S^\lambda_{(u_0,v_0)})$ is given by:

$$E_{(u_0,v_0),\lambda}(u,v) = \frac{||u-u_0||_{1,p}^{pq} + ||v-v_0||_{1,q}^{pq}}{p\ q} - \lambda J(u,v), \qquad (4.38)$$

where $J\colon W_{0,G}^{1,p,q}(\Omega) \to \mathbb{R}$ is defined by

$$J(u,v) = \int_\Omega b(x)G(u(x),v(x))dx.$$

In standard way from the conditions (**G**) and (**b**) it follows that the function $E_{(u_0,v_0),\lambda}$ is of class C^1 and its differential is given by:

$$\begin{aligned}E'_{(u_0,v_0),\lambda}(u,v)(w_1,w_2) &= ||u-u_0||_{1,p}^{pq-p}\langle A_p(u-u_0),w_1\rangle_p + \\ &+||v-v_0||_{1,q}^{pq-q}\langle A_q(v-v_0),w_2\rangle_q - \lambda J'(u,v)(w_1,w_2),\end{aligned} \tag{4.39}$$

where

$$J'(u,v)(w_1,w_2) = \int_\Omega [b(x)G'_u(u,v)w_1(x) + b(x)G'_v(u,v)w_2(x)]dx, \tag{4.40}$$

for all $(w_1,w_2) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$. The critical points of the functional $E_{(u_0,v_0),\lambda}$ are the solutions of problem $((S^\lambda_{(u_0,v_0)})$.

In the sequel we will use the following notations:

(N1) $||\cdot||\colon W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \to \mathbb{R}$ denotes the homogenized Minkowski type norm defined by $||(u,v)|| = (||u||_{1,p}^{pq} + ||v||_{1,q}^{pq})^{\frac{1}{pq}}$, for every $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. Throughout in this section the space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ is considered to be endowed with this norm. We have that the space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ is a separable, uniformly convex, smooth real Banach space. Note that the following inequality holds for every $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$

$$||u||_{1,p} + ||v||_{1,q} \leq 2||(u,v)||. \tag{4.41}$$

(N2) According to the Lions compact embedding theorem, for every $r \in [p,p^\star]$ and $s \in [q,q^\star]$ the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ and $W_0^{1,q}(\Omega) \hookrightarrow L^s(\Omega)$ are compact. Therefore for the fixed numbers $r \in [p,p^\star]$ and $s \in [q,q^\star]$, there exists a positive real number c such that the following inequalities hold for every $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$

$$||u||_r \leq c||u||_{1,p}, \quad ||v||_s \leq c||v||_{1,q}. \tag{4.42}$$

(N3) $\nu_1 := \frac{r}{r-(p_1+1)}$, $\nu_2 := \frac{s}{s-(q_1+1)}$.

(N4) From condition (**G**) it follows the inequaliy:

$$|F(x,y) - F(\bar{x},\bar{y})| \leq k(|x|+|\bar{x}|)^{p_1}|x-\bar{x}| + k(|y|+|\bar{y}|)^{q_1}|y-\bar{y}|, \tag{4.43}$$

for every $(x,y),(\bar{x},\bar{y}) \in \mathbb{R}\times\mathbb{R}$.

In the next as in above two section we denote with $W^{1,p,q}_{0,G}(\Omega)$ the fixed points set of the action $G = id^m \times O(N-m)$ on $W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)$. The norms $\|\cdot\|_{1,p}, \|\cdot\|_{1,q}$ and $\|(\cdot,\cdot)\|$ restricted to $W^{1,p,q}_{0,G}(\Omega)$ we will denote in the same way and we introduce the following notations $J_G = J|_{W^{1,p,q}_{0,G}(\Omega)}, E_{(u_0,v_0),G} = E_{(u_0,v_0),\lambda}|_{W^{1,p,q}_{0,G}(\Omega)}$.

Proposition 4.3 *The function $J_G\colon W^{1,p,q}_{0,G}(\Omega) \to \mathbb{R}$ has the following properties:*

(a) *There exists a positive real number C such that the inequality*

$$|J_G(u,v)| \leq C(||u||^{p_1+1}_{1,p} + ||v||^{q_1+1}_{1,q})$$

holds for every pair $(u,v) \in W^{1,p,q}_{0,G}(\Omega)$.

(b) *J_G is sequentially weakly continuous.*

(c) *For every $(u,v),(w_1,w_2) \in W^{1,p,q}_{0,G}(\Omega)$ we have*

$$|J'_G(u,v)(w_1,w_2)| \leq k(||b||_{\nu_1}||u||^{p_1}_r||w_1||_r + ||b||_{\nu_2}||v||^{q_1}_s||w_2||_s). \tag{4.44}$$

Proof (a) From (4.43) it follows that

$$|F(u(x),v(x))| \leq k|u(x)|^{p_1+1} + k|v(x)|^{q_1+1}$$

for every $x \in \Omega$. Thus

$$|b(x)F(u(x),v(x))| \leq k|b(x)||u(x)|^{p_1+1} + k|b(x)||v(x)|^{q_1+1}. \tag{4.45}$$

Using (4.45) and Hölder's inequality, we get

$$|J_G(u,v)| \leq k||b||_{\nu_1}||u||^{p_1+1}_r + k||b||_{\nu_2}||v||^{q_1+1}_s.$$

Hence, in view of (4.42), we obtain

$$|J_G(u,v)| \leq k||b||_{\nu_1}c^{p_1+1}||u||^{p_1+1}_{1,p} + k||b||_{\nu_2}c^{q_1+1}||v||^{q_1+1}_{1,q}.$$

By taking $C = \max\{k||b||_{\nu_1}c^{p_1+1}, k||b||_{\nu_2}c^{q_1+1}\}$, we obtain the asserted inequality.

(b) In view of (4.43), for every $(u,v),(\bar{u},\bar{v}) \in W^{1,p,q}_{0,G}$ the following inequality holds

$$\begin{aligned}|J_G(u,v) - J_G(\bar{u},\bar{v})| \leq \quad & k\int_\Omega b(x)(|u(x)| + |\bar{u}(x)|)^{p_1}|u(x) - \bar{u}(x)|dx \\ & +k\int_\Omega b(x)(|v(x)| + |\bar{v}(x)|)^{q_1}|v(x) - \bar{v}(x)|dx.\end{aligned}$$

Using Hölder's inequality (note that $\frac{1}{\nu_1}+\frac{p_1}{r}+\frac{1}{r}=1$ and $\frac{1}{\nu_2}+\frac{q_1}{s}+\frac{1}{s}=1$), we get

$$\begin{aligned} |J_G(u,v)-J_G(\bar{u},\bar{v})| &\leq k||b||_{\nu_1}||(|u|+|\bar{u}|)||_r^{p_1}||u-\bar{u}||_r \\ &+ k||b||_{\nu_2}||(|v|+|\bar{v}|)||_s^{q_1}||v-\bar{v}||_s. \end{aligned} \tag{4.46}$$

Now, let $(u_n,v_n)_{i\in\mathbb{N}}\subset W_{0,G}^{1,p.q}(\Omega)$ be a sequence which converges weakly to $(u,v)\in W_{0,G}^{1,p.q}(\Omega)$.

From (4.46) we obtain that the following inequality holds for every $n\in\mathbb{N}$

$$\begin{aligned} |J_G(u_n,v_n)-J_G(u,v)| &\leq k2^{p_1}c^{p_1}M^{p_1}||b||_{\nu_1}||u_n-u||_r \\ &+ k2^{q_1}c^{q_1}M^{q_1}||b||_{\nu_2}||v_n-v||_s. \end{aligned} \tag{4.47}$$

Because the embedding $W_{0,G}^{1,p}(\Omega)\hookrightarrow L^r(\Omega)$ and $W_{0,G}^{1,q}(\Omega)\hookrightarrow L^s(\Omega)$ are compact, follow that $u_n\to u$ and $v_n\to v$ strongly in $L^r(\Omega)$ and $L^s(\Omega)$ respectively. From (4.47) follows that $\lim\limits_{n\to\infty} J(u_n,v_n)=J(u,v)$, which prove the assertion.

(c) Pick $(u,v),(w_1,w_2)\in W_{0,G}^{1,p,q}(\Omega)$. From the Mean Value Theorem and (**G**) we have:

$$|F'((u(x),v(x))(w_1(x),w_2(x))|\leq k|u(x)|^{p_1}|w_1(x)|+k|v(x)|^{q_1}|w_2(x)|. \tag{4.48}$$

From Hölder's inequality it follows that

$$|J_G'(u,v)(w_1,w_2)|\leq k(||b||_{\nu_1}||u||_r^{p_1}||w_1||_r+||b||_{\nu_2}||v||_s^{q_1}||w_2||_s).$$

□

Proposition 4.4 *Let $(u_0,v_0)\in W_{0,G}^{1,p,q}(\Omega)$ and $\lambda>0$. The energy functional $E_{(u_0,v_0),G}$ satisfies the following conditions:*

(a) *$E_{(u_0,v_0),G}$ is coercive.*
(b) *$E_{(u_0,v_0),G}$ verifies the Palais-Smale condition.*
(c) *$E_{(u_0,v_0),G}$ is weakly lower semicontinuous.*

Proof (a) According to assertion (a) of Proposition 4.3, the following inequality holds for every $(u,v)\in W_{0,G}^{1,p,q}(\Omega)$

$$\frac{||u-u_0||_{1,p}^{pq}}{pq}-\lambda C||u||_{1,p}^{p_1+1}+\frac{||v-v_0||_{1,q}^{pq}}{pq}-\lambda C||v||_{1,q}^{q_1+1}\leq E_{(u_0,v_0),G}(u,v). \tag{4.49}$$

Define $g\colon W^{1,p}_{0,G}(\Omega) \to \mathbb{R}$ and $h\colon W^{1,q}_{0,G}(\Omega) \to \mathbb{R}$ by

$$g(u) = \frac{||u-u_0||^{pq}_{1,p}}{pq} - \lambda C||u||^{p_1+1}_{1,p} \text{ and } h(v) = \frac{||v-v_0||^{pq}_{1,q}}{pq} - \lambda C||v||^{q_1+1}_{1,q}.$$

Using these notations, (4.49) can be rewritten as

$$g(u) + h(v) \le E_{(u_0,v_0),G}(u,v), \text{ for all } (u,v) \in W^{1,p,q}_{0,G}(\Omega). \tag{4.50}$$

For $u \neq 0$ we have

$$\begin{aligned} g(u) &= ||u||^p_{1,p}\left(\frac{||u-u_0||^{pq}_{1,p}}{pq||u||^p_{1,p}} - \lambda C||u||^{p_1+1-p}_{1,p}\right) \\ &= ||u||^p_{1,p}\left(\frac{||u-u_0||^{p}_{1,p}}{pq||u||^p_{1,p}}||u-u_0||^{pq-p}_{1,p} - \lambda C||u||^{p_1+1-p}_{1,p}\right). \end{aligned}$$

Since $\lim\limits_{||u||_{1,p}\to\infty} \dfrac{||u-u_0||^p_{1,p}}{||u||^p_{1,p}} = 1$, $\lim\limits_{||u||_{1,p}\to\infty} ||u-u_0||^{pq-p}_{1,p} = +\infty$ (because $pq-p>0$), and $\lim\limits_{||u||_{1,p}\to\infty} ||u||^{p_1+1-p}_{1,p} = 0$ (recall that $p_1 < p-1$), we obtain that $\lim\limits_{||u||_{1,p}\to\infty} g(u) = +\infty$. Hence, g is coercive. A similar argument yields that h is coercive. Since g and h are continuous, it follows from (4.50) that $E_{(u_0,v_0),G}$ is coercive.

(b) Let $\{(u_n,v_n)\}$ be a sequence (PS) sequence in $W^{1,p,q}_{0,G}(\Omega)$ for the function $E_{(u_0,v_0),G}$, that is

(PS_1) $E_{(u_0,v_0),G}(u_n,v_n)$ is bounded,

(PS_2) $E'_{(u_0,v_0),G}(u_n,v_n) \to 0$.

From the coercivity of the function $E_{(u_0,v_0),G}$ follows that the sequence $\{(u_n,v_n)\}$ is bounded in $W^{1,p,q}_{0,G}(\Omega)$. As in Lemma 4.3 with some mirror modifications we obtain that the function $E_{(u_0,v_0),G}$ satisfies the (PS) condition.

(c) The map $(u,v) \in W^{1,p,q}_{0,G}(\Omega) \longmapsto \frac{||(u,v)-(u_0,v_0)||^{pq}}{pq} \in \mathbb{R}$ is weakly lower semicontinuous, since all its lower level sets are weakly closed (recall that $W^{1,p,q}_{0,G}(\Omega)$ is reflexive, since it is uniformly convex). The map $(u,v) \in W^{1,p,q}_{0,G}(\Omega) \longmapsto -\lambda J_G(u,v) \in \mathbb{R}$ is sequentially weakly continuous (according to assertion (b) of Proposition 4.3), hence the map $E_{(u_0,v_0),G}$ is sequentially weakly lower semicontinuous. Since $E_{(u_0,v_0),G}$ is coercive, the Eberlein-Smulyan theorem implies that $E_{(u_0,v_0),G}$ is weakly lower semicontinuous. $\square$

T The main result of this section is the following.

Theorem 4.4 *Assume that the hypotheses* (**G**) *and* (**b**) *hold, and that the function* J_G *is not constant. Then, for every* $\sigma \in]\inf_{W^{1,p,q}_{0,G}(\Omega)} J_G, \sup_{W^{1,p,q}_{0,G}(\Omega)} J_G[$ *and every* $(u_0, v_0) \in J^{-1}(]-\infty, \sigma[)$*, one of the following alternatives is true:*

(A_1) *There exists* $\lambda > 0$ *such that the function* $E_{(u_0,v_0),G}$ *has at least three critical points in* $W^{1,p,q}_{0,G}(\Omega)$*.*

(A_2) *There exists* $(u^*, v^*) \in J_G^{-1}(\sigma)$ *such that, for all* $(u, v) \in J_G^{-1}([\sigma, +\infty[) \setminus \{(u^*, v^*)\}$*, the inequality*

$$\|(u,v) - (u_0, v_0)\| > ||(u^*, v^*) - (u_0, v_0)||$$

holds.

Proof Fix $\sigma \in]\inf_{W^{1,p,q}_{0,G}(\Omega)} J_G, \sup_{W^{1,p,q}_{0,G}(\Omega)} J_G[$, $(u_0, v_0) \in J^{-1}(]-\infty, \sigma[)$, and assume that ($A_1$) does not hold. This implies that for every $\lambda > 0$ the function $E_{(u_0,v_0),G}$ has at most one local minimum in $W^{1,p,q}_{0,G}(\Omega)$. Indeed, the existence of two local minima of $E_{(u_0,v_0),G}$ would imply, by the mountain pass theorem of zero altitude 1.8 (recall that, according to assertion (b) of Proposition 4.4, the function $E_{(u_0,v_0),G}$ satisfies the Palais-Smale condition) that this function has a third critical point. This would contradict our assumption that (A_1) does not hold.

We are going to prove that in this case (A_2) is satisfied. For this purpose we apply the following result that is due to Ricceri ([257], Theorem 1).

Theorem 4.5 *Let* X *be a topological space,* Λ *a real interval, and* $f : X \times \Lambda \to \mathbb{R}$ *a function satisfying the following conditions:*

(i) *For every* $x \in X$*, the function* $f(x, \cdot)$ *is quasi–concave and continuous.*

(ii) *For every* $\lambda \in \Lambda$*, the function* $f(\cdot, \lambda)$ *is lower semicontinuous and each of its local minima is a global minimum.*

(iii) *There exist* $\rho_0 > \sup_\Lambda \inf_X f$ *and* $\lambda_0 \in \Lambda$ *such that* $\{x \in X \ : \ f(x, \lambda_0) \leq \rho_0\}$ *is compact.*

Then,

$$\sup_\Lambda \inf_X f = \inf_X \sup_\Lambda f.$$

Set $\Lambda := [0, +\infty[$ and define the map $f\colon W_{0,G}^{1,p,q}(\Omega) \times \Lambda \to \mathbb{R}$ by

$$f(u,v,\lambda) = E_{(u_0,v_0),G}(u,v) + \lambda\sigma = \frac{||(u,v)-(u_0,v_0)||^{pq}}{pq} + \lambda(\sigma - J_G(u,v)).$$

We consider the space $W_{0,G}^{1,p,q}(\Omega)$ be equipped with the weak topology. Our aim is to show that f satisfies the conditions (i), (ii), and (iii) of Theorem 4.5. Observe that for every $(u,v) \in W_{0,G}^{1,p,q}(\Omega)$ the map $f(u,v,\cdot)\colon \Lambda \to \mathbb{R}$ is afine, thus it is quasi-concave and continuous, that is, condition (i) is fulfilled. Next we prove that condition (ii) is also satisfied. For this fix a real number $\lambda \geq 0$. By assertion (c) of Proposition 4.4 we know that $f(\cdot,\lambda)$ is (weakly) lower semicontinuous. Assume now that $f(\cdot,\lambda)$ has a local minimum which is not global. Since $f(\cdot,\lambda)$ is coercive (by assertion (a) of Proposition 4.4) and weakly lower semicontinuous, it has a global minimum. Thus, $f(\cdot,\lambda)$ has at least two local minima. It follows that $E_{(u_0,v_0),G}$ has at least two local minima, too, which is impossible by the assumption and the remark made at the beginning of the proof. We conclude that condition (ii) holds. To prove that condition (iii) is also satisfied we show first that $\sup\limits_{\Lambda} \inf\limits_{W_{0,G}^{1,p,q}(\Omega)} f(u,v,\lambda) < +\infty$. For this choose $(u_1,v_1) \in W_{0,G}^{1,p,q}(\Omega)$ such that $\sigma < J(u_1,v_1)$. For every $\lambda \in \Lambda$ we have

$$\inf_{W_{0,G}^{1,p,q}(\Omega)} f(u,v,\lambda) \leq f(u_1,v_1,\lambda) \leq \frac{\|(u_1,v_1)-(u_0,v_0)\|^{pq}}{pq}, \tag{4.51}$$

hence $\sup\limits_{\Lambda} \inf\limits_{W_{0,G}^{1,p,q}(\Omega)} f(u,v,\lambda) < +\infty$. For every $\rho_0 > \sup\limits_{\Lambda} \inf\limits_{W_{0,G}^{1,p,q}(\Omega)} f(u,v,\lambda)$ the set $\{(u,v) \in W_{0,G}^{1,p,q}(\Omega) \mid f(u,v,0) \leq \rho_0\}$ is weakly compact, thus condition (iii) is satisfied. Applying Theorem 4.5, we obtain

$$\alpha := \sup_{\Lambda} \inf_{W_{0,G}^{1,p,q}(\Omega)} f = \inf_{W_{0,G}^{1,p,q}(\Omega)} \sup_{\Lambda} f. \tag{4.52}$$

Note that the function $\lambda \in \Lambda \longmapsto \inf\limits_{W_{0,G}^{1,p,q}(\Omega)} f(u,v,\lambda)$ is upper semicontinuous, since for every $(u,v) \in W_{0,G}^{1,p,q}(\Omega)$ the map $f(u,v,\cdot)$ is continuous. Also, the first inequality of (4.51) yields that

$$\lim_{\lambda\to+\infty} \inf_{W_{0,G}^{1,p,q}(\Omega)} f(u,v,\lambda) = -\infty.$$

Thus, the map $\lambda \in \Lambda \longmapsto \inf\limits_{W_{0,G}^{1,p,q}(\Omega)} f(u,v,\lambda)$ is upper bounded and attains its maximum at a point $\lambda^* \in \Lambda$. Hence,

$$\alpha = \sup_{\Lambda} \inf_{W^{1,p,q}_{0,G}(\Omega)} f = \inf_{W^{1,p,q}_{0,G}(\Omega)} \left(\frac{||(u,v)-(u_0,v_0)||^{pq}}{pq} + \lambda^*(\sigma - J_G(u,v)) \right). \tag{4.53}$$

On the other hand, we have for every $(u,v) \in W^{1,p,q}_{0,G}(\Omega)$ that

$$\sup_{\Lambda} f(u,v,\lambda) = \begin{cases} +\infty, & \text{if } \sigma > J_G(u,v) \\ \frac{||(u,v)-(u_0,v_0)||^{pq}}{pq}, & \text{if } \sigma \le J_G(u,v). \end{cases}$$

In view of (4.52) it follows that

$$\alpha = \inf_{J_G^{-1}([\sigma,+\infty[)} \left(\frac{||(u,v)-(u_0,v_0)||^{pq}}{pq} \right).$$

Since the map $(u,v) \in W^{1,p,q}_{0,G}(\Omega) \longmapsto \frac{||(u,v)-(u_0,v_0)||^{pq}}{pq} \in \mathbb{R}$ is weakly lower semicontinuous and coercive, and since the set $J_G^{-1}([\sigma,+\infty[)$ is weakly closed and nonempty, there exists a pair $(u^*,v^*) \in J_G^{-1}([\sigma,+\infty[)$ such that

$$\alpha = \frac{||(u^*,v^*)-(u_0,v_0)||^{pq}}{pq}. \tag{4.54}$$

Observe that $\lambda^* > 0$. Otherwise, $\lambda^* = 0$, which implies $\alpha = 0$, hence $(u_0,v_0) = (u^*,v^*)$, which is impossible since $J_G(u_0,v_0) < \sigma$.

The relations (4.53) and (4.54) yield that

$$\frac{||(u^*,v^*)-(u_0,v_0)||^{pq}}{pq} \le \frac{||(u^*,v^*)-(u_0,v_0)||^{pq}}{pq} + \lambda^*(\sigma - J_G(u^*,v^*)),$$

thus $J_G(u^*,v^*) = \sigma$. This implies (by (4.53)) that (u^*,v^*) is a global minimum of $E_{(u_0,v_0),G}$. By the assumption and the remark at the beginning of the proof this pair is the only global minimum of $E_{(u_0,v_0),G}$. Thus, for every pair $(u,v) \in J_G^{-1}([\sigma,+\infty[) \setminus \{(u^*,v^*)\}$, the inequality $\|(u,v)-(u_0,v_0)\| > ||(u^*,v^*)-(u_0,v_0)||$ holds. We conclude that (A_2) is satisfied. □

Corollary 4.1 *Assume that the hypotheses of* Theorem 4.4 *are fulfilled. If S is a convex dense subset of $W^{1,p,q}_{0,G}(\Omega)$, and if there exists*

$$\sigma \in]\inf_{W^{1,p,q}_{0,G}(\Omega)} J_G, \sup_{W^{1,p,q}_{0,G}(\Omega)} J_G[$$

such that the level set $J_G^{-1}([\sigma,+\infty[)$ is not convex, then there exist $(u_0,v_0) \in J_G^{-1}(]-\infty,\sigma[) \cap S$ and $\lambda > 0$ such that the function $E_{(u_0,v_0),G}$

has at least three critical points in $W^{1,p,q}_{0,G}(\Omega)$*, i.e., there exist* $(u_0, v_0) \in J_G^{-1}(]-\infty, \sigma[) \cap S$ *and* $\lambda > 0$ *such that problem* $((S^{\lambda}_{(u_0,v_0)})$ *has at least three solutions in* $W^{1,p,q}_{0,G}(\Omega)$.

Proof Since J_G is sequentially weakly continuous (by assertion (c) of Proposition 4.3), the level set $M := J^{-1}([\sigma, +\infty[)$ is sequentially weakly closed. Theorem A.1 yields the existence of pairwise distinct pairs $(u_0, v_0) \in S$, $(u_1, v_1), (u_2, v_2) \in M$ such that

$$||(u_1, v_1) - (u_0, v_0)|| = ||(u_2, v_2) - (u_0, v_0)|| = \inf_{(u,v)\in M} ||(u, v) - (u_0, v_0)||.$$

It follows that $(u_0, v_0) \notin M$, that is, $J_G(u_0, v_0) < \sigma$. Also, the above relations show that alternative (A_2) of Theorem 4.4 does not hold. Hence (A_1) must be satisfied, i.e., there exists $\lambda > 0$ such that the function $E_{(u_0,v_0),G}$ has at least three critical points in $W^{1,p,q}_{0,G}(\Omega)$. Using the **Principle of Symmetric Criticality**, we conclude that problem $((S^{\lambda}_{(u_0,v_0)})$ has at least three solutions in $W^{1,p}_0(\Omega) \times W^{1,p}_0(\Omega)$. $\square$

Remark. The assumption of Corollary 4.1 stating that there exists $\sigma \in]\inf_{W^{1,p,q}_{0,G}(\Omega)} J, \sup_{W^{1,p,q}_{0,G}(\Omega)} J[$ such that the level set $J^{-1}([\sigma, +\infty[)$ is not convex is equivalent to the fact that J is not qvasi-concave.

A direct application of Corollary 4.1 is the following.

Theorem 4.6 *Let* $G\colon \mathbb{R}^2 \to [0, +\infty[$ *be a function which is not quasi-concave and which satisfies condition* **(G)**. *Let* $b\colon \Omega \to [0, +\infty[$ *be a* G*-invariant function with* $b(0) > 0$ *and for which condition* **(b)** *holds, and assume that* S *is a convex dense subset of* $W^{1,p,q}_{0,G}(\Omega)$. *Then, there exist* $(u_0, v_0) \in S$ *and* $\lambda > 0$ *with the property that problem* $((S^{\lambda}_{(u_0,v_0)})$ *has at least three solutions in* $W^{1,p}_0(\Omega) \times W^{1,p}_0(\Omega)$.

Proof We are going to apply Corollary 4.1 to the spaces $W^{1,p}_{0,G}(\Omega)$ and $W^{1,q}_{0,G}(\Omega)$ and to the map $J|_{W^{1,p,q}_{0,G}(\Omega)}$. First we prove that $J|_{W^{1,p,q}_{0,G}(\Omega)}$ is not constant. For this we assume, by contradiction, that $J|_{W^{1,p,q}_{0,G}(\Omega)}(u, v) = J|_{W^{1,p,q}_{0,G}(\Omega)}(0, 0) = 0$ for every $(u, v) \in W^{1,p,q}_{0,G}(\Omega)$.
Since $b \geq 0$ and $F \geq 0$, it follows that $b(x)F(u(x), v(x)) = 0$ for every $(u, v) \in W^{1,p,q}_{0,G}(\Omega)$ and a.e. $x \in \Omega$. Since $b(0) > 0$, there exists a real number $R > 0$ such that the closed ball B centered in 0 with radius R is contained in Ω and $b(x) > 0$ for a.e. $x \in B$. Consequently, $G(u(x), v(x)) = 0$ for every $(u, v) \in W^{1,p,q}_{0,G}(\Omega)$ and a.e. $x \in B$. Since F

is not constant, there exist $s_0, t_0 \in \mathbb{R}$ such that $G(s_0, t_0) > 0$. Choosing $u \in W^{1,p}_{0,G}(\Omega)$ and $v \in W^{1,q}_{0,G}(\Omega)$ such that $u(x) = s_0$ and $v(x) = t_0$ for every $x \in B$, we get a contradiction.

We show that $J|_{W^{1,p,q}_{0,G}(\Omega)}$ is not quasi-concave. It follows from above that

$$t := \int_B b(x)dx > 0.$$

The fact that G is not quasi-concave implies the existence of a real number $\rho_0 \in]0, \sup_{\mathbb{R}\times\mathbb{R}} G[$ such that $G^{-1}([\rho_0, +\infty[)$ is not convex. Thus, we find $\rho \in \mathbb{R}$, $\alpha \in]0,1[$, and $(s_i, t_i) \in \mathbb{R}^2$, $i \in \{1,2,3\}$, with the following properties

$$(s_2, t_2) = \alpha(s_1, t_1) + (1-\alpha)(s_3, t_3),\ G(s_1, t_1) > \rho,\ G(s_3, t_3) > \rho,\ G(s_2, t_2) < \rho.$$

Let

$$M := \max\{F(x,y) \mid \ |x| \le \max\{|s_1|, |s_3|\},\ |y| \le \max\{|t_1|, |t_3|\}\}.$$

Choose $R_1 > R$ and $\varepsilon > 0$ such that

$$\|b\|_\infty M \text{meas}(A) < \varepsilon < t|F(s_i, t_i) - \rho|, \text{ for } i \in \{1,2,3\},$$

where $A := \{x \in \Omega : R < |x| < R_1\}$ and meas(A) stays for the Lebesgue measure of A. For $i \in \{1,3\}$ let $u_i, v_i \in C_c^\infty(\Omega)$ be G-invariant functions such that $||u_i||_\infty = |s_i|$, $||v_i||_\infty = |t_i|$,

$$u_i(x) = \begin{cases} s_i, & |x| \le R \\ \\ 0, & |x| \ge R_1, \end{cases} \quad \text{and} \quad v_i(x) = \begin{cases} t_i, & |x| \le R \\ \\ 0, & |x| \ge R_1. \end{cases}$$

Put $(u_2, v_2) := \alpha(u_1, v_1) + (1-\alpha)(u_3, v_3)$. It follows that $u_2, v_2 \in C_c^\infty(\Omega)$, $||u_2||_\infty \le \alpha|s_1| + (1-\alpha)|s_3|$, $||v_2||_\infty \le \alpha|t_1| + (1-\alpha)|t_3|$,

$$u_2(x) = \begin{cases} s_2, & |x| \le R \\ \\ 0, & |x| \ge R_1, \end{cases} \quad \text{and} \quad v_2(x) = \begin{cases} t_2, & |x| \le R \\ \\ 0, & |x| \ge R_1. \end{cases}$$

Let $i \in \{1,2,3\}$. Note that for every $x \in \Omega$ the following inequalities hold

$$|u_i(x)| \le ||u_i||_\infty \le \max\{|s_1|, |s_3|\} \quad \text{and} \quad |v_i(x)| \le ||v_i||_\infty \le \max\{|t_1|, |t_3|\},$$

thus

$$0 \le b(x)G(u_i(x), v_i(x)) \le ||b||_\infty M.$$

Since

$$\begin{aligned}
J|_{W^{1,p,q}_{0,G}(\Omega)}(u_i,v_i) &= \int_B b(x)G(u_i(x),v_i(x))dx + \int_A b(x)G(u_i(x),v_i(x))dx \\
&= G(s_i,t_i)t + \int_A b(x)G(u_i(x),v_i(x))dx,
\end{aligned}$$

we conclude that, for $i \in \{1,3\}$,

$$J|_{W^{1,p,q}_{0,G}(\Omega)}(u_i,v_i) \geq G(s_i,t_i)t - ||b||_\infty M\text{meas}(A) > G(s_i,t_i)t - \epsilon > t\rho.$$

and

$$J|_{W^{1,p,q}_{0,G}(\Omega)}(u_2,v_2) \leq G(s_2,t_2)t + ||b||_\infty M\text{meas}(A) < G(s_2,t_2)t + \epsilon < t\rho.$$

Thus $J|_{W^{1,p,q}_{0,G}(\Omega)}$ is not quasi-concave. Therefore, Corollary 4.1 yields the existence of a pair $(u_0,v_0) \in S$ and of a real number $\lambda > 0$ such that the function $E_{(u_0,v_0),G}|_{W^{1,p,q}_{0,G}(\Omega)}$ has at least three critical points in $W^{1,p,q}_{0,G}(\Omega)$. In order to ensure that these points are also critical points of $E_{(u_0,v_0),G}$, we have to show that $E_{(u_0,v_0),G}$ is G-invariant. For this pick arbitrary $(u,v) \in W^{1,p,q}_{0,G}(\Omega)$ and $g \in G$. Then,

$$\begin{aligned}
&E_{(u_0,v_0),G}(g\cdot u, g\cdot v) = \tfrac{||(g\cdot u,g\cdot v)-(u_0,v_0)||^{pq}}{pq} - \lambda J(g\cdot u, g\cdot v) \\
&= \tfrac{||(g\cdot u,g\cdot v)-(g\cdot u_0,g\cdot v_0)||^{pq}}{pq} - \lambda \int_\Omega b(x)G((g\cdot u)(x),(g\cdot v)(x))dx.
\end{aligned}$$

Using the formula for the change of variable, and taking into account that b is G-invariant and that the elements of G are orthogonal maps (hence the absolute value of the determinant of their matrices is 1), we conclude that

$$\begin{aligned}
E_{(u_0,v_0),G}(g\cdot u, g\cdot v) &= \frac{||(u,v)-(u_0,v_0)||^{pq}}{pq} - \lambda \int_\Omega b(x)G(u(x),v(x))dx \\
&= E_{(u_0,v_0),G}(u,v).
\end{aligned}$$

So, by Principle of Symmetric criticality, every critical point of $E_{(u_0,v_0),G}$ is also a critical point of $E_{(u_0,v_0),\lambda}$. The conclusion follows now from Corollary 4.1. □

Example 4.3 We give an example of a non-constant function $F\colon \mathbb{R}^2 \to [0,+\infty)$ which satisfies the conditions required in the hypotheses of Theorem 4.6. Let $\gamma, \delta \in \mathbb{R}$ be such that $1 < \gamma < p$ and $1 < \delta < q$. Define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$F(u,v) = |u|^\gamma + |v|^\delta.$$

Then, F is convex and of class C^1 and satisfies the condition **(G)**. The relations

$$F(0,0) = F\left(\frac{1}{2}(-1,0) + \frac{1}{2}(1,0)\right) = 0 < 1 = F(-1,0) = F(1,0)$$

show that F is not quasi-concave.

4.6 Comments and historical notes

In the last few years many paper is dedicated to study the existence and multiplicity of solutions for gradient, Hamiltonian and non-variational systems on bounded or unbounded domain. For this see the papers of de Figueiredo and his collaborators.

The motivation to investigate elliptic eigenvalue problems on such domains arises from Mathematical Physics, see for instance Amick [8], Amick and Toland [9]; the mathematical development was initiated by Esteban [107], Grossinho and Tersian [132], Fao and Zhao [110]. Many papers study the existence and multiplicity of solutions for elliptic systems defined on unbounded domains, see Bartsch and de Figueiredo [27], Bartsch and Wang [31], Costa [75], Dinu [90], Grossinho [131] and Kristály [163], [164], [167]. The main tools used in the aforementioned papers are based on Mountain Pass Theorems and on the symmetric version of Mountain Pass Theorem, when the energy functional associated to problem (S_λ) satisfies the (PS) condition.

5

Systems with arbitrary growth nonlinearities

> The universe is not required to be in perfect harmony with human ambition.
>
> Carl Sagan (1934–1996)

In this chapter we are dealing with the elliptic system

$$\begin{cases} -\Delta u = v^p & \text{in} \quad \Omega; \\ -\Delta v = f(u) & \text{in} \quad \Omega; \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $0 < p < \frac{2}{N-2}$, Ω is a bounded domain in $\mathbb{R}^N$, and the continuous nonlinear term f has an arbitrary growth near the origin or at infinity. Here and in the sequel, we use the notation $s^\alpha = \operatorname{sgn}(s)|s|^\alpha$, $\alpha > 0$.

5.1 Introduction

We consider the elliptic system

$$\begin{cases} -\Delta u = g(v) & \text{in} \quad \Omega; \\ -\Delta v = f(u) & \text{in} \quad \Omega; \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{S}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is an open bounded domain with smooth boundary, and $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions.

In the particular case when $g(s) = s^p$, $f(s) = s^q$ $(p, q > 1)$ and $N \geq 3$, system (S) has been widely studied replacing the usual criticality notion (i.e., $p, q \leq \frac{N+2}{N-2}$) by the so-called "critical hyperbola" which involves both parameters p and q, i.e., those pairs of points $(p, q) \in \mathbb{R}^2_+$ which verify

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N}. \tag{CH}$$

Points (p, q) on this curve meet the typical non-compactness phenomenon of Sobolev embeddings and non-existence of solutions for (S) has been pointed out by Mitidieri [211] and van der Vorst [287] via Pohozaev-type arguments. On the other hand, when

$$1 > \frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}, \tag{5.1}$$

the existence of nontrivial solutions for (S) has been proven by de Figueiredo and Felmer [112], Hulshof, Mitidieri and van der Vorst [141]. Note that the latter results work also for nonlinearities $g(s) \sim s^p$ and $f(s) \sim s^q$ as $|s| \to \infty$ with (p, q) fulfilling (5.1). The points verifying (5.1) form a proper region in the first quadrant of the (p, q)-plane situated below the critical hyperbola (CH). Note that (5.1) is verified for any $p, q > 1$ whenever $N = 2$.

In spite of the aforementioned results, the whole region below (CH) is far to be understood from the point of view of existence/multiplicity of solutions for (S). Via a Mountain Pass argument, de Figueiredo and Ruf [118] proved the existence of at least one nontrivial solution to the problem

$$\begin{cases} -\Delta u = v^p & \text{in} \quad \Omega; \\ -\Delta v = f(u) & \text{in} \quad \Omega; \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{$\tilde{S}$}$$

when

$$\begin{cases} 0 < p, & \text{if} \quad N = 2; \\ 0 < p < \frac{2}{N-2}, & \text{if} \quad N \geq 3, \end{cases} \tag{$\tilde{5.1}$}$$

and $f : \mathbb{R} \to \mathbb{R}$ has a suitable superlinear growth at infinity, formulated in the term of the Ambrosetti-Rabinowitz condition. Later, Salvatore [266] guaranteed via the Pohozaev's fibering method the existence of a whole sequence of solutions to $(\tilde{S})$ in a similar context as [118] assuming in addition that the nonlinear term f is odd. Note that in both papers (i.e., [118] and [266]) *no further growth restriction* is required on the nonlinear term f other than the Ambrosetti-Rabinowitz condition. This latter fact is not surprising taking into account that $(\tilde{5.1})$ is actually equivalent to

$$1 > \frac{1}{p+1} > 1 - \frac{2}{N},$$

which is nothing but a "degenerate" case of (5.1) putting formally $q = \infty$, i.e., the growth of f may be arbitrary large.

This chapter is divided into two parts, in both cases we guarantee the existence of infinitely many pairs of distinct solutions to the system $(\tilde{S})$ when $(\tilde{5.1})$ holds. In one hand, in Section 5.2 the nonlinear term f has an *oscillatory behaviour*. Moreover, the nonlinear term f may enjoy an arbitrary growth at infinity (resp., at zero) whenever it oscillates near the origin (resp., at infinity) in a suitable way. In addition, the size of our solutions reflects the oscillatory behaviour of the nonlinear term, see relations (5.6) and (5.12) below; namely, the solutions are small (resp., large) in L^∞-norm and in a suitably chosen Sobolev space whenever the nonlinearity f oscillates near the origin (resp., at infinity). We emphasize that no symmetry condition is required on f. These results are proved by means of Ricceri's variational principle, see Theorem 1.16. On the other hand, in Section 5.3 the odd nonlinear term f fulfills an Ambrosettti-Rabinowitz type condition and the one-parametric fibering method is used in order to prove the existence of a whole sequence of solutions to $(\tilde{S})$.

Note that system $(\tilde{S})$ is equivalent to the Poisson equation

$$\begin{cases} -\Delta(-\Delta u)^{\frac{1}{p}} = f(u) & \text{in} \quad \Omega; \\ u = \Delta u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{P}$$

The suitable functional space where solutions of (P) is going to be sought is

$$X = W^{2,\frac{p+1}{p}}(\Omega) \cap W_0^{1,\frac{p+1}{p}}(\Omega)$$

endowed with the norm

$$\|u\|_X = \left(\int_\Omega |\Delta u|^{\frac{p+1}{p}} dx \right)^{\frac{p}{p+1}}.$$

Due to $(\tilde{5.1})$ one has $\frac{p+1}{p} > 1 + \frac{N-2}{2} = \frac{N}{2}$, therefore $W^{2,\frac{p+1}{p}}(\Omega) \subset\subset C(\Omega)$, so

$$X \subset\subset C(\Omega). \tag{5.2}$$

For further use, we denote by $\kappa_0 > 0$ the best embedding constant of

$X \subset\subset C(\Omega)$. The energy functional associated to the Poisson problem (P) is $E : X \to \mathbb{R}$ defined by

$$E(u) = \frac{p}{p+1}\|u\|_X^{\frac{p+1}{p}} - \mathcal{F}(u) \quad \text{where} \quad \mathcal{F}(u) = \int_\Omega F(u(x))dx.$$

Due to (5.2), the functional E is well-defined, is of class C^1 on X and

$$E'(u)(w) = \int_\Omega (-\triangle u)^{\frac{1}{p}}(-\triangle w)dx - \int_\Omega f(u)wdx, \quad u, w \in X.$$

Note that if $u \in X$ is a critical point of E then it is a weak solution of problem (P); in such a case, the pair $(u, (-\Delta u)^{\frac{1}{p}}) \in X \times X$ is a weak solution of system $(\tilde{S})$. See also [118, Subsection 3.1] and [266, Proposition 2.1]. Moreover, standard regularity arguments show that the pair $(u, (-\Delta u)^{\frac{1}{p}}) \in X \times X$ is actually a strong solution of system $(\tilde{S})$, see [118].

5.2 Elliptic systems with oscillatory terms

First of all, we are going to construct a special element in the space X which will play a crucial role in our proofs. Let $x_0 \in \Omega$ and $R > 0$ be such that $B(x_0, R) \subset \Omega$; here and in the sequel, $B(x_0, a) = \{x \in \mathbb{R}^N : |x - x_0| < a\}$, $a > 0$. Let $0 < r < R$ be fixed. We consider the function $w : \Omega \to \mathbb{R}$ defined by

$$w(x) = \frac{\int_{-\infty}^{-|x-x_0|} \alpha(t)dt}{\int_{-R}^{-r} \alpha(t)dt}, \tag{5.3}$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is given by

$$\alpha(t) = \begin{cases} e^{\frac{1}{(t+R)(t+r)}}, & \text{if} \quad t \in]-R, -r[; \\ 0, & \text{if} \quad t \notin]-R, -r[. \end{cases}$$

It is clear that $w \in C_0^\infty(\Omega) \subset X$; moreover, $w \geq 0$, $\|w\|_\infty = 1$ and

$$w(x) = \begin{cases} 1, & \text{if} \quad x \in B(x_0, r); \\ 0, & \text{if} \quad x \in \Omega \setminus B(x_0, R). \end{cases} \tag{5.4}$$

Lemma 5.1 *Let $\{a_k\}$, $\{b_k\} \subset]0, \infty[$ be two sequences such that $a_k < b_k$, $\lim_{k\to\infty} a_k/b_k = 0$, and $\operatorname{sgn}(s)f(s) \leq 0$ for every $|s| \in [a_k, b_k]$. Let $s_k = (b_k/\kappa_0)^{\frac{p+1}{p}}$. Then,*

a) $\max_{[-b_k,b_k]} F = \max_{[-a_k,a_k]} F \equiv F(\overline{s}_k)$ *with $\overline{s}_k \in [-a_k, a_k]$.*

b) $\|\overline{s}_k w\|_X^{\frac{p+1}{p}} < s_k$ *for $k \in \mathbb{N}$ large enough.*

Proof *a*) It follows from the standard Mean Value Theorem and from the hypotheses that $\mathrm{sgn}(s)f(s) \le 0$ for every $|s| \in [a_k, b_k]$.

b) Since $\lim_{k\to\infty} a_k/b_k = 0$, we may fix $k_0 \in \mathbb{N}$ such that $a_k/b_k < \kappa_0^{-1}\|w\|_X^{-1}$ for $k > k_0$. Then, one has $\|\overline{s}_k w\|_X^{\frac{p+1}{p}} = |\overline{s}_k|^{\frac{p+1}{p}}\|w\|_X^{\frac{p+1}{p}} \le a_k^{\frac{p+1}{p}}\|w\|_X^{\frac{p+1}{p}} < (b_k/\kappa_0)^{\frac{p+1}{p}} = s_k$. □

As we pointed out, the results of this section are based on Ricceri's variational principle, see Theorem 1.16. In our framework concerning problem (P) (thus, system $(\tilde{S})$), $\Psi, \Phi : X \to \mathbb{R}$ are defined by

$$\Psi(u) = \|u\|_X^{\frac{p+1}{p}}, \quad \Phi(u) = -\mathcal{F}(u), \quad u \in X.$$

Standard arguments show that Ψ and Φ are sequentially weakly lower semicontinuous. The energy functional becomes $E = \frac{p}{p+1}\Psi + \Phi$. Moreover the function from (1.24) takes the form

$$\varphi(s) = \inf_{\|u\|_X^{p+1} < s^p} \frac{\sup\{\mathcal{F}(v) : \|v\|_X^{p+1} \le s^p\} - \mathcal{F}(u)}{s - \|u\|_X^{\frac{p+1}{p}}}, \quad s > 0. \tag{5.5}$$

Now, we are in the position to state our first result. Let $f \in C(\mathbb{R}, \mathbb{R})$ and $F(s) = \int_0^s f(t)dt$, $s \in \mathbb{R}$. We assume that:

(H_0^1) $-\infty < \liminf_{s\to 0} \frac{F(s)}{|s|^{\frac{p+1}{p}}} \le \limsup_{s\to 0} \frac{F(s)}{|s|^{\frac{p+1}{p}}} = +\infty$,

(H_0^2) there exist two sequences $\{a_k\}$ and $\{b_k\}$ in $]0, \infty[$ with $b_{k+1} < a_k < b_k$, $\lim_{k\to\infty} b_k = 0$ such that

$$\mathrm{sgn}(s)f(s) \le 0 \quad \text{for every } |s| \in [a_k, b_k], \text{ and}$$

(H_0^3) $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\lim_{k\to\infty} \frac{\max_{[-a_k, a_k]} F}{b_k^{\frac{p+1}{p}}} = 0$.

Remark 5.1 Hypotheses $(H_0^1) - (H_0^2)$ imply an oscillatory behaviour of f near the origin while (H_0^3) is a technical assumption which seems to be indispensable in our arguments.

In the sequel, we provide a concrete example when hypotheses $(H_0^1) - (H_0^3)$ are fulfilled. Let $a_k = k^{-k^{k+1}}$ and $b_k = k^{-k^k}$, $k \ge 2$ and $a_1 = 1$, $b_1 = 2$. It is clear that $b_{k+1} < a_k < b_k$, $\lim_{k\to\infty} a_k/b_k = 0$, and

$\lim_{k\to\infty} b_k = 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(s) = \begin{cases} \varphi_k\left(\frac{s-b_{k+1}}{a_k-b_{k+1}}\right), & s \in [b_{k+1}, a_k],\ k \geq 1; \\ 0, & s \in]a_k, b_k[,\ k \geq 1; \\ 0, & s \in]-\infty, 0]; \\ g(s), & s \in [2, \infty[, \end{cases}$$

where $g : [2, \infty[\to \mathbb{R}$ is *any* continuous function with $g(2) = 0$, and $\varphi_k : [0,1] \to [0, \infty[$ is a sequence of continuous functions such that $\varphi_k(0) = \varphi_k(1) = 0$ and there are some positive constants c_1 and c_2 such that

$$c_1(b_k^{\frac{2p+2}{p}} - b_{k+1}^{\frac{2p+2}{p}})(a_k - b_{k+1})^{-1} \leq \int_0^1 \varphi_k(s)ds \leq c_2(b_k^{\frac{p+2}{p}} - b_{k+1}^{\frac{p+2}{p}})(a_k - b_{k+1})^{-1}.$$

Note that $F(s) = 0$ for every $s \in]-\infty, 0]$ and F is non-decreasing on $[0,2]$, while $c_1 b_k^{\frac{2p+2}{p}} \leq F(a_k) = \max_{[-a_k, a_k]} F \leq c_2 b_k^{\frac{p+2}{p}}$. Due to these inequalities, each hypotheses from above are verified.

Theorem 5.1 *Assume that* $(\tilde{5.1})$ *holds and* $f \in C(\mathbb{R}, \mathbb{R})$ *fulfills* $(H_0^1) - (H_0^3)$. *Then, system* $(\tilde{S})$ *possesses a sequence* $\{(u_k, v_k)\} \subset X \times X$ *of distinct* (*strong*) *solutions which satisfy*

$$\lim_{k\to\infty} \|u_k\|_X = \lim_{k\to\infty} \|v_k\|_X = \lim_{k\to\infty} \|u_k\|_\infty = \lim_{k\to\infty} \|v_k\|_\infty = 0. \tag{5.6}$$

The proof of Theorem 5.1 is based on the following two lemmas; thus, we assume the hypotheses of Theorem 5.1 are fulfilled. Let $\{a_k\}$ and $\{b_k\}$ be as in the hypotheses. We recall from (1.25) that $\delta = \liminf_{s\to 0^+} \varphi(s)$ where φ comes from (5.5).

Lemma 5.2 $\delta = 0$.

Proof By definition, $\delta \geq 0$. Suppose that $\delta > 0$. By the first inequality of (H_0^1), there exist two positive numbers ℓ_0 and ϱ_0 such that

$$F(s) > -\ell_0 |s|^{\frac{p+1}{p}} \quad \text{for every} \quad s \in]-\varrho_0, \varrho_0[. \tag{5.7}$$

Furthermore, let s_k, $\overline{s}_k$ be as in Lemma 5.1 and let $\overline{w}_k = \overline{s}_k w \in E$, where w is defined in (5.3). By (H_0^3) and condition $\lim_{k\to\infty} \frac{\overline{s}_k}{b_k} = 0$ ($|\overline{s}_k| \leq a_k$), there exists $k_0 \in \mathbb{N}$ such that for $k > k_0$ we have

$$m(\Omega)\frac{F(\overline{s}_k)}{b_k^{\frac{p+1}{p}}} + \left(\frac{\delta}{2}\|w\|_X^{\frac{p+1}{p}} + m(\Omega)\ell_0\right)\left(\frac{|\overline{s}_k|}{b_k}\right)^{\frac{p+1}{p}} < \frac{\delta}{2\kappa_0^{\frac{p+1}{p}}}. \tag{5.8}$$

Let $v \in E$ be arbitrarily fixed with $\|v\|_X^{\frac{p+1}{p}} \leq s_k$. Thus, due to the embedding from (5.2), we have $\|v\|_\infty \leq b_k$. Due to Lemma 5.1 *a*), we obtain

$$F(v(x)) \leq \max_{[-b_k, b_k]} F = F(\overline{s}_k) \quad \text{for every } x \in \Omega.$$

Since $0 \leq |\overline{w}_k(x)| \leq |\overline{s}_k| < \varrho_0$ for large $k \in \mathbb{N}$ and for all $x \in \Omega$, taking into account (5.7) and (5.8), it follows that

$$\begin{aligned}
\sup_{\|v\|_X^{p+1} \leq s_k^p} \mathcal{F}(v) - \mathcal{F}(\overline{w}_k) &= \sup_{\|v\|_X^{\frac{p+1}{p}} \leq s_k} \int_\Omega F(v)dx - \int_\Omega F(\overline{w}_k)dx \\
&\leq m(\Omega)F(\overline{s}_k) + m(\Omega)\ell_0 |\overline{s}_k|^{\frac{p+1}{p}} \\
&< \frac{\delta}{2}(s_k - \|\overline{w}_k\|_X^{\frac{p+1}{p}}).
\end{aligned}$$

Since $\|\overline{w}_k\|_X^{\frac{p+1}{p}} < s_k$ (cf. Lemma 5.1 *b*)), and $s_k \to 0$ as $k \to \infty$, we obtain

$$\delta \leq \liminf_{k\to\infty} \varphi(s_k) \leq \liminf_{k\to\infty} \frac{\sup_{\|v\|_X^{p+1} \leq s_k^p} \mathcal{F}(v) - \mathcal{F}(\overline{w}_k)}{s_k - \|\overline{w}_k\|_X^{\frac{p+1}{p}}} \leq \frac{\delta}{2},$$

contradiction. This proves our claim. $\square$

Lemma 5.3 0 *is not a local minimum of* $E = \frac{p}{p+1}\Psi + \Phi$.

Proof Let $\ell_0 > 0$ and $\varrho_0 > 0$ from the proof of Lemma 5.2, and $x_0 \in \Omega$ and $r, R > 0$ from the definition of the function w, see (5.3). Let $\mathcal{L}_0 > 0$ be such that

$$r^N \omega_N \mathcal{L}_0 - \frac{p}{p+1}\|w\|_X^{\frac{p+1}{p}} - (R^N - r^n)\omega_N \ell_0 > 0, \tag{5.9}$$

where ω_N is the volume of the N-dimensional unit ball. By the right hand side of (H_0^1) we deduce the existence of a sequence $\{s_k^0\} \subset]-\varrho_0, \varrho_0[$ converging to zero such that

$$F(s_k^0) > \mathcal{L}_0 |s_k^0|^{\frac{p+1}{p}}. \tag{5.10}$$

Let $w_k^0 = s_k^0 w \in E$. Due to (5.4), (5.7), (5.9) and (5.10), we have

$$\begin{aligned} E(w_k^0) &= \frac{p}{p+1}\|w_k^0\|_X^{\frac{p+1}{p}} - \int_\Omega F(w_k^0) \\ &= \frac{p}{p+1}\|w\|_X^{\frac{p+1}{p}}|s_k^0|^{\frac{p+1}{p}} - \int_{B(x_0,r)} F(w_k^0) - \int_{B(x_0,R)\setminus B(x_0,r)} F(w_k^0) \\ &\leq \frac{p}{p+1}\|w\|_X^{\frac{p+1}{p}}|s_k^0|^{\frac{p+1}{p}} - F(s_k^0)m(B(x_0,r)) + \ell_0(m(B(0,R)) - m(B(0,r)))|s_k^0|^{\frac{p+1}{p}} \\ &\leq |s_k^0|^{\frac{p+1}{p}}\left(\frac{p}{p+1}\|w\|_X^{\frac{p+1}{p}} - r^N\omega_N\mathcal{L}_0 + (R^N - r^n)\omega_N\ell_0\right) \\ &< 0 = E(0). \end{aligned}$$

Since $\|w_k^0\|_X \to 0$ as $k \to \infty$, 0 is not a local minimum of E, as claimed. □

Proof of Theorem 5.1. Applying Theorem 1.16 with $\lambda = \frac{p}{p+1}$ (see Lemma 5.2), we can exclude condition $(A1)$ (see Lemma 5.3). Therefore there exists a sequence $\{u_k\} \subset X$ of pairwise distinct critical points of $E = \frac{p}{p+1}\Psi + \Phi$ such that

$$\lim_{k\to\infty} \|u_k\|_X = 0. \tag{5.11}$$

Thus, $\{(u_k, v_k)\} = \{(u_k, (-\Delta u_k)^{\frac{1}{p}})\} \subset X \times X$ is a sequence of distinct pairs of solutions to the system $(\tilde{S})$.

It remains to prove (5.6). First, due to (5.2) and (5.11), we have that $\lim_{k\to\infty} \|u_k\|_\infty = 0$. For every $k \in \mathbb{N}$ let $m_k \in [-\|u_k\|_\infty, \|u_k\|_\infty] =: J_k$ such that $|f(m_k)| = \max_{s\in J_k} |f(s)|$. Note that $\mathrm{diam} J_k \to 0$ as $k \to \infty$; thus $\lim_{k\to\infty} m_k = 0$ which implies that $\lim_{k\to\infty} f(m_k) = 0$. On the other hand, from the second equation of system $(\tilde{S})$ we have that

$$\|v_k\|_X^{\frac{p+1}{p}} = \int_\Omega |\Delta v_k|^{\frac{p+1}{p}} dx = \int_\Omega |f(u_k)|^{\frac{p+1}{p}} dx \leq |f(m_k)|^{\frac{p+1}{p}} m(\Omega),$$

which implies that $\lim_{k\to\infty} \|v_k\|_X = 0$. Using again (5.2) we have that $\lim_{k\to\infty} \|v_k\|_\infty = 0$.

In the sequel, we state a perfect counterpart of Theorem 5.1 when the nonlinearity f has an oscillation at infinity. We assume that:

(H_∞^1) $-\infty < \liminf_{|s|\to\infty} \frac{F(s)}{|s|^{\frac{p+1}{p}}} \leq \limsup_{|s|\to\infty} \frac{F(s)}{|s|^{\frac{p+1}{p}}} = +\infty$,

(H_∞^2) there exist two sequences $\{a_k\}$ and $\{b_k\}$ in $]0,\infty[$ with $a_k < b_k < a_{k+1}$ and $\lim_{k\to\infty} b_k = \infty$ such that

$$\mathrm{sgn}(s)f(s) \leq 0 \quad \text{for every } |s| \in [a_k, b_k], \text{ and}$$

(H_∞^3) $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\lim_{k\to\infty} \frac{\max_{[-a_k,a_k]} F}{b_k^{\frac{p+1}{p}}} = 0.$

Remark 5.2 Assumptions $(H_\infty^1)-(H_\infty^2)$ imply an oscillatory behaviour of f at infinity. A concrete example is described in the sequel when hypotheses $(H_\infty^1) - (H_\infty^3)$ are fulfilled. Let $a_k = k^{k^k}$ and $b_k = k^{k^{k+1}}$ $(k \geq 2)$ and $a_1 = 5$, $b_1 = 10$. Clearly, one has $a_k < b_k < a_{k+1}$, $\lim_{k\to\infty} a_k/b_k = 0$, and $\lim_{k\to\infty} b_k = \infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(s) = \begin{cases} \varphi_k\left(\frac{s-b_k}{a_{k+1}-b_k}\right), & s \in [b_k, a_{k+1}],\ k \geq 1; \\ 0, & s \in]a_k, b_k[,\ k \geq 1; \\ g(s), & s \in [-5, 5]; \\ 0, & s \in]-\infty, -5[, \end{cases}$$

where $g : [-5, 5] \to \mathbb{R}$ is *any* continuous function with $g(\pm 5) = 0$, and $\varphi_k : [0, 1] \to [0, \infty[$ is a sequence of continuous functions such that $\varphi_k(0) = \varphi_k(1) = 0$ and there are some constants $c_1, c_2 > 0$ such that

$$c_1(b_{k+1}^{\frac{3p+1}{3p}} - b_k^{\frac{3p+1}{3p}})(a_{k+1}-b_k)^{-1} \leq \int_0^1 \varphi_k(s)ds \leq c_2(b_{k+1}^{\frac{2p+1}{2p}} - b_k^{\frac{2p+1}{2p}})(a_{k+1}-b_k)^{-1}.$$

Note that $F(s) = 0$ for every $s \in]-\infty, -5]$ and F is non-decreasing on $[5, \infty[$. Moreover, for $k \in \mathbb{N}$ large enough we have

$$c_1(b_k^{\frac{3p+1}{3p}} - 10^{\frac{3p+1}{3p}}) + \int_0^5 g(s)ds \leq F(a_k) = \max_{[-a_k,a_k]} F \leq c_2(b_k^{\frac{2p+1}{2p}} - 10^{\frac{2p+1}{2p}}) + \int_0^5 g(s)ds.$$

Now, an easy computation shows the hypotheses from above are verified.

Theorem 5.2 *Assume that* $(\tilde{5.1})$ *holds and* $f \in C(\mathbb{R}, \mathbb{R})$ *fulfills* $(H_\infty^1) - (H_\infty^3)$. *Then, system* $(\tilde{S})$ *possesses a sequence* $\{(u_k, v_k)\} \subset X \times X$ *of distinct* (*strong*) *solutions which satisfy*

$$\lim_{k\to\infty} \|u_k\|_X = \lim_{k\to\infty} \|v_k\|_X = \lim_{k\to\infty} \|u_k\|_\infty = \lim_{k\to\infty} \|v_k\|_\infty = \infty. \quad (5.12)$$

The proof of Theorem 5.2 is similar to that of Theorem 5.1. Let $\{a_k\}$ and $\{b_k\}$ be from Theorem 5.2 and $\gamma = \liminf_{s\to+\infty} \varphi(s)$ from (1.25) where φ comes from (5.5).

Lemma 5.4 $\gamma = 0$.

Proof It is clear that $\gamma \geq 0$. Suppose that $\gamma > 0$. Due to the left-hand side of (H_∞^1), one can find two positive numbers ℓ_∞ and ϱ_∞ such that

$$F(s) > -\ell_\infty |s|^{\frac{p+1}{p}} \quad \text{for every} \quad |s| > \varrho_\infty. \tag{5.13}$$

Let s_k, $\overline{s}_k$ be as in Lemma 5.1. By the fact that $\lim_{k\to\infty} b_k = \infty$, hypothesis (H_∞^3) and condition $\lim_{k\to\infty} \frac{\overline{s}_k}{b_k} = 0$ $(-a_k \leq \overline{s}_k \leq a_k)$, there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$ we have

$$m(\Omega)\frac{\max_{[-\varrho_\infty,\varrho_\infty]}|F| + F(\overline{s}_k)}{b_k^{\frac{p+1}{p}}} + \left(\frac{\gamma}{2}\|w\|_X^{\frac{p+1}{p}} + m(\Omega)\ell_\infty\right)\left(\frac{|\overline{s}_k|}{b_k}\right)^{\frac{p+1}{p}} < \frac{\gamma}{2\kappa_0^{\frac{p+1}{p}}}. \tag{5.14}$$

Let $\overline{w}_k = \overline{s}_k w \in X$, where w is defined in (5.3). A similar estimation as in Lemma 5.2 gives throughout relations (5.13) and (5.14) that

$$\begin{aligned}
\sup_{\|v\|_X^{p+1} \leq s_k^p} \mathcal{F}(v) - \mathcal{F}(\overline{w}_k) &= \sup_{\|v\|_X^{\frac{p+1}{p}} \leq s_k} \int_\Omega F(v)dx - \int_\Omega F(\overline{w}_k)dx \\
&= \sup_{\|v\|_X^{\frac{p+1}{p}} \leq s_k} \int_\Omega F(v)dx - \int_{\{|w_k(x)|>\varrho_\infty\}} F(\overline{w}_k)dx \\
&\quad - \int_{\{|w_k(x)|\leq\varrho_\infty\}} F(\overline{w}_k)dx \\
&\leq m(\Omega)F(\overline{s}_k) + m(\Omega)\ell_\infty|\overline{s}_k|^{\frac{p+1}{p}} + m(\Omega)\max_{[-\varrho_\infty,\varrho_\infty]}|F| \\
&< \frac{\gamma}{2}(s_k - \|\overline{w}_k\|_X^{\frac{p+1}{p}}).
\end{aligned}$$

Since $s_k \to +\infty$,

$$\gamma \leq \liminf_{k\to\infty} \varphi(s_k) \leq \liminf_{k\to\infty} \frac{\sup_{\|v\|_X^{p+1}\leq s_k^p} \mathcal{F}(v) - \mathcal{F}(\overline{w}_k)}{s_k - \|\overline{w}_k\|_X^{\frac{p+1}{p}}} \leq \frac{\gamma}{2},$$

which contradicts $\gamma > 0$. $\square$

Lemma 5.5 $E = \frac{p}{p+1}\Psi + \Phi$ *is not bounded from below on* X.

Proof Let ℓ_∞ and ϱ_∞ from the proof of Lemma 5.4, and let $\mathcal{L}_\infty > 0$ be such that

$$r^N \omega_N \mathcal{L}_\infty - \frac{p}{p+1}\|w\|_X^{\frac{p+1}{p}} - (R^N - r^n)\omega_N \ell_\infty > 0, \tag{5.15}$$

where r and R are from the definition of the function w, see (5.3). By the second part of (H_∞^1) we deduce the existence of a sequence $\{s_k^\infty\} \subset \mathbb{R}$

with $\lim_{k\to\infty}|s_k^\infty| = \infty$ and

$$F(s_k^\infty) > \mathcal{L}_\infty |s_k^\infty|^{\frac{p+1}{p}}. \tag{5.16}$$

Let $w_k^\infty = s_k^\infty w \in X$. We clearly have that

$$E(w_k^\infty) = \frac{p}{p+1}\|w\|_X^{\frac{p+1}{p}} |s_k^\infty|^{\frac{p+1}{p}} - F(s_k^\infty)\omega_N r^N - \int_{B(x_0,R)\setminus B(x_0,r)} F(w_k^\infty).$$

For abbreviation, we choose the set $D = B(x_0,R)\setminus B(x_0,r)$. Then, on account of (5.13) we have

$$\begin{aligned}
\int_D F(w_k^\infty) &= \int_{D\cap\{|w_k^\infty(x)|>\varrho_\infty\}} F(w_k^\infty) + \int_{D\cap\{|w_k^\infty(x)|\le\varrho_\infty\}} F(w_k^\infty)\\
&\ge -\ell_\infty \int_{D\cap\{|w_k^\infty(x)|>\varrho_\infty\}} |w_k^\infty|^{\frac{p+1}{p}} - (R^n - r^N)\omega_N \max_{[-\varrho_\infty,\varrho_\infty]}|F|\\
&\ge -(R^n - r^N)\omega_N \left(\ell_\infty |s_k^\infty|^{\frac{p+1}{p}} + \max_{[-\varrho_\infty,\varrho_\infty]}|F|\right).
\end{aligned}$$

Consequently, due to (5.16) and the above estimation, we have

$$\begin{aligned}
E(w_k^\infty) \le\ & |s_k^\infty|^{\frac{p+1}{p}} \left(\frac{p}{p+1}\|w\|_X^{\frac{p+1}{p}} - r^N\omega_N\mathcal{L}_\infty + (R^N - r^n)\omega_N\ell_\infty\right)\\
&+(R^N - r^n)\omega_N \max_{[-\varrho_\infty,\varrho_\infty]}|F|.
\end{aligned}$$

Since $\lim_{k\to\infty}|s_k^\infty| = \infty$, due to (5.15), we have $\lim_{k\to\infty}E(w_k^\infty) = -\infty$; consequently, $\inf_X E = -\infty$. $\square$

Proof of Theorem 5.2. In Theorem 1.16 we may choose $\lambda = \frac{p}{p+1}$ (see Lemma 5.4). On account of Lemma 5.5 the alternative $(B1)$ can be excluded. Therefore, there exists a sequence $\{u_k\} \subset X$ of distinct critical points of $E = \frac{p}{p+1}\Psi + \Phi$ such that

$$\lim_{k\to\infty}\|u_k\|_X = \infty. \tag{5.17}$$

Thus, $\{(u_k, v_k)\} = \{(u_k, (-\Delta u_k)^{\frac{1}{p}})\} \subset X \times X$ is a sequence of distinct pairs of solutions to the system $(\tilde{S})$.

We now prove the rest of (5.12). Assume that for every $k \in \mathbb{N}$ we have $\|v_k\|_\infty \le M$ for some $M > 0$. In particular, from the first equation of system $(\tilde{S})$ we obtain that

$$\|u_k\|_X^{\frac{p+1}{p}} = \int_\Omega |\Delta u_k|^{\frac{p+1}{p}} dx = \int_\Omega |v_k|^{p+1} dx \le M^{p+1} m(\Omega),$$

which contradicts relation (5.17). Consequently, $\lim_{k\to\infty}\|v_k\|_\infty = \infty$. But, this fact and (5.2) give at once that $\lim_{k\to\infty}\|v_k\|_X = \infty$ as well.

Assume finally that for every $k \in \mathbb{N}$ we have $\|u_k\|_\infty \le M'$ for some $M' > 0$. The second equation of system $(\tilde{S})$ shows that

$$\|v_k\|_X^{\frac{p+1}{p}} = \int_\Omega |\Delta v_k|^{\frac{p+1}{p}} dx = \int_\Omega |f(u_k)|^{\frac{p+1}{p}} dx \le m(\Omega) \max_{s\in[-M',M']} |f(s)|^{\frac{p+1}{p}},$$

which contradicts the fact that $\lim_{k\to\infty} \|v_k\|_X = \infty$. The proof is complete.

5.3 Elliptic systems with mountain pass geometry

Similarly to the previous section, we consider again the system

$$\begin{cases} -\Delta u = v^p & \text{in} \quad \Omega; \\ -\Delta v = f(u) & \text{in} \quad \Omega; \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases} \qquad (\tilde{S})$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $F(s) = \displaystyle\int_0^s f(t)dt$. We assume that

(H_1) there exist constants $\theta > 1 + \dfrac{1}{p}$ and $s_0 \ge 0$ such that

$$0 < \theta F(s) \le f(s)s \text{ for all } |s| \ge s_0;$$

(H_2) $f(s) = o(s^{\frac{1}{p}})$ for $s \to 0$.

Remark 5.3 Note that hypothesis (H_1) is a sort of Ambrosetti-Rabinowitz condition, see [7]. In particular, (H_1) implies that at infinity, $|f|$ grows faster than $s \mapsto |s|^{\frac{1}{p}}$.

Let $S = \{u \in X : \|u\|_X = 1\}$. For any $v \in S$, we consider in $\lambda \in \mathbb{R}$ the algebraic equation

$$|\lambda|^{\frac{p+1}{p}} - \int_\Omega f(\lambda v)\lambda v dx = 0. \qquad (5.18)$$

Remark 5.4 Let $f(s) = s^q$ with $pq \ne 1$. A direct calculation implies that (5.18) has exactly two solutions $\lambda_\pm(v) = \pm\left(\displaystyle\int_\Omega |v|^{q+1}\right)^t$, where $t = \left(\dfrac{1}{p} - q\right)^{-1}$. Moreover, $\lambda_\pm(v) \in C^1(S)$.

In view of this remark, for the general case, it is natural to assume that

(H_S) there are selections $\lambda_\pm(v) \in C^1(S)$ among the solutions of (5.18).

The main result of this section reads as follows.

Theorem 5.3 *Assume that* $(\tilde{5.1})$ *holds and* $f \in C(\mathbb{R},\mathbb{R})$ *fulfills* (H_1), (H_2) *and* (H_S). *Then, system* $(\tilde{S})$ *possesses a nontrivial solution. If* f *is odd, then* $(\tilde{S})$ *has infinitely many pairs of solutions.*

To prove Theorem 5.3, we first show that (H_1) and (H_2) imply that for every $v \in S$, equation (5.18) has at least two opposite sign solutions $\lambda_-(v) < 0 < \lambda_+(v)$ as we anticipated in Remark 5.4 for the particular case $f(s) = s^q$ with $pq \neq 1$.

We associate with E a functional $\widetilde{E}$ defined on $\mathbb{R} \times X$ by

$$\widetilde{E}(\lambda, v) = E(\lambda v) = \frac{p}{p+1}|\lambda|^{\frac{p+1}{p}} \int_\Omega |\Delta v|^{\frac{p+1}{p}} dx - \int_\Omega F(\lambda v) dx.$$

Clearly, the restriction of $\widetilde{E}$ to $\mathbb{R} \times S$, still denoted by $\widetilde{E}$, becomes

$$\widetilde{E}(\lambda, v) = \frac{p}{p+1}|\lambda|^{\frac{p+1}{p}} - \int_\Omega F(\lambda v) dx.$$

On account of Theorem 1.28, one can prove that if $(\lambda, v) \in (\mathbb{R} \setminus \{0\}) \times S$ is a conditionally critical point of the functional $\widetilde{E}$ on $\mathbb{R} \times S$ then the vector $u = \lambda v$ is a nonzero critical point of the functional E, that is, $E'(u) = 0$. In particular, the conditionally critical point of the functional $\widetilde{E}$ implies that $\widetilde{E}'_\lambda(\lambda, v) = 0$, which takes the form

$$|\lambda|^{\frac{p+1}{p}-2}\lambda - \int_\Omega f(\lambda v) v dx = 0$$

or equivalently, for $\lambda \neq 0$,

$$|\lambda|^{\frac{p+1}{p}} - \int_\Omega f(\lambda v)\lambda v dx = 0. \tag{5.19}$$

Setting

$$\varphi_v(\lambda) = 1 - |\lambda|^{-\frac{p+1}{p}} \int_\Omega f(\lambda v)\lambda v dx,$$

the following lemma holds.

Lemma 5.6 *Assume that* $(\tilde{5.1})$ *holds and* $f \in C(\mathbb{R},\mathbb{R})$ *fulfills* (H_1), (H_2). *Then, for all* $v \in S$

(i) $\lim\limits_{\lambda \to 0} \varphi_v(\lambda) = 1$,

(ii) $\lim_{|\lambda|\to+\infty} \varphi_v(\lambda) = -\infty$.

Proof Fix $v \in S$ arbitrarily.

(i) By relation (5.2), v is a nontrivial bounded continuous function. Thus, there exists $M > 0$ such that $|v(x)| \leq M$ for all $x \in \Omega$. Now, by hypothesis (H_2), for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|s| \leq \delta$ implies $\dfrac{|f(s)|}{|s|^{\frac{1}{p}}} < \varepsilon$. Fixing λ small enough, $|\lambda| < \dfrac{\delta}{M}$, it follows that

$$|f(\lambda v(x))| < \varepsilon |\lambda v(x)|^{\frac{1}{p}} \text{ for any } x \in \Omega,$$

hence

$$\lim_{\lambda\to 0} |\lambda|^{-\frac{p+1}{p}} \int_\Omega f(\lambda v)\lambda v dx = 0.$$

Consequently,

$$\lim_{\lambda\to 0} \varphi_v(\lambda) = 1 - \lim_{\lambda\to 0} |\lambda|^{-\frac{p+1}{p}} \int_\Omega f(\lambda v)\lambda v dx = 1,$$

i.e., (i) holds.

(ii) One can write that

$$|\lambda|^{-\frac{p+1}{p}} \int_\Omega f(\lambda v)\lambda v dx = |\lambda|^{-\frac{p+1}{p}} \left\{ \int_{\Omega_\lambda^-} f(\lambda v)\lambda v dx + \int_{\Omega_\lambda^+} f(\lambda v)\lambda v dx \right\}, \tag{5.20}$$

where $\Omega_\lambda^- = \{x \in \Omega : \ |\lambda v(x)| < s_0\}$ and $\Omega_\lambda^+ = \{x \in \Omega : \ |\lambda v(x)| \geq s_0\}$, s_0 being the positive constant from (H_1). Clearly, the boundedness of $f(\lambda v)\lambda v$ on Ω_λ^- implies that

$$\lim_{|\lambda|\to+\infty} |\lambda|^{-\frac{p+1}{p}} \int_{\Omega_\lambda^-} f(\lambda v)\lambda v dx = 0 \tag{5.21}$$

while by (H_1) it follows that

$$|\lambda|^{-\frac{p+1}{p}} \int_{\Omega_\lambda^+} f(\lambda v)\lambda v dx \geq \theta |\lambda|^{-\frac{p+1}{p}} \int_{\Omega_\lambda^+} F(\lambda v) dx$$

$$\geq c_1 \theta |\lambda|^{-\frac{p+1}{p}} \int_{\Omega_\lambda^+} |\lambda v|^\theta dx.$$

Denoted by λ^* a real positive number such that $\Omega_{\lambda^*}^+ \neq \emptyset$. For $|\lambda| \geq \lambda^*$

it follows that $\Omega^+_{\lambda^*} \subset \Omega^+_\lambda$. Thus, one can fix a positive constant $c_2 > 0$ such that for $|\lambda| \geq \lambda^*$ we have

$$|\lambda|^{-\frac{p+1}{p}} \int_{\Omega^+_\lambda} f(\lambda v)\lambda v dx \geq c_1\theta |\lambda|^{\theta-\frac{p+1}{p}} \int_{\Omega^+_{\lambda^*}} |v|^\theta dx \geq c_2 |\lambda|^{\theta-\frac{p+1}{p}}.$$

As $\theta > \dfrac{p+1}{p}$, we conclude that

$$\lim_{|\lambda|\to+\infty} |\lambda|^{-\frac{p+1}{p}} \int_{\Omega^+_\lambda} f(\lambda v)\lambda v dx = +\infty. \tag{5.22}$$

Consequently,

$$\lim_{|\lambda|\to+\infty} \varphi_v(\lambda) = 1 - \lim_{|\lambda|\to+\infty} |\lambda|^{-\frac{p+1}{p}} \int_\Omega f(\lambda v)\lambda v dx = -\infty,$$

i.e., (ii) holds. $\square$

Lemma 5.6 implies that equation $\varphi_v(\lambda) = 0$ has at least two nontrivial opposite sign solutions for any $v \in S$. By regularity assumption (H_S) there exist two selections $\lambda_\pm(v)$ which are of class C^1 on S. Then, the functional $\widehat{E}_\pm : S \to \mathbb{R}$ defined by $\widehat{E}_\pm(v) = \widetilde{E}(\lambda_\pm(v), v)$ is of class C^1 and on account of (5.19), it becomes

$$\widehat{E}_\pm(v) = \frac{p}{p+1} \int_\Omega f(\lambda_\pm(v)v)\lambda_\pm(v) v dx - \int_\Omega F(\lambda_\pm(v)v)dx.$$

Proposition 5.1 *Each critical point $v \in S$ of $\widehat{E}_\pm$ provides a conditionally critical point $(\lambda_\pm(v), v) \in (\mathbb{R}\backslash\{0\})\times S$ of $\widetilde{E}$. In particular, for a minimum point $v \in S$ of $\widehat{E}_\pm$ relative to S, the point $(\lambda_\pm(v), v) \in (\mathbb{R}\backslash\{0\})\times S$ is a conditionally critical point of $\widetilde{E}$.*

Proof First of all, we notice that $\varphi_v(\lambda_\pm(v)) = 0$; therefore, $\widetilde{E}'_\lambda(\lambda_\pm(v), v) = 0$. On the other hand, by assumption, we have that

$$0 \in \widehat{E}'_\pm(v) + N_S(v). \tag{5.23}$$

Therefore, a simple calculation shows that $-\widetilde{E}'_v(\lambda_\pm(v), v) \in N_S(v)$. Consequently, we have that

$$-\widetilde{E}'(\lambda_\pm(v), v) \in \{0\} \times N_S(v) = N_{(\mathbb{R}\backslash\{0\})\times S}(\lambda_\pm(v), v),$$

i.e., $(\lambda_\pm(v), v) \in (\mathbb{R} \setminus \{0\}) \times S$ is a conditionally critical point of the functional $\widetilde{E}$.

Now, let $v \in S$ be a relative minimum of $\widehat{E}_\pm$ to S. In particular, this fact implies that

$$\langle \widehat{E}'_\pm(v), w\rangle = 0 \quad \text{for all } w \in T_v(S). \tag{5.24}$$

Here, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and X^*. Note that relation (5.24) is nothing but relation (5.23), see Appendix A. $\square$

In view of Theorem 1.28 and Proposition 5.1, it is enough to find critical points of $\widehat{E}_\pm$ on S. In the sequel, we deal only with the functional $\widehat{E}_+$; analogously, it is possible to consider the functional $\widehat{E}_-$.

Due to (H_1), the functional $\widehat{E}_+$ is bounded from below. Indeed, if $\Omega^+_{\lambda_+}$ and $\Omega^-_{\lambda_+}$ are defined as in Lemma 5.6, by (H_1), there exists a real constant M such that

$$\begin{aligned}
\widehat{E}_+(v) &= \int_\Omega \left(\frac{p}{p+1} f(\lambda_+(v)v)\lambda_+(v)v - F(\lambda_+(v)v)\right) dx \\
&= \int_{\Omega^-_{\lambda_+}} \left(\frac{p}{p+1} f(\lambda_+(v)v)\lambda_+(v)v - F(\lambda_+(v)v)\right) dx \\
&\quad + \int_{\Omega^+_{\lambda_+}} \left(\frac{p}{p+1} f(\lambda_+(v)v)\lambda_+(v)v - F(\lambda_+(v)v)\right) dx \\
&\geq M + \left(\frac{p}{p+1}\theta - 1\right) \int_{\Omega^+_{\lambda_+}} F(\lambda_+(v)v) dx \geq M.
\end{aligned}$$

Lemma 5.7 *Assume that* $(\tilde{5.1})$ *holds and* $f \in C(\mathbb{R}, \mathbb{R})$ *fulfills* (H_1), (H_2) *and* (H_S). *If* $\{v_k\} \subset S$ *is such that the sequence* $\{\widehat{E}_+(v_k)\}$ *is bounded from above, then the sequence* $\{\lambda_+(v_k)\}$ *is contained in a compact interval* $[c, C]$ *for some* $c, C > 0$.

Proof Let $\{v_k\} \subset S$ be such that $\{\widehat{E}_+(v_k)\}$ is bounded from above. By (5.2), up to subsequence,

$$v_k \rightharpoonup v \text{ weakly in } X \text{ and uniformly in } \Omega. \tag{5.25}$$

For simplicity, denote by $\{\lambda_k\}$ the sequence $\{\lambda_+(v_k)\}$.

We first assume by contradiction that, up to subsequence, $\lambda_k \to 0$. By (5.25), $\lambda_k v_k \to 0$ uniformly in Ω. Arguing as in the proof of (i) of Lemma 5.6, by (H_2) it follows that

$$\lim_{k\to\infty} \lambda_k^{-\frac{p+1}{p}} \int_\Omega f(\lambda_k v_k)\lambda_k v_k dx = 0,$$

hence, by (5.19) we easily obtain the contradiction, i.e., there exists $c > 0$ such that $\lambda_k \geq c$ for every $k \in \mathbb{N}$.

Now, we assume by contradiction, passing to a subsequence if necessarily, that $\lambda_k \to +\infty$. We introduce the sets

$$\Omega_k^- = \{x \in \Omega : |\lambda_k v_k(x)| < s_0\} \text{ and } \Omega_k^+ = \{x \in \Omega : |\lambda_k v_k(x)| \geq s_0\}.$$

By (5.19), for any $k \in \mathbb{N}$, we have that

$$\lambda_k^{-\frac{p+1}{p}} \int_{\Omega_k^-} f(\lambda_k v_k)\lambda_k v_k dx + \lambda_k^{-\frac{p+1}{p}} \int_{\Omega_k^+} f(\lambda_k v_k)\lambda_k v_k dx = 1. \tag{5.26}$$

Since the sequence $\{\lambda_k v_k\}$ is uniformly bounded on the set Ω_k^-, we have

$$\lim_{k\to\infty} \lambda_k^{-\frac{p+1}{p}} \int_{\Omega_k^-} f(\lambda_k v_k)\lambda_k v_k dx = 0 \tag{5.27}$$

and

$$\left\{ \int_{\Omega_k^-} F(\lambda_k v_k)dx \right\} \text{ is bounded.} \tag{5.28}$$

In particular, by (5.26) and (5.27) it follows that

$$\lim_{k\to\infty} \lambda_k^{-\frac{p+1}{p}} \int_{\Omega_k^+} f(\lambda_k v_k)\lambda_k v_k dx = 1.$$

Due to (H_1), we have

$$\int_{\Omega_k^+} f(\lambda_k v_k)\lambda_k v_k dx \geq \theta \int_{\Omega_k^+} F(\lambda_k v_k)dx.$$

Consequently, passing to a subsequence if necessarily, we have that

$$\lim_{k\to\infty} \lambda_k^{-\frac{p+1}{p}} \int_{\Omega_k^+} F(\lambda_k v_k)dx = l, \quad 0 \leq l \leq \frac{1}{\theta}. \tag{5.29}$$

On the other hand, we may observe that

$$\widehat{E}_+(v_k) = \lambda_k^{\frac{p+1}{p}} \left(\frac{p}{p+1} - \lambda_k^{-\frac{p+1}{p}} \int_{\Omega_k^+} F(\lambda_k v_k)dx \right) - \int_{\Omega_{\lambda_k}^-} F(\lambda_k v_k)dx.$$

Therefore, by $\theta > \dfrac{p+1}{p}$, (5.28) and (5.29) we obtain that $\lim_{k\to\infty} \widehat{E}_+(v_k) = +\infty$ which contradicts the boundedness of the sequence $\{\widehat{E}_+(v_k)\}$. $\square$

Proposition 5.2 *Assume that* $(\tilde{5.1})$ *holds and* $f \in C(\mathbb{R}, \mathbb{R})$ *fulfills* (H_1), (H_2) *and* (H_S). *Then* $\widehat{E}_\pm$ *attains its infimum on* S.

Proof We prove the statement only for $\widehat{E}_+$; in a similar way, we can prove that also the functional $\widehat{E}_-$ attains its infimum on S.

Let $\{v_k\} \subset S$ be a sequence such that $\widehat{E}_+(v_k) \to m_+ = \inf_S \widehat{E}_+$. There exists $\overline{v}_+ \in X \setminus \{0\}$ with the property $\|\overline{v}_+\|_X \leq 1$ such that, up to a subsequence, (5.25) holds. Due to Lemma 5.7 the corresponding sequence $\{\lambda_+(v_k)\}$ converges, up to subsequence, to a real number $\overline{\lambda}_+ > 0$. By using (5.25) and passing to the limit in (5.19), the pair $(\overline{\lambda}_+, \overline{v}_+)$ still solves (5.19). Thus, $\overline{\lambda}_+ = \lambda_+(\overline{v}_+)$ with $\overline{v}_+ \neq 0$. In conclusion, we have that $\lambda_+(v_k) \to \lambda_+(\overline{v}_+)$. Moreover, by (5.25), we conclude that

$$\widehat{E}_+(v_k) \to \widehat{E}_+(\overline{v}_+) = m_+.$$

It remains to prove that $\|\overline{v}_+\|_X = 1$. To see this, we argue by contradiction, i.e., we assume that $\|\overline{v}_+\|_X < 1$. For any $t > 0$, by using relation (5.19), we deduce that

$$\begin{aligned}
\frac{d}{dt}\widehat{E}_+(t\overline{v}_+) &= \frac{d}{dt}\left[\frac{p}{p+1}(\lambda_+(t\overline{v}_+))^{\frac{p+1}{p}} - \int_\Omega F(\lambda_+(t\overline{v}_+)t\overline{v}_+)dx\right] \\
&= \left[(\lambda_+(t\overline{v}_+))^{\frac{1}{p}} - \int_\Omega f(\lambda_+(t\overline{v}_+)t\overline{v}_+)t\overline{v}_+dx\right] \\
&\times \frac{d}{dt}\lambda_+(t\overline{v}_+)\overline{v}_+ - \int_\Omega f(\lambda_+(t\overline{v}_+)t\overline{v}_+)\lambda_+(t\overline{v}_+)\overline{v}_+dx \\
&= -\frac{1}{t}(\lambda_+(t\overline{v}_+))^{\frac{p+1}{p}} < 0.
\end{aligned}$$

Therefore, the function $t \mapsto \widehat{E}_+(t\overline{v}_+)$ is decreasing with respect to t, $t > 0$. Now, let $t_0 > 1$ such that $t_0\|\overline{v}_+\|_X = 1$. Then $t_0\overline{v}_+ \in S$, but $\widehat{E}_+(t_0\overline{v}_+) < \widehat{E}_+(\overline{v}_+) = m_+ = \inf_S \widehat{E}_+$, contradiction. □

Proof of Theorem 5.3. On account of Propositions 5.1, 5.2 and Theorem 1.28, we conclude that $\overline{u}_+ = \lambda_+(\overline{v}_+)\overline{v}_+$ and $\overline{u}_- = \lambda_-(\overline{v}_-)\overline{v}_-$ are two nontrivial critical points of E, thus, they correspond to two nontrivial pairs of solutions of system $(\tilde{S})$. Note that these pairs of solutions may coincide.

If f is odd, we necessarily have $\lambda_+(\overline{v}_+) = -\lambda_-(\overline{v}_-)$ and $\widehat{E} := \widehat{E}_+ = \widehat{E}_-$. Moreover, $\overline{v}_+ = \overline{v}_-$; consequently, $\overline{u}_+$ and $\overline{u}_- = -\overline{u}_+$ provide two opposite sign pairs of solutions to system $(\tilde{S})$. Moreover, $\widehat{E}$ is even, bounded from below, of class C^1 and it satisfies the (PS)-condition on S. Then, applying Theorem 1.10 with $M = S$, together with Example C.2, we conclude that $\widehat{E}$ has infinitely many critical points $\{v_k\} \subset S$ with $\lim_{k\to\infty} \widehat{E}(v_k) = \sup_S \widehat{E} = +\infty$. On account of the first part of Proposition 5.1, there exists a sequence of different conditionally critical points of $\widetilde{E}$. In view of Theorem 1.28, E has a sequence of geometrically different critical points $\pm u_1, \pm u_2, \ldots, \pm u_k, \ldots$ with $u_k(x) = \lambda(v_k)v_k$ such that $E(u_k) \to +\infty$. This concludes the proof.

5.4 Comments and perspectives

1. We assume $(\tilde{5.1})$ holds and consider the perturbed system

$$\begin{cases} -\Delta u = v^p & \text{in} \quad \Omega; \\ -\Delta v = f(u) + \varepsilon g(u) & \text{in} \quad \Omega; \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{$\tilde{S}_\varepsilon$}$$

where $g : \mathbb{R} \to \mathbb{R}$ is any continuous function with $g(0) = 0$. We predict that for every $n \in \mathbb{N}$ there exists $\varepsilon_n > 0$ such that for every $\varepsilon \in [-\varepsilon_n, \varepsilon_n]$, system $(\tilde{S}_\varepsilon)$ has at least n distinct pairs of solutions whenever $f : \mathbb{R} \to \mathbb{R}$ verifies the set of assumptions from Theorems 5.1 or 5.2. This statement is not unexpected taking into account the recent papers of Anello and Cordaro [12] and Kristály [168] where a prescribed number of solutions were guaranteed for certain elliptic problems of *scalar* type whenever the parameter in the front of the perturbation is small enough. In both papers (i.e., [12] and [168]) the uniform Lipschitz truncation function $h_a : \mathbb{R} \to \mathbb{R}$ $(a > 0)$, $h_a(s) = \min(a, \max(s, 0))$ plays a key role, fulfilling as well the so-called *Markovian property* concerning the superposition operators: for any $u \in W_0^{1,r}(\Omega)$ one also has $h_a \circ u \in W_0^{1,r}(\Omega)$, where $r > 1$. Note however that a similar property is not available any longer replacing the space $W_0^{1,r}(\Omega)$ by a higher order Sobolev space $W^{2,r}(\Omega)$; in particular, the Markovian property is not valid for $X = W^{2,\frac{p+1}{p}}(\Omega) \cap W_0^{1,\frac{p+1}{p}}(\Omega)$.

2. Based on $(\tilde{5.1})$, the embedding $X \subset\subset C(\Omega)$ is essential in our investigations, see the proof of Lemmas 5.2 and 5.4. Is it possible to obtain similar conclusions as in Theorems 5.1 and 5.2 by omitting $(\tilde{5.1})$ and considering the whole region below the critical hyperbola?

6

Scalar Field Equations with Oscillatory Terms

The mathematics are distinguished by a particular privilege, that is, in the course of ages, they may always advance and can never recede.

Edward Gibbon (1737–1794), *Decline and Fall of the Roman Empire*

In this chapter we study the existence of multiple solutions for two classes of quasilinear problems: a double eigenvalue problem for a scalar field equation and a scalar field system with generalized boundary conditions. The nonlinear boundary conditions studied here recover the standard Dirichlet, Neumann, or periodic boundary conditions.

6.1 Introduction

Let $h_p : \mathbb{R}^n \to \mathbb{R}^n$ be the homeomorphism defined by $h_p(x) = |x|^{p-2}x$ for all $x \in \mathbb{R}^N$, where $p > 1$ is fixed. Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^N$. For $T > 0$, let $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory mapping satisfying:

(F_1) for a.e. $t \in [0,T]$, the function $F(t,\cdot)$ is continuously differentiable;
(F_2) the mapping $F(\cdot,x) : [0,T] \to \mathbb{R}$ is measurable for each $x \in \mathbb{R}^n$, $F(\cdot,0) \in L^1(0,T)$, and for each $M > 0$ there exists $\alpha_M \in L^1(0,T)$ such that for all $x \in \mathbb{R}^n$ with $|x| < M$,

$$|\nabla F(t,x)| \le \alpha_M(t) \qquad \text{for a.e. } t \in [0,T];$$

Let $j : \mathbb{R}^n \times \mathbb{R}^n \to]-\infty,+\infty]$ be a function having the following properties:

(J_1) $D(j) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : j(x,y) < +\infty\} \neq \emptyset$ is a closed convex cone with $D(j) \neq \{(0,0)\}$;
(J_2) j is a convex and lower semi-continuous (shortly, l.s.c.) function.

The first problem studied in this chapter is the following.

Let $\gamma > 0$ be arbitrary. For λ, $\mu > 0$ we consider the following double eigenvalue problem involving the p-Laplace operator:

$$(P_{\lambda,\mu}) \qquad \begin{cases} -[h_p(u')]' + \gamma h_p(u) = \lambda \nabla F(t,u) \text{ a.e. } t \in [0,T], \\ \Big(h_p(u')(0), -h_p(u')(T)\Big) \in \mu \partial j(u(0), u(T)), \end{cases}$$

where $u : [0,T] \to \mathbb{R}^n$ is of class C^1 and $h_p(u')$ is absolutely continuous. Note, that $\nabla F(t,\eta)$ denotes the gradient of $F(t,\cdot)$ at $\eta \in \mathbb{R}^n$, while ∂j denotes the subdifferential of j in the sense of convex analysis.

Next, we assume that:
(C1) $F : \mathbb{R}^n \to \mathbb{R}$ is of class C^1,
(C2) $S \subset \mathbb{R}^n \times \mathbb{R}^n$ is a closed convex set wit $\{(x,x) \ : \ x \in \mathbb{R}^n\} \subset S$,
(C3) $\gamma_1, \ldots, \gamma_n \in L^\infty(]0,T[,\mathbb{R})$ are so that $\operatorname{essinf} \gamma_i > 0$, for $i = \overline{1,n}$ and put $\gamma = (\gamma_1, \ldots, \gamma_n)$,
(C4) $\alpha \in L^1(]0,T[,\mathbb{R})$ is so that $\alpha(t) \geq 0$ a.e. in $]0,T[$.

The second problem studied in this chapter is the following:

$$(S) \qquad \begin{cases} -[h_p(u')]' + \gamma \circ h_p(u) = \alpha(t) \nabla F(u) \\ (h_p(u')(0), -h_p(u')(T)) \in N_S(u(0), u(T)), \end{cases}$$

where $N_S(x,y)$ denotes the normal cone of S at $(x,y) \in S$.

6.2 Multiple solutions of a double eigenvalue problem

In this section we prove a multiplicity result for the problem $(P_{\lambda,\mu})$. For this purpose we suppose that the following conditions are fulfilled:

(F_3) $\displaystyle\lim_{|x|\to\infty} \frac{F(t,x) - F(t,0)}{|x|^p} \leq 0$ uniformly for a.e. $t \in [0,T]$.
(F_4) $\displaystyle\lim_{|x|\to 0} \frac{F(t,x) - F(t,0)}{|x|^p} \leq 0$ uniformly for a.e. $t \in [0,T]$;
(F_5) there exists $s_0 \in \mathbb{R}^N$ such that $\displaystyle\int_0^T \Big(F(t,s_0) - F(t,0)\Big)dt > 0$.

(J_3) $j(0,0) = 0$, $j(x,y) \geq 0$ for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

We consider the Sobolev space $W^{1,p} = W^{1,p}(0,T;\mathbb{R}^n)$ equipped with the norm

$$\|u\|_\eta = \left(\|u'\|_{L^p}^p + \eta\|u\|_{L^p}^p\right)^{1/p},$$

where $\eta > 0$, and denote by $\|\cdot\|_{L^p}$ the norm of $L^p = L^p(0,T;\mathbb{R}^n)$, that is,

$$\|u\|_{L^p} = \left(\int_0^T |u(t)|^p dt\right)^{1/p}.$$

We consider the space of continuous functions $C = C([0,T];\mathbb{R}^n)$ endowed with the norm

$$\|u\|_C = \max\{|u(t)| : t \in [0,T]\}.$$

For $\gamma > 0$, we consider $\varphi_\gamma : W^{1,p} \to \mathbb{R}$ defined by

$$\varphi_\gamma(u) := \frac{1}{p}\left(\|u'\|_{L^p}^p + \gamma\|u\|_{L^p}^p\right) \text{ for all } u \in W^{1,p}.$$

Note that φ_γ is convex, $\varphi_\gamma \in C^1(W^{1,p};\mathbb{R})$ and for all $u, v \in W^{1,p}$,

$$\langle\varphi_\gamma'(u), v\rangle = \int_0^T (h_p(u'), v')dt + \gamma\int_0^T (h_p(u), v)dt\,, \tag{6.1}$$

where $(\cdot,\cdot)$ denotes the usual inner product in $\mathbb{R}^n$.

We define the function $J : W^{1,p} \to]-\infty, +\infty]$ by

$$J(u) = j(u(0), u(T)) \text{ for all } u \in W^{1,p}.$$

Then J is a proper, convex and l.s.c. function. We also observe that

$$D(J) = \{u \in W^{1,p} : (u(0), u(T)) \in D(j)\}.$$

We consider the functional $\hat{\mathcal{F}} : C \to \mathbb{R}$ defined by

$$\hat{\mathcal{F}}(v) = -\int_0^T F(t,v)dt + \int_0^T F(t,0)dt \text{ for all } v \in C$$

and $\mathcal{F} : W^{1,p} \to \mathbb{R}$ defined by $\mathcal{F} = \hat{\mathcal{F}}\Big|_{W^{1,p}}$. The functional $\mathcal{F}$ is sequentially weakly continuous, since the embedding $W^{1,p} \hookrightarrow C$ is compact.

We also have for all $u, v \in W^{1,p}$,

$$\langle \mathcal{F}'(u), v\rangle = -\int_0^T (\nabla F(t,u), v)dt\,. \tag{6.2}$$

Note that for $1 \le r < p$ and $p < q < p^*$ the embeddings $L^p \hookrightarrow L^r, W^{1,p} \hookrightarrow L^q$, $W^{1,p} \hookrightarrow C$ are continuous, hence there exist constants $C_{r,p}, \hat{C}_{q,p}, \hat{c} > 0$ such that

$$\|u\|_{L^r} \le C_{r,p}\|u\|_{L^p}, \qquad \|u\|_{L^q} \le \hat{C}_{q,p}\|u\|_{W^{1,p}}, \quad \|u\|_C \le \hat{c}\|u\|_{W^{1,p}}$$

for all $u \in W^{1,p}$.

The energy functional $I : [0,\infty)\times[0,\infty)\times W^{1,p} \to]-\infty,\infty]$ associated to the problem $(P_{\lambda,\mu})$ is defined by

$$I(\lambda,\mu,u) = \varphi_\gamma(u) + \lambda\mathcal{F}(u) + \mu J(u).$$

The functional I is of Szulkin type and J' denote the derivative in the sense of Definition A.4.

The following result is due to Jebelean and Moroşanu [148], Proposition 3.2.

Proposition 6.1 *Assume that $F : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ satisfies (F_1) and (F_2) and $j : \mathbb{R}^n\times\mathbb{R}^n \to]-\infty,+\infty]$ satisfies (J_1) and (J_2). If $u \in W^{1,p}$ is a critical point of $I(\lambda,\mu,\cdot)$, then u is a solution of $(P_{\lambda,\mu})$.*

Proof Assume that $u \in W^{1,p}$ is a critical point of $I(\lambda,\mu,\cdot)$. Then, for all $v \in W^{1,p}$,

$$\lambda\langle \mathcal{F}'(u), v-u\rangle + \varphi_\gamma(v) - \varphi_\gamma(u) + \mu\Big(J(v) - J(u)\Big) \ge 0\,.$$

Let $w \in W^{1,p}$ and set $v = u + sw$, where $s > 0$. Dividing by s and then letting $s \to 0^+$, we obtain for all $w \in W^{1,p}$,

$$\lambda\langle \mathcal{F}'(u), w\rangle + \langle \varphi'_\gamma(u), w\rangle + \mu J'(u;w) \ge 0\,,$$

where $J'(u;w)$ means the derivative in the sense of Definition A.4.

By the definition of J we deduce that for every $w \in W^{1,p}$,

$$\lambda\langle \mathcal{F}'(u), w\rangle + \langle \varphi'_\gamma(u), w\rangle + \mu j'((u(0),u(T));(w(0),w(T))) \ge 0\,. \tag{6.3}$$

Since $C_0^\infty(0,T;\mathbb{R}^n) \subset W^{1,p}$, it follows that for all $w \in C_0^\infty(0,T;\mathbb{R}^n)$,

$$\lambda\langle \mathcal{F}'(u), w\rangle + \langle \varphi'_\gamma(u), w\rangle = 0\,.$$

Combining relations (6.1) and (6.2) we have for all $w \in C_0^\infty(0,T;\mathbb{R}^n)$,

$$\int_0^T (h_p(u'),w')dt = \int_0^T (-\gamma h_p(u) + \nabla F(t,u), w)dt\,. \tag{6.4}$$

For any $u \in W^{1,p}$, we obtain

$$h_p(u), h_p(u') \in L^{p'}(0,T;\mathbb{R}^n) \text{ with } \frac{1}{p}+\frac{1}{p'}=1. \tag{6.5}$$

But $\nabla F(\cdot,u) \in L^1(0,T)$ (by (F_2)), hence relations (6.4) and (6.5) yield $h_p(u') \in W^{1,1}$ and

$$-\Big(h_p(u')\Big)' = -\gamma h_p(u) + \nabla F(t,u) \text{ for a.e. } t \in [0,T]. \tag{6.6}$$

Since h_p is a homeomorphism and $h_p(u') \in W^{1,1}$, it follows that u is of class C^1.

In order to show that u satisfies the boundary condition of problem $(P_{\lambda,\mu})$ we note that by (6.3) and (6.6) we have for all $w \in W^{1,p}$,

$$\mu j'((u(0),u(T));(w(0),w(T))) \geq (h_p(u')(0),w(0)) - (h_p(u')(T),w(T))\,.$$

Therefore

$$\mu j'((u(0),u(T));(x,y)) \geq (h_p(u')(0),w(0)) - (h_p(u')(T),w(T))\,.$$

From Lemma A.2 it follows that

$$\Big(h_p(u')(0), -h_p(u')(T)\Big) \in \mu\partial j(u(0),u(T)).$$

We conclude that u is a solution of problem $(P_{\lambda,\mu})$. □

We introduce the constant $\eta_1 = \eta_1(p,\gamma) > 0$ by setting

$$\eta_1 = \inf\left\{\frac{\|u'\|_{L^p}^p + \gamma\|u\|_{L^p}^p}{\|u\|_{L^p}^p} : u \in W^{1,p}\setminus\{0\}, u \in D(J)\right\}.$$

Since $\eta_1 > 0$, we obtain

$$2^{-1/p}\|u\|_{\eta_1} \leq (\|u'\|_{L^p}^p + \gamma\|u\|_{L^p}^p)^{1/p} \leq \|u\|_{\eta_1} \text{ for all } u \in D(J). \tag{6.7}$$

Lemma 6.1 [148, Lemma 4.3] *Assume that $\{u_n\} \subset D(J)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p}$ and*

$$\liminf_{n\to\infty}\Big(\langle\varphi_\gamma'(u_n), u-u_n\rangle + \mu J'(u_n;u-u_n)\Big) \geq 0\,. \tag{6.8}$$

Then $\{u_n\}$ has a subsequence which converges strongly in $W^{1,p}$.

Proof Note that $u \in D(J)$, since the closed convex set $D(J)$ is weakly closed in $(W^{1,p}, \|\cdot\|_{\eta_1})$. Hence, $J'(u_n; u - u_n) < \infty$ for all $n \in \mathbb{N}$. Combining relation (6.8) and the lower semi-continuity of J, we have

$$\begin{aligned}
&\limsup_{n\to\infty}\langle\varphi'_\gamma(u_n), u_n - u\rangle \leq \\
&-\liminf_{n\to\infty}\Big(\langle\varphi'_\gamma(u_n), u - u_n\rangle + \mu J'(u_n; u - u_n)\Big) \\
&+\mu\limsup_{n\to\infty} J'(u_n; u - u_n) \\
&\leq \mu\limsup_{n\to\infty}(J(u) - J(u_n)) = \mu(J(u) - \liminf_{n\to\infty} J(u_n)) \leq 0.
\end{aligned}$$

Hence,

$$\limsup_{n\to\infty}\langle\varphi'_\gamma(u_n), u_n - u\rangle \leq 0. \tag{6.9}$$

From the expression (6.1) of φ'_γ we have

$$\begin{aligned}
\langle\varphi'_{\eta_1}(u_n), u_n - u\rangle &= \langle\varphi'_\gamma(u_n), u_n - u\rangle \\
&\quad +(\eta_1 - \gamma)\int_0^T (h_p(u_n), u_n - u)dt\,.
\end{aligned} \tag{6.10}$$

Taking into account that $u_n \rightharpoonup u$ weakly in $W^{1,p}$ and that the embedding $W^{1,p} \hookrightarrow C$ is compact, we obtain for a subsequence of $\{u_n\}$ (denoted again by $\{u_n\}$) that

$$\lim_{n\to\infty}\int_0^T (h_p(u_n), u_n - u)dt = 0. \tag{6.11}$$

Then, by relations (6.9), (6.10) and (6.11) we have

$$\limsup_{n\to\infty}\langle\varphi'_{\eta_1}(u_n), u_n - u\rangle \leq 0. \tag{6.12}$$

On the other hand, for each positive integer n,

$$0 \leq (\|u_n\|_{\eta_1}^{p-1} - \|u\|_{\eta_1}^{p-1})(\|u_n\|_{\eta_1} - \|u\|_{\eta_1}). \tag{6.13}$$

Using Hölder's inequality, we obtain

$$(\|u_n\|_{\eta_1}^{p-1} - \|u\|_{\eta_1}^{p-1})(\|u_n\|_{\eta_1} - \|u\|_{\eta_1}) \leq \langle\varphi'_{\eta_1}(u_n) - \varphi'_{\eta_1}(u), u_n - u\rangle.$$

Thus, by relation (6.13) we deduce that

$$\lim_{n\to\infty}\|u_n\|_{\eta_1} = \|u\|_{\eta_1}.$$

Since $(W^{1,p}, \|\cdot\|_{\eta_1})$ is uniformly convex, it follows that $\{u_n\}$ converges strongly to u. □

Remark 6.1 Let $\varepsilon > 0$ be arbitrary. From (F_1), (F_2) and (F_3) it follows that there exists $\delta_1 > 0$ (depending on ε) such that

$$F(t,x) - F(t,0) \le \varepsilon |x|^p + \alpha_{\delta_1}(t)\delta_1 \qquad \text{for all } x \in \mathbb{R}^n, \text{ a.e. } t \in [0,T].$$

Then

$$\mathcal{F}(u) \ge -\varepsilon \|u\|_{L^p}^p - \delta_1 \|\alpha_{\delta_1}\|_{L^1(0,T)} \qquad \text{for all } u \in W^{1,p}. \tag{6.14}$$

Proposition 6.2 *Assume that* $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *satisfies* (F_1), (F_2) *and* (F_3) *and that* $j : \mathbb{R}^n \times \mathbb{R}^n \to]-\infty, +\infty]$ *satisfies* (J_1) *and* (J_2). *Then the following properties hold:*

(1) $I(\lambda, \mu, \cdot)$ *is weakly sequentially lower semi-continuous on* $W^{1,p}$ *for each* $\lambda > 0$, $\mu \ge 0$;
(2) $\lim\limits_{\|u\|_{\eta_1} \to +\infty} I(\lambda, \mu, u) = +\infty$ *for each* $\lambda > 0, \mu \ge 0$;
(3) $\mathcal{E}(\lambda, \mu, \cdot)$ *satisfies the* (PS) *condition for each* $\lambda, \mu > 0$.

Proof (1) The function $I(\lambda, \mu, \cdot)$ is weakly sequentially l.s.c on $W^{1,p}$, because $\mathcal{F}$ is weakly sequentially l.s.c., while φ_γ and J are convex and l.s.c., hence they are also weakly sequentially l.s.c.

(2) We first observe that

$$\|u\|_{L^p}^p \le \frac{1}{\eta_1} \|u\|_{\eta_1}^p \qquad \text{for all } u \in W^{1,p}.$$

In (6.14) we choose $\varepsilon < \frac{\eta_1}{2\lambda p}$. Since the embedding $L^p \hookrightarrow L^1$ is continuous and (6.7) holds, we have for all $u \in D(J)$,

$$\begin{aligned} I(\lambda, \mu, u) &\ge \frac{1}{p}\Big(\|u'\|_{L^p}^p + \gamma \|u\|_{L^p}^p\Big) - \lambda \varepsilon \|u\|_{L^p}^p \\ &- \lambda \delta_1 \|\alpha_{\delta_1}\|_{L^1(0,T)} + \mu J(u) \\ &\ge \frac{\eta_1 - 2\varepsilon\lambda p}{2\eta_1 p} \|u\|_{\eta_1}^p - \lambda\delta_1 \|\alpha_{\delta_1}\|_{L^1(0,T)} + \mu J(u). \end{aligned}$$

Since J is convex and l.s.c., it is bounded from below by an affine functional and then there exist constants c_1, c_2, $c_3 > 0$ such that for all $u \in D(J)$,

$$I(\lambda, \mu, u) \ge \frac{\eta_1 - 2\varepsilon\lambda p}{2\eta_1 p} \|u\|_{\eta_1}^p - \lambda\delta_1 \|\alpha_{\delta_1}\|_{L^1(0,T)} - c_1|u(0)| - c_2|u(T)| - c_3.$$

By the continuity of the embedding $W^{1,p} \hookrightarrow C$ we have for all $u \in W^{1,p}$,

$$I(\lambda, \mu, u) \ge c_4 \|u\|_{\eta_1}^p - c_5 \|u\|_{\eta_1} - c_6,$$

where c_4, c_5, c_6 are positive constants. Since $p > 1$, it follows that $I(\lambda, \mu, u) \to +\infty$ as $\|u\|_{\eta_1} \to +\infty$.

(3) Let $\{u_n\}$ in $W^{1,p}$ be a sequence satisfying $I(\lambda, \mu, u_n) \to c$. Then, for all $v \in W^{1,p}$,

$$\lambda\langle \mathcal{F}'(u_n), v - u_n\rangle + \varphi_\gamma(v) - \varphi_\gamma(u_n) + \mu J(v) - \mu J(u_n) \geq -\varepsilon_n \|v - u_n\|_{\eta_1},$$

where $\{\varepsilon_n\} \subset [0, \infty)$ with $\varepsilon_n \to 0$. We have a subsequence $\{u_n\} \subset D(J)$ (we just eliminate the finite number of elements of the sequence which do not belong to $D(J)$), since $\mu > 0$ and $I(\lambda, \mu, u_n) \to c$.

But $I(\lambda, \mu, \cdot)$ is coercive, hence $\{u_n\}$ is bounded in $W^{1,p}$. The embedding $W^{1,p} \hookrightarrow C$ is compact, then we can find a subsequence, still denoted by $\{u_n\}$, which is weakly convergent to a point $u \in W^{1,p}$ and strongly in C.

In the above inequality we take $v = u_n + s(u - u_n)$, with $s > 0$, then divide both sides of the inequality by s and let $s \searrow 0$. So, for all positive integer n

$$\lambda\langle \mathcal{F}'(u_n), u - u_n\rangle + \langle \varphi'_\gamma(u_n), u - u_n\rangle + \mu J'(u_n; u - u_n) \geq -\varepsilon_n \|u - u_n\|_{\eta_1}.$$

By the upper semi-continuity of $\mathcal{F}'$, it follows that

$$\liminf_{n\to\infty} \Big(\langle \varphi'_\gamma(u_n), u - u_n\rangle + \mu J'(u_n; u - u_n) \Big) \geq 0.$$

By Lemma 6.1 we deduce that $\{u_n\}$ has a subsequence that converges strongly to $u \in W^{1,p}$. □

Remark 6.2 From (F_1), (F_2), (F_3) and (F_4) it follows that for each $\varepsilon > 0$ there exist $\delta_\varepsilon, \bar{\delta}_\varepsilon > 0$ such that

$$F(t,x) - F(t,0) \leq \varepsilon |x|^p + \frac{\alpha_{\delta_\varepsilon}(t)}{\bar{\delta}_\varepsilon^{r-1}} |x|^r \qquad \text{for all } x \in \mathbb{R}^n, \text{ a.e. } t \in [0,T],$$

where $r \geq 1$. Then, by using the continuity of the embedding $W^{1,p} \hookrightarrow C$, we obtain

$$\mathcal{F}(u) \geq -\varepsilon \|u\|^p_{L^p} - \frac{\hat{c}^r \|\alpha_{\delta_\varepsilon}\|_{L^1(0,T)}}{\bar{\delta}_\varepsilon^{r-1}} \|u\|^r_\gamma \qquad \text{for all } u \in W^{1,p}. \quad (6.15)$$

Remark 6.3 If $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfies (F_1) and (F_4), then $0 = \nabla F(t,0)$ for a.e. $t \in [0,T]$.

The main result of this section is the following.

Theorem 6.1 *Let $F : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ be a function satisfying $(F_1)-(F_5)$ and let $j : \mathbb{R}^n \times \mathbb{R}^n \to]-\infty,+\infty]$ be a function satisfying $(J_1)-(J_3)$. Then for each fixed $\mu > 0$, there exists an open interval $\Lambda_\mu \subset]0,+\infty[$ such that for each $\lambda \in \Lambda_\mu$, the problem $(P_{\lambda,\mu})$ has at least two nontrivial solutions.*

Proof Let $\mu > 0$ be fixed. We define the function $g :]0,+\infty[\to \mathbb{R}$ by

$$g(t) = \sup\{-\mathcal{F}(u) \ : \ \varphi_\gamma(u) + \mu J(u) \le t \ \}, \quad \text{for all} \quad t > 0.$$

Using (6.15) for $r \in]p,p^*[$ it follows that for all $u \in W^{1,p}$ we have

$$-\mathcal{F}(u) \le \frac{\varepsilon}{\gamma}\|u\|_\gamma^p + \frac{\hat{c}^r\|\alpha_{\delta_\varepsilon}\|_{L^1(0,T)}}{\bar{\delta}_\varepsilon^{r-1}}\|u\|_\gamma^r.$$

Since $p < r$, we have

$$\lim_{t\to 0^+}\frac{g(t)}{t} = 0.$$

Using (F_5) we define $u_0(t) = s_0$ for a.e. $t \in [0,T]$. Then $u_0 \in W^{1,p}\setminus\{0\}$ and $-\mathcal{F}(u_0) > 0$. Due to the convergence relation above, it is possible to choose a real number t_0 such that $0 < t_0 < \varphi_\gamma(u_0) + \mu J(u_0)$ and

$$\frac{g(t_0)}{t_0} < [\varphi_\gamma(u_0) + \mu J(u_0)]^{-1} \cdot (-\mathcal{F}(u_0)).$$

We choose $\rho_0 > 0$ such that

$$g(t_0) < \rho_0 < [\varphi_\gamma(u_0) + \mu J(u_0)]^{-1} \cdot (-\mathcal{F}(u_0))t_0. \tag{6.16}$$

We apply Theorem 1.22 to the space $W^{1,p}$, the interval $\Lambda =]0,+\infty[$ and the functions $E_1, E_2 : W^{1,p} \to \mathbb{R}, h : \Lambda \to \mathbb{R}$ defined by

$$E_1(u) = \varphi_\gamma(u), \zeta_1(u) = \mu J(u), I_2(u) = E_2(u) = \mathcal{F}(u), h(\lambda) = \rho_0\lambda.$$

By Proposition 6.2, the assumptions (i) and (ii) of Theorem 1.22 are fulfilled.

We prove now the minimax inequality

$$\sup_{\lambda\in\Lambda}\inf_{u\in W^{1,p}}\Big(\varphi_\gamma(u) + \mu J(u) + \lambda\mathcal{F}(u) + \rho_0\lambda\Big) <$$
$$\inf_{u\in W^{1,p}}\sup_{\lambda\in\Lambda}\Big(\varphi_\gamma(u) + \mu J(u) + \lambda\mathcal{F}(u) + \rho_0\lambda\Big).$$

The function

$$\lambda \mapsto \inf_{u\in W^{1,p}}\Big(\varphi_\gamma(u) + \mu J(u) + \lambda\mathcal{F}(u) + \rho_0\lambda\Big)$$

is upper semi-continuous on Λ. Since

$$\inf_{u\in W^{1,p}}\Big(\varphi_\gamma(u)+\mu J(u)+\lambda\mathcal{F}(u)+\rho_0\lambda\Big)\le\varphi_\gamma(u_0)+\mu J(u_0)+\lambda\mathcal{F}(u_0)+\rho_0\lambda$$

and $\rho_0<-\mathcal{F}(u_0)$, it follows that

$$\lim_{\lambda\to+\infty}\inf_{u\in W^{1,p}}\Big(\varphi_\gamma(u)+\mu J(u)+\lambda\mathcal{F}(u)+\rho_0\lambda\Big)=-\infty.$$

Thus we can find $\overline{\lambda}\in\Lambda$ such that

$$\begin{aligned}\beta_1 &:= \sup_{\lambda\in\Lambda}\inf_{u\in W^{1,p}}\Big(\varphi_\gamma(u)+\mu J(u)+\lambda\mathcal{F}(u)+\rho_0\lambda\Big)\\ &= \inf_{u\in W^{1,p}}\Big(\varphi_\gamma(u)+\mu J(u)+\overline{\lambda}\mathcal{F}(u)+\rho_0\overline{\lambda}\Big).\end{aligned}$$

In order to prove that $\beta_1<t_0$, we distinguish two cases:

I. If $0\le\overline{\lambda}<\dfrac{t_0}{\rho_0}$, we have

$$\beta_1\le\varphi_\gamma(0)+\mu J(0)+\overline{\lambda}\mathcal{F}(0)+\rho_0\overline{\lambda}=\overline{\lambda}\rho_0<t_0.$$

II. If $\overline{\lambda}\ge\dfrac{t_0}{\rho_0}$, then we use $\rho_0<-\mathcal{F}(u_0)$ and the inequality (6.16) to get

$$\begin{aligned}\eta_1 &\le \varphi_\gamma(u_0)+\mu J(u_0)+\overline{\lambda}\mathcal{F}(u_0)+\rho_0\overline{\lambda}\\ &\le \varphi_\gamma(u_0)+\mu J(u_0)+\frac{t_0}{\rho_0}(\rho_0+\mathcal{F}(u_0))<t_0\,.\end{aligned}$$

From $g(t_0)<\rho_0$ it follows that for all $u\in W^{1,p}$ with $\varphi_\gamma(u)+\mu J(u)\le t_0$ we have $-\mathcal{F}(u)<\rho_0$. Hence

$$t_0\le\inf\left\{\varphi_{\mathcal{G}}(u)+\mu J(u):\ -\mathcal{F}(u)\ge\rho_0\right\}.$$

On the other hand,

$$\begin{aligned}\beta_2 &= \inf_{u\in W^{1,p}}\sup_{\lambda\in\Lambda}(\varphi_{\mathcal{G}}(u)+\mu J(u)+\lambda\mathcal{F}(u)+\rho_0\lambda)\\ &= \inf\left\{\varphi_{\mathcal{G}}(u)+\mu J(u):-\mathcal{F}(u)\ge\rho_0\right\}.\end{aligned}$$

We conclude that

$$\beta_1<t_0\le\beta_2.$$

Hence, assumption (iii) from Theorem 1.22 holds. Thus, there exists an open interval $\Lambda_\mu\subseteq]0,\infty)$ such that for each $\lambda\in\Lambda_\mu$ the function $\varphi_{\mathcal{G}}+\mu J+\lambda\mathcal{F}$ has at least three critical points in $W^{1,p}$. By Proposition 6.1 it follows that these critical points are solutions of $(P_{\lambda,\mu})$. Since $\nabla F(t,0)=0$ for a.e. $t\in[0,T]$, we get that at least two of the above solutions are nontrivial. $\square$

Remark 6.4 The two conditions from (J_3) can be replaced by
(J_3') $j(x,y) \geq j(0,0)$ for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Then all the proofs above can be adapted by considering

$$J(u) = j(u(0), u(T)) - j(0,0).$$

Corollary 6.1 *Let $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be a function satisfying $(F_1) - (F_5)$ and let $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a positive, convex and Gâteaux differentiable function with $b(0,0) = 0$. Assume that $S \subset \mathbb{R}^n \times \mathbb{R}^n$ is a nonempty closed convex cone with $S \neq \{(0,0)\}$, whose normal cone we denote by N_S. Then, for each fixed $\gamma, \mu > 0$, there exists an open interval $\Lambda_0 \subset]0, +\infty[$ such that for each $\lambda \in \Lambda_0$, the following problem*
$(\hat{P}_{\lambda,\mu})$

$$\begin{cases} -[h_p(u')]' + \gamma h_p(u) = \lambda \bar{\partial} F(t,u) \ \text{a.e. } t \in [0,T], \\ (u(0), u(T)) \in S, \\ \Big(h_p(u')(0), -h_p(u')(T)\Big) \in \mu \nabla b(u(0), u(T)) + \mu N_S(u(0), u(T)), \end{cases}$$

has at least two nontrivial solutions.

Proof The statement follows by applying Theorem 6.1 to the function F and the convex function $j : \mathbb{R}^n \times \mathbb{R}^n \to] - \infty, +\infty]$ defined by

$$j(x,y) = b(x,y) + I_S(x,y), \text{ for all } (x,y) \in \mathbb{R}^n \times \mathbb{R}^n,$$

where

$$I_S(x,y) = \begin{cases} 0, \text{ if } (x,y) \in S \\ +\infty, \text{ if } (x,y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus S, \end{cases}$$

is the indicator function of the cone S.

Note, that in this case $D(j) = S$ and j satisfies the conditions $(J_1) - (J_3)$. Moreover,

$$\partial j(x,y) = \nabla b(x,y) + \partial I_S(x,y) = \nabla b(x,y) + N_S(x,y) \text{ for all } (x,y) \in S.$$

□

Example 6.1 Various possible choices of b and S from Corollary 6.1 recover some classical boundary conditions. For instance:

(a) $b = 0$ and $S = \{(x,x) : x \in \mathbb{R}^n\}$ we get periodic boundary conditions $u(0) = u(T), u'(0) = u'(T)$;

(b) $b = 0$ and $S = \mathbb{R}^n \times \mathbb{R}^n$ we get Neumann type boundary conditions $u'(0) = u'(T) = 0$;

(c) $b(z) = \frac{1}{2}(Az, z)_{R^{2N}}$, $z \in \mathbb{R}^{2N}$, where A is a symmetric, positive $2N \times 2N$ real valued matrix, and $S = \mathbb{R}^n \times \mathbb{R}^n$; we get the following mixed boundary conditions

$$\begin{pmatrix} h_p(u')(0) \\ -h_p(u')(T) \end{pmatrix} = A \begin{pmatrix} u(0) \\ u(T) \end{pmatrix}.$$

For these choices of F, b and S it follows by Corollary 6.1 that for each fixed $\gamma, \mu > 0$, there exists an open interval $\Lambda_0 \subset]0, +\infty[$ such that for each $\lambda \in \Lambda_0$ the problem $(\hat{P}_{\lambda,\mu})$ has at least two nontrivial solutions.

6.3 Scalar field systems with nonlinear oscillatory terms

To formulate the second problem we assume that:

(C1) $F : \mathbb{R}^n \to \mathbb{R}$ is of class C^1,

(C2) $S \subset \mathbb{R}^n \times \mathbb{R}^n$ is a closed convex set wit $\{(x, x) \ : \ x \in \mathbb{R}^n\} \subset S$,

(C3) $\gamma_1, \ldots, \gamma_n \in L^\infty(]0, T[, \mathbb{R})$ are so that essinf $\gamma_i > 0$, for all $i = 1, \ldots, n$ and put $\gamma = (\gamma_1, \ldots, \gamma_n)$,

(C4) $\alpha \in L^1(]0, T[, \mathbb{R})$ is so that $\alpha(t) \geq 0$ a.e. in $]0, T[$.

We study the existence and multiplicity of the solutions of the system:

$$(S) \qquad \begin{cases} -[h_p(u')]' + \gamma \circ h_p(u) = \alpha(t) \nabla F(u) \\ \\ (h_p(u')(0), -h_p(u')(T)) \in N_S(u(0), u(T)), \end{cases}$$

where $N_S(x, y)$ denotes the normal cone of S at $(x, y) \in S$.

The first step in the study of (S) is to establish the corresponding energy function. For this some preparation is needed:

- For every $i \in \{1, \ldots, n\}$ denote by X_i the Sobolev space $W^{1,p}(\,]0, T[\,, \mathbb{R})$ equipped with the norm $||\cdot||_i$, where

$$||f||_i = \left(\int_0^T \gamma_i(t) |f(t)|^p \mathrm{d}t + \int_0^T |f'(t)|^p \mathrm{d}t \right)^{\frac{1}{p}}.$$

Since essinf$\gamma_\mathrm{i} > 0$ it follows that $||\cdot||_i$ is equivalent to the usual norm on $W^{1,p}(\,]0, T[\,, \mathbb{R})$.

- X is the Sobolev space $W^{1,p}(\,]0,T[\,,\mathbb{R}^n)$ equipped with the usual norm
$$||u|| = \left(\int_0^T |u(t)|^p \mathrm{d}t + \int_0^T |u'(t)|^p \mathrm{d}t\right)^{1/p}.$$
We observe that X is isomorphic to $W^{1,p}(\,]0,T[\,,\mathbb{R})^n$.
- Let Y be the real Banach space $C([0,T],\mathbb{R}^n)$ endowed with the supremum norm $||\cdot||_s$. For simplicity we will denote the supremum norm on every space $C([0,T],\mathbb{R}^m)$, $m \in \mathbb{N}$, with the same symbol $||\cdot||_s$.
- Put
$$\Sigma := \{u \in X \mid (u(0), u(T)) \in S\}.$$
Note that Σ is a closed convex subset of X containing the constant functions.
- Define $E_1 \colon Y \to \mathbb{R}$ by
$$E_1(u) = \int_0^T \alpha(t)F(u(t))\mathrm{d}t, \quad \text{for every} \quad u \in Y,$$
$E_2 \colon X \to \mathbb{R}$ by
$$E_2(u) = \frac{1}{p}(||u_1||_1^p + \cdots + ||u_n||_n^p), \quad \text{for every} \quad u \in X,$$
and $\zeta_1 \colon X \to]-\infty,+\infty]$ by
$$\zeta_1(u) = \begin{cases} 0, & u \in \Sigma \\ +\infty, & u \notin \Sigma. \end{cases}$$

The notations used above are inspired from Theorem 1.23.

Lemma 6.2 *The following assertions hold:*

(a) *The above defined maps $E_1 \colon X \to \mathbb{R}$ and $E_2 \colon X \to \mathbb{R}$ are of class C^1 and $\langle E_1'(u); v\rangle = \int_0^T \alpha(t)f(u(t))v(t)dt$, for every $u, v \in X$.*

(b) *The map $\zeta_1 \colon X \to]-\infty,+\infty]$ is convex, proper, and lower semicontinuous.*

Define now $I_1 \colon X \to]-\infty,+\infty]$ and $I_2 \colon X \to \mathbb{R}$ by

$$I_1(u) = E_1(u) + \zeta_1(u), \quad I_2(u) = E_2(u) \qquad \text{for every } u \in X. \tag{6.17}$$

The energy functional associated to the system (S) is given by $I_1 + I_2 : X \to]-\infty,+\infty]$ is given by $I_1 + I_2 = (E_1 + E_2) + \zeta_1$.

Our aim is to apply Theorem 1.23 to the maps I_1 and I_2 defined in (6.17). Indeed,

- X, Y are real Banach spaces, X is reflexive, and X is compactly embedded in Y.
- $E_1\colon Y \to \mathbb{R}$, $E_2\colon X \to \mathbb{R}$ are locally Lipschitz, and ζ_1 is convex, proper, and lower semi-continuous (according to Lemma 6.2).
- $\Psi = E_2\colon X \to \mathbb{R}$ is weakly sequentially lower semi-continuous (being convex and continuous) and coercive.
- Since $\inf_X \Psi = 0$ and since Σ contains the constant functions, condition (1.36) is satisfied.

Furthermore we have to introduce some suitable subsets of $\mathbb{R}^n$. For this note that, since X_i is embedded in $C([0,T],\mathbb{R})$, there exist $c_i > 0$, $i = \overline{1,n}$, such that

$$||f||_s \le c_i ||f||_i, \quad \text{for every } f \in X_i. \tag{6.18}$$

For every $r > 0$, let

$$\begin{aligned} A(r) &:= \left\{ x \in \mathbb{R}^n : \ \frac{1}{p}\sum_{i=1}^n \frac{1}{c_i^p}|x_i|^p \le r \right\} \\ B(r) &:= \left\{ x \in \mathbb{R}^n : \ \frac{1}{p}\sum_{i=1}^n |x_i|^p \int_0^T \gamma_i(t)\mathrm{d}t \le r \right\}. \end{aligned} \tag{6.19}$$

Remark 6.5 1) For every $r > 0$ the inclusion $B(r) \subseteq A(r)$ holds. To see this, we observe that relation (6.18) implies

$$1 \le c_i \left(\int_0^T \gamma_i(t)\mathrm{d}t \right)^{1/p}, \qquad \text{for } i = \overline{1,n}.$$

Pick now an arbitrary $x \in B(r)$. Then

$$\frac{1}{p}\sum_{i=1}^n \frac{1}{c_i^p}|x_i|^p \le \frac{1}{p}\sum_{i=1}^n |x_i|^p \int_0^T \gamma_i(t)\mathrm{d}t \le r,$$

hence $x \in A(r)$.

2) Since the map $x \in \mathbb{R}^n \longmapsto \sum_{i=1}^n |x_i|^p \int_0^T \gamma_i(t)\mathrm{d}t \in \mathbb{R}$ is convex, we have for every $r > 0$,

$$\mathrm{int\,B(r)} = \left\{ \mathrm{x} \in \mathbb{R}^{\mathrm{n}} : \ \frac{1}{\mathrm{p}}\sum_{\mathrm{i}=1}^{\mathrm{n}} |\mathrm{x_i}|^{\mathrm{p}} \int_0^{\mathrm{T}} \gamma_{\mathrm{i}}(\mathrm{t})\mathrm{dt} < \mathrm{r} \right\}.$$

Theorem 6.2 *Let $r > 0$ be such that*

$$\min_{x\in A(r)} F(x) = \min_{x\in \mathrm{int B(r)}} F(x)\,.$$

Then the following assertions hold:

(a) *$\varphi(r) = 0$, where φ is defined by relation (1.37).*
(b) *Problem (P) has a solution $u \in \Sigma$ satisfying $\Psi(u) < r$.*

Proof (a) We have $\varphi(r) \geq 0$, by definition. To show the converse inequality choose $x^0 \in \mathrm{int\,B(r)}$ so that

$$F(x^0) = \min_{x\in \mathrm{int B(r)}} F(x) = \min_{x\in A(r)} F(x),$$

and let $u^0\colon X \to \mathbb{R}^n$ be the function taking the constant value x^0. For every $i \in \{1,\ldots,n\}$ we have

$$||u_i^0||_i = |x_i^0| \left(\int_0^T \gamma_i(t)\mathrm{d}t\right)^{1/p}.$$

Therefore

$$\psi(u^0) = \frac{1}{p}\sum_{i=1}^n |x_i^0|^p \int_0^T \gamma_i(t)\mathrm{d}t < r,$$

hence $u^0 \in I_2^{-1}(\,]-\infty, r[)$. Since $I_2^{-1}(\,]-\infty,r])$ is convex and closed in the norm topology, it is closed also in the weak topology, hence

$$\overline{(I_2^{-1}(]-\infty,r[))_w} \subseteq I_2^{-1}(\,]-\infty,r]).$$

Pick now an arbitrary element $v \in \overline{(I_2^{-1}(]-\infty,r[))_w}$. Then $I_2(v) \leq r$. Thus, by (6.18), for all $t \in [0,T]$

$$\frac{1}{p}\sum_{i=1}^n \frac{1}{c_i^p}|v_i(t)|^p \leq \frac{1}{p}\sum_{i=1}^n \frac{1}{c_i^p}||v_i||_s^p \leq \frac{1}{p}\sum_{i=1}^n ||v_i||_i^p = I_2(v) \leq r\,.$$

We conclude that $v(t) \in A(r)$ for every $t \in [0,T]$. Hence

$$F(x^0) \leq F(v(t)),\ \text{for every } t \in [0,T].$$

It follows that

$$E_1(u^0) = \int_0^T \alpha(t)F(x^0)\mathrm{d}t \leq \int_0^T \alpha(t)F(v(t))\mathrm{d}t = E_1(v) \leq I_1(v).$$

Since $v \in \overline{(I_2^{-1}(]-\infty,r[))_w}$ was chosen arbitrarily, we conclude that

$$\inf_{v\in\overline{(I_2^{-1}(]-\infty,r[))_w}} I_1(v) = I_1(u^0).$$

This implies, according to the definition of φ in (1.37),

$$\varphi(r) \leq \frac{I_1(u^0) - I_1(u^0)}{r - I_2(u^0)} = 0,$$

hence $\varphi(r) = 0$.

(b) Since $\varphi(r) = 0$, we can apply Theorem 1.23 for $\lambda = 1$, and conclude that the map $I_1 + I_2$ has a critical point u lying in Σ and such that $I_2(u) < r$. The assertion follows from Proposition 6.1. $\square$

Theorem 6.3 *Assume that:*

(1) *There exists a sequence $(r_k)_{k\in\mathbb{N}}$ of positive reals such that* $\lim r_k = +\infty$ *and*

$$\min_{x\in A(r_k)} F(x) = \min_{x\in \mathrm{intB(r_k)}} F(x), \text{ for every } k \in \mathbb{N}.$$

(2) *The following inequality holds*

$$\liminf_{|x|\to+\infty} \frac{F(x)\int_0^T \alpha(t)\mathrm{d}t}{\sum_{i=1}^n |x_i|^p \int_0^T \gamma_i(t)\mathrm{d}t} < -\frac{1}{p}.$$

Then problem (P) has an unbounded sequence of solutions.

Proof Assumption (1) implies, according to Theorem 6.2(a), that $\varphi(r_k) = 0$, for every $k \in \mathbb{N}$. Let γ be defined as in relation (1.38). Since $\varphi(r) \geq 0$ for every $r > 0$, we conclude that

$$\gamma = \liminf_{r\to+\infty} \varphi(r) = 0.$$

Applying Theorem 1.23(a) for $\lambda = 1$, we conclude that either assertion (b1) or (b2) of this theorem must hold. Next we show that (b1) is not satisfied, that is, we prove that $I_1 + I_2$ is unbounded below. For this fix a real number q such that

$$\liminf_{|x|\to+\infty} \frac{F(x)\int_0^T \alpha(t)\mathrm{d}t}{\sum_{i=1}^n |x_i|^p \int_0^T \gamma_i(t)\mathrm{d}t} < q < -\frac{1}{p}.$$

Choose now a sequence $(x^k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ such that $\lim |x^k| = +\infty$ and

$$\frac{F(x^k)\int_0^T \alpha(t)\mathrm{d}t}{\sum_{i=1}^n |x_i^k|^p \int_0^T \gamma_i(t)\mathrm{d}t} < q, \text{ for every } k \in \mathbb{N}.$$

For every $k \in \mathbb{N}$ denote by $u^k : X \to \mathbb{R}^n$ the constant function taking the value x^k. Then the following relations hold for every $k \in \mathbb{N}$

$$\begin{aligned} I_1(u^k) + I_2(u^k) &= F(x^k) \int_0^T \alpha(t)\mathrm{d}t + \frac{1}{p} \sum_{i=1}^n |x_i^k|^p \int_0^T \gamma_i(t)\mathrm{d}t \\ &< \left(q + \frac{1}{p} \right) \sum_{i=1}^n |x_i^k|^p \int_0^T \gamma_i(t)\mathrm{d}t. \end{aligned}$$

Since $|x^k| \to +\infty$, $\int_0^T \gamma_i(t)\mathrm{d}t > 0$, for every $i \in \{1, \dots, n\}$, and $q + \frac{1}{p} < 0$, we conclude that $\lim_{k\to\infty}(I_1(u^k) + I_2(u^k)) = -\infty$, thus $I_1 + I_2$ is unbounded below. The assertion follows now from Theorem 1.23(b2), the definition of I_2, and Proposition 6.1. □

Theorem 6.4 *Assume that:*

(1) *There exists a sequence* $(r_k)_{k\in\mathbb{N}}$ *of positive reals such that* $\lim r_k = 0$ *and*

$$\min_{x \in A(r_k)} F(x) = \min_{x \in \mathrm{int B(r_k)}} F(x), \text{ for every } k \in \mathbb{N}.$$

(2) *The following inequality holds*

$$\liminf_{x \to 0_n} \frac{F(x) \int_0^T \alpha(t)\mathrm{d}t}{\sum_{i=1}^n |x_i|^p \int_0^T \gamma_i(t)\mathrm{d}t} < -\frac{1}{p}.$$

Then problem (S) *has a sequence of pairwise distinct solutions which converges strongly to the zero function* $\theta_X \in X$.

Proof We first observe that θ_X is the only global minimum of I_2. Assumption (1) and Theorem 6.2(a) imply that $\varphi(r_k) = 0$, for every $k \in \mathbb{N}$. Let δ be defined as in relation (1.39). Since $\varphi(r) \geq 0$ for every $r > 0$, we conclude that

$$\delta = \liminf_{r \to 0^+} \varphi(r) = 0.$$

Applying Theorem 1.23(c) for $\lambda = 1$, we conclude that either assertion (c1) or (c2) of this theorem must hold. Next we show that (c1) is not satisfied, that is, we prove that θ_X is not a local minimum of $I_1 + I_2$. For this fix a real number q such that

$$\liminf_{x \to 0_n} \frac{F(x) \int_0^T \alpha(t)\mathrm{d}t}{\sum_{i=1}^n |x_i|^p \int_0^T \gamma_i(t)\mathrm{d}t} < q < -\frac{1}{p}.$$

Choose now a sequence $(x^k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ such that $\lim x^k = 0_n$ and

$$\frac{F(x^k)\int_0^T \alpha(t)\mathrm{d}t}{\sum_{i=1}^n |x_i^k|^p \int_0^T \gamma_i(t)\mathrm{d}t} < q, \text{ for every } k \in \mathbb{N}.$$

For every $k \in \mathbb{N}$ denote by $u^k \colon X \to \mathbb{R}^n$ the constant function taking the value x^k. Then the following relations hold for every $k \in \mathbb{N}$

$$\begin{aligned} I_1(u^k) + I_2(u^k) &= F(x^k)\int_0^T \alpha(t)\mathrm{d}t + \frac{1}{p}\sum_{i=1}^n |x_i^k|^p \int_0^T \gamma_i(t)\mathrm{d}t \\ &< \left(q + \frac{1}{p}\right)\sum_{i=1}^n |x_i^k|^p \int_0^T \gamma_i(t)\mathrm{d}t \\ &\le 0 = I_1(\theta_X) + I_2(\theta_X). \end{aligned}$$

We have that $\lim ||u^k|| = 0$, thus θ_X is not a local minimum of $I_1 + I_2$. Theorem 1.23(c2) and Proposition 6.1 imply now the existence of a sequence $(\tilde{u}^k)$ of pairwise distinct solutions of (S) such that $\lim I_2(\tilde{u}^k) = 0$. Since I_2 is a norm on X which is equivalent to the norm $||\cdot||$, we conclude that $(\tilde{u}^k)$ converges strongly to θ_X in X. $\square$

6.4 Applications

We specialize now some of the data of the previous section in order to obtain applications of Theorems 6.3 and 6.4. We assume that $n \ge 1$ is a natural number, $T = 1$, $p > 1$ is a real number, $S \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is a set satisfying condition (C2), γ_i is the function taking the constant value 1, for every $i \in \{1, \dots, n\}$, and $\alpha \in L^1(\,]0,T[\,,\mathbb{R})$ is so that $\int_0^1 \alpha(t)\,dt > 1$ and $\alpha(t) \ge 0$ a.e. in $]0,1[$. In this case every norm $||\cdot||_i$, $i = \overline{1,n}$, reduces to the usual norm on $W^{1,p}(\,]0,1[\,,\mathbb{R})$, and all the constants c_i, $i = \overline{1,n}$, in (6.18) can be considered to be equal to a suitable real number $c > 0$. Furthermore, we assume in this section that $\mathbb{R}^n$ is endowed with the p-norm

$$|x| = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}.$$

Thus, for $r > 0$, the sets $A(r)$ and $B(r)$ defined in (6.19) become

$$A(r) = \left\{x \in \mathbb{R}^n : \frac{1}{p}\cdot\frac{1}{c^p}\,|x|^p \le r\right\} \quad B(r) = \left\{x \in \mathbb{R}^n : \frac{1}{p}\,|x|^p \le r\right\}.$$

Example 6.2 We give now an application of Theorem 6.3. In order to define $F\colon \mathbb{R}^n \to \mathbb{R}$ we consider a function $f\colon [0,+\infty[\to [0,+\infty[$ with the following properties:

(i) f is bijective,
(ii) f is strictly increasing,
(iii) $\displaystyle\lim_{t\to+\infty} \frac{f(t+\pi)}{f(t)} > c^p$,
(iv) f^{-1} is locally Lipschitz.

Note that the above properties imply that $f(0) = 0$, f^{-1} is strictly increasing, and

$$\lim_{t\to+\infty} f(t) = \lim_{t\to+\infty} f^{-1}(t) = +\infty.$$

An example of a function f satisfying properties (i)–(iv) is

$$t \in [0,+\infty[\longmapsto a^t - 1 \in [0,+\infty[,$$

where the real $a > 1$ is chosen so that $a^\pi > c^p$.

Define $F\colon \mathbb{R}^n \to \mathbb{R}$ by

$$F(x) = \frac{1}{p}\,|x|^p \sin f^{-1}\left(|x|^p\right), \text{ for every } x \in \mathbb{R}^n.$$

A straightforward argument yields that F is of class C^1 (that is, satisfies condition (C1)).

We next show that the assumptions (1) and (2) of Theorem 6.3 are fulfilled. Indeed, to check (1), we observe that by (iii) and for every sufficiently large $k \in \mathbb{N}$,

$$\frac{f((2k+1)\pi)}{f(2k\pi)} > c^p. \tag{6.20}$$

For these values of k put

$$r_k := \frac{f((2k+1)\pi)}{pc^p}\,.$$

Then $\lim r_k = +\infty$. Furthermore,

$$\min_{x\in A(r_k)} F(x) = \min_{|x|^p\le f((2k+1)\pi)} F(x). \tag{6.21}$$

If $f(2k\pi) \le |x|^p \le f((2k+1)\pi)$, then

$$2k\pi \le f^{-1}\left(|x|^p\right) \le (2k+1)\pi,$$

hence

$$\sin\left(f^{-1}\left(|x|^p\right)\right) \ge 0,.$$

This means that $F \geq 0$. Taking into account that $F(0_n) = 0$, it follows that

$$\min_{|x|^p \leq f((2k+1)\pi)} F(x) = \min_{|x|^p \leq f(2k\pi)} F(x). \tag{6.22}$$

On the other hand, if $x \in \mathbb{R}^n$ is so that $|x|^p \leq f(2k\pi)$ then, in view of (6.20), we have

$$\frac{1}{p}|x|^p \leq \frac{1}{p} f(2k\pi) < \frac{f((2k+1)\pi)}{pc^p} = r_k. \tag{6.23}$$

Using (6.21), (6.22), and (6.23), we conclude that

$$\min_{x \in A(r_k)} F(x) = \min_{x \in \mathrm{int B(r_k)}} F(x)\,,$$

hence assumption (1) of Theorem 6.3 is fulfilled. For assumption (2) we observe that

$$\begin{aligned}
\liminf_{|x| \to +\infty} \frac{F(x) \int_0^1 \alpha(t)\mathrm{d}t}{\sum_{i=1}^n |x_i|^p} &= \liminf_{|x| \to +\infty} \frac{1}{p} \sin\left(f^{-1}\left(|x|^p\right)\right) \int_0^1 \alpha(t)\mathrm{d}t \\
&= -\frac{1}{p} \int_0^1 \alpha(t)\mathrm{d}t < -\frac{1}{p}\,.
\end{aligned}$$

Theorem 6.3 yields now that problem (P) has an unbounded sequence of solutions.

Example 6.3 To get an application of Theorem 6.4 consider a function $f\colon\,]0,+\infty[\to]0,+\infty[$ with the following properties:

(i) f is surjective,
(ii) f is strictly increasing,
(iii) f is differentiable on $]0,+\infty[$,
(iv) the map $t \in]0,+\infty[\mapsto \frac{f'\left(\frac{1}{t}\right)}{t} \in \mathbb{R}$ is bounded on every interval $]0,a[$, $a > 0$,
(v) $\displaystyle\lim_{t \to +\infty} \frac{f^{-1}(t+\pi)}{f^{-1}(t)} > c^p$.

Under these assumptions, f^{-1} is strictly increasing and $\lim_{t \to +\infty} f^{-1}(t) = +\infty$. An example for a function satisfying properties (i)–(v) is

$$t \in]0,+\infty[\longmapsto \log_a(t+1) \in]0,+\infty[,$$

where the real $a > 1$ is chosen so that $a^\pi > c^p$.

Define $g\colon [0,+\infty[\to \mathbb{R}$ by

$$g(t) = \begin{cases} t\sin f\,(1/t)\,, & t \neq 0 \\ \\ 0, & t = 0. \end{cases}$$

This map has the following properties:

- g is continuous on $[0,+\infty[$,
- g is differentiable on $]0,+\infty[$ and g is not differentiable in 0,
- g' is bounded on every interval $]0,a[$, $a > 0$. This follows from property (iv) of f and the fact that for every $t > 0$ we have $g'(t) = \sin f\,(1/t) - \cos f\,(1/t)\,\frac{f'(1/t)}{t}$.

Define now $F\colon \mathbb{R}^n \to \mathbb{R}$ by

$$F(x) = \frac{1}{p}\, g(|x|^p), \quad \text{for every } x \in \mathbb{R}^n.$$

Since $p > 1$, it follows that F is of class C^1.

We next show that F satisfies conditions (1) and (2) of Theorem 6.4. To verify (1), we observe that (by (v)) and for every sufficiently large $k \in \mathbb{N}$

$$\frac{f^{-1}((2k+1)\pi)}{f^{-1}(2k\pi)} > c^p. \tag{6.24}$$

For these values of k put

$$r_k := \frac{1}{pc^p} \cdot \frac{1}{f^{-1}(2k\pi)}\,.$$

Then $\lim r_k = 0$. Furthermore,

$$\min_{x\in A(r_k)} F(x) = \min_{|x|^p \leq \frac{1}{f^{-1}(2k\pi)}} F(x). \tag{6.25}$$

If $\frac{1}{f^{-1}((2k+1)\pi)} \leq |x|^p \leq \frac{1}{f^{-1}(2k\pi)}$, then

$$2k\pi \leq f\left(\frac{1}{|x|^p}\right) \leq (2k+1)\pi,$$

hence

$$\sin f\left(\frac{1}{|x|^p}\right) \geq 0\,.$$

Therefore $F \geq 0$. Taking into account that $F(0_n) = 0$, it follows that

$$\min_{|x|^p \leq \frac{1}{f^{-1}(2k\pi)}} F(x) = \min_{|x|^p \leq \frac{1}{f^{-1}((2k+1)\pi)}} F(x). \tag{6.26}$$

On the other hand, if $x \in \mathbb{R}^n$ is so that $|x|^p \leq \frac{1}{f^{-1}((2k+1)\pi)}$, then, in view of (6.24), we have that

$$\frac{1}{p}|x|^p \leq \frac{1}{p}\frac{1}{f^{-1}((2k+1)\pi)} < \frac{1}{pc^p} \cdot \frac{1}{f^{-1}(2k\pi)} = r_k. \tag{6.27}$$

Using relations (6.25), (6.26), and (6.27), we conclude that

$$\min_{x \in A(r_k)} F(x) = \min_{x \in \mathrm{intB(r_k)}} F(x),$$

hence assumption (1) of Theorem 6.4 is fulfilled. For assumption (2) we observe that

$$\begin{aligned}\liminf_{x\to 0_n} \frac{F(x)\int_0^1 \alpha(t)\mathrm{d}t}{\sum_{i=1}^n |x_i|^p} &= \liminf_{x\to 0_n} \frac{1}{p}\sin f\left(\frac{1}{|x|^p}\right)\int_0^1 \alpha(t)\mathrm{d}t \\ &= -\frac{1}{p}\int_0^1 \alpha(t)\mathrm{d}t < -\frac{1}{p}.\end{aligned}$$

According to Theorem 6.4, problem (P) has a sequence of pairwise distinct solutions which converges strongly to the zero function $\theta_X \in X$.

6.5 Comments and historical notes

In the last few years, many papers were dedicated to the study of p-Laplacian systems with various types of boundary conditions. We mention here Manásevich and Mawhin [197], [198], Mawhin [205], Gasinski and Papageorgiu [124], Jebelean and Moroşanu [147], [148]. The methods used in these papers to prove the existence or multiple solutions of p-Laplacian systems are based on degree theory, minimax result, fixed point theorems, or on continuation methods of Leray-Schauder type. Jebelean [146] used the symmetric version of the mountain pass theorem to prove the existence of infinitely many solutions for problem $(P_{\lambda,\mu})$. Proposition 6.1 and Lemma 6.1 appear in the paper by Jebelean and Moroşanu [148], while Theorem 6.2 is due to Lisei, Moroşanu, and Varga [193]. The results contained in Section 6.3 are due to Breckner and Varga [45].

7

Competition Phenomena in Dirichlet Problems

> The purpose of models is not to fit the data but to sharpen the questions.
>
> Samuel Karlin (1924–2007)

Having infinitely many solutions for a given equation, after a 'small' perturbation of it, one might be expected to find still many solutions for the perturbed equation; moreover, once the perturbation tends to zero, the number of solutions for the perturbed equation should tend to infinity. Such phenomenon is well-known in the case of the equation $\sin x = c$ with $c \in (-1, 1)$ fixed, and its perturbation $\sin x = c + \varepsilon x$, $x \in \mathbb{R}$; the perturbed equation has more and more solutions as $|\varepsilon|$ decreases to 0. This natural phenomenon has been first exploited in an abstract framework by Krasnosel'skii [161]. More precisely, by using topological methods, Krasnosel'skii asserts the existence of more and more critical points of an even functional of class C^1 perturbed by a non-even term tending to zero, the critical points of the perturbed functional being the solutions for the studied equation.

The purpose of this chapter is to study the number and behaviour of solutions to a Dirichlet problem which involves an *oscillatory nonlinearity* and a *pure power term*.

7.1 Introduction

We consider the multiplicity of solutions to the problem

$$\begin{cases} -\triangle u = \lambda a(x) u^p + f(u) & \text{in} \quad \Omega, \\ u \geq 0, \ u \not\equiv 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{P$_\lambda$}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, $f : [0, \infty) \to \mathbb{R}$ *oscillates* near the *origin* or at *infinity*, and $p > 0$, $\lambda \in \mathbb{R}$. Denoting formally 0^+ or $+\infty$ by the common symbol L (standing for a limit point), we assume on the continuous nonlinearity f that

(f_1^L) $\quad -\infty < \liminf_{s \to L} \frac{F(s)}{s^2} \leq \limsup_{s \to L} \frac{F(s)}{s^2} = +\infty;$

(f_2^L) $\quad l_L := \liminf_{s \to L} \frac{f(s)}{s} < 0,$

where $F(s) = \int_0^s f(t)dt$. One can easily observe that f has an *oscillatory behaviour at* L.

While oscillatory right hand sides usually produce infinitely many distinct solutions, the additional term involving u^p may alter the situation radically. Our purpose is to fully describe this phenomenon, showing that the number of distinct nontrivial solutions to problem (P_λ) is strongly influenced by u^p and depends on λ whenever one of the following two cases holds:

- $p \leq 1$ and f oscillates near the origin;
- $p \geq 1$ and f oscillates at infinity (p may be critical or even supercritical). The coefficient $a \in L^\infty(\Omega)$ is allowed to change its sign, while its size is relevant only for the threshold value $p = 1$ when the behaviour of $f(s)/s$ also plays a crucial role in both cases.

7.2 Effects of the competition

In this section we present the main results of this chapter, establishing as well some interesting connections between the behaviour of certain algebraic equations and PDEs.

We notice that our problem (P_λ) can be compared with elliptic problems involving the so-called *concave-convex* nonlinearities, see Ambrosetti-Brézis-Cerami [5], de Figueiredo-Gossez-Ubilla [117], [116]. In such a case, the sublinear term u^p and the superlinear term $f(u) = u^q$ compete with each other, where $0 \leq p < 1 < q \leq (N+2)/(N-2) = 2^* - 1$. As a consequence of this competition, problem (P_λ) has at least two positive solutions for small $\lambda > 0$ and no positive solution for large λ. Since $u \mapsto -\Delta u$ is a linear map, the above statement is well reflected by the algebraic equation

$$s = \lambda s^p + s^q, \quad s > 0. \tag{$E_{p,q}^\lambda$}$$

Indeed, there exists a $\lambda^* > 0$ such that for every $\lambda \in (0, \lambda^*)$ equation $(E_{p,q}^\lambda)$ has two solutions, $(E_{p,q}^{\lambda^*})$ has one solution, and for $\lambda > \lambda^*$ equation $(E_{p,q}^\lambda)$ has no solution, see Figure 7.1.

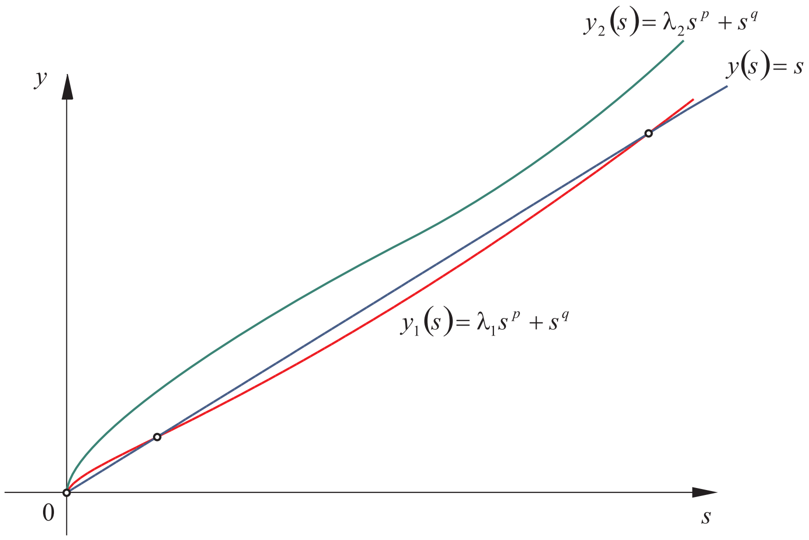

Fig. 7.1. Let $0 \le p < 1 < q$. For every $\lambda_1 < \lambda^* < \lambda_2$, the algebraic equation $(E_{p,q}^{\lambda_1})$ has two solutions, while $(E_{p,q}^{\lambda_2})$ has no solution.

Equations involving oscillatory terms usually give infinitely many distinct solutions, see Kristály [168], Kristály-Moroşanu-Tersian [173], Omari-Zanolin [229], Saint Raymond [265]. However, surprising facts may occur even in simple cases; indeed, if $p = 1$ and we consider the oscillatory function $f(s) = f_\mu(s) = \mu \sin s$ ($\mu \in \mathbb{R}$), problem (P_λ) has only the trivial solution whenever $(|\lambda| \cdot \|a\|_{L^\infty} + |\mu|)\lambda_1(\Omega) < 1$, where $\lambda_1(\Omega)$ denotes the principal eigenvalue of $-\triangle$ on $H_0^1(\Omega)$, and $\|\cdot\|_{L^\infty}$ is the $L^\infty(\Omega)$-norm. In order to obtain infinitely many distinct solutions for (P_0) we have to consider a class of functions having a suitable oscillatory behaviour; this class consists from functions which fulfill (f_1^L) and (f_2^L), $L \in \{0^+, +\infty\}$.

In the sequel, we state our main results, treating separately the two cases, i.e., when f oscillates near the *origin,* and at *infinity,* respectively. The coefficient $a \in L^\infty(\Omega)$ is allowed to be *indefinite* (i.e., it may change its sign), suggested by several recent works, including Alama-Tarantello [3], [2], Berestycki-Cappuzzo Dolcetta-Nirenberg [35], de Figueiredo-Gossez-Ubilla [117], [116], Servadei [270].

A. Oscillation near the origin. Let $f \in C([0, \infty), \mathbb{R})$ and $F(s) = \int_0^s f(t)dt$, $s \ge 0$. We assume

(f_1^0) $-\infty < \liminf_{s\to 0^+} \frac{F(s)}{s^2}$; $\limsup_{s\to 0^+} \frac{F(s)}{s^2} = +\infty$;

(f_2^0) $l_0 := \liminf_{s\to 0^+} \frac{f(s)}{s} < 0$.

Remark 7.1 Hypotheses (f_1^0) and (f_2^0) imply an oscillatory behaviour of f near the origin. Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $0 < \alpha < 1 < \alpha + \beta$, and $\gamma \in (0,1)$. The function $f_0 : [0,\infty) \to \mathbb{R}$ defined by $f_0(0) = 0$ and $f_0(s) = s^\alpha(\gamma + \sin s^{-\beta})$, $s > 0$, verifies (f_1^0) and (f_2^0), respectively.

Theorem 7.1 (Case $p \geq 1$) *Assume $a \in L^\infty(\Omega)$ and let $f \in C([0,\infty), \mathbb{R})$ satisfy (f_1^0) and (f_2^0). If*

(i) *either $p = 1$ and $\lambda a(x) < \lambda_0$ a.e. $x \in \Omega$ for some $0 < \lambda_0 < -l_0$,*

(ii) *or $p > 1$ and $\lambda \in \mathbb{R}$ is arbitrary,*

then there exists a sequence $\{u_i^0\}_i \subset H_0^1(\Omega)$ of distinct weak solutions of problem (P_λ) *such that*

$$\lim_{i\to\infty} \|u_i^0\|_{H_0^1} = \lim_{i\to\infty} \|u_i^0\|_{L^\infty} = 0. \tag{7.1}$$

Remark 7.2 (i) If $l_0 = -\infty$, then (i) holds for *every* $\lambda \in \mathbb{R}$. For instance, this may happen for f_0 from Remark 7.1.

(ii) Notice that $p > 1$ may be *critical* or even *supercritical* in Theorem 7.1(ii). Having a suitable nonlinearity oscillating near the origin, Theorem 7.1 roughly says that the term defined by $s \mapsto s^p$ ($s \geq 0$) does not affect the number of distinct solutions of (P_λ) whenever $p > 1$; this is also the case for certain values of $\lambda \in \mathbb{R}$ when $p = 1$. A similar relation may be stated as before for both the equation ($E_{p,q}^\lambda$) and the elliptic problem involving concave-convex nonlinearities. Namely, the thesis of Theorem 7.1 is nicely illustrated by the equation

$$s = \lambda s^p + f_0(s),\ s \geq 0, \tag{E_0}$$

where f_0 is the function appearing in Remark 7.1. Since $l_0 = -\infty$, for *every* $\lambda \in \mathbb{R}$ and $p \geq 1$, equation (E_0) has *infinitely many* distinct positive solutions.

On the other hand, this phenomenon dramatically changes when $p < 1$. In this case, the term $s \mapsto s^p$ ($s \geq 0$) may compete with the function f_0 near the origin such that the number of distinct solutions of (E_0) becomes *finite* for many values of λ; this fact happens when $0 < p < \alpha$ (α is the number defined in Remark 7.1). However, the number of distinct solutions to (E_0) becomes greater and greater if $|\lambda|$ gets smaller and smaller as a simple (graphical) argument shows. See Figure 7.2.

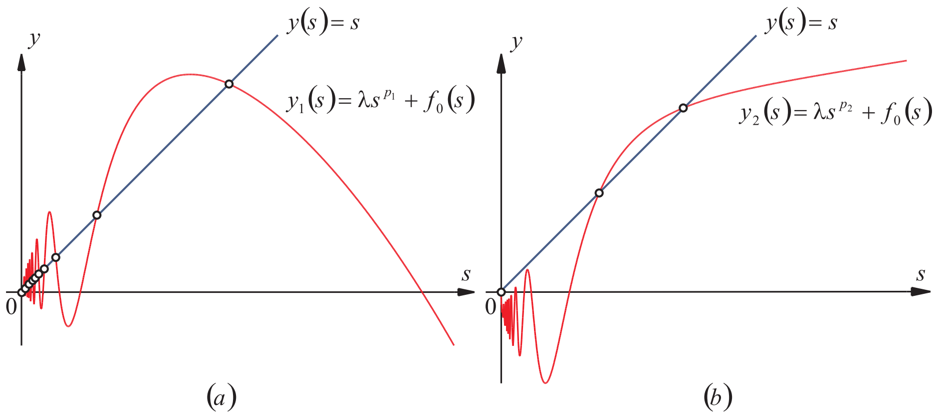

Fig. 7.2. For any $\lambda \in \mathbb{R}$ and $p = p_1 > 1$, (E_0) has infinitely many distinct solutions, see (a). However, for many $\lambda \in \mathbb{R}$ and $p = p_2 < 1$, (E_0) has only finitely many distinct solutions, see (b).

In the language of our Dirichlet problem (P_λ), the latter statement is perfectly described by the following result.

Theorem 7.2 (Case $0 < p < 1$) *Assume $a \in L^\infty(\Omega)$. Let $f \in C([0,\infty),\mathbb{R})$ satisfy (f_1^0) and (f_2^0), and $0 < p < 1$. Then, for every $k \in \mathbb{N}$, there exists $\lambda_k^0 > 0$ such that (P_λ) has at least k distinct weak solutions $\{u_{1,\lambda}^0, ..., u_{k,\lambda}^0\} \subset H_0^1(\Omega)$ whenever $\lambda \in [-\lambda_k^0, \lambda_k^0]$. Moreover,*

$$\|u_{i,\lambda}^0\|_{H_0^1} < i^{-1} \quad \textit{and} \quad \|u_{i,\lambda}^0\|_{L^\infty} < i^{-1} \quad \textit{for any } i = \overline{1,k};\ \lambda \in [-\lambda_k^0, \lambda_k^0]. \tag{7.2}$$

B. Oscillation at infinity. Let $f \in C([0,\infty),\mathbb{R})$. We assume

(f_1^∞) $-\infty < \liminf_{s\to\infty} \frac{F(s)}{s^2}$; $\limsup_{s\to\infty} \frac{F(s)}{s^2} = +\infty$;

(f_2^∞) $l_\infty := \liminf_{s\to\infty} \frac{f(s)}{s} < 0$.

Remark 7.3 Hypotheses (f_1^∞) and (f_2^∞) imply an oscillatory behaviour of f at infinity. Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $1 < \alpha$, $|\alpha - \beta| < 1$, and $\gamma \in (0,1)$. Then, the function $f_\infty : [0,\infty) \to \mathbb{R}$ defined by $f_\infty(s) = s^\alpha(\gamma + \sin s^\beta)$ verifies the hypotheses (f_1^∞) and (f_2^∞), respectively.

The counterpart of Theorem 7.1 can be stated as follows.

Theorem 7.3 (Case $p \leq 1$) *Assume $a \in L^\infty(\Omega)$. Let $f \in C([0,\infty),\mathbb{R})$ satisfy (f_1^∞) and (f_2^∞) with $f(0)=0$. If*

(i) *either $p = 1$ and $\lambda a(x) < \lambda_\infty$ a.e. $x \in \Omega$ for some $0 < \lambda_\infty < -l_\infty$,*

(ii) *or $p < 1$ and $\lambda \in \mathbb{R}$ is arbitrary,*

then there exists a sequence $\{u_i^\infty\}_i \subset H_0^1(\Omega)$ of distinct, weak solutions of problem (P_λ) *such that*

$$\lim_{i\to\infty} \|u_i^\infty\|_{L^\infty} = \infty. \tag{7.3}$$

Remark 7.4 If

$$\sup_{s\in[0,\infty)} \frac{|f(s)|}{1+s^{2^*-1}} < \infty \tag{7.4}$$

then we also have $\lim_{i\to\infty} \|u_i^\infty\|_{H_0^1} = \infty$ in Theorem 7.3. For details, see Section 7.5.

Remark 7.5 A similar observation can be made as in Remark 7.2. Indeed, when f oscillates at infinity, Theorem 7.3 shows that the term defined by $s \mapsto s^p$ $(s \geq 0)$ does not affect the number of distinct solutions of (P_λ) whenever $p < 1$. This is also the case for certain values of $\lambda \in \mathbb{R}$ when $p = 1$. A similar phenomenon occurs in the equation

$$s = \lambda s^p + f_\infty(s), \ \ s \geq 0, \tag{E_∞}$$

where f_∞ is the function defined in Remark 7.3. Since $l_\infty = -\infty$, for *every* $\lambda \in \mathbb{R}$ and $p \leq 1$, equation (E_∞) has *infinitely many* distinct positive solutions.

On the other hand, when $p > 1$, the term $s \mapsto s^p$ $(s \geq 0)$ may dominate the function f_∞ at infinity. In particular, when $\alpha < p$, the number of distinct solutions of (E_∞) may become *finite* for many values of λ (here, α is the number defined in Remark 7.3). The positive finding is that the number of distinct solutions for (E_∞) increases whenever $|\lambda|$ decreases to zero. See Figure 7.3.

In view of this observation, we obtain a natural counterpart of Theorem 7.2.

Theorem 7.4 (Case $p > 1$) *Assume $a \in L^\infty(\Omega)$. Let $f \in C([0,\infty),\mathbb{R})$ satisfy (f_1^∞) and (f_2^∞) with $f(0)=0$, and $p > 1$. Then, for every $k \in \mathbb{N}$, there exists $\lambda_k^\infty > 0$ such that* (P_λ) *has at least k distinct weak solutions $\{u_{1,\lambda}^\infty, \ldots, u_{k,\lambda}^\infty\} \subset H_0^1(\Omega)$ whenever $\lambda \in [-\lambda_k^\infty, \lambda_k^\infty]$. Moreover,*

$$\|u_{i,\lambda}^\infty\|_{L^\infty} > i-1 \quad \textit{for any } i = \overline{1,k}; \ \ \lambda \in [-\lambda_k^\infty, \lambda_k^\infty]. \tag{7.5}$$

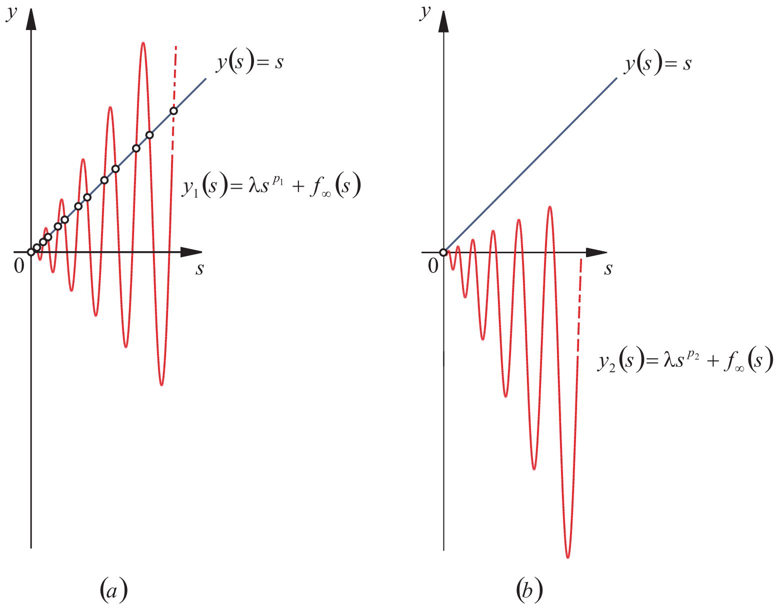

Fig. 7.3. For any $\lambda \in \mathbb{R}$ and $p = p_1 < 1$, (E_∞) has infinitely many distinct solutions, see (a). However, for many $\lambda \in \mathbb{R}$ and $p = p_2 > 1$, (E_∞) has only finitely many (or, even none) distinct solutions, see (b).

Remark 7.6 If f verifies (7.4) and $p \le 2^* - 1$ in Theorem 7.4, then

$$\|u_{i,\lambda}^\infty\|_{H_0^1} > i - 1 \quad \text{for any } i = \overline{1,k};\ \lambda \in [-\lambda_k^\infty, \lambda_k^\infty].$$

We conclude this section by stating a result for a model problem which involves concave-convex nonlinearities and an oscillatory term. We consider the problem

$$\begin{cases} -\triangle u = \lambda u^p + \mu u^q + f(u),\ u \ge 0 & \text{on} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{$\mathrm{P}_{\lambda,\mu}$}$$

where $0 < p < 1 < q$, and $\lambda, \mu \in \mathbb{R}$. The following result proves that the number of solutions $(\mathrm{P}_{\lambda,\mu})$ is influenced

(i) by the sublinear term when f oscillates near the origin (with no effect of the superlinear term); and alternatively,
(ii) by the superlinear term when f oscillates at infinity (with no effect of the sublinear term).

More precisely, applying Theorems 7.2 and 7.4, we have

Theorem 7.5 *Let $f \in C([0,\infty),\mathbb{R})$ and $0 < p < 1 < q$.*

(i) *If (f_1^0) and (f_2^0) hold, then for every $k \in \mathbb{N}$ and $\mu \in \mathbb{R}$, there exists $\lambda_{k,\mu} > 0$ such that $(\mathrm{P}_{\lambda,\mu})$ has at least k distinct weak solutions in $H_0^1(\Omega)$ whenever $\lambda \in [-\lambda_{k,\mu}, \lambda_{k,\mu}]$.*

(ii) *If (f_1^∞) and (f_2^∞) hold with $f(0) = 0$, then for every $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, there exists $\mu_{k,\lambda} > 0$ such that $(\mathrm{P}_{\lambda,\mu})$ has at least k distinct weak solutions in $H_0^1(\Omega)$ whenever $\mu \in [-\mu_{k,\lambda}, \mu_{k,\lambda}]$.*

7.3 A general location result

We consider the problem

$$\begin{cases} -\triangle u + K(x)u = h(x,u), \quad u \geq 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{P_h^K}$$

and assume that

(H_K): $K \in L^\infty(\Omega)$, $\mathrm{essinf}_\Omega K > 0$;

(H_h^1): $h : \Omega \times [0,\infty) \to \mathbb{R}$ is a Carathéodory function, $h(x,0) = 0$ for a.e. $x \in \Omega$, and there is $M_h > 0$ such that $|h(x,s)| \leq M_h$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$;

(H_h^2): there are $0 < \delta < \eta$ such that $h(x,s) \leq 0$ for a.e. $x \in \Omega$ and all $s \in [\delta,\eta]$.

We extend the function h by $h(x,s) = 0$ for a.e. $x \in \Omega$ and $s \leq 0$. We introduce the energy functional $\mathcal{E} : H_0^1(\Omega) \to \mathbb{R}$ associated with problem (P_h^K), defined by

$$\mathcal{E}(u) = \frac{1}{2}\|u\|_{H_0^1}^2 + \frac{1}{2}\int_\Omega K(x)u^2dx - \int_\Omega H(x,u(x))dx, \quad u \in H_0^1(\Omega),$$

where $H(x,s) = \int_0^s h(x,t)dt$, $s \in \mathbb{R}$. Due to hypothesis (H_h^1), it is easy to see that $\mathcal{E}$ is well-defined. Moreover, standard arguments show that $\mathcal{E}$ is of class C^1 on $H_0^1(\Omega)$.

Finally, considering the number $\eta \in \mathbb{R}$ from (H_h^2), we introduce the set

$$W^\eta = \{u \in H_0^1(\Omega) : \|u\|_{L^\infty} \leq \eta\}.$$

Since $h(x,0) = 0$, then 0 is clearly a solution of (P_h^K). In the sequel, under some general assumptions, we guarantee the existence of a (possible trivial) weak solution of (P_h^K) which is indispensable in our further investigations (see Sections 7.4 and 7.5).

Theorem 7.6 *Assume that* (H_K), (H_h^1), (H_h^2) *hold. Then*

(i) *the functional* $\mathcal{E}$ *is bounded from below on* W^η *and its infimum is attained at some* $\tilde{u} \in W^\eta$*;*
(ii) $\tilde{u}(x) \in [0, \delta]$ *for a.e.* $x \in \Omega$*;*
(iii) $\tilde{u}$ *is a weak solution of* (P_h^K).

Proof. (i) Due to (H_h^1) and by using Hölder's and Poincaré's inequalities, the functional $\mathcal{E}$ is bounded from below on the whole space $H_0^1(\Omega)$. In addition, one can easily see that $\mathcal{E}$ is sequentially weak lower semicontinuous and the set W^η is convex and closed in $H_0^1(\Omega)$, thus weakly closed. Combining these facts, there is an element $\tilde{u} \in W^\eta$ which is a minimum point of $\mathcal{E}$ over W^η.

(ii) Let $A = \{x \in \Omega : \tilde{u}(x) \notin [0, \delta]\}$ and suppose that $m(A) > 0$. Here and in the sequel, $m(\cdot)$ denotes the Lebesgue measure. Define the function $\gamma : \mathbb{R} \to \mathbb{R}$ by $\gamma(s) = \min(s_+, \delta)$, where $s_+ = \max(s, 0)$. Now, set $w = \gamma \circ \tilde{u}$. Since γ is a Lipschitz function and $\gamma(0) = 0$, the theorem of Marcus-Mizel [201] shows that $w \in H_0^1(\Omega)$. Moreover, $0 \le w(x) \le \delta$ for a.e. Ω. Consequently, $w \in W^\eta$.

We introduce the sets $A_1 = \{x \in A : \tilde{u}(x) < 0\}$ and $A_2 = \{x \in A : \tilde{u}(x) > \delta\}$. Thus, $A = A_1 \cup A_2$, and we have that $w(x) = \tilde{u}(x)$ for all $x \in \Omega \setminus A$, $w(x) = 0$ for all $x \in A_1$, and $w(x) = \delta$ for all $x \in A_2$. Moreover, we have

$$
\begin{aligned}
&\mathcal{E}(w) - \mathcal{E}(\tilde{u}) = \\
&= \frac{1}{2}\left[\|w\|_{H_0^1}^2 - \|\tilde{u}\|_{H_0^1}^2\right] + \frac{1}{2}\int_\Omega K(x)\left[w^2 - \tilde{u}^2\right] - \int_\Omega [H(x, w) - H(x, \tilde{u})] \\
&= -\frac{1}{2}\int_A |\nabla \tilde{u}|^2 + \frac{1}{2}\int_A K(x)[w^2 - \tilde{u}^2] - \int_A [H(x, w) - H(x, \tilde{u})].
\end{aligned}
$$

Since $\mathrm{essinf}_\Omega K > 0$, one has

$$\int_A K(x)[w^2 - \tilde{u}^2] = -\int_{A_1} K(x)\tilde{u}^2 + \int_{A_2} K(x)[\delta^2 - \tilde{u}^2] \le 0.$$

Due to the fact that $h(x, s) = 0$ for all $s \le 0$, one has

$$\int_{A_1} [H(x, w) - H(x, \tilde{u})] = 0.$$

By the mean value theorem, for a.e. $x \in A_2$, there exists $\theta(x) \in [\delta, \tilde{u}(x)] \subseteq [\delta, \eta]$ such that

$$H(x, w(x)) - H(x, \tilde{u}(x)) = H(x, \delta) - H(x, \tilde{u}(x)) = h(x, \theta(x))(\delta - \tilde{u}(x)).$$

Thus, on account of (H_h^2), one has

$$\int_{A_2} [H(x,w) - H(x,\tilde{u})] \geq 0.$$

Consequently, every term of the expression $\mathcal{E}(w) - \mathcal{E}(\tilde{u})$ is non-positive. On the other hand, since $w \in W^\eta$, then $\mathcal{E}(w) \geq \mathcal{E}(\tilde{u}) = \inf_{W^\eta} \mathcal{E}$. So, every term in $\mathcal{E}(w) - \mathcal{E}(\tilde{u})$ should be zero. In particular,

$$\int_{A_1} K(x)\tilde{u}^2 = \int_{A_2} K(x)[\tilde{u}^2 - \delta^2] = 0.$$

Due to (H_K), we necessarily have $m(A) = 0$, contradicting our assumption.

(iii) Let us fix $v \in C_0^\infty(\Omega)$ and let $\varepsilon_0 = (\eta - \delta)/(\|v\|_{C_0} + 1) > 0$. Define the function $E : [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}$ by $E(\varepsilon) = \mathcal{E}(\tilde{u} + \varepsilon v)$ with $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. Due to (ii), for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, the element $\tilde{u} + \varepsilon v$ belongs to the set W^η. Consequently, due to (i), one has $E(\varepsilon) \geq E(0)$ for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. Since E is differentiable at 0 and $E'(0) = 0$ it follows that $\langle \mathcal{E}'(\tilde{u}), v\rangle_{H_0^1} = 0$. Since $v \in C_0^\infty(\Omega)$ is arbitrary, and the set $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, we obtain that $\tilde{u}$ is a weak solution of (P_h^K). $\square$

We conclude this section by constructing a special function which will be useful in the proof of our theorems. In the sequel, let $B(x_0, r) \subset \Omega$ be the N-dimensional ball with radius $r > 0$ and center $x_0 \in \Omega$. For $s > 0$, define

$$z_s(x) = \begin{cases} 0, & \text{if} \quad x \in \Omega \setminus B(x_0, r); \\ s, & \text{if} \quad x \in B(x_0, r/2); \\ \frac{2s}{r}(r - |x - x_0|), & \text{if} \quad x \in B(x_0, r) \setminus B(x_0, r/2). \end{cases} \tag{7.6}$$

It is clear that $z_s \in H_0^1(\Omega)$. Moreover, we have $\|z_s\|_{L^\infty} = s$ and

$$\|z_s\|_{H_0^1}^2 = \int_\Omega |\nabla z_s|^2 = 4r^{N-2}(1 - 2^{-N})\omega_N s^2 \equiv C(r,N)s^2 > 0, \tag{7.7}$$

where ω_N is the volume of $B(0,1) \subset \mathbb{R}^N$.

Notation. For every $\eta > 0$, we define the truncation function $\tau_\eta : [0, \infty) \to \mathbb{R}$ by $\tau_\eta(s) = \min(\eta, s)$, $s \geq 0$.

7.4 Nonlinearities with oscillation near the origin

Since parts (i) and (ii) of Theorem 7.1 will be treated simultaneously, we consider again the problem from the previous Section

$$\begin{cases} -\triangle u + K(x)u = h(x,u), \quad u \geq 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \qquad (\mathrm{P}_h^K)$$

where the potential $K : \Omega \to \mathbb{R}$ fulfills (H_K). The function $h : \Omega \times [0,\infty) \to \mathbb{R}$ is Carathéodory, and we assume

(H_0^0): $h(x,0) = 0$ for a.e. $x \in \Omega$, and there exists $s_0 > 0$ such that

$$\sup_{s\in[0,s_0]} |h(\cdot,s)| \in L^\infty(\Omega);$$

(H_1^0): $-\infty < \liminf_{s\to 0^+} \frac{H(x,s)}{s^2}$ and $\limsup_{s\to 0^+} \frac{H(x,s)}{s^2} = +\infty$ uniformly for a.e. $x \in \Omega$; here, $H(x,s) = \int_0^s h(x,t)dt$;

(H_2^0): there are two sequences $\{\delta_i\}$, $\{\eta_i\}$ with $0 < \eta_{i+1} < \delta_i < \eta_i$, $\lim_{i\to\infty} \eta_i = 0$, and $h(x,s) \leq 0$ for a.e. $x \in \Omega$ and for every $s \in [\delta_i, \eta_i]$, $i \in \mathbb{N}$.

Theorem 7.7 *Assume* (H_K), (H_0^0), (H_1^0) *and* (H_2^0) *hold. Then there exists a sequence* $\{u_i^0\}_i \subset H_0^1(\Omega)$ *of distinct weak solutions of problem* (P_h^K) *such that*

$$\lim_{i\to\infty} \|u_i^0\|_{H_0^1} = \lim_{i\to\infty} \|u_i^0\|_{L^\infty} = 0. \tag{7.8}$$

Proof Without any loss of generality, we may assume that $\{\delta_i\}_i, \{\eta_i\}_i \subset (0, s_0)$, where $s_0 > 0$ comes from (H_0^0). For every $i \in \mathbb{N}$, define the truncation function $h_i : \Omega \times [0,\infty) \to \mathbb{R}$ by

$$h_i(x,s) = h(x, \tau_{\eta_i}(s)) \tag{7.9}$$

and let $\mathcal{E}_i : H_0^1(\Omega) \to \mathbb{R}$ be the energy functional associated with the problem $(\mathrm{P}_{h_i}^K)$. Let $H_i(x,s) = \int_0^s h_i(x,t)dt$.

Due to hypotheses (H_0^0) and (H_2^0), the function h_i verifies the assumptions of Theorem 7.6 for every $i \in \mathbb{N}$ with $[\delta_i, \eta_i]$. Consequently, for every $i \in \mathbb{N}$, there exists $u_i^0 \in W^{\eta_i}$ such that

$$u_i^0 \text{ is the minimum point of the functional } \mathcal{E}_i \text{ on } W^{\eta_i}, \tag{7.10}$$

$$u_i^0(x) \in [0, \delta_i] \text{ for a.e. } x \in \Omega, \tag{7.11}$$

$$u_i^0 \text{ is a weak solution of } (\mathrm{P}_{h_i}^K). \tag{7.12}$$

Thanks to (7.9), (7.11) and (7.12), u_i^0 is a weak solution not only for $(\mathrm{P}_{h_i}^K)$ but also for the problem (P_h^K).

Now, we prove that there are infinitely many distinct elements in the sequence $\{u_i^0\}_i$. To see this, we first prove that

$$\mathcal{E}_i(u_i^0) < 0 \quad \text{for all } i \in \mathbb{N}; \tag{7.13}$$

$$\lim_{i\to\infty} \mathcal{E}_i(u_i^0) = 0. \tag{7.14}$$

The left part of (H_1^0) implies the existence of some $l_0^h > 0$ and $\zeta \in (0, \eta_1)$ such that

$$\mathrm{essinf}_{x\in\Omega} H(x,s) \geq -l_0^h s^2 \text{ for all } s \in (0,\zeta). \tag{7.15}$$

Let $L_0^h > 0$ be large enough so that

$$\frac{1}{2}C(r,N) + \left(\frac{1}{2}\|K\|_{L^\infty} + l_0^h\right) m(\Omega) < L_0^h (r/2)^N \omega_N, \tag{7.16}$$

where $r > 0$ and $C(r,N) > 0$ come from (7.7). Taking into account the right part of (H_1^0), there is a sequence $\{\tilde{s}_i\}_i \subset (0,\zeta)$ such that $\tilde{s}_i \leq \delta_i$ and

$$\mathrm{essinf}_{x\in\Omega} H(x,\tilde{s}_i) > L_0^h \tilde{s}_i^2 \quad \text{for all } i \in \mathbb{N}. \tag{7.17}$$

Let $i \in \mathbb{N}$ be a fixed number and let $z_{\tilde{s}_i} \in H_0^1(\Omega)$ be the function from (7.6) corresponding to the value $\tilde{s}_i > 0$. Then $z_{\tilde{s}_i} \in W^{\eta_i}$, and on account of (7.7), (7.17) and (7.15), one has

$$\begin{aligned}
\mathcal{E}_i(z_{\tilde{s}_i}) &= \frac{1}{2}\|z_{\tilde{s}_i}\|_{H_0^1}^2 + \frac{1}{2}\int_\Omega K(x) z_{\tilde{s}_i}^2 - \int_\Omega H_i(x, z_{\tilde{s}_i}(x))dx \\
&= \frac{1}{2}C(r,N)\tilde{s}_i^2 + \frac{1}{2}\int_\Omega K(x) z_{\tilde{s}_i}^2 \\
&\quad - \int_{B(x_0,r/2)} H(x,\tilde{s}_i)dx - \int_{B(x_0,r)\setminus B(x_0,r/2)} H(x, z_{\tilde{s}_i}(x))dx \\
&\leq \left[\frac{1}{2}C(r,N) + \frac{1}{2}\|K\|_{L^\infty} m(\Omega) - L_0^h (r/2)^N \omega_N + l_0^h m(\Omega)\right] \tilde{s}_i^2.
\end{aligned}$$

Consequently, using (7.10) and (7.16), we obtain that

$$\mathcal{E}_i(u_i^0) = \min_{W^{\eta_i}} \mathcal{E}_i \leq \mathcal{E}_i(z_{\tilde{s}_i}) < 0 \tag{7.18}$$

which proves in particular (7.13). Now, let us prove (7.14). For every $i \in \mathbb{N}$, by using the mean value theorem, (7.9), (H_0^0) and (7.11), we have

$$\mathcal{E}_i(u_i^0) \geq -\int_\Omega H_i(x, u_i^0(x))dx \geq -\|\sup_{s\in[0,s_0]} |h(\cdot,s)|\|_{L^\infty} m(\Omega)\delta_i.$$

Due to $\lim_{i\to\infty}\delta_i = 0$, the above inequality and (7.18) leads to (7.14).

On account of (7.9) and (7.11), we observe that

$$\mathcal{E}_i(u_i^0) = \mathcal{E}_1(u_i^0) \quad \text{for all } i \in \mathbb{N}.$$

Combining this relation with (7.13) and (7.14), we see that the sequence $\{u_i^0\}_i$ contains infinitely many distinct elements.

It remains to prove relation (7.8). The former limit easily follows by (7.11), i.e. $\|u_i^0\|_{L^\infty} \leq \delta_i$ for all $i \in \mathbb{N}$, combined with $\lim_{i\to\infty}\delta_i = 0$. For the latter limit, we use (7.18), (H_0^0), (7.9) and (7.11), obtaining for all $i \in \mathbb{N}$ that

$$\begin{aligned} \frac{1}{2}\|u_i^0\|^2_{H_0^1} &\leq \frac{1}{2}\|u_i^0\|^2_{H_0^1} + \frac{1}{2}\int_\Omega K(x)(u_i^0)^2 \\ &< \int_\Omega H_i(x, u_i^0(x)) \leq \|\sup_{s\in[0,s_0]}|h(\cdot,s)|\|_{L^\infty} m(\Omega)\delta_i \end{aligned}$$

which concludes the proof of Theorem 7.7. □

Proof of Theorem 7.1. (i) *Case* $p = 1$. Let $\lambda \in \mathbb{R}$ as in the hypothesis, i.e., $\lambda a(x) < \lambda_0$ a.e. $x \in \Omega$ for some $0 < \lambda_0 < -l_0$. Let us choose $\tilde{\lambda}_0 \in (\lambda_0, -l_0)$ and

$$K(x) = \tilde{\lambda}_0 - \lambda a(x) \quad \text{and } h(x,s) = \tilde{\lambda}_0 s + f(s) \text{ for all } (x,s) \in \Omega\times[0,\infty). \tag{7.19}$$

Note that $\mathrm{essinf}_\Omega K \geq \tilde{\lambda}_0 - \lambda_0 > 0$, so ($\mathrm{H}_K$) is satisfied. Due to ($f_1^0$) and ($f_2^0$), we have $f(0) = 0$. Thus, (H_0^0) clearly holds. Moreover, since $H(x,s)/s^2 = \tilde{\lambda}_0/2 + F(s)/s^2$, $s > 0$, hypothesis (f_1^0) implies (H_1^0). Finally, since $l_0 < -\tilde{\lambda}_0$, there exists a sequence $\{s_i\}_i \subset (0,1)$ converging to 0 such that $f(s_i)/s_i < -\tilde{\lambda}_0$ for all $i \in \mathbb{N}$. Consequently, by using the continuity of f, we may choose two sequences $\{\delta_i\}_i, \{\eta_i\}_i \subset (0,1)$ such that $0 < \eta_{i+1} < \delta_i < s_i < \eta_i$, $\lim_{i\to\infty}\eta_i = 0$, and $\tilde{\lambda}_0 s + f(s) \leq 0$ for all $s \in [\delta_i, \eta_i]$ and $i \in \mathbb{N}$. Therefore, (H_2^0) holds too. It remains to apply Theorem 7.7, observing that (P_h^K) is equivalent to problem (P_λ) via the choice (7.19).

(ii) *Case* $p > 1$. Let $\lambda \in \mathbb{R}$ be arbitrary fixed. Let us also fix a number $\lambda_0 \in (0, -l_0)$ and choose

$$K(x) = \lambda_0 \quad \text{and } h(x,s) = \lambda a(x)s^p + \lambda_0 s + f(s) \text{ for all } (x,s) \in \Omega\times[0,\infty). \tag{7.20}$$

Clearly, (H_K) is satisfied. Since $a \in L^\infty(\Omega)$, a simple argument yields that (H_0^0) also holds. Moreover, since $p > 1$ and $H(x,s)/s^2 = \lambda a(x)s^{p-1}/(p+$

$1) + \lambda_0/2 + F(s)/s^2$, $s > 0$, hypothesis (f_1^0) implies (H_1^0). Note that for a.e $x \in \Omega$ and every $s \in [0, \infty)$, we have

$$h(x,s) \leq |\lambda| \cdot \|a\|_{L^\infty} s^p + \lambda_0 s + f(s) \equiv \tilde{h}_0(s). \tag{7.21}$$

Due to (f_2^0),

$$\liminf_{s \to 0^+} \frac{\tilde{h}_0(s)}{s} = \lambda_0 + l_0 < 0.$$

In particular, there exists a sequence $\{s_i\}_i \subset (0,1)$ converging to 0 such that $\tilde{h}_0(s_i) < 0$ for all $i \in \mathbb{N}$. Consequently, by using the continuity of $\tilde{h}_0$, we can choose two sequences $\{\delta_i\}_i, \{\eta_i\}_i \subset (0,1)$ such that $0 < \eta_{i+1} < \delta_i < s_i < \eta_i$, $\lim_{i\to\infty} \eta_i = 0$, and $\tilde{h}_0(s) \leq 0$ for all $s \in [\delta_i, \eta_i]$ and $i \in \mathbb{N}$. Therefore, by using (7.21), hypothesis (H_2^0) holds. Now, we can apply Theorem 7.7; problem (P_h^K) is equivalent to problem (P_λ) through the choice (7.20). In both cases (i.e., (i) and (ii)), relation (7.1) is implied by (7.8). This completes the proof of Theorem 7.1. $\square$

Proof of Theorem 7.2. The proof is divided into four steps.

Step 1. Let $\lambda_0 \in (0, -l_0)$. On account of (f_1^0), there exists a sequence $\{s_i\}_i \subset (0,1)$ converging to 0, such that $f(s_i)/s_i < -\lambda_0$. For every $\lambda \in \mathbb{R}$ define the functions $h^\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}$ by

$$h^\lambda(x,s) = \lambda a(x) s^p + \lambda_0 s + f(s) \text{ for all } (x,s) \in \Omega \times [0,\infty);$$

$$\tilde{h}(\lambda, s) = |\lambda| \cdot \|a\|_{L^\infty} s^p + \lambda_0 s + f(s) \text{ for all } s \in [0, \infty).$$

Since $\tilde{h}(0, s_i) = \lambda_0 s_i + f(s_i) < 0$ and due to the continuity of $\tilde{h}$, we can choose three sequences $\{\delta_i\}_i, \{\eta_i\}_i, \{\lambda_i\}_i \subset (0,1)$ such that $0 < \eta_{i+1} < \delta_i < s_i < \eta_i$, $\lim_{i\to\infty} \eta_i = 0$, and for every $i \in \mathbb{N}$,

$$\tilde{h}(\lambda, s) \leq 0 \quad \text{for all} \quad \lambda \in [-\lambda_i, \lambda_i] \quad \text{and} \quad s \in [\delta_i, \eta_i]. \tag{7.22}$$

Clearly, we may assume that

$$\delta_i \leq \min\{i^{-1}, 2^{-1} i^{-2} [1 + \|a\|_{L^1} + m(\Omega) \max_{s\in[0,1]} |f(s)|]^{-1}\}, \quad i \in \mathbb{N}. \tag{7.23}$$

Since $h^\lambda(x,s) \leq \tilde{h}(\lambda, s)$ for a.e. $x \in \Omega$ and all $(\lambda, s) \in \mathbb{R} \times [0,\infty)$, on account of (7.22), for every $i \in \mathbb{N}$, we have

$$h^\lambda(x,s) \leq 0 \quad \text{for a.e. } x \in \Omega \text{ and all } \lambda \in [-\lambda_i, \lambda_i],\ s \in [\delta_i, \eta_i]. \tag{7.24}$$

For every $i \in \mathbb{N}$ and $\lambda \in [-\lambda_i, \lambda_i]$, let $h_i^\lambda : \Omega \times [0,\infty) \to \mathbb{R}$ be defined by

$$h_i^\lambda(x,s) = h^\lambda(x, \tau_{\eta_i}(s)) \tag{7.25}$$

and $K(x) = \lambda_0$. Let also $\mathcal{E}_{i,\lambda} : H_0^1(\Omega) \to \mathbb{R}$ be the energy functional associated with the problem $(\mathrm{P}^K_{h_i^\lambda})$, i.e.,

$$\mathcal{E}_{i,\lambda}(u) = \frac{1}{2}\|u\|^2_{H_0^1} + \frac{1}{2}\int_\Omega K(x)u^2 - \int_\Omega \left(\int_0^{u(x)} h_i^\lambda(x,s)ds\right) dx. \quad (7.26)$$

Then, for every $i \in \mathbb{N}$ and $\lambda \in [-\lambda_i, \lambda_i]$, the function h_i^λ verifies the hypotheses of Theorem 7.6; see (7.24) for $(\mathrm{H}^2_{h_i^\lambda})$. Therefore, for every $i \in \mathbb{N}$ and $\lambda \in [-\lambda_i, \lambda_i]$

$$\text{there exists } u^0_{i,\lambda} \in W^{\eta_i} \text{ such that } \mathcal{E}_{i,\lambda}(u^0_{i,\lambda}) = \min_{W^{\eta_i}} \mathcal{E}_{i,\lambda}; \quad (7.27)$$

$$u^0_{i,\lambda}(x) \in [0, \delta_i] \text{ for a.e. } x \in \Omega, \quad (7.28)$$

$$u^0_{i,\lambda} \text{ is a weak solution of } (\mathrm{P}^K_{h_i^\lambda}). \quad (7.29)$$

Due to the definition of the functions h_i^λ and K, $u^0_{i,\lambda}$ is a weak solution not only for $(\mathrm{P}^K_{h_i^\lambda})$, see (7.25), (7.28) and (7.29), but also for our initial problem (P_λ) once we guarantee that $u^0_{i,\lambda} \not\equiv 0$.

Step 2. For $\lambda = 0$, the function $h_i^\lambda = h_i^0$ verifies the hypotheses of Theorem 7.7; more precisely, h_i^0 is precisely the function appearing in (7.9) and $\mathcal{E}_i := \mathcal{E}_{i,0}$ is the energy functional associated with problem $(\mathrm{P}^K_{h_i^0})$. Consequently, besides (7.27)-(7.29), the elements $u_i^0 := u^0_{i,0}$ also verify

$$\mathcal{E}_i(u_i^0) = \min_{W^{\eta_i}} \mathcal{E}_i \le \mathcal{E}_i(z_{\tilde{s}_i}) < 0 \text{ for all } i \in \mathbb{N}, \quad (7.30)$$

where $z_{\tilde{s}_i} \in W^{\eta_i}$ come from the proof of Theorem 7.7, see (7.18).

Step 3. Let $\{\theta_i\}_i$ be a sequence with negative terms such that $\lim_{i\to\infty}\theta_i = 0$. On account of (7.30), up to a subsequence, we may assume that

$$\theta_i < \mathcal{E}_i(u_i^0) \le \mathcal{E}_i(z_{\tilde{s}_i}) < \theta_{i+1}. \quad (7.31)$$

Let

$$\lambda_i' = \frac{(p+1)(\theta_{i+1} - \mathcal{E}_i(z_{\tilde{s}_i}))}{\|a\|_{L^1}+1} \text{ and } \lambda_i'' = \frac{(p+1)(\mathcal{E}_i(u_i^0) - \theta_i)}{\|a\|_{L^1}+1}, \ i \in \mathbb{N}.$$

Fix $k \in \mathbb{N}$. On account of (7.31),

$$\lambda_k^0 = \min(\lambda_1, ..., \lambda_k, \lambda_1', ..., \lambda_k', \lambda_1'', ..., \lambda_k'') > 0.$$

Then, for every $i \in \{1, ..., k\}$ and $\lambda \in [-\lambda_k^0, \lambda_k^0]$ we have

$$\begin{aligned}
\mathcal{E}_{i,\lambda}(u_{i,\lambda}^0) &\leq \mathcal{E}_{i,\lambda}(z_{\tilde{s}_i}) && \text{(see (7.27))} \\
&= \frac{1}{2}\|z_{\tilde{s}_i}\|_{H_0^1}^2 - \frac{\lambda}{p+1}\int_\Omega a(x) z_{\tilde{s}_i}^{p+1} - \int_\Omega F(z_{\tilde{s}_i}(x))dx \\
&= \mathcal{E}_i(z_{\tilde{s}_i}) - \frac{\lambda}{p+1}\int_\Omega a(x) z_{\tilde{s}_i}^{p+1} \\
&< \theta_{i+1}, && \text{(see the choice of } \lambda_i' \text{ and } \tilde{s}_i \leq \delta_i < 1)
\end{aligned}$$

and taking into account that $u_{i,\lambda}^0$ belongs to W^{η_i}, and u_i^0 is the minimum point of $\mathcal{E}_i$ over the set W^{η_i}, see relation (7.30), we have

$$\begin{aligned}
\mathcal{E}_{i,\lambda}(u_{i,\lambda}^0) &= \mathcal{E}_i(u_{i,\lambda}^0) - \frac{\lambda}{p+1}\int_\Omega a(x)(u_{i,\lambda}^0)^{p+1} \\
&\geq \mathcal{E}_i(u_i^0) - \frac{\lambda}{p+1}\int_\Omega a(x)(u_{i,\lambda}^0)^{p+1} \\
&> \theta_i. && \text{(see the choice of } \lambda_i'' \text{ and (7.28))}
\end{aligned}$$

In conclusion, for every for every $i \in \{1, ..., k\}$ and $\lambda \in [-\lambda_k^0, \lambda_k^0]$ we have

$$\theta_i < \mathcal{E}_{i,\lambda}(u_{i,\lambda}^0) < \theta_{i+1} < 0,$$

thus

$$\mathcal{E}_{1,\lambda}(u_{1,\lambda}^0) < ... < \mathcal{E}_{k,\lambda}(u_{k,\lambda}^0) < 0.$$

But, $u_{i,\lambda}^0 \in W^{\eta_1}$ for every $i \in \{1, ..., k\}$, so $\mathcal{E}_{i,\lambda}(u_{i,\lambda}^0) = \mathcal{E}_{1,\lambda}(u_{i,\lambda}^0)$, see relation (7.25). Therefore, from above, we obtain that for every $\lambda \in [-\lambda_k^0, \lambda_k^0]$,

$$\mathcal{E}_{1,\lambda}(u_{1,\lambda}^0) < ... < \mathcal{E}_{1,\lambda}(u_{k,\lambda}^0) < 0 = \mathcal{E}_{1,\lambda}(0).$$

These inequalities show that the elements $u_{1,\lambda}^0, ..., u_{k,\lambda}^0$ are distinct (and non-trivial) whenever $\lambda \in [-\lambda_k^0, \lambda_k^0]$.

Step 4. It remains to prove conclusion (7.2). The former relation follows directly by (7.28) and (7.23). To check the latter, we observe that for every $i \in \{1, ..., k\}$ and $\lambda \in [-\lambda_k^0, \lambda_k^0]$,

$$\mathcal{E}_{1,\lambda}(u_{i,\lambda}^0) = \mathcal{E}_{i,\lambda}(u_{i,\lambda}^0) < \theta_{i+1} < 0.$$

Consequently, for every $i \in \{1, ..., k\}$ and $\lambda \in [-\lambda_k^0, \lambda_k^0]$, by a mean value theorem we obtain

$$\begin{aligned}
\frac{1}{2}\|u_{i,\lambda}^0\|_{H_0^1}^2 &< \frac{\lambda}{p+1}\int_\Omega a(x)(u_{i,\lambda}^0)^{p+1} + \int_\Omega F(u_{i,\lambda}^0(x))dx \\
&\le \left[\frac{1}{p+1}\|a\|_{L^1} + m(\Omega)\max_{s\in[0,1]}|f(s)|\right]\delta_i \quad \text{(see (7.28) and } \delta_{\mathrm{i}}, \lambda_k^0 \le 1) \\
&< 2^{-1}i^{-2}, \qquad\qquad \text{(see (7.23))}
\end{aligned}$$

which concludes the proof of Theorem 7.2. □

7.5 Nonlinearities with oscillation at infinity

In order to prove Theorems 7.3 and 7.4 we follow more or less the technique of the previous Section. However, for completeness, we give all the details. We consider again the problem (P_h^K), where the Carathéodory function $h : \Omega \times [0, \infty) \to \mathbb{R}$ fulfills

(H_0^∞): $h(x, 0) = 0$ for a.e. $x \in \Omega$, and for every $s \ge 0$,

$$\sup_{t\in[0,s]} |h(\cdot, t)| \in L^\infty(\Omega);$$

(H_1^∞): $-\infty < \liminf_{s\to\infty} \frac{H(x,s)}{s^2}$ and $\limsup_{s\to\infty} \frac{H(x,s)}{s^2} = +\infty$ uniformly for a.e. $x \in \Omega$; here, $H(x, s) = \int_0^s h(x, t)dt$;

(H_2^∞): there are two sequences $\{\delta_i\}$, $\{\eta_i\}$ with $0 < \delta_i < \eta_i < \delta_{i+1}$, $\lim_{i\to\infty} \delta_i = +\infty$, and $h(x, s) \le 0$ for a.e. $x \in \Omega$ and for every $s \in [\delta_i, \eta_i]$, $i \in \mathbb{N}$.

Theorem 7.8 *Assume* (H_K), (H_0^∞), (H_1^∞) *and* (H_2^∞) *hold. Then there exists a sequence* $\{u_i^\infty\}_i \subset H_0^1(\Omega)$ *of distinct weak solutions of problem* (P_h^K) *such that*

$$\lim_{i\to\infty} \|u_i^\infty\|_{L^\infty} = \infty. \tag{7.32}$$

Proof. For any $i \in \mathbb{N}$, we introduce the truncation function $h_i : \Omega \times [0, \infty) \to \mathbb{R}$ by

$$h_i(x, s) = h(x, \tau_{\eta_i}(s)). \tag{7.33}$$

Let $\mathcal{E}_i : H_0^1(\Omega) \to \mathbb{R}$ be the energy functional associated with problem $(\mathrm{P}_{h_i}^K)$. As before, let $H_i(x, s) = \int_0^s h_i(x, t)dt$.

On account of hypotheses (H_0^∞) and (H_2^∞), h_i fulfills the assumptions of Theorem 7.6 for every $i \in \mathbb{N}$ with $[\delta_i, \eta_i]$. Thus, for every $i \in \mathbb{N}$, there is an element $u_i^\infty \in W^{\eta_i}$ such that

$$u_i^\infty \text{ is the minimum point of the functional } \mathcal{E}_i \text{ on } W^{\eta_i}, \tag{7.34}$$

$$u_i^\infty(x) \in [0, \delta_i] \text{ for a.e. } x \in \Omega, \tag{7.35}$$

$$u_i^\infty \text{ is a weak solution of } (\mathrm{P}_{h_i}^K). \tag{7.36}$$

Due to (7.36), (7.33) and (7.35), u_i^∞ is also a weak solution to problem (P_h^K).

We prove that there are infinitely many distinct elements in the sequence $\{u_i^\infty\}_i$. To this end, it is enough to show that

$$\lim_{i\to\infty} \mathcal{E}_i(u_i^\infty) = -\infty. \tag{7.37}$$

Indeed, let us assume that in the sequence $\{u_i^\infty\}_i$ there are only finitely many distinct elements, say $\{u_1^\infty, ..., u_{i_0}^\infty\}$ for some $i_0 \in \mathbb{N}$. Consequently, due to (7.33), the sequence $\{\mathcal{E}_i(u_i^\infty)\}_i$ reduces to at most the finite set $\{\mathcal{E}_{i_0}(u_1^\infty), ..., \mathcal{E}_{i_0}(u_{i_0}^\infty)\}$, which contradicts relation (7.37).

Now, we prove (7.37). By (H_1^∞), there exist $l_\infty^h > 0$ and $\zeta > 0$ such that

$$\mathrm{essinf}_{x\in\Omega} H(x,s) \geq -l_\infty^h s^2 \text{ for all } s > \zeta. \tag{7.38}$$

Fix $L_\infty^h > 0$ large enough such that

$$\frac{1}{2}C(r,N) + \left(\frac{1}{2}\|K\|_{L^\infty} + l_\infty^h\right) m(\Omega) < L_\infty^h (r/2)^N \omega_N, \tag{7.39}$$

where $r > 0$ and $C(r,N) > 0$ are from (7.7). Due to the right hand side of (H_1^∞), one can fix a sequence $\{\tilde{s}_i\}_i \subset (0,\infty)$ such that $\lim_{i\to\infty} \tilde{s}_i = \infty$, and

$$\mathrm{essinf}_{x\in\Omega} H(x, \tilde{s}_i) > L_\infty^h \tilde{s}_i^2 \ \text{ for all } i \in \mathbb{N}. \tag{7.40}$$

Since $\lim_{i\to\infty} \delta_i = \infty$, see ($\mathrm{H}_2^\infty$), we can choose a subsequence $\{\delta_{m_i}\}_i$ of $\{\delta_i\}_i$ such that $\tilde{s}_i \leq \delta_{m_i}$ for all $i \in \mathbb{N}$. Let $i \in \mathbb{N}$ be fixed and let $z_{\tilde{s}_i} \in H_0^1(\Omega)$ be the function from (7.6) corresponding to the value $\tilde{s}_i > 0$. Then $z_{\tilde{s}_i} \in W^{\eta_{m_i}}$, and on account of (7.7), (7.40) and (7.38), we have

$$\begin{aligned}
\mathcal{E}_{m_i}(z_{\tilde{s}_i}) &= \frac{1}{2}\|z_{\tilde{s}_i}\|^2_{H_0^1} + \frac{1}{2}\int_\Omega K(x) z^2_{\tilde{s}_i} - \int_\Omega H_{m_i}(x, z_{\tilde{s}_i}(x))dx \\
&= \frac{1}{2}C(r,N)\tilde{s}_i^2 + \frac{1}{2}\int_\Omega K(x) z^2_{\tilde{s}_i} - \int_{B(x_0,r/2)} H(x,\tilde{s}_i)dx \\
&\quad - \int_{(B(x_0,r)\setminus B(x_0,r/2))\cap\{z_{\tilde{s}_i}>\zeta\}} H(x, z_{\tilde{s}_i}(x))dx \\
&\quad - \int_{(B(x_0,r)\setminus B(x_0,r/2))\cap\{z_{\tilde{s}_i}\le\zeta\}} H(x, z_{\tilde{s}_i}(x))dx \\
&\le \left[\frac{1}{2}C(r,N) + \frac{1}{2}\|K\|_{L^\infty} m(\Omega) - L^h_\infty (r/2)^N \omega_N + l^h_\infty m(\Omega)\right]\tilde{s}_i^2 \\
&\quad + \|\sup_{s\in[0,\zeta]} |h(\cdot,s)|\|_{L^\infty} m(\Omega)\zeta.
\end{aligned}$$

The above estimate, relation (7.39) and $\lim_{i\to\infty}\tilde{s}_i = \infty$ clearly show that

$$\lim_{i\to\infty}\mathcal{E}_{m_i}(z_{\tilde{s}_i}) = -\infty. \tag{7.41}$$

On the other hand, using (7.34), we have

$$\mathcal{E}_{m_i}(u^\infty_{m_i}) = \min_{W^{\eta_{m_i}}} \mathcal{E}_{m_i} \le \mathcal{E}_{m_i}(z_{\tilde{s}_i}). \tag{7.42}$$

Therefore, on account of (7.41), we have

$$\lim_{i\to\infty}\mathcal{E}_{m_i}(u^\infty_{m_i}) = -\infty. \tag{7.43}$$

Note that the sequence $\{\mathcal{E}_i(u_i^\infty)\}_i$ is non-increasing. Indeed, let $i,k\in\mathbb{N}$, $i<k$. Then, due to (7.33), we have

$$\mathcal{E}_i(u_i^\infty) = \min_{W^{\eta_i}}\mathcal{E}_i = \min_{W^{\eta_i}}\mathcal{E}_k \ge \min_{W^{\eta_k}}\mathcal{E}_k = \mathcal{E}_k(u_k^\infty).$$

Combining this fact with (7.43), we obtain (7.37).

Now, we prove (7.32). Arguing by contradiction assume there exists a subsequence $\{u^\infty_{k_i}\}_i$ of $\{u^\infty_i\}_i$ such that for all $i\in\mathbb{N}$, we have $\|u^\infty_{k_i}\|_{L^\infty}\le M$ for some $M>0$. In particular, $\{u^\infty_{k_i}\}_i \subset W^{\eta_l}$ for some $l\in\mathbb{N}$. Thus, for every $k_i\ge l$, we have

$$\begin{aligned}
\mathcal{E}_l(u_l^\infty) &= \min_{W^{\eta_l}}\mathcal{E}_l = \min_{W^{\eta_l}}\mathcal{E}_{k_i} \\
&\ge \min_{W^{\eta_{k_i}}}\mathcal{E}_{k_i} = \mathcal{E}_{k_i}(u^\infty_{k_i}) \\
&\ge \min_{W^{\eta_l}}\mathcal{E}_{k_i} && (\text{cf. hypothesis, } u^\infty_{k_i}\in W^{\eta_l}) \\
&= \mathcal{E}_l(u_l^\infty).
\end{aligned}$$

As a consequence,

$$\mathcal{E}_{k_i}(u_{k_i}^\infty) = \mathcal{E}_l(u_l^\infty) \quad \text{for all } i \in \mathbb{N}. \tag{7.44}$$

Since the sequence $\{\mathcal{E}_i(u_i^\infty)\}_i$ is non-increasing, on account of (7.44), one can find a number $i_0 \in \mathbb{N}$ such that $\mathcal{E}_i(u_i^\infty) = \mathcal{E}_l(u_l^\infty)$ for all $i \geq i_0$. This fact contradicts (7.37), which concludes the proof of Theorem 7.8. $\square$

Proof of Theorem 7.3. (i) *Case* $p = 1$. Let us fix $\lambda \in \mathbb{R}$ as in the hypothesis, i.e., $\lambda a(x) < \lambda_\infty$ a.e. $x \in \Omega$ for some $0 < \lambda_\infty < -l_\infty$. Fix also $\tilde{\lambda}_\infty \in (\lambda_\infty, -l_\infty)$ and let

$$K(x) = \tilde{\lambda}_\infty - \lambda a(x) \;\text{ and } h(x,s) = \tilde{\lambda}_\infty s + f(s) \text{ for all } (x,s) \in \Omega \times [0,\infty). \tag{7.45}$$

It is clear that $\mathrm{essinf}_\Omega K \geq \tilde{\lambda}_\infty - \lambda_\infty > 0$, so (H_K) is satisfied. Since $f(0) = 0$, (H_0^∞) holds too. Note that $H(x,s)/s^2 = \tilde{\lambda}_\infty/2 + F(s)/s^2$, $s > 0$; thus, hypothesis (f_1^∞) implies (H_1^∞). Since $l_\infty < -\tilde{\lambda}_\infty$, there is a sequence $\{s_i\}_i \subset (0,\infty)$ converging to $+\infty$ such that $f(s_i)/s_i < -\tilde{\lambda}_\infty$ for all $i \in \mathbb{N}$. By using the continuity of f, we may fix two sequences $\{\delta_i\}_i, \{\eta_i\}_i \subset (0,\infty)$ such that $0 < \delta_i < s_i < \eta_i < \delta_{i+1}$, $\lim_{i\to\infty} \delta_i = \infty$, and $\tilde{\lambda}_\infty s + f(s) \leq 0$ for all $s \in [\delta_i, \eta_i]$ and $i \in \mathbb{N}$. Therefore, (H_2^∞) is also fulfilled. Now, we are in the position to apply Theorem 7.8. Throughout the choice (7.45), (P_h^K) is equivalent to problem (P_λ) which concludes the proof.

(ii) *Case* $p < 1$. Let $\lambda \in \mathbb{R}$ be fixed arbitrarily and fix also a number $\lambda_\infty \in (0, -l_\infty)$. Now, let us choose

$$K(x) = \lambda_\infty \;\text{ and } h(x,s) = \lambda a(x)s^p + \lambda_\infty s + f(s) \text{ for all } (x,s) \in \Omega \times [0,\infty). \tag{7.46}$$

Hypothesis (H_K) is clearly satisfied. Due to the fact that $a \in L^\infty(\Omega)$, hypothesis (H_0^∞) holds too. Since $p < 1$ and $H(x,s)/s^2 = \lambda a(x) s^{p-1}/(p+1) + \lambda_\infty/2 + F(s)/s^2$, $s > 0$, hypothesis (f_1^∞) implies (H_1^∞). For a.e. $x \in \Omega$ and every $s \in [0,\infty)$, we have

$$h(x,s) \leq |\lambda| \cdot \|a\|_{L^\infty} s^p + \lambda_\infty s + f(s) \equiv \tilde{h}_\infty(s). \tag{7.47}$$

Thanks to (f_2^∞), we have

$$\liminf_{s\to\infty} \frac{\tilde{h}_\infty(s)}{s} = \lambda_\infty + l_\infty < 0.$$

Therefore, one can fix a sequence $\{s_i\}_i \subset (0,\infty)$ converging to $+\infty$ such that $\tilde{h}_\infty(s_i) < 0$ for all $i \in \mathbb{N}$. Now, by using the continuity of $\tilde{h}_\infty$, one can fix two sequences $\{\delta_i\}_i, \{\eta_i\}_i \subset (0,\infty)$ such that $0<\delta_i<s_i<\eta_i<\delta_{i+1}$, $\lim_{i\to\infty}\delta_i=\infty$, and $\tilde{h}_\infty(s)\leq 0$ for all $s\in[\delta_i,\eta_i]$ and $i\in\mathbb{N}$. Thus, by using (7.47), hypothesis (H_2^∞) holds. Now, we can apply Theorem 7.8, observing that problem (P_h^K) is equivalent to problem (P_λ) through the choice (7.46). Finally, in both cases (i.e., (i) and (ii)), (7.32) implies relation (7.3). This concludes the proof of Theorem 7.3. □

Proof of Remark 7.4. Assume that (7.4) holds. By contradiction, let us assume that there exists a bounded subsequence $\{u_{k_i}^\infty\}_i$ of $\{u_i^\infty\}_i$ in $H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is continuously embedded into $L^t(\Omega)$, $t\in[1,2^*]$, after an elementary estimate, we obtain that the sequence $\{\mathcal{E}_{k_i}(u_{k_i}^\infty)\}_i$ is bounded. Since the sequence $\{\mathcal{E}_i(u_i^\infty)\}_i$ is non-increasing, it will be bounded as well, which contradicts (7.37). □

Proof of Theorem 7.4. The proof is divided into five steps.

Step 1. Let $\lambda_\infty\in(0,-l_\infty)$. Due to (f_1^∞), we may fix a sequence $\{s_i\}_i\subset(0,\infty)$ converging to ∞, such that $f(s_i)/s_i<-\lambda_\infty$. For every $\lambda\in\mathbb{R}$, let us define two functions $h^\lambda:\Omega\times\mathbb{R}\to\mathbb{R}$ and $\tilde{h}:\mathbb{R}^2\to\mathbb{R}$ by

$$h^\lambda(x,s)=\lambda a(x)s^p+\lambda_\infty s+f(s) \text{ for all } (x,s)\in\Omega\times[0,\infty);$$

$$\tilde{h}(\lambda,s)=|\lambda|\cdot\|a\|_{L^\infty}s^p+\lambda_\infty s+f(s) \text{ for all } s\in[0,\infty).$$

Note that $\tilde{h}(0,s_i)=\lambda_\infty s_i+f(s_i)<0$. Due to the continuity of $\tilde{h}$, we can fix three sequences $\{\delta_i\}_i,\{\eta_i\}_i\subset(0,\infty)$ and $\{\lambda_i\}_i\subset(0,1)$ such that $0<\delta_i<s_i<\eta_i<\delta_{i+1}$, $\lim_{i\to\infty}\delta_i=\infty$, and for every $i\in\mathbb{N}$,

$$\tilde{h}(\lambda,s)\leq 0 \quad \text{for all} \quad \lambda\in[-\lambda_i,\lambda_i] \quad \text{and} \quad s\in[\delta_i,\eta_i]. \tag{7.48}$$

Without any loss of generality, we may assume that

$$\delta_i\geq i,\quad i\in\mathbb{N}. \tag{7.49}$$

Note that $h^\lambda(x,s)\leq\tilde{h}(\lambda,s)$ for a.e. $x\in\Omega$ and all $(\lambda,s)\in\mathbb{R}\times[0,\infty)$. Taking into account of (7.48), for every $i\in\mathbb{N}$, we have

$$h^\lambda(x,s)\leq 0 \quad \text{for a.e. } x\in\Omega \text{ and all } \lambda\in[-\lambda_i,\lambda_i],\ s\in[\delta_i,\eta_i]. \tag{7.50}$$

For any $i\in\mathbb{N}$ and $\lambda\in[-\lambda_i,\lambda_i]$, let $h_i^\lambda:\Omega\times[0,\infty)\to\mathbb{R}$ be defined by

$$h_i^\lambda(x,s)=h^\lambda(x,\tau_{\eta_i}(s)) \tag{7.51}$$

and $K(x)=\lambda_\infty$. Let $\mathcal{E}_{i,\lambda}:H_0^1(\Omega)\to\mathbb{R}$ be the energy functional associated with the problem $(\mathrm{P}_{h_i^\lambda}^K)$, which is formally the same as in (7.26). Note that for every $i\in\mathbb{N}$ and $\lambda\in[-\lambda_i,\lambda_i]$, the function h_i^λ fulfills the hypotheses of Theorem 7.6; see (7.50) for $(\mathrm{H}^2_{h_i^\lambda})$. Consequently, for every $i \in \mathbb{N}$ and $\lambda \in [-\lambda_i, \lambda_i]$

$$\text{there exists } \tilde{u}^\infty_{i,\lambda} \in W^{\eta_i} \text{ with } \mathcal{E}_{i,\lambda}(\tilde{u}^\infty_{i,\lambda}) = \min_{W^{\eta_i}} \mathcal{E}_{i,\lambda}; \tag{7.52}$$

$$\tilde{u}^\infty_{i,\lambda}(x) \in [0, \delta_i] \text{ for a.e. } x \in \Omega, \tag{7.53}$$

$$\tilde{u}^\infty_{i,\lambda} \text{ is a weak solution of } (\mathrm{P}^K_{h_i^\lambda}). \tag{7.54}$$

On account of the definition of the functions h_i^λ and K, and relations (7.54) and (7.53), $\tilde{u}^\infty_{i,\lambda}$ is also a weak solution for our initial problem (P_λ) once we have $\tilde{u}^\infty_{i,\lambda} \not\equiv 0$.

Step 2. Note that for $\lambda = 0$, the function $h_i^\lambda = h_i^0$ verifies the hypotheses of Theorem 7.8; in fact, h_i^0 is the function appearing in (7.33) and $\mathcal{E}_i := \mathcal{E}_{i,0}$ is the energy functional associated with problem $(\mathrm{P}^K_{h_i^0})$. Denoting $u_i^\infty := \tilde{u}^\infty_{i,0}$, we also have

$$\mathcal{E}_{m_i}(u^\infty_{m_i}) = \min_{W^{\eta_{m_i}}} \mathcal{E}_{m_i} \le \mathcal{E}_{m_i}(z_{\tilde{s}_i}); \tag{7.55}$$

$$\lim_{i\to\infty} \mathcal{E}_{m_i}(u^\infty_{m_i}) = -\infty, \tag{7.56}$$

where the special subsequence $\{u^\infty_{m_i}\}_i$ of $\{u_i^\infty\}_i$ and $z_{\tilde{s}_i} \in W^{\eta_{m_i}}$ appear in the proof of Theorem 7.8, see relations (7.42) and (7.43), respectively.

Step 3. Let us fix a sequence $\{\theta_i\}_i$ with negative terms such that $\lim_{i\to\infty} \theta_i = -\infty$. Due to (7.55) and (7.56), up to a subsequence, we may assume that

$$\theta_{i+1} < \mathcal{E}_{m_i}(u^\infty_{m_i}) \le \mathcal{E}_{m_i}(z_{\tilde{s}_i}) < \theta_i. \tag{7.57}$$

For any $i \in \mathbb{N}$, define

$$\lambda_i' = \frac{(p+1)(\theta_i - \mathcal{E}_{m_i}(z_{\tilde{s}_i}))}{\delta_{m_i}^{p+1}(\|a\|_{L^1}+1)} \quad \text{and} \quad \lambda_i'' = \frac{(p+1)(\mathcal{E}_{m_i}(u^\infty_{m_i}) - \theta_{i+1})}{\delta_{m_i}^{p+1}(\|a\|_{L^1}+1)}\,.$$

Fix $k \in \mathbb{N}$. Thanks to (7.57),

$$\lambda_k^\infty = \min(\lambda_1, ..., \lambda_k, \lambda_1', ..., \lambda_k', \lambda_1'', ..., \lambda_k'') > 0.$$

Therefore, for every $i \in \{1, ..., k\}$ and $\lambda \in [-\lambda_k^\infty, \lambda_k^\infty]$ we have

$$\begin{aligned}
\mathcal{E}_{m_i,\lambda}(\tilde{u}^\infty_{m_i,\lambda}) &\le \mathcal{E}_{m_i,\lambda}(z_{\tilde{s}_i}) && \text{(see (7.52))}\\
&= \frac{1}{2}\|z_{\tilde{s}_i}\|^2_{H_0^1} - \frac{\lambda}{p+1}\int_\Omega a(x) z_{\tilde{s}_i}^{p+1} - \int_\Omega F(z_{\tilde{s}_i}(x))dx\\
&= \mathcal{E}_{m_i}(z_{\tilde{s}_i}) - \frac{\lambda}{p+1}\int_\Omega a(x) z_{\tilde{s}_i}^{p+1}\\
&< \theta_i, && \text{(see the choice of } \lambda_i' \text{ and } \tilde{s}_i \le \delta_{m_i})
\end{aligned}$$

and since $\tilde{u}^\infty_{m_i,\lambda}$ belongs to $W^{\eta_{m_i}}$, and $u^\infty_{m_i}$ is the minimum point of $\mathcal{E}_{m_i}$ over the set $W^{\eta_{m_i}}$, see relation (7.55), we have

$$\begin{aligned}
\mathcal{E}_{m_i,\lambda}(\tilde{u}^\infty_{m_i,\lambda}) &= \mathcal{E}_{m_i}(\tilde{u}^\infty_{m_i,\lambda}) - \frac{\lambda}{p+1}\int_\Omega a(x)(\tilde{u}^\infty_{m_i,\lambda})^{p+1} \\
&\geq \mathcal{E}_{m_i}(u^\infty_{m_i}) - \frac{\lambda}{p+1}\int_\Omega a(x)(\tilde{u}^\infty_{m_i,\lambda})^{p+1} \\
&> \theta_{i+1}. \qquad \text{(see the choice of } \lambda''_i \text{ and (7.53))}
\end{aligned}$$

Consequently, for every $i \in \{1, ..., k\}$ and $\lambda \in [-\lambda^\infty_k, \lambda^\infty_k]$ we have

$$\theta_{i+1} < \mathcal{E}_{m_i,\lambda}(\tilde{u}^\infty_{m_i,\lambda}) < \theta_i < 0, \tag{7.58}$$

therefore

$$\mathcal{E}_{m_k,\lambda}(\tilde{u}^\infty_{m_k,\lambda}) < ... < \mathcal{E}_{m_1,\lambda}(\tilde{u}^\infty_{m_1,\lambda}) < 0. \tag{7.59}$$

Note that $\tilde{u}^\infty_{m_i,\lambda} \in W^{\eta_{m_k}}$ for every $i \in \{1, ..., k\}$, so $\mathcal{E}_{m_i,\lambda}(\tilde{u}^\infty_{m_i,\lambda}) = \mathcal{E}_{m_k,\lambda}(\tilde{u}^\infty_{m_i,\lambda})$, see relation (7.51). From above, for every $\lambda \in [-\lambda^\infty_k, \lambda^\infty_k]$, we have

$$\mathcal{E}_{m_k,\lambda}(\tilde{u}^\infty_{m_k,\lambda}) < ... < \mathcal{E}_{m_k,\lambda}(\tilde{u}^\infty_{m_1,\lambda}) < 0 = \mathcal{E}_{m_k,\lambda}(0).$$

In particular, the elements $\tilde{u}^\infty_{m_1,\lambda}, ..., \tilde{u}^\infty_{m_k,\lambda}$ are distinct (and non-trivial) whenever $\lambda \in [-\lambda^\infty_k, \lambda^\infty_k]$.

Step 4. Assume that $k \geq 2$ and fix $\lambda \in [-\lambda^\infty_k, \lambda^\infty_k]$. We prove that

$$\|\tilde{u}^\infty_{m_i,\lambda}\|_{L^\infty} > \delta_{m_{i-1}} \text{ for all } i \in \{2, ..., k\}. \tag{7.60}$$

Let us assume that there exists an element $i_0 \in \{2, ..., k\}$ such that $\|\tilde{u}^\infty_{m_{i_0},\lambda}\|_{L^\infty} \leq \delta_{m_{i_0-1}}$. Since $\delta_{m_{i_0-1}} < \eta_{m_{i_0-1}}$, then $\tilde{u}^\infty_{m_{i_0},\lambda} \in W^{\eta_{m_{i_0-1}}}$. Thus, on account of (7.52) and (7.51), we have

$$\mathcal{E}_{m_{i_0-1},\lambda}(\tilde{u}^\infty_{m_{i_0-1},\lambda}) = \min_{W^{\eta_{m_{i_0-1}}}} \mathcal{E}_{m_{i_0-1},\lambda} \leq \mathcal{E}_{m_{i_0-1},\lambda}(\tilde{u}^\infty_{m_{i_0},\lambda}) = \mathcal{E}_{m_{i_0},\lambda}(\tilde{u}^\infty_{m_{i_0},\lambda}),$$

which contradicts (7.59). Therefore, (7.60) holds true.

Step 5. Let $u^\infty_{i,\lambda} := \tilde{u}^\infty_{m_i,\lambda}$ for any $i \in \{1, ..., k\}$ and $\lambda \in [-\lambda^\infty_k, \lambda^\infty_k]$; these elements verify all the requirements of Theorem 7.4. Indeed, since $\mathcal{E}_{m_1,\lambda}(u^\infty_{1,\lambda}) = \mathcal{E}_{m_1,\lambda}(\tilde{u}^\infty_{m_1,\lambda}) < 0 = \mathcal{E}_{m_1,\lambda}(0)$, then $\|u^\infty_{1,\lambda}\|_{L^\infty} > 0$, which proves (7.5) for $i = 1$. If $k \geq 2$, then on account of Step 4, (7.49) and $m_i \geq i$, for every $i \in \{2, ..., k\}$, we have

$$\|u^\infty_{i,\lambda}\|_{L^\infty} > \delta_{m_{i-1}} \geq m_{i-1} \geq i - 1,$$

i.e., relation (7.5) holds true. This ends the proof of Theorem 7.4. $\square$

Proof of Remark 7.6. Due to (7.4), there exists a $C > 0$ such that $|f(s)| \le C(1 + s^{2^*-1})$ for all $s \ge 0$. We denote by $S_t > 0$ the Sobolev embedding constant of the continuous embedding $H_0^1(\Omega) \hookrightarrow L^t(\Omega)$, $t \in [1, 2^*]$. Without any loss of generality, we may assume that for every $i \in \mathbb{N}$,

$$\theta_i < -\frac{1}{p+1}\|a\|_{L^\infty} S_{p+1}^{p+1}(i-1)^{p+1} - C[S_1(i-1) + S_{2^*}^{2^*}(i-1)^{2^*}], \quad (7.61)$$

where the sequence $\{\theta_i\}_i$ comes from Step 3 of the proof of Theorem 7.4.

Fix $\lambda \in [-\lambda_k^\infty, \lambda_k^\infty]$ and assume that there exists $i_0 \in \{1, ..., k\}$ such that $\|u_{i_0,\lambda}^\infty\|_{H_0^1} \le i_0 - 1$. On account of (7.58), we have in particular $\mathcal{E}_{m_{i_0},\lambda}(u_{i_0,\lambda}^\infty) < \theta_{i_0}$. Consequently, we have

$$\begin{aligned}
\frac{1}{2}\|u_{i_0,\lambda}^\infty\|_{H_0^1}^2 &= \mathcal{E}_{m_{i_0},\lambda}(u_{i_0,\lambda}^\infty) + \frac{\lambda}{p+1}\int_\Omega a(x)(u_{i_0,\lambda}^\infty)^{p+1} + \int_\Omega F(u_{i_0,\lambda}^\infty(x))dx \\
&< \theta_{i_0} + \frac{|\lambda|}{p+1}\|a\|_{L^\infty} S_{p+1}^{p+1}\|u_{i_0,\lambda}^\infty\|_{H_0^1}^{p+1} \\
&\quad + C[S_1\|u_{i_0,\lambda}^\infty\|_{H_0^1} + S_{2^*}^{2^*}\|u_{i_0,\lambda}^\infty\|_{H_0^1}^{2^*}] \\
&\le \theta_{i_0} + \frac{1}{p+1}\|a\|_{L^\infty} S_{p+1}^{p+1}(i_0-1)^{p+1} \\
&\quad + C[S_1(i_0-1) + S_{2^*}^{2^*}(i_0-1)^{2^*}] \qquad (\lambda_k^\infty \le 1) \\
&< 0, \qquad (\text{see } (7.61))
\end{aligned}$$

contradiction. □

7.6 Perturbation from symmetry

For details related to the main notions and properties used in this section we refer to Appendix D.

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. For some fixed $r > 0$, consider the following nonlinear eigenvalue problem: find $(u, \lambda) \in H_0^1(\Omega) \times \mathbb{R}$ such that

$$\begin{cases} f(x,u) \in L^1_{\mathrm{loc}}(\Omega)\,, \\ -\Delta u = \lambda f(x,u) \quad \text{in } \Omega\,, \\ \displaystyle\int_\Omega |\nabla u|^2 dx = r^2\,, \end{cases} \qquad (7.62)$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that the following conditions hold:

$(f1)$ $f(x, -s) = -f(x, s)$, for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$;

($f2$) there exist $a \in L^1(\Omega)$, $b \in \mathbb{R}$ and $0 \le p < \frac{2N}{N-2}$ (if $N > 2$) such that

$$0 < sf(x,s) \le a(x) + b|s|^p\,, \qquad F(x,s) \le a(x) + b|s|^p\,,$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}\backslash\{0\}$, where $F(x,s) = \int_0^s f(x,t)dt$;

($f3$) $\sup\limits_{|s|\le t} |f(x,s)| \in L^1_{\mathrm{loc}}(\Omega)$, for every $t > 0$.

We point out that if $N = 1$, then in condition ($f2$) the term $b|s|^p$ can be substituted by any continuous function $\varphi(s)$ of s, while, if $N = 2$, the same term can be substituted by $\exp(\varphi(s))$, with $\varphi(s)s^{-2} \to 0$ as $|s| \to \infty$.

Our first result in this section is the following.

Theorem 7.9 *Assume that hypotheses ($f1$) − ($f3$) hold. Then problem (7.62) admits a sequence $(\pm u_n, \lambda_n)$ of distinct solutions.*

Our main purpose is to study what happens when the energy functional is affected by an *arbitrary* perturbation which destroys the symmetry.

Consider the problem: find $(u,\lambda) \in H^1_0(\Omega) \times \mathbb{R}$ such that

$$\begin{cases} f(x,u),\ g(x,u) \in L^1_{\mathrm{loc}}(\Omega)\,, \\ -\Delta u = \lambda\,(f(x,u) + g(x,u)) \quad \text{in } \Omega\,, \\ \displaystyle\int_\Omega |\nabla u|^2 dx = r^2\,, \end{cases} \tag{7.63}$$

where $g : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function. We make no symmetry assumption on g, but we impose only

($g1$) $0 < sg(x,s) \le a(x) + b|s|^p$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R} \setminus \{0\}$;

($g2$) $\sup\limits_{|s|\le t} |g(x,s)| \in L^1_{\mathrm{loc}}(\Omega)$, for every $t > 0$;

($g3$) $G(x,s) \le C_g\,(1 + |s|^p)$, for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, for some $C_g > 0$, where $G(x,s) = \int_0^s g(x,t)dt$.

The next result shows that the number of solutions of problem (7.63) becomes greater and greater, as the perturbation tends to zero.

Theorem 7.10 *Assume that hypotheses ($f1$)−($f3$) and ($g1$)−($g3$) hold. Then, for every positive integer n, there exists $\varepsilon_n > 0$ such that problem (7.63) admits at least n distinct solutions, provided that ($g3$) holds for $C_g = \varepsilon_n$.*

We prove theorems 7.9 and 7.10 by a variational argument. First we set

$$S_r = \left\{ u \in H_0^1(\Omega) : \int_\Omega |\nabla u|^2 dx = r^2 \right\}$$

and we study the critical points on S_r of the even continuous functional $I : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I(u) = -\int_\Omega F(x,u)dx\,.$$

We point out that if conditions $(f2)$, $(f3)$ are substituted by the more standard condition $0 < sf(x,s) \le a_1(x)|s| + b|s|^p$ with $a_1 \in L^{\frac{2N}{N+2}}(\Omega)$, then I is of class C^1 and Theorem 7.9 can be found in [249, Theorem 8.17]. Under our assumptions, f could have the form $f(x,s) = \alpha(x)\gamma(s)$ with $\alpha \in L^1(\Omega)$, $\alpha \ge 0$, $\gamma \in C_c(\mathbb{R})$, γ odd and $s\gamma(s) \ge 0$ for any $s \in \mathbb{R}$. In such a case, I is clearly continuous, but not locally Lipschitz.

When f and g are subjected to the standard condition we have mentioned, results like Theorem 7.10 go back to Krasnoselskii [161]. For perturbation results, quite different from ours, where the perturbed problem still has infinitely many solutions, we refer the reader to [249, 280]. In a nonsmooth setting, a result in the line of Theorem 7.10 has been proved in [81] when f and g satisfy the standard condition, but the function u is subjected to an obstacle, so that the equation becomes a variational inequality.

We first observe that from our assumption $(f2)$ it follows that $I(u) < 0$ and $\sup I_r(u) = 0$, where $I_r = I_{|S_r}$.

We start with the following preliminary result.

Lemma 7.1 *The following properties hold true:*

(a) *if $u \in S_r$ satisfies $|dI_r|(u) < +\infty$, then $f(x,u) \in L^1_{\mathrm{loc}}(\Omega) \cap H^{-1}(\Omega)$ and there exists $\mu \in \mathbb{R}$ such that*

$$\|\mu\Delta u + f(x,u)\|_{H^{-1}} \le |dI_r|(u)\,;$$

(b) *the functional I_r satisfies $(PS)_c$ for any $c < 0$;*

(c) *if $u \in S_r$ is a critical point of I_r, then there exists $\lambda > 0$ such that (u, λ) is a solution of problem (7.62).*

Proof (*a*) Set

$$I_{r,est}(w) = \begin{cases} I(w) & \text{if } w \in S_r\,, \\ +\infty & \text{if } w \in H_0^1(\Omega) \setminus S_r\,. \end{cases}$$

Then $|dI_{r,est}|(u) = |dI_r|(u)$. We also deduce that there exists $\alpha \in \partial I_{r,est}(u)$ with $\|\alpha\|_{H^{-1}} \le |dI_{r,est}|(u)$, where ∂ stands for the generalized gradient of $I_{r,est}$. Taking into account $(f2)$, we obtain

$$I^0(u;0) \le 0\,, \qquad I^0(u;2u) \le -2\int_\Omega f(x,u)u dx < +\infty\,.$$

Actually, the same proof shows a stronger fact, namely that

$$\overline{I}^0(u;0) \le 0\,, \qquad \overline{I}^0(u;2u) \le -2\int_\Omega f(x,u)u dx < +\infty\,.$$

Thus, there are $\beta \in \partial I(u)$ and $\mu \in \mathbb{R}$ such that $\alpha = \beta - \mu\Delta u$. We conclude that $f(x,u) \in L^1_{\rm loc}(\Omega) \cap H^{-1}(\Omega)$ and $\beta = -f(x,u)$. Then (a) follows.

(b) Let $c < 0$ and let (u_n) be a $(PS)_c$–sequence for I_r. By the previous point, we have $f(x,u_n) \in L^1_{\rm loc}(\Omega) \cap H^{-1}(\Omega)$ and there exists a sequence (μ_n) in $\mathbb{R}$ with

$$\|\mu_n \Delta u_n + f(x,u_n)\|_{H^{-1}} \to 0\,.$$

Up to a subsequence, (u_n) is convergent to some u weakly in $H_0^1(\Omega)$ and a.e. From $(f2)$ it follows $I(u) = c < 0$, hence $u \ne 0$. Again by $(f2)$ and Lebesgue's dominated convergence theorem, we deduce that

$$0 < \int_\Omega f(x,u)u dx = \lim_n \int_\Omega f(x,u_n)u_n dx = \lim_n \mu_n \int_\Omega |\nabla u_n|^2 dx\,.$$

Therefore, up to a further subsequence, (μ_n) is convergent to some $\mu > 0$ and

$$\left\|\Delta u_n + \frac{1}{\mu} f(x,u_n)\right\|_{H^{-1}} \to 0\,.$$

This shows that (u_n) is precompact in $H_0^1(\Omega)$ and (b) follows.

(c) Arguing as in (b), we find that $f(x,u) \in L^1_{\rm loc}(\Omega) \cap H^{-1}(\Omega)$ and that there exists $\mu > 0$ with $\mu\Delta u + f(x,u) = 0$. This concludes the proof. □

Lemma 7.2 *There exists a sequence (b_n) of essential values of I_r strictly increasing to* 0.

Proof Let $\psi :]-\infty, 0[\to \mathbb{R}$ be an increasing diffeomorphism. From Lemma 7.1 it follows that $\psi \circ I_r$ satisfies condition $(PS)_c$ for every $c \in \mathbb{R}$. Then the set $\{u \in S_r : \psi \circ I_r(u) \leq b\}$ has finite genus for every $b \in \mathbb{R}$. For any integer $n \geq 1$, set

$$c_n := \inf_{S \in \Gamma_n} \sup_{u \in S} \psi \circ I_r(u)\,,$$

where $\Gamma_n := \{S | subset S_r;\ S \in \mathcal{F},\, \gamma(S) \geq n\}$ and $\mathcal{F}$ denotes the family of closed and symmetric subsets of S_r with respect to the origin. Since $c_n \to +\infty$ as $n \to \infty$, it follows that there exists a sequence (b'_n) of essential values of $\psi \circ I_r$ strictly increasing to $+\infty$. Then $b_n = \psi^{-1}(b'_n)$ has the required properties. $\square$

Proof of Theorem 7.9. Using Lemma 7.1, we deduce that each b_n is a critical value of I_r. Again from Lemma 7.1, we conclude that there exists a sequence $(\pm u_n, \lambda_n)$ of solutions of problem (7.62) with $I(u_n) = b_n$ strictly increasing to 0. This concludes the proof of Theorem 7.9.

Let us now introduce the continuous functional $J_r : S_r \to \mathbb{R}$ defined by

$$J(u) = I(u) - \int_\Omega G(x,u)dx\,.$$

Lemma 7.3 *For every $\eta > 0$, there exists $\varepsilon > 0$ such that $\sup_{u \in S_r} |I_r(u) - J_r(u)| < \eta$, provided that $(g3)$ holds for $C_g = \varepsilon$.*

Proof By Sobolev inclusions, we have for any $u \in S_r$,

$$0 \leq I_r(u) - J_r(u) = \int_\Omega G(x,u)dx \leq C_g \int_\Omega (1 + |u|^p)dx < \eta\,,$$

if g is chosen as in the hypothesis. $\square$

Proof of Theorem 7.10. As in the proof of Theorem 7.9, let us consider a strictly increasing sequence (b_n) of essential values of I_r such that $b_n \to 0$ as $n \to \infty$. Given $n \geq 1$, take some $\delta > 0$ with $b_n + \delta < 0$ and $2(b_j - b_{j-1}) < \delta$ for $j = 2, \ldots, n$. Thus, for any $j = 1, \ldots, n$, there exists $\eta_j > 0$ such that $\sup_{u \in S_r} |I_r(u) - J_r(u)| < \eta_j$ implies the existence of an essential value $c_j \in]b_j - \delta, b_j + \delta[$ of J_r. We now apply Lemma 7.3 for $\eta = \min\{\eta_1, \ldots, \eta_n\}$. Thus we obtain $\varepsilon_n > 0$ such that $\sup_{u \in S_r} |I_r(u) -$

$J_r(u)| < \eta$, if $(g3)$ holds with $C_g = \varepsilon_n$. It follows that J_r has at least n distinct essential values $c_1, \ldots, c_n$ in the interval $]-\infty, 0[$.

Now Lemma 1 can be clearly adapted to the functional J_r. Then we find $u_1, \ldots, u_n \in S_r$ and $\lambda_1, \ldots, \lambda_n > 0$ such that each (u_j, λ_j) is a solution of Problem (7.62) with $J_r(u_j) = c_j$.

7.7 Historical notes, comments

A. Historical notes. Let us consider the equation

$$\begin{cases} -\triangle u = f(x,u) + \lambda g(x,u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{E$_\lambda$}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded open domain, while $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions. We assume the unperturbed equation (E_0) has *infinitely many* distinct solutions. Then, the main question is:

(q) *Fixing $k \in \mathbb{N}$, can one find a number $\lambda_k > 0$ such that the perturbed equation* (E_λ) *has at least k distinct solutions whenever $\lambda \in [-\lambda_k, \lambda_k]$?*

Two different classes of results are available in the literature answering affirmatively question (q):

1) *Perturbation of symmetric problems.* Assume $f(x,s) = -f(x,-s)$ for every $(x,s) \in \Omega \times \mathbb{R}$. It is well-known that if the energy functional has the mountain pass geometry, problem (E_0) has infinitely many solutions, due to the symmetric version of the Mountain Pass theorem, see Ambrosetti-Rabinowitz [7]. Furthermore, question (q) was fully answered by Li-Liu [186] for arbitrarily continuous nonlinearity g, following the topological approach developed by Degiovanni-Lancelotti [81] and Degiovanni-Rădulescu [85];

2) *Perturbation of oscillatory problems.* Assume $f(x,\cdot)$ oscillates near the origin or at infinity, uniformly with respect to $x \in \Omega$. Special kinds of oscillations produce infinitely many solutions for (E_0), as shown by Omari-Zanolin [229], and Saint Raymond [265]. Concerning the perturbed problem, Anello-Cordaro [12] answered question (q), by using Theorem 1.17.

B. Comments. In this chapter we presented a third, *direct method* for answering question (q) whenever the nonlinear term $f(x,\cdot)$ belongs to a wide class of oscillatory functions and $g(x,s) = s^p$. We fully described the effect of the term g to the oscillatory function f. Namely, the number of distinct solutions of (E_λ) is sensitive when:

- $p \leq 1$ and f oscillates near the origin;
- $p \geq 1$ and f oscillates at infinity.

Notice that g can be any continuous function with $g(x, 0) = 0$, and our method works also for other elliptic problems.

8
Problems to Part I

> I hope that seeing the excitement of solving this problem will make young mathematicians realize that there are lots and lots of other problems in mathematics which are going to be just as challenging in the future.
>
> Andrew Wiles (b. 1953)

Problem 8.1 Show that any function $f : \mathbb{R} \to \mathbb{R}$ of class C^1 having two local minima has necessarily a third critical point.

Problem 8.2 Show that the $(PS)_c$-condition implies the $(C)_c$-condition for any function $f : X \to \mathbb{R}$ of class C^1 and $c \in \mathbb{R}$, see Remark 1.3. Construct a function which shows that the converse does not hold in general.

Problem 8.3 Prove that every function $f : T^2 \to \mathbb{R}$ of class C^1 has at least 3 critical points. Here $T^2 = S^1 \times S^1$ is the 2-dimensional torus. [Hint: Prove that $\mathrm{cat}(T^2) = 3$ and apply Theorem 1.9.]

Problem 8.4 Let X be a real Banach space and assume that f is a nonnegative C^1 function on X satisfying the Palais–Smale condition or even a weaker form: every sequence (u_n) in X such that $\sup_n f(u_n) < \infty$ and $\|f'(u_n)\| \to 0$, is bounded in X. Prove that f is coercive, that is,

$$f(u) \to +\infty \quad \text{as } \|u\| \to \infty.$$

Problem 8.5 Let X be a real Banach space and assume that f is a nonnegative C^1 function on X satisfying the Palais–Smale condition. Prove that every minimizing sequence of f has a convergent subsequence.

Problem 8.6 Let X be a real Banach space and assume that f is a C^1 function on X which is bounded below and satisfies the Palais–Smale condition. Suppose that all the critical points of f lie in $\{u \in X;\ \|u\| < R\}$. Set

$$M(r) = \inf\{f(u);\ \|u\| = r\}\,.$$

Prove that for $r > R$, $M(r)$ is strictly increasing and continuous from the right.

Problem 8.7 Let Ω be a bounded open set in $\mathbb{R}^N$. For every $f \in L^2(\Omega)$ and all $\lambda \in \mathbb{R}$, define the functional

$$E_\lambda(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \lambda u^2\right)\, dx - \int_\Omega f u\, dx\,, \qquad u \in H_0^1(\Omega)\,.$$

a) Prove that if λ is not an eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$, then E_λ satisfies the Palais–Smale condition in $H_0^1(\Omega)$.

b) Prove that if λ is an eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$ and if $f = 0$, then E_λ does not satisfy the Palais–Smale condition in $H_0^1(\Omega)$.

Problem 8.8 Let X be a real Banach space and let U be an open subset of X. Let Φ be a real C^1 bounded below function on $\overline{U}$. Set

$$a = \inf_{\overline{U}} \Phi\,.$$

Assume F is a closed subset of U such that $\operatorname{dist}(F, \partial U) > 0$ (no assumption if $U = X$) and

$$a = \inf_{F} \Phi\,.$$

Prove that there exists a sequence (x_n) in U such that

$$\Phi(x_n)\to a, \quad \Phi'(x_n)\to 0 \ \text{ in } X^* \quad \text{and } \operatorname{dist}(x_n, F)\to 0.$$

Problem 8.9 In Theorem 1.3, instead of the assumption that f is bounded from below, assume that

$$f(x) + \varphi(d(x, x_0)) \geq 0 \qquad \text{for every } x \in X,$$

for some $x_0 \in X$ and some function $\varphi(t) = o(t)$ as $t\to+\infty$. Prove that the conclusion of Theorem 1.3 still holds.

Problem 8.10 (Caristi's fixed point theorem). Let (X,d) be a complete metric space and let $f : X \to X$ be a function. Assume that $\Psi : X \to \mathbb{R}$ is a lower semicontinuous function such that

$$d(x, f(x)) \le \Psi(x) - \Psi(f(x)) \qquad \text{for all } x \in X.$$

Prove that f has a fixed point.

Problem 8.11 Let Φ be a C^1 function on $\mathbb{R}^N$ such that

$$m = \liminf_{\|x\| \to \infty} \Phi(x) \quad \text{is finite.}$$

Prove that there exists a sequence of points (x_m) in $\mathbb{R}^N$ such that $\|x_m\| \to \infty$, $\Phi(x_m) \to m$, $\|\Phi'(x_m)\| \to 0$ and $\Phi'(x_m)$ is parallel to x_m ($\Phi'(x_m)$ may be zero).

Problem 8.12 Let X be a real Banach space and assume that $f : X \to \mathbb{R}$ is a function of class C^1 such that

$$\inf_{\|u-e_0\|=\rho} f(u) \ge \alpha > \max\{f(e_0), f(e_1)\}$$

for some $\alpha \in \mathbb{R}$ and $e_0 \ne e_1 \in X$ with $0 < \rho < \|e_0 - e_1\|$. Set

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = e_0,\ \gamma(1) = e_1\}.$$

Prove that the conclusion of Theorem 1.7 can be strengthened as follows: there exists a sequence (u_n) in X such that $f(u_n) \to c$ and $(1 + \|u_n\|)\, \|f'(u_n)\| \to 0$ as $n \to \infty$.

Problem 8.13 Prove the same result if $(1 + \|u\|)$ is replaced by $\varphi(\|u\|)$, where $\varphi \ge 1$ and $\int_0^\infty \frac{dt}{\varphi(t)} = +\infty$.

Problem 8.14 Let X be a real Banach space and assume that $f, g : X \to \mathbb{R}$ are functions of class C^1 satisfying the Palais–Smale condition with $f \ge g$ on X. Let K be a compact metric space and let K^* be a nonempty closed subset of K, $K^* \ne K$. Define

$$\mathcal{P} = \{p \in C(K, X);\ p = p^* \text{ on } K^*\},$$

where p^* is a fixed continuous map on K. Assume that

$$c = \inf_{p \in \mathcal{P}} \max_{t \in K} f(p(t)) = \inf_{p \in \mathcal{P}} \max_{t \in K} g(p(t)) > \inf_{p \in \mathcal{P}} \max_{t \in K^*} f(p^*(t)).$$

Prove that f and g have a common critical point where both functions equal c.

Problem 8.15 Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and assume that p is a real number such that $1 < p < (N+2)/(N-2)$ if $N \geq 3$ and $1 < p < \infty$ if $N \in \{1, 2\}$.

Prove that the nonlinear problem

$$\begin{cases} -\Delta u + u = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

has a solution.

Problem 8.16 Establish the same existence result for the problem

$$\begin{cases} -\Delta u + u = f(x, u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$f(x, u) = o\left(|u|^{(N+2)/(N-2)}\right) \qquad \text{as } u \to +\infty, \text{ uniformly in } x \in \overline{\Omega}.$$

Problem 8.17 Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and assume that f is a C^1 function on $[0, 1]$ such that $f > 0$ on $(0, 1)$ and $f(0) = f'(0) = f(1) = 0$. Suppose that $\underline{u} \in C_0^1(\overline{\Omega})$ satisfies $0 < \underline{u} \leq 1$ in Ω and $-\Delta \underline{u} \leq f(\underline{u})$ in Ω.

Consider the nonlinear elliptic problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.1}$$

Prove that if $\underline{u}$ is not a solution of (8.1), then there are at least two solutions u_1 and u_2 of problem (8.1) with $0 < u_1 < u_2 < 1$ in Ω.

Problem 8.18 Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, $\lambda \in \mathbb{R}$, and $p > 1$. For any integer $j \geq 1$, set

$$\Gamma_j := \{A \subset H_0^1(\Omega);\ A \text{ is compact, symmetric, } 0 \notin A,\ \gamma(A) \geq j\}\,.$$

Define the functional

$$E(u) := \frac{1}{2} \int_\Omega [|\nabla u|^2 - \lambda u^2] dx - \frac{1}{p+1} \int_\Omega |u|^{p+1}\, dx\,, \qquad u \in H_0^1(\Omega)\,.$$

Prove that

$$\inf_{A\in\Gamma_j}\max_{u\in A} E(u) = -\infty \quad \text{and} \quad \sup_{A\in\Gamma_j}\min_{u\in A} E(u) = +\infty\,.$$

Problem 8.19 Let $(X, \|\cdot\|)$ be a real Banach space with $\|\cdot\|$ of class C^1 on $X \setminus \{0\}$. Denote by $S = \{v \in X \mid \|v\| = 1\ \}$ the unit sphere and let $J \subset \mathbb{R}$ a nonempty open set. Let $f : X \to \mathbb{R}$ be a function of class C^1 on $X \setminus \{0\}$. We associate to f a function $\tilde{f} : J \times X \to \mathbb{R}$ defined by

$$\tilde{f}(t, v) = f(tv)\,,$$

where $(t, v) \in J \times S$ and $J \subset \mathbb{R}$ is an nonempty set. Let $H : \mathbb{R} \times X \to \mathbb{R}$ be a C^1 function satisfying

$$\langle H'_v, v\rangle \neq tH'_t \quad \text{if} \quad H(t, v) = c.$$

If $(t, v) \in J \times X$ with $tv \neq 0$ is a critical point of $\tilde{f}$, then $u = tv$ is a critical point of f.

Problem 8.20 Let $(X, \|\cdot\|)$ be a real Banach space with $\|\cdot\|$ of class C^1 on $X \setminus \{0\}$. Denote by $S = \{v \in X : \|v\| = 1\ \}$ the unit sphere and let $J \subset \mathbb{R}$ a nonempty open set. Let $f : X \to \mathbb{R}$ be a function of class C^1 on $X \setminus \{0\}$. Suppose that for all $v \in S$ the number

$$\hat{f}(v) = \max_{t\in J} f(tv)$$

is finite, and $\hat{f}(v) > f(0)$ if $0 \in J$. Assume that $\hat{f} : S \to \mathbb{R}$ is of class C^1. Then to every critical point $v \in S$ of $\hat{f}$, there correspond a critical point $u = tv$ of f with $t \in J \setminus \{0\}$ such that $f(u) = \hat{f}(v)$.

PART II

Variational Principles in Geometry

9
Sublinear Problems on Riemann Manifolds

If only I had the theorems!
Then I should find the proofs
easily enough.

Bernhard Riemann
(1826–1866)

In this chapter we are concerned with elliptic problems defined on compact Riemannian manifolds. This study is motivated by the Emden-Fowler equation arising in Mathematical Physics; after a suitable transformation, one obtain a new problem defined on the standard unit sphere $\mathbb{S}^d$, $d \geq 3$. This problem has been extensively studied in the case when the nonlinear term is superlinear and (sub)critical at infinity. Our aim is to complete these results by considering nonlinear terms of *sublinear* type and *oscillatory* type.

9.1 Introduction

Consider the parametrized Emden-Fowler equation

$$(\mathrm{EF})_\lambda \qquad -\triangle v = \lambda |x|^{\alpha-2} K(x/|x|) f(|x|^{-\alpha} v), \quad x \in \mathbb{R}^{d+1} \setminus \{0\},$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, K is smooth on the d-dimensional unit sphere $\mathbb{S}^d$, $d \geq 3$, $\alpha \in \mathbb{R}$, and $\lambda > 0$ is a parameter. The equation $(\mathrm{EF})_\lambda$ has been extensively studied in the *pure superlinear* case, i.e., when f has the form $f(t) = |t|^{p-1}t$, $p > 1$, see for instance Cotsiolis-Iliopoulos [79], Vázquez-Véron [288]. In these papers, the authors obtained existence and multiplicity of solutions for $(\mathrm{EF})_\lambda$, applying either minimization or minimax methods. Note that in the pure superlinear case the presence of the parameter $\lambda > 0$ is not relevant due to the rescalling technique. The solutions of $(\mathrm{EF})_\lambda$ are being sought in the particular form

$$v(x) = v(|x|, x/|x|) = u(r,\sigma) = r^\alpha u(\sigma), \tag{9.1}$$

where $(r,\sigma) \in (0,\infty) \times \mathbb{S}^d$ are the spherical coordinates in $\mathbb{R}^{d+1} \setminus \{0\}$. This type of transformation is also used by Bidaut-Véron-Véron [39], where the asymptotics of a special form of $(\mathrm{EF})_\lambda$ has been studied. Throughout (9.1), the equation $(\mathrm{EF})_\lambda$ reduces to

$$(9.1)_\lambda \qquad -\Delta_h u + \alpha(1-\alpha-d)u = \lambda K(\sigma) f(u), \ \sigma \in \mathbb{S}^d,$$

where Δ_h denotes the Laplace-Beltrami operator on $(\mathbb{S}^d, h)$ and h is the canonical metric induced from $\mathbb{R}^{d+1}$.

Note that when $\alpha = -d/2$ or $\alpha = -d/2+1$, and $f(t) = |t|^{\frac{4}{d-2}}t$, the existence of a smooth solution $u > 0$ of $(9.1)_\lambda$ can be viewed as an affirmative answer to the famous Yamabe problem on $\mathbb{S}^d$ (see also the Nirenberg problem); for these topics we refer the reader to Aubin [19], Cotsiolis-Iliopoulos [78], Hebey[136], and references therein. In these cases the right hand side of $(9.1)_\lambda$ involves the Sobolev critical exponent.

The purposes of this chapter is to guarantee *multiple* solutions of $(\mathrm{EF})_\lambda$ for certain $\lambda > 0$ when the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ has a

(i) sublinear growth at infinity; or
(ii) oscillatory behaviour near zero or at infinity.

9.2 Existence of two solutions

Since $1 - d < \alpha < 0$ implies the coercivity of the operator $u \mapsto -\Delta_h u + \alpha(1-\alpha-d)u$, the form of $(9.1)_\lambda$ motivates the study of the following general *eigenvalue problem,* which constitutes the main objective of this section: *Find $\lambda \in (0,\infty)$ and $u \in H_1^2(M)$ such that*

$$(\mathrm{P}_\lambda) \qquad -\Delta_g u + \alpha(\sigma)u = \tilde{K}(\lambda,\sigma) f(u), \ \sigma \in M,$$

where we assume

(A_1) *(M,g) is a smooth compact d-dimensional Riemannian manifold without boundary, $d \geq 3$;*
(A_2) *$\alpha \in C^\infty(M)$ and $\tilde{K} \in C^\infty((0,\infty) \times M)$ are positive functions;*
(f_1) *$f : \mathbb{R} \to \mathbb{R}$ is locally Hölder continuous and sublinear at infinity, that is,*

$$\lim_{|s|\to\infty} \frac{f(s)}{s} = 0. \tag{9.2}$$

A typical case when (9.2) holds is

($f_1^{q,c}$) *There exist $q \in (0,1)$ and $c > 0$ such that $|f(s)| \le c|s|^q$ for every $s \in \mathbb{R}$.*

As usual, Δ_g is the Laplace-Beltrami operator on (M,g); its expression in local coordinates is $\Delta_g u = g^{ij}(\partial_{ij}u - \partial u^k_{ij}\partial_k u)$. For every $u \in C^\infty(M)$, set

$$\|u\|^2_{H^2_\alpha} = \int_M \langle \nabla u, \nabla u\rangle d\sigma_g + \int_M \alpha(\sigma)\langle u, u\rangle d\sigma_g,$$

where $\langle\cdot,\cdot\rangle$ is the inner product on covariant tensor fields associated to g, ∇u is the covariant derivative of u, and $d\sigma_g$ is the Riemannian measure. The Sobolev space $H^2_\alpha(M)$ is defined as the completion of $C^\infty(M)$ with respect to the norm $\|\cdot\|_{H^2_\alpha}$. Clearly, $H^2_\alpha(M)$ is a Hilbert space endowed with the inner product

$$\langle u_1, u_2\rangle_{H^2_\alpha} = \int_M \langle \nabla u_1, \nabla u_2\rangle d\sigma_g + \int_M \alpha(\sigma)\langle u_1, u_2\rangle d\sigma_g, \quad u_1, u_2 \in H^2_\alpha(M).$$

Since α is positive, the norm $\|\cdot\|_{H^2_\alpha}$ is equivalent with the standard norm $\|\cdot\|_{H^2_1}$; actually, the latter norm is nothing but $\|\cdot\|_{H^2_\alpha}$ with $\alpha = 1$. Moreover, we have

$$\min\{1, \min_M \alpha^{1/2}\}\|u\|_{H^2_1} \le \|u\|_{H^2_\alpha} \le \max\{1, \|\alpha\|^{1/2}_{L^\infty}\}\|u\|_{H^2_1}, \quad u \in H^2_\alpha(M). \tag{9.3}$$

Note that $H^2_\alpha(M)$ is compactly embedded into $L^p(M)$ for every $p \in [1, 2d/(d-2))$; the Sobolev embedding constant will be denoted by $S_p > 0$.

The presence of the parameter $\lambda > 0$ in (P_λ) is indispensable. Indeed, if we consider a sublinear function at infinity which is, in addition, uniformly Lipschitz (with Lipschitz constant $L > 0$), and $\tilde{K}(\lambda, \sigma) = \lambda K(\sigma)$, with $K \in C^\infty(M)$ positive, one can prove that for $0 < \lambda < \frac{1}{L}\frac{\min_M \alpha}{\max_M K} := \lambda_L$ we have only the $u = u_\lambda = 0$ solution of (P_λ), as the standard contraction principle on the Hilbert space $H^2_1(M)$ shows. For a concrete example, let us consider the function $f(s) = \ln(1+s^2)$ and assume that $\frac{K(\sigma)}{\alpha(\sigma)} = \text{const.} = \mu_0 \in (0,\infty)$. Then, for every $0 < \lambda < \frac{\min_M \alpha}{\mu_0 \max_M \alpha}$, problem ($\mathrm{P}_\lambda$) has only the trivial solution; however, when $\lambda > \frac{5}{4\mu_0}$, problem ($\mathrm{P}_\lambda$) has *three* distinct constant solutions which are precisely the fixed points of the function $s \mapsto \lambda\mu_0 \ln(1+s^2)$. Note that one of the solutions is the trivial one.

Let $\lambda > 0$. The energy functional $\mathcal{E}_\lambda : H_1^2(M) \to \mathbb{R}$ associated with problem (P_λ) is

$$\mathcal{E}_\lambda(u) = \frac{1}{2}\|u\|^2_{H^2_\alpha} - \int_M \tilde{K}(\lambda,\sigma)F(u(\sigma))d\sigma_g, \tag{9.4}$$

where $F(s) = \int_0^s f(\tau)d\tau$. Due to our initial assumptions (A_1), (A_2) and (f_1), the functional $\mathcal{E}_\lambda$ is well-defined, it belongs to $C^1(H_1^2(M),\mathbb{R})$, and its critical points are precisely the weak (so, classical) solutions of problem (P_λ). By (f_1), for every $\varepsilon > 0$ sufficiently small there is $c(\varepsilon) > 0$ such that $|f(s)| \le \varepsilon|s| + c(\varepsilon)$ for every $s \in \mathbb{R}$. Thus, for every $u \in H_1^2(M)$, we have

$$\mathcal{E}_\lambda(u) \ge \frac{1}{2}(1 - \varepsilon\|\tilde{K}(\lambda,\cdot)\|_{L^\infty} S_2^2)\|u\|^2_{H^2_\alpha} - c(\varepsilon)\|\tilde{K}(\lambda,\cdot)\|_{L^\infty} S_1\|u\|_{H^2_\alpha}.$$

Therefore, the functional $\mathcal{E}_\lambda$ is coercive and bounded from below on $H_1^2(M)$. Moreover, it satisfies the standard Palais-Smale condition, see Zeidler [296, Example 38.25].

Our first result concerns the case when $f : \mathbb{R} \to \mathbb{R}$ fulfills the hypothesis

(f_2) *There exists* $\nu_0 > 1$ *such that* $\lim_{s\to 0} \frac{f(s)}{|s|^{\nu_0}} = 0.$

Before to state this result, let us note that the usual norm on the space $L^p(M)$ will be denoted by $\|\cdot\|_p$, $p \in [1,\infty]$.

Theorem 9.1 *Assume that* $f : \mathbb{R} \to \mathbb{R}$ *fulfills* (f_1), (f_2) *and* $\sup_{s\in\mathbb{R}} F(s) > 0$. *Assume also that* (A_1) *and* (A_2) *are verified with* $\tilde{K}(\lambda,\sigma) = \lambda K(\sigma)$, *and* $K \in C^\infty(M)$ *is positive. Then*

(i) for every $\lambda > \lambda^* := \frac{1}{2}\frac{\|\alpha\|_{L^1}}{\|K\|_{L^1}}\left(\max_{s\neq 0}\frac{F(s)}{s^2}\right)^{-1}$ *problem* (P_λ) *has at least two distinct, nontrivial solutions* $u^1_\lambda, u^2_\lambda \in H_1^2(M)$, *where* u^1_λ *is the global minimum of the energy functional associated with* (P_λ)*;*

(ii) if $(f_1^{q,c})$ *holds then* $\|u^1_\lambda\|_{H_1^2} = o(\lambda^{\frac{1}{1-r}})$ *for every* $r \in (q,1)$, *but* $\|u^1_\lambda\|_{H_1^2} \neq O(\lambda^{\frac{1}{1-\mu}})$ *for any* $\mu \in (1,(d+2)/(d-2))$ *as* $\lambda \to \infty$.

A direct consequence of Theorem 9.1 applied for $(9.1)_\lambda$ is the following

Theorem 9.2 *Assume that* $1-d < \alpha < 0$. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function as in Theorem* 9.1 *and* $K \in C^\infty(\mathbb{S}^d)$ *positive. Then, for every* $\lambda > \lambda^*$ *problem* $(\mathrm{EF})_\lambda$ *has at least two distinct, nontrivial solutions.*

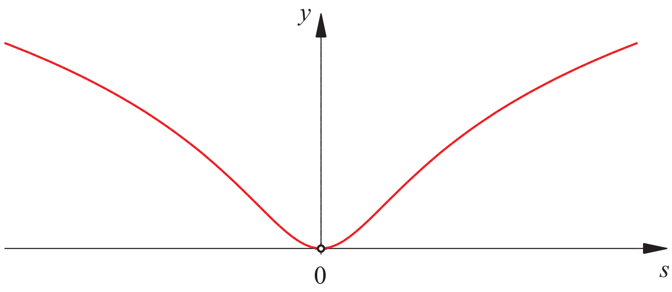

Fig. 9.1. The function $f(s) = \ln(1+s^2)$.

Example 9.1 Let $f(s) = \ln(1+s^2)$, see Figure 9.1. One can apply Theorems 9.1–9.2.

Proof of Theorem 9.1. Due to (9.2) and (f_2), the term $F(s)/s^2$ tends to 0 as $|s| \to \infty$ and $s \to 0$, respectively. Since there exists $s_0 \in \mathbb{R}$ such that $F(s_0) > 0$, the number λ^* is well-defined. Now, fix a number $s^* \in \mathbb{R}\setminus\{0\}$ such that $\frac{F(s^*)}{(s^*)^2} = \max_{s\neq 0} \frac{F(s)}{s^2}$. Therefore, $\lambda^* = \frac{1}{2}\frac{\|\alpha\|_{L^1}}{\|K\|_{L^1}}\frac{(s^*)^2}{F(s^*)}$.

Fix $\lambda > \lambda^*$. Let us first consider the constant function $u_{s^*}(\sigma) = s^*$, $\sigma \in M$. One has

$$\begin{aligned}\mathcal{E}_\lambda(u_{s^*}) &= \frac{1}{2}\|u_{s^*}\|^2_{H^2_\alpha} - \lambda\int_M K(\sigma)F(u_{s^*}(\sigma))d\sigma_g \\ &= \frac{1}{2}\|\alpha\|_{L^1}(s^*)^2 - \lambda\|K\|_{L^1}F(s^*) = (\lambda^*-\lambda)\|K\|_{L^1}F(s^*) \\ &< 0.\end{aligned}$$

Thus, $\inf_{H^2_1(M)}\mathcal{E}_\lambda \leq \mathcal{E}_\lambda(u_{s^*}) < 0$. Since $\mathcal{E}_\lambda$ verifies the (PS)-condition and it is bounded from below, one can find $u^1_\lambda \in H^2_1(M)$ such that $\mathcal{E}_\lambda(u^1_\lambda) = \inf_{H^2_1(M)}\mathcal{E}_\lambda$, see Theorem 1.4. Therefore, $u^1_\lambda \in H^2_1(M)$ is the first solution of (P_λ) and $u^1_\lambda \neq 0$, since $\mathcal{E}_\lambda(0) = 0$.

Now, we prove that for every $\lambda > \lambda^*$ the functional $\mathcal{E}_\lambda$ has the standard Mountain Pass geometry. First, due to (f_1) and (f_2), one can fix two numbers $C > 0$ and $1 < \nu < \frac{d+2}{d-2}$ such that

$$|F(s)| \leq C|s|^{\nu+1}, \quad s \in \mathbb{R}.$$

Since $\nu + 1 < \frac{2d}{d-2}$ (thus $H^2_\alpha(M) \hookrightarrow L^{\nu+1}(M)$ is continuous), one has

$$\mathcal{E}_\lambda(u) \geq \frac{1}{2}\|u\|^2_{H^2_\alpha} - \lambda C\|K\|_{L^\infty}S^{\nu+1}_{\nu+1}\|u\|^{\nu+1}_{H^2_\alpha}, \quad u \in H^2_1(M). \tag{9.5}$$

Let us take $\rho_\lambda > 0$ so small such that

$$\rho_\lambda < \min\left\{(2\lambda C\|K\|_{L^\infty} S_{\nu+1}^{\nu+1})^{\frac{1}{1-\nu}} \min\{1, \|\alpha\|^{-1/2}\}, |s^*|(\mathrm{Vol}_g(M))^{1/2}\right\}.$$

Consequently, by (9.5), for every $u \in H_1^2(M)$ complying with $\|u\|_{H_1^2} = \rho_\lambda$, we have

$$\begin{aligned}
\mathcal{E}_\lambda(u) &\geq \left(\frac{1}{2} - \lambda C\|K\|_{L^\infty} S_{\nu+1}^{\nu+1}\|u\|_{H_\alpha^2}^{\nu-1}\right)\|u\|_{H_\alpha^2}^2 \\
&\geq \left(\frac{1}{2} - \lambda C\|K\|_{L^\infty} S_{\nu+1}^{\nu+1} \max\{1, \|\alpha\|_{L^\infty}^{(\nu-1)/2}\}\rho_\lambda^{\nu-1}\right)\min\{1, \min_M \alpha\}\rho_\lambda^2 \\
&\equiv \eta(\rho_\lambda) > 0.
\end{aligned}$$

By construction, one has $\|u_{s^*}\|_{H_1^2} = |s^*|(\mathrm{Vol}_g(M))^{1/2} > \rho_\lambda$ and from above $\mathcal{E}_\lambda(u_{s^*}) < 0 = \mathcal{E}_\lambda(0)$. Since $\mathcal{E}_\lambda$ satisfies the (PS)-condition, one can apply the mountain pass theorem. Thus, there exists an element $u_\lambda^2 \in H_1^2(M)$ such that $\mathcal{E}_\lambda'(u_\lambda^2) = 0$ and $\mathcal{E}_\lambda(u_\lambda^2) \geq \eta(\rho_\lambda) > 0$. In particular, $u_\lambda^2 \neq 0$, and the elements u_λ^1 and u_λ^2 are distinct. This ends (i).

Now, we assume that $(f_1^{c,q})$ holds. Since $\mathcal{E}_\lambda(u_\lambda^1) < 0$, then

$$\frac{1}{2}\|u_\lambda^1\|_{H_\alpha^2}^2 - \lambda\frac{c}{q+1}\|K\|_{L^\infty} S_2^{q+1}(\mathrm{Vol}_g(M))^{(1-q)/2}\|u_\lambda^1\|_{H_\alpha^2}^{q+1} < 0.$$

In particular, $\|u_\lambda^1\|_{H_1^2} = O(\lambda^{\frac{1}{1-q}})$ as $\lambda \to \infty$. Therefore, for every $r \in (q, 1)$, one has $\|u_\lambda^1\|_{H_1^2} = o(\lambda^{\frac{1}{1-r}})$ as $\lambda \to \infty$.

Let us assume that $\|u_\lambda^1\|_{H_1^2} = O(\lambda^{\frac{1}{1-\mu}})$ for some $\mu \in (1, (d+2)/(d-2))$ as $\lambda \to \infty$. Consequently, $\|u_\lambda^1\|_{H_\alpha^2} \to 0$ as $\lambda \to \infty$. On the other hand, since $\mathcal{E}_\lambda(u_\lambda^1) \leq (\lambda^* - \lambda)\|K\|_{L^1}F(t^*)$, then $\mathcal{E}_\lambda(u_\lambda^1) \to -\infty$ as $\lambda \to \infty$. On account of (9.5), we have

$$\left(\frac{1}{2} - \lambda C\|K\|_{L^\infty} S_{\mu+1}^{\mu+1}\|u_\lambda^1\|_{H_\alpha^2}^{\mu-1}\right)\|u_\lambda^1\|_{H_\alpha^2}^2 \to -\infty$$

as $\lambda \to \infty$. Therefore, the expression $\lambda\|u_\lambda^1\|_{H_\alpha^2}^{\mu-1}$ necessarily tends to ∞ as $\lambda \to \infty$. But this contradicts the initial assumption. This completes (ii). □

Proof of Theorem 9.2. Let us choose $(M, g) = (\mathbb{S}^d, h)$, and $\alpha(\sigma) := \alpha(1 - \alpha - d)$ for every $\sigma \in \mathbb{S}^d$ in Theorem 9.1. Thus, for every $\lambda > \lambda^*$, problem $(9.1)_\lambda$ has at least two distinct, nontrivial solutions $u_\lambda^1, u_\lambda^2 \in H_1^2(\mathbb{S}^d)$. On account of (9.1), the elements $v_\lambda^i(x) = |x|^\alpha u_\lambda^i(x/|x|)$, $i \in$

$\{1, 2\}$, are solutions of $(\mathrm{EF})_\lambda$. □

Now, instead of (f_2) we consider a weaker assumption, namely:

(f_2') $\lim_{s\to 0} \frac{f(s)}{s} = 0$.

The following result shows that (f_2') is still enough to prove a similar multiplicity result as Theorem 9.1, but we loose unfortunately the precise location of the eigenvalues. More precisely, we have

Theorem 9.3 *Assume that $f : \mathbb{R} \to \mathbb{R}$ fulfills (f_1), (f_2') and $\sup_{s\in\mathbb{R}} F(s) > 0$. Assume also that (A_1) and (A_2) are verified with $\tilde{K}(\lambda, \sigma) = \lambda K(\sigma)$.*

Then there exist a nonempty open interval $\Lambda \subset (0, \infty)$ and a number $\gamma > 0$ such that for every $\lambda \in \Lambda$ problem (P_λ) has at least two distinct, nontrivial solutions $u_\lambda^1, u_\lambda^2 \in H_1^2(M)$ and $\|u_\lambda^i\|_{H_1^2} < \gamma$, $i \in \{1, 2\}$.

Similarly as in Theorem 9.2, we have

Theorem 9.4 *Assume that $1 - d < \alpha < 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function as in Theorem 9.3 and $K \in C^\infty(\mathbb{S}^d)$ positive. Then, there exists a nonempty open interval $\Lambda \subset (0, \infty)$ such that for every $\lambda \in \Lambda$ problem $(\mathrm{EF})_\lambda$ has at least two distinct, nontrivial solutions.*

Example 9.2 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(s) = 0$ for $s \le 0$, $f(s) = \frac{s}{\ln s}$ for $s \in (0, e^{-1}]$ and $f(s) = -e^{-1}$ for $s > e^{-1}$, see Figure 9.2. Note that f satisfies (f_2') but not (f_2) for any $\nu_0 > 1$. Therefore, one can apply Theorems 9.3 and 9.4 (but not Theorems 9.1 and 9.2).

In order to prove Theorems 9.3 and 9.4, we introduce the functionals $\mathcal{N}, \mathcal{F} : H_1^2(M) \to \mathbb{R}$ defined by

$$\mathcal{N}(u) = \frac{1}{2}\|u\|_{H_\alpha^2}^2 \quad \text{and} \quad \mathcal{F}(u) = \int_M K(\sigma)F(u(\sigma))d\sigma_g, \quad u \in H_1^2(M). \tag{9.6}$$

Proposition 9.1 $\lim_{\rho\to 0^+} \frac{\sup\{\mathcal{F}(u):\mathcal{N}(u)<\rho\}}{\rho} = 0$.

Proof. Due to (f_2'), for an arbitrarily small $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $|f(s)| < \varepsilon(2\|K\|_{L^\infty} S_2^2)^{-1}|s|$ for every $|s| < \delta(\varepsilon)$. On account of (f_1), one may fix $1 < \nu < \frac{d+2}{d-2}$ and $c(\varepsilon) > 0$ such that $|f(s)| < c(\varepsilon)|s|^\nu$ for every $|s| \ge \delta(\varepsilon)$. Combining these two facts, after an integration, we obtain

$$|F(s)| \le \varepsilon(4\|K\|_{L^\infty} S_2^2)^{-1} s^2 + c(\varepsilon)(\nu+1)^{-1}|s|^{\nu+1} \quad \text{for every } s \in \mathbb{R}.$$

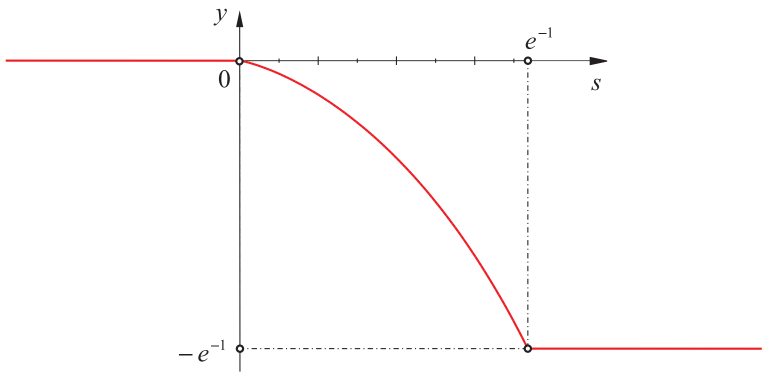

Fig. 9.2. The function from Example 9.2.

Fix a $\rho > 0$ and any element $u \in H_\alpha^2(M)$ complying with $\mathcal{N}(u) < \rho$. Due to the above estimation, we have

$$\begin{aligned}\mathcal{F}(u) &\le \frac{\varepsilon}{4}\|u\|^2_{H^2_\alpha} + \frac{c(\varepsilon)}{\nu+1}\|K\|_{L^\infty} S^{\nu+1}_{\nu+1}\|u\|^{\nu+1}_{H^2_\alpha} \\ &< \frac{\varepsilon}{2}\rho + \frac{c(\varepsilon)}{\nu+1}\|K\|_{L^\infty} S^{\nu+1}_{\nu+1}(2\rho)^{\frac{\nu+1}{2}} = \frac{\varepsilon}{2}\rho + c'(\varepsilon)\rho^{\frac{\nu+1}{2}}.\end{aligned}$$

Thus there exists $\rho(\varepsilon) > 0$ such that for every $0 < \rho < \rho(\varepsilon)$, we have

$$0 \le \frac{\sup\{\mathcal{F}(u) : \mathcal{N}(u) < \rho\}}{\rho} \le \frac{\varepsilon}{2} + c'(\varepsilon)\rho^{\frac{\nu-1}{2}} < \varepsilon,$$

which completes the proof. □

Proof of Theorem 9.3. Let $X = H_1^2(M)$, and the functionals $\mathcal{N}, \mathcal{F}$ defined in (9.6). Note that $\mathcal{E}_\lambda = \mathcal{N} - \lambda\mathcal{F}$. We know already that for every $\lambda > 0$ the functional $\mathcal{E}_\lambda = \mathcal{N} - \lambda\mathcal{F}$ is coercive and satisfies the Palais-Smale condition. Moreover, since the embedding $H_1^2(M) \hookrightarrow L^p(M)$ is compact, $1 \le p < 2d/(d-2)$, the functional $\mathcal{F}$ is sequentially weakly continuous; thus, $\mathcal{E}_\lambda$ is sequentially weakly lower semicontinuous.

Due to (9.2) and (f_2'), the term $F(s)/s^2$ tends to 0 as $|s| \to \infty$ and $s \to 0$, respectively. Since there exists $s_0 \in \mathbb{R}$ such that $F(s_0) > 0$, we may fix a number $s^* \neq 0$ such that $\frac{F(s^*)}{(s^*)^2} = \max_{s\neq 0}\frac{F(s)}{s^2}$. Therefore, λ^* appearing in Theorem 9.1 is well-defined and $\lambda^* = \frac{1}{2}\frac{\|\alpha\|_{L^1}}{\|K\|_{L^1}}\frac{(s^*)^2}{F(s^*)}$.

Now, let us choose $u_0 = 0$, and $u_1(\sigma) = u_{s^*}(\sigma) = s^*$ for every $\sigma \in M$. Fixing $\varepsilon \in (0, 1)$, due to Proposition 9.1, one may choose $\rho > 0$ such that

$$\frac{\sup\{\mathcal{F}(u) : \mathcal{N}(u) < \rho\}}{\rho} < \frac{\varepsilon}{\lambda^*}; \tag{9.7}$$

$$\rho < \frac{1}{2}(s^*)^2\|\alpha\|_{L^1}.$$

Note that $\frac{\varepsilon}{\lambda^*} < \frac{1}{\lambda^*} = \frac{\mathcal{F}(u_1)}{\mathcal{N}(u_1)}$ and $\frac{1}{2}(s^*)^2\|\alpha\|_{L^1} = \mathcal{N}(u_1)$. Therefore, by choosing

$$\overline{a} = \frac{1+\varepsilon}{\frac{\mathcal{F}(u_1)}{\mathcal{N}(u_1)} - \frac{\sup\{\mathcal{F}(u):\mathcal{N}(u)<\rho\}}{\rho}}, \tag{9.8}$$

all the hypotheses of Theorem 1.13 are verified. Consequently, there is a nonempty open interval $\Lambda \subset [0, \overline{a}]$ and a number $\gamma > 0$ such that for every $\lambda \in \Lambda$, the functional $\mathcal{E}_\lambda$ has at least three distinct critical points in $H_1^2(M)$ having $\|\cdot\|_{H_1^2}$-norm less than γ. This ends the proof of Theorem 9.3. □

Proof of Theorem 9.4. Similar to Theorem 9.2. □

9.3 Existence of many global minima

In order to obtain a new kind of multiplicity result concerning (P_λ) (specially, $(9.1)_\lambda$ and $(\mathrm{EF})_\lambda$), we require:

(f_3) *There exists $\mu_0 \in (0, \infty)$ such that the global minima of the function $s \mapsto \tilde{F}_{\mu_0}(s) := \frac{1}{2}s^2 - \mu_0 F(s)$ has at least $m \geq 2$ connected components.*

Note that (f_3) implies that the function $s \mapsto \tilde{F}_{\mu_0}(s)$ has at least $m-1$ local maxima. Thus, the function $s \mapsto \mu_0 f(s)$ has at least $2m-1$ fixed points. In particular, if for some $\lambda > 0$ one has $\frac{\tilde{K}(\lambda,\sigma)}{\alpha(\sigma)} = \mu_0$ for every $\sigma \in M$, then problem (P_λ) has at least $2m-1 \geq 3$ constant solutions. On the other hand, the following general result can be shown.

Theorem 9.5 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function which fulfills (f_1) and (f_3). Assume that (A_1) and (A_2) are verified with $\tilde{K}(\lambda, \sigma) = \lambda K(\sigma) + \mu_0\alpha(\sigma)$, and $K \in C^\infty(M)$ is positive. Then*

(i) for every $\eta > \max\{0, \|\alpha\|_{L^1} \min_{s\in\mathbb{R}} \tilde{F}_{\mu_0}(s)\}$ there exists a number $\tilde{\lambda}_\eta > 0$ such that for every $\lambda \in (0, \tilde{\lambda}_\eta)$ problem (P_λ) has at least $m+1$ solutions $u_\lambda^{1,\eta}, \ldots, u_\lambda^{m+1,\eta} \in H_1^2(M)$;

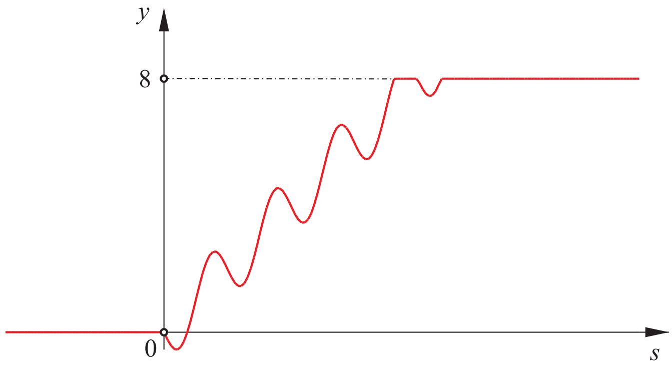

Fig. 9.3. The function $f(s) = \min\{s_+ - \sin(\pi s_+), 2(m-1)\}$ from Example 9.3 for $m = 5$.

(ii) if $(f_1^{q,c})$ *holds then for each* $\lambda \in (0, \tilde{\lambda}_\eta)$ *there is a set* $I_\lambda \subset \{1, \ldots, m+1\}$ *with* $\mathrm{card}(I_\lambda) = m$ *such that*

$$\|u_\lambda^{i,\eta}\|_{H_1^2} < \frac{s_{\eta,q,c}}{\min\{1, \min_M \alpha^{1/2}\}}, \qquad i \in I_\lambda,$$

where $s_{\eta,q,c} > 0$ *is the greatest solution of the equation*

$$\frac{1}{2}s^2 - \frac{\mu_0 c\|\alpha\|_{L^1}^{(1-q)/2}}{q+1}s^{q+1} - \eta = 0, \qquad s > 0.$$

A consequence of Theorem 9.5 in the context of $(\mathrm{EF})_\lambda$ reads as follows.

Theorem 9.6 *Assume that* $1 - d < \alpha < 0$. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function as in Theorem* 9.5 *and* $K \in C^\infty(\mathbb{S}^d)$ *be a positive function. Then, there exists* $\lambda_0 > 0$ *such that for every* $\lambda \in (0, \lambda_0)$ *problem*

$$-\triangle v = |x|^{\alpha-2}[\lambda K(x/|x|) + \mu_0\alpha(1-\alpha-d)]f(|x|^{-\alpha}v), \quad x \in \mathbb{R}^{d+1} \setminus \{0\}$$

has at least $m+1$ *solutions.*

Example 9.3 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(s) = \min\{s_+ - \sin(\pi s_+), 2(m-1)\}$ where $m \in \mathbb{N} \setminus \{1\}$ is fixed and we use the notation $s_+ = \max\{s, 0\}$. Clearly, (f_1) is verified, while for $\mu_0 = 1$, the assumption (f_3) is also fulfilled. Indeed, the function $s \mapsto \tilde{F}_1(s)$ has precisely m global minima; they are $0, 2, \ldots, 2(m-1)$. Moreover, $\min_{s\in\mathbb{R}} \tilde{F}_1(s) = 0$. Therefore, one can apply Theorems 9.5 and 9.6.

In order to prove Theorems 9.5 and 9.6, throughout relation (9.6), we define the functional $\mathcal{N}_{\mu_0} : H_1^2(M) \to \mathbb{R}$ by

$$\mathcal{N}_{\mu_0}(u) = \mathcal{N}(u) - \mu_0 \int_M \alpha(\sigma)F(u(\sigma))d\sigma_g, \quad u \in H_1^2(M).$$

Proposition 9.2 *The set of all global minima of the functional $\mathcal{N}_{\mu_0}$ has at least m connected components in the weak topology on $H_1^2(M)$.*

Proof. First, for every $u \in H_1^2(M)$ we have

$$\begin{aligned}
\mathcal{N}_{\mu_0}(u) &= \frac{1}{2}\|u\|^2_{H^2_\alpha} - \mu_0 \int_M \alpha(\sigma)F(u(\sigma))d\sigma_g \\
&= \frac{1}{2}\int_M |\nabla u|^2 d\sigma_g + \int_M \alpha(\sigma)\tilde{F}_{\mu_0}(u(\sigma))d\sigma_g \\
&\geq \|\alpha\|_{L^1} \inf_{s\in\mathbb{R}} \tilde{F}_{\mu_0}(s).
\end{aligned}$$

Moreover, if we consider $u(\sigma) = u_{\tilde{s}}(\sigma) = \tilde{s}$ for a.e. $\sigma \in M$, where $\tilde{s} \in \mathbb{R}$ is a minimum point of the function $s \mapsto \tilde{F}_{\mu_0}(s)$, then we have equality in the previous estimation. Thus,

$$\inf_{u\in H_1^2(M)} \mathcal{N}_{\mu_0}(u) = \|\alpha\|_{L^1} \inf_{s\in\mathbb{R}} \tilde{F}_{\mu_0}(s).$$

Moreover, if $u \in H_1^2(M)$ is not a constant function, then $|\nabla u|^2 = g^{ij}\partial_i u \partial_j u > 0$ on a positive measured set of the manifold M. In this case, we have

$$\mathcal{N}_{\mu_0}(u) = \frac{1}{2}\int_M |\nabla u|^2 d\sigma_g + \int_M \alpha(\sigma)\tilde{F}_{\mu_0}(u(\sigma))d\sigma_g > \|\alpha\|_{L^1} \inf_{s\in\mathbb{R}} \tilde{F}_{\mu_0}(s).$$

Consequently, between the sets

$$\mathrm{Min}(\mathcal{N}_{\mu_0}) = \{u \in H_1^2(M) : \mathcal{N}_{\mu_0}(u) = \inf_{u\in H_1^2(M)} \mathcal{N}_{\mu_0}(u)\}$$

and

$$\mathrm{Min}(\tilde{F}_{\mu_0}) = \{s \in \mathbb{R} : \tilde{F}_{\mu_0}(s) = \inf_{s\in\mathbb{R}} \tilde{F}_{\mu_0}(s)\}$$

there is a one-to-one correspondence. Indeed, let θ be the function that associates to every number $t \in \mathbb{R}$ the equivalence class of those functions which are a.e. equal to t in the whole manifold M. Then $\theta : \mathrm{Min}(\tilde{F}_{\mu_0}) \to \mathrm{Min}(\mathcal{N}_{\mu_0})$ is actually a homeomorphism between $\mathrm{Min}(\tilde{F}_{\mu_0})$ and $\mathrm{Min}(\mathcal{N}_{\mu_0})$, where the set $\mathrm{Min}(\mathcal{N}_{\mu_0})$ is considered with the relativization of the weak topology on $H_1^2(M)$. On account of the hypothesis (f_3), the set $\mathrm{Min}(\tilde{F}_{\mu_0})$ contains at least $m \geq 2$ connected components. Therefore, the same is true for the set $\mathrm{Min}(\mathcal{N}_{\mu_0})$, which completes the proof. $\square$

Proof of Theorem 9.5. Let us choose $X = H_1^2(M)$, $\mathcal{N} = \mathcal{N}_{\mu_0}$ and $\mathcal{G} = -\mathcal{F}$ in Theorem 1.18. Due to Proposition 9.2 and to basic properties of the functions $\mathcal{N}_{\mu_0}$, $\mathcal{F}$, all the hypotheses of Theorem 1.18 are satisfied.

Then, for every $\eta > \max\{0, \|\alpha\|_{L^1} \min_{s\in\mathbb{R}} \tilde{F}_{\mu_0}(s)\} \left(\geq \inf_{u\in H_1^2(M)} \mathcal{N}_{\mu_0}(u)\}\right)$ there is a number $\tilde{\lambda}_\eta > 0$ such that for every $\lambda \in (0, \tilde{\lambda}_\eta)$ the function $\mathcal{N}_{\mu_0} - \lambda\mathcal{F}$ has at least $m+1$ critical points; let us denote them by $u_\lambda^{1,\eta}, \ldots, u_\lambda^{m+1,\eta} \in H_1^2(M)$. Clearly, they are solutions of problem (P_λ), which concludes (i).

We know in addition that m elements from $u_\lambda^{1,\eta}, \ldots, u_\lambda^{m+1,\eta}$ belong to the set $\mathcal{N}_{\mu_0}^{-1}((-\infty, \eta))$. Let $\tilde{u}$ be such an element, i.e.,

$$\mathcal{N}_{\mu_0}(\tilde{u}) = \frac{1}{2}\|\tilde{u}\|_{H_\alpha^2}^2 - \mu_0 \int_M \alpha(\sigma) F(\tilde{u}(\sigma)) d\sigma_g < \eta. \tag{9.9}$$

Assume that $(f_1^{q,c})$ holds. Then $|F(s)| \leq \frac{c}{q+1}|s|^{q+1}$ for every $s \in \mathbb{R}$. By using the Hölder inequality, one has

$$\int_M \alpha(\sigma)|\tilde{u}(\sigma)|^{q+1} d\sigma_g \leq \|\alpha\|_{L^1}^{(1-q)/2} \|\tilde{u}\|_{H_\alpha^2}^{q+1}. \tag{9.10}$$

Since $\eta > 0$, the equation

$$\frac{1}{2}s^2 - \frac{\mu_0 c\|\alpha\|_{L^1}^{(1-q)/2}}{q+1}|s|^{q+1} - \eta = 0, \tag{9.11}$$

always has a positive solution. On account of (9.9) and (9.10), the number $\|\tilde{u}\|_{H_\alpha^2}$ is less then the greatest solution $s_{\eta,q,c} > 0$ of the equation (9.11). It remains to apply (9.3), which concludes (ii). $\square$

Proof of Theorem 9.6. It follows directly by Theorem 9.5. $\square$

9.4 Comments and perspectives

10

Critical Problems on Spheres

Facts are the air of scientists.
Without them you can never
fly.

Linus Pauling (1901–1994)

10.1 Introduction

We consider the nonlinear elliptic problem

$$-\Delta_h u = f(u) \quad \text{on } \mathbb{S}^d, \tag{P}$$

where $\Delta_h u = \mathrm{div}_h(\nabla u)$ denotes the Laplace-Beltrami operator acting on $u : \mathbb{S}^d \to \mathbb{R}$, $(\mathbb{S}^d, h)$ is the unit sphere, h being the canonical metric induced from $\mathbb{R}^{d+1}$.

Consider a continuous nonlinearity $f : \mathbb{R} \to \mathbb{R}$ that satisfies (f_1^L) and (f_2^L) from Section 7.1, where L denotes 0^+ or $+\infty$. As we pointed out before, these assumptions imply that f has an *oscillatory behaviour at* L. In particular, a whole sequence of distinct, *constant* solutions for (P) appears as zeros of the function $s \mapsto f(s)$, $s > 0$.

The purpose of this section is to investigate the existence of *non-*constant solutions for (P) under the assumptions (f_1^L) and (f_2^L). This problem will be achieved by constructing sign-changing solutions for (P). We prove two multiplicity results corresponding to $L = 0^+$ and $L = +\infty$, respectively; the 'piquancy' is that not only infinitely many sign-changing solutions for (P) are guaranteed but we also give a lower estimate of the number of those sequences of solutions for (P) whose elements in different sequences are mutually symmetrically distinct.

In order to handle this problem, solutions for (P) are being sought in the standard Sobolev space $H_1^2(\mathbb{S}^d)$. We say that $u \in H_1^2(\mathbb{S}^d)$ is a weak solution for (P) if

$$\int_{\mathbb{S}^d} \langle \nabla u, \nabla v \rangle d\sigma_h = \int_{\mathbb{S}^d} f(u) v d\sigma_h \quad \text{for all } v \in H_1^2(\mathbb{S}^d).$$

Since we are interested in the existence of infinitely many sign-changing solutions, it seems some kind of symmetry hypothesis on the nonlinearity f is indispensable; namely, we assume that f is odd in an arbitrarily small neighborhood of the origin whenever $L = 0^+$, and f is odd on the whole $\mathbb{R}$ whenever $L = +\infty$. In the case $L = 0^+$ no further assumption on f is needed at infinity (neither symmetry nor growth of f; in particular, f may have even a *super*critical growth). However, when $L = +\infty$, we have to control the growth of f; we assume $f(s) = O(s^{\frac{d+2}{d-2}})$ as $s \to \infty$, that is, f has an *asymptotically critical growth at infinity.* In both cases ($L = 0^+$ and $L = +\infty$), the energy functional $\mathcal{E} : H_1^2(\mathbb{S}^d) \to \mathbb{R}$ associated with (P) is well-defined, which is the key tool in order to achieve our results.

The first task is to construct certain subspaces of $H_1^2(\mathbb{S}^d)$ containing invariant functions under special actions defined by means of carefully chosen subgroups of the orthogonal group $O(d+1)$. A particular form of this construction has been first exploited by Ding [89]. In our case, every nontrivial element from these subspaces of $H_1^2(\mathbb{S}^d)$ changes the sign. The main feature of these subspaces of $H_1^2(\mathbb{S}^d)$ is based on the symmetry properties of their elements: no nontrivial element from one subspace can belong to another subspace, i.e., elements from distinct subspaces are distinguished by their symmetries. Consequently, guaranteeing nontrivial solutions for (P) in distinct subspaces of $H_1^2(\mathbb{S}^d)$ of the above type, these elements cannot be compared with each other. In the next subsection we show by an explicit construction that the minimal number of these subspaces of $H_1^2(\mathbb{S}^d)$ is $s_d = [d/2] + (-1)^{d+1} - 1$. Here, $[\cdot]$ denotes the integer function.

10.2 Group-theoretical argument

Let $d \geq 5$ and $s_d = [d/2] + (-1)^{d+1} - 1$. For every $i \in \{1, ..., s_d\}$, we define

$$G_{d,i} = \begin{cases} O(i+1) \times O(d-2i-1) \times O(i+1), & \text{if} \quad i \neq \frac{d-1}{2}, \\ O(\frac{d+1}{2}) \times O(\frac{d+1}{2}), & \text{if} \quad i = \frac{d-1}{2}. \end{cases}$$

Let us denote by $\langle G_{d,i}; G_{d,j}\rangle$ the group generated by $G_{d,i}$ and $G_{d,j}$. The key result of this section is

Proposition 10.1 *For every $i, j \in \{1, ..., s_d\}$ with $i \neq j$, the group $\langle G_{d,i}; G_{d,j}\rangle$ acts transitively on $\mathbb{S}^d$.*

Proof. Without loosing the generality, we may assume that $i < j$. The proof is divided into three steps. For abbreviation, we introduce the notation $0_k = (0, ..., 0) \in \mathbb{R}^k$, $k \in \{1, ..., d+1\}$.

Step 1. The group $\langle G_{d,i}; G_{d,j}\rangle$ acts transitively on $\mathbb{S}^{d-j-1} \times \{0_{j+1}\}$. When $j = \frac{d-1}{2}$, the proof is trivial since $O(\frac{d+1}{2})$ acts transitively on $\mathbb{S}^{\frac{d-1}{2}}$. Assume so that $j \neq \frac{d-1}{2}$. We show that for every $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{S}^{d-j-1}$ with $\sigma_1 \in \mathbb{R}^{i+1}$, $\sigma_2 \in \mathbb{R}^{j-i}$, $\sigma_3 \in \mathbb{R}^{d-2j-1}$, and $\omega \in \mathbb{S}^j$ fixed arbitrarily, there exists $g_{ij} \in \langle G_{d,i}; G_{d,j}\rangle$ such that

$$g_{ij}(\omega, 0_{d-j}) = (\sigma, 0_{j+1}). \tag{10.1}$$

Since $O(j+1)$ acts transitively on $\mathbb{S}^j$, for every $\tilde{\sigma}_2 \in \mathbb{R}^{j-i}$ with the property that $(\sigma_1, \tilde{\sigma}_2) \in \mathbb{S}^j$, there exists an element $g_j \in O(j+1)$ such that

$$g_j \omega = (\sigma_1, \tilde{\sigma}_2). \tag{10.2}$$

Note that $|\sigma_1|^2 + |\tilde{\sigma}_2|^2 = 1$ and $|\sigma_1|^2 + |\sigma_2|^2 + |\sigma_3|^2 = 1$; so $|\tilde{\sigma}_2|^2 = |\sigma_2|^2 + |\sigma_3|^2$.

If $\tilde{\sigma}_2 = 0_{j-i}$ then $\sigma_2 = 0_{j-i}$ and $\sigma_3 = 0_{d-2j-1}$; thus, $\sigma = (\sigma_1, 0_{d-j-i-1})$. Let $g_{ij} := g_j \times id_{\mathbb{R}^{d-j}} \in G_{d,j}$. Then, due to (10.2), we have

$$g_{ij}(\omega, 0_{d-j}) = (g_j\omega, 0_{d-j}) = (\sigma_1, 0_{j-i}, 0_{d-j}) = (\sigma_1, 0_{d-i}) = (\sigma, 0_{j+1}),$$

which proves (10.1).

If $\tilde{\sigma}_2 \neq 0_{j-i}$, let $r = |\tilde{\sigma}_2| > 0$. Since $O(d-2i-1)$ acts transitively on $\mathbb{S}^{d-2i-2}$ (thus, also on the sphere $r\mathbb{S}^{d-2i-2}$), then there exists $g_i \in O(d-2i-1)$ such that $g_i(\tilde{\sigma}_2, 0_{d-j-i-1}) = (\sigma_2, \sigma_3, 0_{j-i}) \in r\mathbb{S}^{d-2i-2}$. Let

$$\tilde{g}_i = id_{\mathbb{R}^{i+1}} \times g_i \times id_{\mathbb{R}^{i+1}} \in G_{d,i} \quad \text{and} \quad \tilde{g}_j = g_j \times id_{\mathbb{R}^{d-j}} \in G_{d,j}.$$

Then $g_{ij} := \tilde{g}_i\tilde{g}_j \in \langle G_{d,i}; G_{d,j}\rangle$ and on account of (10.2) and $i+1 < d-j$ (since $i < j \leq s_d$), we have

$$\tilde{g}_i\tilde{g}_j(\omega, 0_{d-j}) = \tilde{g}_i(g_j\omega, 0_{d-j}) = \tilde{g}_i(\sigma_1, \tilde{\sigma}_2, 0_{d-j}) = (\sigma_1, g_i(\tilde{\sigma}_2, 0_{d-j-i-1}), 0_{i+1})$$

$$= (\sigma_1, \sigma_2, \sigma_3, 0_{j-i}, 0_{i+1}) = (\sigma, 0_{j+1}),$$

i.e., relation (10.1).

Now, let $\overline{\sigma}, \tilde{\sigma} \in \mathbb{S}^{d-j-1}$. Then, fixing $\omega \in \mathbb{S}^j$, on account of (10.1), there are $g_1, g_2 \in \langle G_{d,i}; G_{d,j}\rangle$ such that $g_1(\omega, 0_{d-j}) = (\overline{\sigma}, 0_{j+1})$ and $g_2(\omega, 0_{d-j}) = (\tilde{\sigma}, 0_{j+1})$. Consequently, $g_2 g_1^{-1} \in \langle G_{d,i}; G_{d,j}\rangle$ and $g_2 g_1^{-1}(\overline{\sigma}, 0_{j+1}) = (\tilde{\sigma}, 0_{j+1})$, i.e., the group $\langle G_{d,i}; G_{d,j}\rangle$ acts transitively on $\mathbb{S}^{d-j-1} \times \{0_{j+1}\}$.

Step 2. *The group $\langle G_{d,i}; G_{d,j}\rangle$ acts transitively on $\mathbb{S}^{d-i-1} \times \{0_{i+1}\}$.* We can proceed in a similar way as in Step 1; however, for the reader's convenience, we sketch the proof. We show that for every $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{S}^{d-i-1}$ with $\sigma_1 \in \mathbb{R}^{i+1}$, $\sigma_2 \in \mathbb{R}^{d-j-i-1}$, $\sigma_3 \in \mathbb{R}^{j-i}$, and $\omega \in \mathbb{S}^{d-j-1}$ fixed arbitrarily, there is $g_{ij} \in \langle G_{d,i}; G_{d,j}\rangle$ such that

$$g_{ij}(\omega, 0_{j+1}) = (\sigma, 0_{i+1}). \tag{10.3}$$

Let $\tilde{\sigma}_2 \in \mathbb{R}^{d-j-i-1}$ be such that $|\sigma_1|^2 + |\tilde{\sigma}_2|^2 = 1$. Then, due to Step 1, there exists $\tilde{g}_{ij} \in \langle G_{d,i}; G_{d,j}\rangle$ such that $\tilde{g}_{ij}(\omega, 0_{j+1}) = (\sigma_1, \tilde{\sigma}_2, 0_{j+1})$.

If $\tilde{\sigma}_2 = 0_{d-j-i-1}$ then $\sigma_2 = 0_{d-j-i-1}$ and $\sigma_3 = 0_{j-i}$; thus, (10.3) is verified with the choice $g_{ij} := \tilde{g}_{ij} \in \langle G_{d,i}; G_{d,j}\rangle$.

If $\tilde{\sigma}_2 \neq 0_{d-j-i-1}$ then let $r = |\tilde{\sigma}_2| > 0$. Since $O(d-2i-1)$ acts transitively on $\mathbb{S}^{d-2i-2}$ (thus, also on the sphere $r\mathbb{S}^{d-2i-2}$), then there exists $g_i \in O(d-2i-1)$ such that $g_i(\tilde{\sigma}_2, 0_{j-i}) = (\sigma_2, \sigma_3) \in r\mathbb{S}^{d-2i-2}$. Let $\tilde{g}_i = id_{\mathbb{R}^{i+1}} \times g_i \times id_{\mathbb{R}^{i+1}} \in G_{d,i}$. Then

$$\tilde{g}_i \tilde{g}_{ij}(\omega, 0_{j+1}) = \tilde{g}_i(\sigma_1, \tilde{\sigma}_2, 0_{j+1}) = (\sigma_1, g_i(\tilde{\sigma}_2, 0_{j-i}), 0_{i+1}) = (\sigma, 0_{i+1}).$$

Consequently, $g_{ij} := \tilde{g}_i \tilde{g}_{ij} \in \langle G_{d,i}; G_{d,j}\rangle$ verifies (10.3). Now, following the last part of Step 1, our claim follows.

Step 3. (Proof concluded) *The group $\langle G_{d,i}; G_{d,j}\rangle$ acts transitively on $\mathbb{S}^d$.*

We show that for every $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in S^d$ with $\sigma_1 \in \mathbb{R}^{i+1}$, $\sigma_2 \in \mathbb{R}^{d-j-i-1}$, $\sigma_3 \in \mathbb{R}^{j+1}$, and $\omega \in S^{d-i-1}$ fixed arbitrarily, there is $g_{ij} \in \langle G_{d,i}; G_{d,j}\rangle$ such that

$$g_{ij}(\omega, 0_{i+1}) = \sigma. \tag{10.4}$$

Let $\tilde{\sigma}_3 \in \mathbb{R}^{j-i}$ such that $|\tilde{\sigma}_3| = |\sigma_3|$. Then, due to Step 2, there exists $\tilde{g}_{ij} \in \langle G_{d,i}; G_{d,j}\rangle$ such that $\tilde{g}_{ij}(\omega, 0_{i+1}) = (\sigma_1, \sigma_2, \tilde{\sigma}_3, 0_{i+1})$.

If $\tilde{\sigma}_3 = 0_{j-i}$ then $\sigma_3 = 0_{j+1}$ and (10.4) is verified by choosing $g_{ij} := \tilde{g}_{ij} \in \langle G_{d,i}; G_{d,j}\rangle$.

If $\tilde{\sigma}_3 \neq 0_{j-i}$, let $r = |\tilde{\sigma}_3| = |\sigma_3| > 0$. Since $O(j+1)$ acts transitively on $\mathbb{S}^j$, there exists $g_j \in O(j+1)$ such that $g_j(\tilde{\sigma}_3, 0_{i+1}) = \sigma_3 \in r\mathbb{S}^j$. Let us fix the element $\tilde{g}_j = id_{\mathbb{R}^{d-j}} \times g_j \in G_{d,j}$. Then

$$\tilde{g}_j \tilde{g}_{ij}(\omega, 0_{i+1}) = \tilde{g}_j(\sigma_1, \sigma_2, \tilde{\sigma}_3, 0_{i+1}) = (\sigma_1, \sigma_2, g_j(\tilde{\sigma}_3, 0_{i+1})) = (\sigma_1, \sigma_2, \sigma_3) = \sigma.$$

Consequently, $g_{ij} := \tilde{g}_j \tilde{g}_{ij} \in \langle G_{d,i}; G_{d,j}\rangle$ verifies (10.4).

Now, let $\overline{\sigma}, \tilde{\sigma} \in \mathbb{S}^d$. Then, fixing $\omega \in \mathbb{S}^{d-i-1}$, on account of (10.4), there are $g_1, g_2 \in \langle G_{d,i}; G_{d,j}\rangle$ such that $g_1(\omega, 0_{i+1}) = \overline{\sigma}$ and $g_2(\omega, 0_{i+1}) = \tilde{\sigma}$. Consequently, $g_2 g_1^{-1} \in \langle G_{d,i}; G_{d,j}\rangle$ and $g_2 g_1^{-1}(\overline{\sigma}) = \tilde{\sigma}$, i.e., the group $\langle G_{d,i}; G_{d,j}\rangle$ acts transitively on $\mathbb{S}^d$. This completes the proof. □

Let $d \geq 5$ and fix $G_{d,i}$ for some $i \in \{1, ..., s_d\}$. We define the function $\tau_i : \mathbb{S}^d \to \mathbb{S}^d$ associated to $G_{d,i}$ by

$$\tau_i(\sigma) = \begin{cases} (\sigma_3, \sigma_2, \sigma_1), & \text{if} \quad i \neq \frac{d-1}{2}, \text{ and } \sigma = (\sigma_1, \sigma_2, \sigma_3) \text{ with } \sigma_1, \sigma_3 \in \mathbb{R}^{i+1}, \ \sigma_2 \in \mathbb{R}^{d-2i-1}; \\ (\sigma_3, \sigma_1), & \text{if} \quad i = \frac{d-1}{2}, \text{ and } \sigma = (\sigma_1, \sigma_3) \text{ with } \sigma_1, \sigma_3 \in \mathbb{R}^{\frac{d+1}{2}}. \end{cases}$$

The explicit form of the groups $G_{d,i}$ and the functions τ_i can be seen in Table 1 for dimensions $d = 5, ..., 12$. It is clear by construction that $\tau_i \notin G_{d,i}$, $\tau_i G_{d,i} \tau_i^{-1} = G_{d,i}$ and $\tau_i^2 = id_{\mathbb{R}^{d+1}}$.

d	s_d	$G_{d,i}$; $i \in \{1, ..., s_d\}$	τ_i; $i \in \{1, ..., s_d\}$
5	2	$G_{5,1} = O(2) \times O(2) \times O(2)$	$\tau_1(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$; $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}^2$
		$G_{5,2} = O(3) \times O(3)$	$\tau_2(\sigma_1, \sigma_2) = (\sigma_2, \sigma_1)$; $\sigma_1, \sigma_2 \in \mathbb{R}^3$
6	1	$G_{6,1} = O(2) \times O(3) \times O(2)$	$\tau_1(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$; $\sigma_1, \sigma_3 \in \mathbb{R}^2, \sigma_2 \in \mathbb{R}^3$
7	3	$G_{7,1} = O(2) \times O(4) \times O(2)$	$\tau_1(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$; $\sigma_1, \sigma_3 \in \mathbb{R}^2, \sigma_2 \in \mathbb{R}^4$
		$G_{7,2} = O(3) \times O(2) \times O(3)$	$\tau_2(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$; $\sigma_1, \sigma_3 \in \mathbb{R}^3, \sigma_2 \in \mathbb{R}^2$
		$G_{7,3} = O(4) \times O(4)$	$\tau_3(\sigma_1, \sigma_2) = (\sigma_2, \sigma_1)$; $\sigma_1, \sigma_2 \in \mathbb{R}^4$
8	2	$G_{8,1} = O(2) \times O(5) \times O(2)$	$\tau_1(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$; $\sigma_1, \sigma_3 \in \mathbb{R}^2, \sigma_2 \in \mathbb{R}^5$
		$G_{8,2} = O(3) \times O(3) \times O(3)$	$\tau_2(\sigma_1, \sigma_2, \sigma_3) = (\sigma_2, \sigma_1, \sigma_3)$; $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}^3$
9	4	$G_{9,1} = O(2) \times O(6) \times O(2)$	$\tau_1(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$; $\sigma_1, \sigma_3 \in \mathbb{R}^2, \sigma_2 \in \mathbb{R}^6$
		$G_{9,2} = O(3) \times O(4) \times O(3)$	$\tau_2(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$; $\sigma_1, \sigma_3 \in \mathbb{R}^3, \sigma_2 \in \mathbb{R}^4$
		$G_{9,3} = O(4) \times O(2) \times O(4)$	$\tau_3(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$; $\sigma_1, \sigma_3 \in \mathbb{R}^4, \sigma_2 \in \mathbb{R}^2$
		$G_{9,4} = O(5) \times O(5)$	$\tau_4(\sigma_1, \sigma_2) = (\sigma_2, \sigma_1)$; $\sigma_1, \sigma_2 \in \mathbb{R}^5$
10	3	$G_{10,1} = O(2) \times O(7) \times O(2)$	$\tau_1(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$; $\sigma_1, \sigma_3 \in \mathbb{R}^2, \sigma_2 \in \mathbb{R}^7$
		$G_{10,2} = O(3) \times O(5) \times O(3)$	$\tau_2(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$; $\sigma_1, \sigma_3 \in \mathbb{R}^3, \sigma_2 \in \mathbb{R}^5$
		$G_{10,3} = O(4) \times O(3) \times O(4)$	$\tau_3(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$; $\sigma_1, \sigma_3 \in \mathbb{R}^4, \sigma_2 \in \mathbb{R}^3$
11	5	$G_{11,1} = O(2) \times O(8) \times O(2)$	$\tau_1(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$; $\sigma_1, \sigma_3 \in \mathbb{R}^2, \sigma_2 \in \mathbb{R}^8$
		$G_{11,2} = O(3) \times O(6) \times O(3)$	$\tau_2(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$; $\sigma_1, \sigma_3 \in \mathbb{R}^3, \sigma_2 \in \mathbb{R}^6$
		$G_{11,3} = O(4) \times O(4) \times O(4)$	$\tau_3(\sigma_1, \sigma_2, \sigma_3) = (\sigma_2, \sigma_1, \sigma_3)$; $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}^4$
		$G_{11,4} = O(5) \times O(2) \times O(5)$	$\tau_4(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$; $\sigma_1, \sigma_3 \in \mathbb{R}^5$, $\sigma_2 \in \mathbb{R}^2$
		$G_{11,5} = O(6) \times O(6)$	$\tau_5(\sigma_1, \sigma_2) = (\sigma_2, \sigma_1)$; $\sigma_1, \sigma_2 \in \mathbb{R}^6$
12	4	$G_{12,1} = O(2) \times O(9) \times O(2)$	$\tau_1(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$; $\sigma_1, \sigma_3 \in \mathbb{R}^2, \sigma_2 \in \mathbb{R}^9$
		$G_{12,2} = O(3) \times O(7) \times O(3)$	$\tau_2(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$; $\sigma_1, \sigma_3 \in \mathbb{R}^3, \sigma_2 \in \mathbb{R}^7$
		$G_{12,3} = O(4) \times O(5) \times O(4)$	$\tau_3(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$; $\sigma_1, \sigma_3 \in \mathbb{R}^4, \sigma_2 \in \mathbb{R}^5$
		$G_{12,4} = O(5) \times O(3) \times O(5)$	$\tau_4(\sigma_1, \sigma_2, \sigma_3) = (\sigma_3, \sigma_2, \sigma_1)$; $\sigma_1, \sigma_3 \in \mathbb{R}^5, \sigma_2 \in \mathbb{R}^3$

Table 1.

Inspired by [30], [32], we introduce the action of the group $G_{d,i}^{\tau_i} = \langle G_{d,i}, \tau_i\rangle \subset O(d+1)$ on the space $H_1^2(\mathbb{S}^d)$. Due to the above properties of τ_i, only two types of elements in $G_{d,i}^{\tau_i}$ can be distinguished; namely,

$\tilde{g} = g \in G_{d,i}$, and $\tilde{g} = \tau_i g \in G_{d,i}^{\tau_i} \setminus G_{d,i}$ (with $g \in G_{d,i}$), respectively. Therefore, the action $G_{d,i}^{\tau_i} \times H_1^2(\mathbb{S}^d) \to H_1^2(\mathbb{S}^d)$ given by

$$gu(\sigma) = u(g^{-1}\sigma), \quad (\tau_i g)u(\sigma) = -u(g^{-1}\tau_i^{-1}\sigma), \tag{10.5}$$

for $g \in G_{d,i}$, $u \in H_1^2(\mathbb{S}^d)$ and $\sigma \in \mathbb{S}^d$, is well-defined, continuous and linear. We define the subspace of $H_1^2(\mathbb{S}^d)$ containing all symmetric points with respect to the compact group $G_{d,i}^{\tau_i}$, i.e.,

$$H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d) = \{u \in H_1^2(\mathbb{S}^d) : \tilde{g}u = u \text{ for every } \tilde{g} \in G_{d,i}^{\tau_i}\}.$$

For further use, we also introduce

$$H_{G_{d,i}}(\mathbb{S}^d) = \{u \in H_1^2(\mathbb{S}^d) : gu = u \text{ for every } g \in G_{d,i}\},$$

where the action of the group $G_{d,i}$ on $H_1^2(\mathbb{S}^d)$ is defined by the first relation of (10.5).

Remark 10.1 Every nonzero element of the space $H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$ changes sign. To see this, let $u \in H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d) \setminus \{0\}$. Due to the $G_{d,i}^{\tau_i}$-invariance of u and (10.5) we have $u(\sigma) = -u(\tau_i^{-1}\sigma)$ for every $\sigma \in \mathbb{S}^d$. Since $u \neq 0$, it should change the sign.

The next result shows us how can we construct mutually distinct subspaces of $H_1^2(\mathbb{S}^d)$ which cannot be compared by symmetrical point of view.

Theorem 10.1 *For every $i, j \in \{1, ..., s_d\}$ with $i \neq j$, one has*

(i) $H_{G_{d,i}}(\mathbb{S}^d) \cap H_{G_{d,j}}(\mathbb{S}^d) = \{$*constant functions on* $\mathbb{S}^d\}$;
(ii) $H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d) \cap H_{G_{d,j}^{\tau_j}}(\mathbb{S}^d) = \{0\}$.

Proof. (i) Let $u \in H_{G_{d,i}}(\mathbb{S}^d) \cap H_{G_{d,j}}(\mathbb{S}^d)$. In particular, u is both $G_{d,i}$- and $G_{d,j}$-invariant, i.e. $g_i u = g_j u = u$ for every $g_i \in G_{d,i}$ and $g_j \in G_{d,j}$, respectively. Consequently, u is also $\langle G_{d,i}, G_{d,j}\rangle$-invariant; thus, $u(\sigma) = u(g_{ij}\sigma)$ for every $g_{ij} \in \langle G_{d,i}, G_{d,j}\rangle$ and $\sigma \in \mathbb{S}^d$. Due to Proposition 10.1, for every fixed $\sigma \in \mathbb{S}^d$, the orbit of $g_{ij}\sigma$ is the whole sphere $\mathbb{S}^d$ whenever g_{ij} runs through $\langle G_{d,i}, G_{d,j}\rangle$. Therefore, u should be constant.

(ii) Let $u \in H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d) \cap H_{G_{d,j}^{\tau_j}}(\mathbb{S}^d)$. The second relation of (10.5) shows that $u(\sigma) = -u(\tau_i^{-1}\sigma) = -u(\tau_j^{-1}\sigma)$, $\sigma \in \mathbb{S}^d$. But, due to (i), u is constant. Thus, u should be 0. $\square$

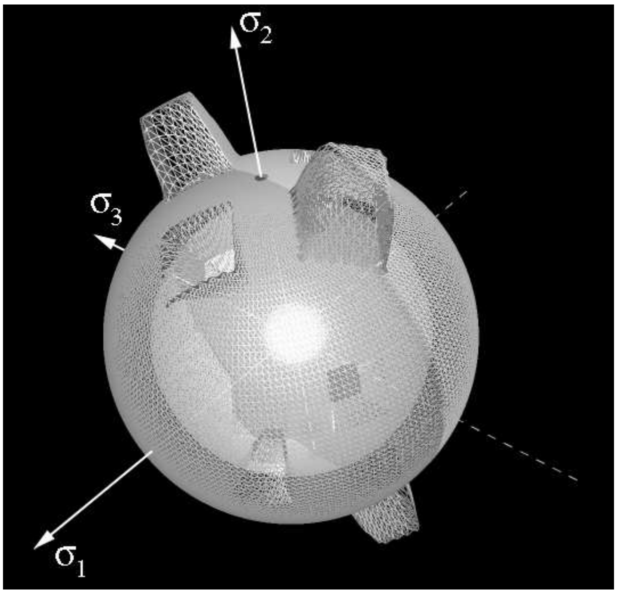

Fig. 10.1. The image of the function $w : \mathbb{S}^d \to \mathbb{R}$ from (10.6) with parameters $r = 0.2, R = 1.5, s = 0.4$; the value $w(\sigma)$ is represented (radially) on the line determined by $0 \in \mathbb{R}^{d+1}$ and $\sigma \in \mathbb{S}^d$, the 'zero altitude' being $c\sigma$, i.e., the sphere $c\mathbb{S}^d$, with $c = 1.3$. The union of those 8 disconnected holes on the sphere $\mathbb{S}^d$ where the function w takes values s and $(-s)$ corresponds to the $G_{d,i}^{\tau_i}$–invariant set D_i. (Note that the figure describes the case $i \neq \frac{d-1}{2}$. When $i = \frac{d-1}{2}$ the coordinate σ_2 vanishes and the figure becomes simpler.)

To conclude this section, we construct explicit functions belonging to $H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$ which is of interest in its own right as well. Before we give the class of functions we are speaking about, we say that a set $D \subset \mathbb{S}^d$ is $G_{d,i}^{\tau_i}$-invariant, if $\tilde{g}D \subseteq D$ for every $\tilde{g} \in G_{d,i}^{\tau_i}$.

Proposition 10.2 *Let $i \in \{1, ..., s_d\}$ and $s > 0$ be fixed. Then there exist a number $C_i > 0$ and a $G_{d,i}^{\tau_i}$-invariant set $D_i \subset \mathbb{S}^d$ with* $\mathrm{Vol}_h(D_i) > 0$, *both independent on the number s, and a function $w \in H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$ such that*

i) $\|w\|_{L^\infty} \leq s$;
ii) $|\nabla w(\sigma)| \leq C_i s$ *for a.e.* $\sigma \in \mathbb{S}^d$;
iii) $|w(\sigma)| = s$ *for every* $\sigma \in D_i$.

An explicit function $w : \mathbb{S}^d \to \mathbb{R}$ fulfilling all the requirements of Proposition 10.2 is given by

$$\begin{aligned} w(\sigma) &= \frac{8s}{(R-r)}\mathrm{sgn}(|\sigma_1|-|\sigma_3|)\max\left(0, \min\left(\frac{R-r}{8}, \frac{R-r}{4}-\right.\right. \\ &\left.\left. - \max\left(\left||\sigma_1|+|\sigma_3|-\frac{R+3r}{4}\right|, \left|||\sigma_1|-|\sigma_3||-\frac{R+3r}{4}\right|\right)\right)\right) \quad (10.6)\end{aligned}$$

where $R > r$, and $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{S}^d$ with $\sigma_1, \sigma_3 \in \mathbb{R}^{i+1}$, $\sigma_2 \in \mathbb{R}^{d-2i-1}$ whenever $i \neq \frac{d-1}{2}$, and $\sigma = (\sigma_1, \sigma_3) \in \mathbb{S}^d$ with $\sigma_1, \sigma_3 \in \mathbb{R}^{\frac{d+1}{2}}$ whenever $i = \frac{d-1}{2}$. The $G_{d,i}^{\tau_i}$-invariant set $D_i \subset \mathbb{S}^d$ can be defined as

$$D_i = \left\{\sigma \in \mathbb{S}^d : \left||\sigma_1|+|\sigma_3|-\frac{R+3r}{4}\right| \leq \frac{R-r}{8}, \left|||\sigma_1|-|\sigma_3||-\frac{R+3r}{4}\right| \leq \frac{R-r}{8}\right\}.$$

The geometrical image of the function w from (10.6) is shown by Figure 10.1.

10.3 Arbitrarily small solutions

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $F(s) = \int_0^s f(t)dt$. We assume that

(f_1^0) $-\infty < \liminf_{s\to 0^+} \frac{F(s)}{s^2} \leq \limsup_{s\to 0^+} \frac{F(s)}{s^2} = +\infty$;
(f_2^0) $\liminf_{s\to 0^+} \frac{f(s)}{s} < 0$.

Theorem 10.2 *Let $d \geq 5$ and $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which is odd in an arbitrarily small neighborhood of the origin, verifying (f_1^0) and (f_2^0). Then there exist at least $s_d = [d/2] + (-1)^{d+1} - 1$ sequences $\{u_k^i\}_k \subset H_1^2(\mathbb{S}^d)$, $i \in \{1, ..., s_d\}$, of sign-changing weak solutions of* (P) *distinguished by their symmetry properties. In addition,*

$$\lim_{k\to\infty} \|u_k^i\|_{L^\infty} = \lim_{k\to\infty} \|u_k^i\|_{H_1^2} = 0 \quad \textit{for every} \quad i \in \{1, ..., s_d\}.$$

Example 10.1 Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha + \beta > 1 > \alpha > 0$, and $\gamma \in (0, 1)$. Then, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(0) = 0$ and $f(s) = |s|^{\alpha-1}s(\gamma + \sin|s|^{-\beta})$ near the origin (but $s \neq 0$) and extended in an arbitrarily way to the whole $\mathbb{R}$, verifies both (f_1^0) and (f_2^0).

In order to prove Theorem 10.2, we need some propositions. Throughout this section we assume the hypotheses of Theorem 10.2 are fulfilled.

Let $\tilde{s} > 0$ be so small that f is odd on $[-\tilde{s}, \tilde{s}]$, and let us define $\tilde{f}(s) = \mathrm{sgn}(s)f(\min(|s|, \tilde{s}))$. Clearly, $\tilde{f}$ is continuous and odd on $\mathbb{R}$. Define also $\tilde{F}(s) = \int_0^s \tilde{f}(t)dt$, $s \in \mathbb{R}$.

On account of (f_2^0), one may fix $c_0 > 0$ such that

$$\liminf_{s\to 0^+} \frac{f(s)}{s} < -c_0 < 0. \tag{10.7}$$

In particular, there is a sequence $\{\overline{s}_k\}_k \subset (0, \tilde{s})$ converging (decreasingly) to 0, such that

$$\tilde{f}(\overline{s}_k) = f(\overline{s}_k) < -c_0 s_k. \tag{10.8}$$

Let us define the functions

$$\psi(s) = \tilde{f}(s) + c_0 s \quad \text{and} \quad \Psi(s) = \int_0^s \psi(t)dt = \tilde{F}(s) + \frac{c_0}{2}s^2, \;\; s \in \mathbb{R}. \tag{10.9}$$

Due to (10.8), $\psi(\overline{s}_k) < 0$; so, there are two sequences $\{a_k\}_k$, $\{b_k\}_k \subset (0, \tilde{s})$, both converging to 0, such that $b_{k+1} < a_k < \overline{s}_k < b_k$ for every $k \in \mathbb{N}$ and

$$\psi(s) \leq 0 \text{ for every } s \in [a_k, b_k]. \tag{10.10}$$

Since $c_0 > 0$, see (10.7), the norm

$$\|u\|_{c_0} = \left(\int_{\mathbb{S}^d} |\nabla u|^2 d\sigma_h + c_0 \int_{\mathbb{S}^d} u^2 d\sigma_h \right)^{1/2} \tag{10.11}$$

is equivalent to the standard norm $\|\cdot\|_{H_1^2}$. Now, we define $\mathcal{E} : H_1^2(\mathbb{S}^d) \to \mathbb{R}$ by

$$\mathcal{E}(u) = \frac{1}{2}\|u\|_{c_0}^2 - \int_{\mathbb{S}^d} \Psi(u(\sigma))d\sigma_h,$$

which is well-defined since ψ has a subcritical growth, and $H_1^2(\mathbb{S}^d)$ is compactly embedded into $L^p(\mathbb{S}^d)$, $p \in [1, 2^*)$, see Hebey [136, Theorem 2.9, p. 37]. Moreover, $\mathcal{E}$ belongs to $C^1(H_1^2(\mathbb{S}^d))$, it is even, and it coincides with the energy functional associated to (P) on the set $B^\infty(\tilde{s}) = \{u \in L^\infty(\mathbb{S}^d) : \|u\|_{L^\infty} \leq \tilde{s}\}$ because the functions f and $\tilde{f}$ coincide on $[-\tilde{s}, \tilde{s}]$.

From now on, we fix $i \in \{1, ..., s_d\}$ and the corresponding subspace $H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$ of $H_1^2(\mathbb{S}^d)$ introduced in the previous section. Let us denote by $\mathcal{E}_i$ the restriction of the functional $\mathcal{E}$ to $H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$ and for every $k \in \mathbb{N}$, consider the set

$$T_k^i = \{u \in H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d) : \; \|u\|_{L^\infty} \leq b_k\}, \tag{10.12}$$

where b_k is from (10.10).

Proposition 10.3 *The functional $\mathcal{E}_i$ is bounded from below on T_k^i and its infimum m_k^i on T_k^i is attained at $u_k^i \in T_k^i$. Moreover, $m_k^i = \mathcal{E}_i(u_k^i) < 0$ for every $k \in \mathbb{N}$.*

Proof. For every $u \in T_k^i$ we have

$$\mathcal{E}_i(u) = \frac{1}{2}\|u\|_{c_0}^2 - \int_{\mathbb{S}^d} \Psi(u(\sigma))d\sigma_h \geq - \max_{[-b_k,b_k]} \Psi \cdot \mathrm{Vol}_h(\mathbb{S}^d) > -\infty.$$

It is clear that T_k^i is convex and closed, thus weakly closed in $H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$. Let $m_k^i = \inf_{T_k^i} \mathcal{E}_i$, and $\{u_n\}_n \subset T_k^i$ be a minimizing sequence of $\mathcal{E}_i$ for m_k^i. Then, for large $n \in \mathbb{N}$, we have

$$\frac{1}{2}\|u_n\|_{c_0}^2 \leq m_k^i + 1 + \max_{[-b_k,b_k]} \Psi \cdot \mathrm{Vol}_h(\mathbb{S}^d),$$

thus $\{u_n\}_n$ is bounded in $H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$. Up to a subsequence, $\{u_n\}_n$ weakly converges in $H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$ to some $u_k^i \in T_k^i$. Since ψ has a subcritical growth, by using the compactness of the embedding $H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d) \subset H_1^2(\mathbb{S}^d) \hookrightarrow L^p(\mathbb{S}^d)$, $1 \leq p < 2^*$, one can conclude the sequentially weak continuity of the function $u \mapsto \int_{\mathbb{S}^d} \Psi(u(\sigma))d\sigma_h$, $u \in H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$. Consequently, $\mathcal{E}_i$ is sequentially weak lower semicontinuous. Combining this fact with the weak closedness of the set T_k^i, we obtain $\mathcal{E}_i(u_k^i) = m_k^i = \inf_{T_k^i} \mathcal{E}_i$.

The next task is to prove that $m_k^i < 0$ for every $k \in \mathbb{N}$. First, due to (10.9) and (f_1^0), we have

$$-\infty < \liminf_{s\to 0^+} \frac{\Psi(s)}{s^2} \leq \limsup_{s\to 0^+} \frac{\Psi(s)}{s^2} = +\infty. \tag{10.13}$$

Therefore, the left-hand side of (10.13) and the evenness of Ψ implies the existence of $\underline{l} > 0$ and $\varrho \in (0, \tilde{s})$ such that

$$\Psi(s) \geq -\underline{l}s^2 \text{ for every } s \in (-\varrho, \varrho). \tag{10.14}$$

Let $D_i \subset \mathbb{S}^d$ and $C_i > 0$ be from Proposition 10.2 (which depend only on $G_{d,i}$ and τ_i), and fix a number $\bar{l} > 0$ large enough such that

$$\bar{l}\mathrm{Vol}_h(D_i) > \left(\underline{l} + \frac{c_0}{2}\right)\mathrm{Vol}_h(\mathbb{S}^d) + \frac{C_i^2}{2}, \tag{10.15}$$

$c_0 > 0$ being from (10.7). Taking into account the right-hand side of (10.13), there is a sequence $\{s_k\}_k \subset (0, \varrho)$ such that $s_k \leq b_k$ and $\Psi(s_k) = \Psi(-s_k) > \bar{l}s_k^2$ for every $k \in \mathbb{N}$.

Let $w_k := w_{s_k} \in H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$ be the function from Proposition 10.2 corresponding to the value $s_k > 0$. Then $w_k \in T_k^i$ and one has

$$\begin{aligned} \mathcal{E}_i(w_k) &= \frac{1}{2}\|w_k\|_{c_0}^2 - \int_{\mathbb{S}^d} \Psi(w_k(\sigma))d\sigma_h \\ &\leq \frac{1}{2}\left(C_i^2 + c_0\mathrm{Vol}_h(\mathbb{S}^d)\right)s_k^2 - \int_{D_i} \Psi(w_k(\sigma))d\sigma_h - \int_{\mathbb{S}^d\setminus D_i} \Psi(w_k(\sigma))d\sigma_h. \end{aligned}$$

On account of Proposition 10.2 iii), we have

$$\int_{D_i} \Psi(w_k(\sigma))d\sigma_h = \Psi(s_k)\mathrm{Vol}_h(D_i) > \bar{l}\mathrm{Vol}_h(D_i)s_k^2.$$

On the other hand, due to relation (10.14) and Proposition 10.2 i), we have

$$\int_{\mathbb{S}^d\setminus D_i} \Psi(w_k(\sigma))d\sigma_h \geq -\underline{l}\int_{\mathbb{S}^d\setminus D_i} w_k^2(\sigma)d\sigma_h > -\underline{l}\mathrm{Vol}_h(\mathbb{S}^d)s_k^2.$$

Combining (10.15) with the above estimations, we obtain that $m_k^i = \inf_{T_k^i} \mathcal{E}_i \leq \mathcal{E}_i(w_k) < 0$, which proves our claim. $\square$

Proposition 10.4 *Let $u_k^i \in T_k^i$ from Proposition 10.3. Then, $\|u_k^i\|_{L^\infty} \leq a_k$. (The number a_k is from* (10.10)*.)*

Proof. Let $A = \{\sigma \in \mathbb{S}^d : u_k^i(\sigma) \notin [-a_k, a_k]\}$ and suppose that $\mathrm{meas}(A) > 0$. Define the function $\gamma(s) = \mathrm{sgn}(s)\min(|s|, a_k)$ and set $w_k = \gamma \circ u_k^i$. Since γ is Lipschitz continuous, then $w_k \in H_1^2(\mathbb{S}^d)$, see Hebey [136, Proposition 2.5, p. 24].

We first claim that $w_k \in H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$. To see this, it suffices to prove that $\tilde{g}w_k = w_k$ for every $\tilde{g} \in G_{d,i}^{\tau_i}$. First, let $\tilde{g} = g \in G_{d,i}$. Since $gu_k^i = u_k^i$, we have

$$gw_k(\sigma) = w_k(g^{-1}\sigma) = (\gamma\circ u_k^i)(g^{-1}\sigma) = \gamma(u_k^i(g^{-1}\sigma)) = \gamma(u_k^i(\sigma)) = w_k(\sigma)$$

for every $\sigma \in \mathbb{S}^d$. Now, let $\tilde{g} = \tau_i g \in G_{d,i}^{\tau_i} \setminus G_{d,i}$ (with $g \in G_{d,i}$). Since γ is an odd function and $(\tau_i g)u_k^i = u_k^i$, on account of (10.5) we have

$$\begin{aligned} (\tau_i g)w_k(\sigma) &= -w_k(g^{-1}\tau_i^{-1}\sigma) = -(\gamma \circ u_k^i)(g^{-1}\tau_i^{-1}\sigma) \\ &= \gamma(-u_k^i(g^{-1}\tau_i^{-1}\sigma)) = \gamma((\tau_i g)u_k^i(\sigma)) = \gamma(u_k^i(\sigma)) \\ &= w_k(\sigma) \end{aligned}$$

for every $\sigma \in \mathbb{S}^d$. In conclusion, the claim is true, and $w_k \in H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$. Moreover, $\|w_k\|_{L^\infty} \leq a_k$. Consequently, $w_k \in T_k^i$.

We introduce the sets

$$A_1 = \{\sigma \in A : \; u_k^i(\sigma) < -a_k\} \qquad \text{and} \qquad A_2 = \{\sigma \in A : \; u_k^i(\sigma) > a_k\}.$$

Thus, $A = A_1 \cup A_2$, and we have that $w_k(\sigma) = u_k^i(\sigma)$ for all $\sigma \in \mathbb{S}^d \setminus A$, $w_k(\sigma) = -a_k$ for all $\sigma \in A_1$, and $w_k(\sigma) = a_k$ for all $\sigma \in A_2$. Moreover,

$$\mathcal{E}_i(w_k) - \mathcal{E}_i(u_k^i) =$$

$$\begin{aligned}
&= -\frac{1}{2}\int_A |\nabla u_k^i|^2 d\sigma_h + \frac{c_0}{2}\int_A [w_k^2 - (u_k^i)^2] d\sigma_h - \int_A [\Psi(w_k) - \Psi(u_k^i)] d\sigma_h \\
&= -\frac{1}{2}\int_A |\nabla u_k^i(\sigma)|^2 d\sigma_h + \frac{c_0}{2}\int_A [a_k^2 - (u_k^i(\sigma))^2] d\sigma_h \\
&\quad - \int_{A_1} [\Psi(-a_k) - \Psi(u_k^i(\sigma))] d\sigma_h - \int_{A_2} [\Psi(a_k) - \Psi(u_k^i(\sigma))] d\sigma_h.
\end{aligned}$$

Note that $\int_A [w_k^2 - (u_k^i)^2] d\sigma_h \le 0$. Next, by the mean value theorem, for a.e. $\sigma \in A_2$, there exists $\theta_k(\sigma) \in [a_k, b_k]$ such that $\Psi(a_k) - \Psi(u_k^i(\sigma)) = \psi(\theta_k(\sigma))(a_k - u_k^i(\sigma))$. Thus, on account of (10.10), one has

$$\int_{A_2} [\Psi(a_k) - \Psi(u_k^i(\sigma))] d\sigma_h \ge 0.$$

In the same way, using the oddness of ψ, we conclude that

$$\int_{A_1} [\Psi(-a_k) - \Psi(u_k^i(\sigma))] d\sigma_h \ge 0.$$

In conclusion, every term of the expression $\mathcal{E}_i(w_k) - \mathcal{E}_i(u_k^i)$ is nonpositive. On the other hand, since $w_k \in T_k^i$, then $\mathcal{E}_i(w_k) \ge \mathcal{E}_i(u_k^i) = \inf_{T_k^i} \mathcal{E}_i$. So, every term in $\mathcal{E}_i(w_k) - \mathcal{E}_i(u_k^i)$ should be zero. In particular,

$$\int_A |\nabla u_k^i(\sigma)|^2 d\sigma_h = \int_A [a_k^2 - (u_k^i(\sigma))^2] d\sigma_h = 0.$$

These equalities imply that meas(A) should be 0, contradicting our initial assumption. □

Proposition 10.5 $\lim_{k\to\infty} m_k^i = \lim_{k\to\infty} \|u_k^i\|_{L^\infty} = \lim_{k\to\infty} \|u_k^i\|_{H_1^2} = 0.$

Proof. Using Proposition 10.4, we have that $\|u_k^i\|_{L^\infty} \le a_k < \tilde{s}$ for a.e. $\sigma \in \mathbb{S}^d$. Therefore, we readily have that $\lim_{k\to\infty} \|u_k^i\|_{L^\infty} = 0$.

Moreover, the mean value theorem shows that

$$\begin{aligned} m_k^i &= \mathcal{E}_i(u_k^i) \geq -\int_{\mathbb{S}^d} \Psi(u_k^i(\sigma))d\sigma_h \geq -\max_{[-\tilde{s},\tilde{s}]}|\psi| \int_{\mathbb{S}^d} |u_k^i(\sigma)|d\sigma_h \\ &\geq -\max_{[-\tilde{s},\tilde{s}]}|\psi|\mathrm{Vol}_h(\mathbb{S}^d)a_k. \end{aligned}$$

Since $\lim_{k\to\infty} a_k = 0$, we have $\lim_{k\to\infty} m_k \geq 0$. On the other hand, $m_k < 0$ for every $k \in \mathbb{N}$, see Proposition 10.3, which implies $\lim_{k\to\infty} m_k^i = 0$.

Note that

$$\frac{\|u_k^i\|_{c_0}^2}{2} = m_k^i + \int_{\mathbb{S}^d} \Psi(u_k^i(\sigma))d\sigma_h \leq m_k^i + \max_{[-\tilde{s},\tilde{s}]}|\psi|\mathrm{Vol}_h(\mathbb{S}^d)a_k,$$

thus $\lim_{k\to\infty} \|u_k^i\|_{c_0} = 0$. But $\|\cdot\|_{c_0}$ and $\|\cdot\|_{H_1^2}$ are equivalent norms. □

Now, we prove the key result of this section where the non-smooth principle of symmetric criticality for Szulkin-type functions plays a crucial role.

Proposition 10.6 *u_k^i is a weak solution of* (P) *for every $k \in \mathbb{N}$.*

Proof. We divide the proof into two parts. First, let

$$T_k = \{u \in H_1^2(\mathbb{S}^d) : \ \|u\|_{L^\infty} \leq b_k\}.$$

Step 1. $\langle \mathcal{E}'(u_k^i), w - u_k^i\rangle_{H_1^2} \geq 0$ for every $w \in T_k$.
The set T_k is closed and convex in $H_1^2(\mathbb{S}^d)$. Let ζ_{T_k} be the indicator function of the set T_k (i.e., $\zeta_{T_k}(u) = 0$ if $u \in T_k$, and $\zeta_{T_k}(u) = +\infty$, otherwise). We define the Szulkin-type functional $\mathcal{I}_k : H_1^2(\mathbb{S}^d) \to \mathbb{R}\cup\{+\infty\}$ by $\mathcal{I}_k = \mathcal{E}+\zeta_{T_k}$, see Section 1.6. We deduce that $\mathcal{E}$ is of class $C^1(H_1^2(\mathbb{S}^d))$, ζ_{T_k} is convex, lower semicontinuous and proper. On account of (10.12), we have that $T_k^i = T_k \cap H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$; therefore, the restriction of ζ_{T_k} to $H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$ is precisely the indicator function $\zeta_{T_k^i}$ of the set T_k^i. Since u_k^i is a local minimum point of $\mathcal{E}_i$ relative to T_k^i (see Proposition 10.3), then u_k^i is a critical point of the functional $\mathcal{I}_k^i := \mathcal{E}_i + \zeta_{T_k^i}$ in the sense of Szulkin [282, p. 78], that is,

$$0 \in \mathcal{E}_i'(u_k^i) + \partial\zeta_{T_k^i}(u_k^i) \quad \text{in} \quad (H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d))^*, \tag{10.16}$$

where $\partial\zeta_{T_k^i}$ stands for the subdifferential of the convex function $\zeta_{T_k^i}$.

Since $\mathcal{E}$ is even, by means of (10.5) one can easily check that it is $G_{d,i}^{\tau_i}$-invariant. The function ζ_{T_k} is also $G_{d,i}^{\tau_i}$-invariant since $\tilde{g}T_k \subseteq T_k$ for every $\tilde{g} \in G_{d,i}^{\tau_i}$ (we use again (10.5)). Finally, since $G_{d,i}^{\tau_i} \subset O(d+1)$ is compact, and $\mathcal{E}_i$ and $\zeta_{T_k^i}$ are the restrictions of $\mathcal{E}$ and ζ_{T_k} to $H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$, respectively, we may apply – via relation (10.16) – the principle of symmetric criticality proved by Kobayaski-Ôtani, see Theorem 1.27. Thus, we obtain

$$0 \in \mathcal{E}'(u_k^i) + \partial\zeta_{T_k}(u_k^i) \quad \text{in} \quad (H_1^2(\mathbb{S}^d))^*.$$

Consequently, for every $w \in H_1^2(\mathbb{S}^d)$, we have

$$\langle \mathcal{E}'(u_k^i), w - u_k^i\rangle_{H_1^2} + \zeta_{T_k}(w) - \zeta_{T_k}(u_k^i) \geq 0,$$

which implies our claim.

Step 2. (Proof concluded) u_k^i *is a weak solution of* (P).
By Step 1, we have

$$\int_{\mathbb{S}^d} \langle \nabla u_k^i, \nabla(w-u_k^i)\rangle d\sigma_h + c_0\int_{\mathbb{S}^d} u_k^i(w-u_k^i)d\sigma_h - \int_{\mathbb{S}^d} \psi(u_k^i)(w-u_k^i)d\sigma_h \geq 0, \quad \forall\, w \in T_k.$$

Recall from (10.9) that $\psi(s) = \tilde{f}(s) + c_0 s$, $s \in \mathbb{R}$. Moreover, f and $\tilde{f}$ coincide on $[-\tilde{s}, \tilde{s}]$ and $u_k^i(\sigma) \in [-a_k, a_k] \subset (-\tilde{s}, \tilde{s})$ for a.e. $\sigma \in \mathbb{S}^d$ (see Proposition 10.4). Consequently, the above inequality reduces to

$$\int_{\mathbb{S}^d} \langle \nabla u_k^i, \nabla(w - u_k^i)\rangle d\sigma_h - \int_{\mathbb{S}^d} f(u_k^i)(w - u_k^i)d\sigma_h \geq 0, \quad \forall\, w \in T_k. \tag{10.17}$$

Let us define the function $\gamma(s) = \mathrm{sgn}(s)\min(|s|, b_k)$, and fix $\varepsilon > 0$ and $v \in H_1^2(\mathbb{S}^d)$ arbitrarily. Since γ is Lipschitz continuous, $w_k = \gamma\circ(u_k^i+\varepsilon v)$ belongs to $H_1^2(\mathbb{S}^d)$, see Hebey [136, Proposition 2.5, p. 24]. The explicit expression of w_k is

$$w_k(\sigma) = \begin{cases} -b_k, & \text{if} \quad \sigma \in \{u_k^i + \varepsilon v < -b_k\} \\ u_k^i(\sigma) + \varepsilon v(\sigma), & \text{if} \quad \sigma \in \{-b_k \leq u_k^i + \varepsilon v < b_k\} \\ b_k, & \text{if} \quad \sigma \in \{b_k \leq u_k^i + \varepsilon v\}. \end{cases}$$

Therefore, $w_k \in T_k$. Taking $w = w_k$ as a test function in (10.17), we obtain

$$\begin{aligned} 0 \quad \leq \quad & -\int_{\{u_k^i+\varepsilon v<-b_k\}} |\nabla u_k^i|^2 + \int_{\{u_k^i+\varepsilon v<-b_k\}} f(u_k^i)(b_k + u_k^i) \\ & +\varepsilon\int_{\{-b_k\leq u_k^i+\varepsilon v<b_k\}} \langle \nabla u_k^i, \nabla v\rangle - \varepsilon\int_{\{-b_k\leq u_k^i+\varepsilon v<b_k\}} f(u_k^i)v \\ & -\int_{\{b_k\leq u_k^i+\varepsilon v\}} |\nabla u_k^i|^2 - \int_{\{b_k\leq u_k^i+\varepsilon v\}} f(u_k^i)(b_k - u_k^i). \end{aligned}$$

After a suitable rearrangement of the terms in this inequality, we obtain that

$$\begin{aligned}
0 \quad \le \quad & \varepsilon \int_{\mathbb{S}^d} \langle \nabla u_k^i, \nabla v \rangle - \varepsilon \int_{\mathbb{S}^d} f(u_k^i) v \\
& - \int_{\{u_k^i + \varepsilon v < -b_k\}} |\nabla u_k^i|^2 - \int_{\{b_k \le u_k^i + \varepsilon v\}} |\nabla u_k^i|^2 \\
& + \int_{\{u_k^i + \varepsilon v < -b_k\}} f(u_k^i)(b_k + u_k^i + \varepsilon v) + \int_{\{b_k \le u_k^i + \varepsilon v\}} f(u_k^i)(-b_k + u_k^i + \varepsilon v) \\
& - \varepsilon \int_{\{u_k^i + \varepsilon v < -b_k\}} \langle \nabla u_k^i, \nabla v \rangle - \varepsilon \int_{\{b_k \le u_k^i + \varepsilon v\}} \langle \nabla u_k^i, \nabla v \rangle.
\end{aligned}$$

Let $M_k = \max_{[-a_k, a_k]} |f|$. Since $u_k^i(\sigma) \in [-a_k, a_k] \subset [-b_k, b_k]$ for a.e. $\sigma \in \mathbb{S}^d$, we have

$$\int_{\{u_k^i + \varepsilon v < -b_k\}} f(u_k^i)(b_k + u_k^i + \varepsilon v) \le -\varepsilon M_k \int_{\{u_k^i + \varepsilon v < -b_k\}} v$$

and

$$\int_{\{b_k \le u_k^i + \varepsilon v\}} f(u_k^i)(-b_k + u_k^i + \varepsilon v) \le \varepsilon M_k \int_{\{b_k \le u_k^i + \varepsilon v\}} v.$$

Using the above estimates and dividing by $\varepsilon > 0$, we obtain

$$\begin{aligned}
0 \quad \le \quad & \int_{\mathbb{S}^d} \langle \nabla u_k^i, \nabla v \rangle d\sigma_h - \int_{\mathbb{S}^d} f(u_k^i) v d\sigma_h \\
& - M_k \int_{\{u_k^i + \varepsilon v < -b_k\}} v d\sigma_h + M_k \int_{\{b_k \le u_k^i + \varepsilon v\}} v d\sigma_h \\
& - \int_{\{u_k^i + \varepsilon v < -b_k\}} \langle \nabla u_k^i, \nabla v \rangle d\sigma_h - \int_{\{b_k \le u_k^i + \varepsilon v\}} \langle \nabla u_k^i, \nabla v \rangle d\sigma_h.
\end{aligned}$$

Now, letting $\varepsilon \to 0^+$, and taking into account Proposition 10.4 (i.e., $-a_k \le u_k^i(\sigma) \le a_k$ for a.e. $\sigma \in \mathbb{S}^d$), we have

$$\text{meas}(\{u_k^i + \varepsilon v < -b_k\}) \to 0 \quad \text{and} \quad \text{meas}(\{b_k \le u_k^i + \varepsilon v\}) \to 0,$$

respectively. Consequently, the above inequality reduces to

$$0 \quad \le \quad \int_{\mathbb{S}^d} \langle \nabla u_k^i, \nabla v \rangle d\sigma_h - \int_{\mathbb{S}^d} f(u_k^i) v d\sigma_h.$$

Putting $(-v)$ instead of v, we see that u_k^i is a weak solution of (P), which completes the proof. $\square$

Proof of Theorem 10.2. Fix $i \in \{1, ..., s_d\}$. Combining Propositions 10.3 and 10.5, one can see that there are infinitely many distinct elements in the sequence $\{u_k^i\}_k$. These elements are weak solutions of (P) as Proposition 10.6 shows, and they change sign, see Remark 10.1. Moreover, due to Theorem 10.1 (ii), solutions in different spaces $H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$, $i \in \{1, ..., s_d\}$, cannot be compared from symmetrical point of view. The $L^\infty-$ and H_1^2–asymptotic behaviour of the sequences of solutions are described in Proposition 10.5. □

10.4 Arbitrarily large solutions

Instead of (f_1^0) and (f_2^0), respectively, we assume

(f_1^∞) $\quad -\infty < \liminf_{s\to\infty} \frac{F(s)}{s^2} \leq \limsup_{s\to\infty} \frac{F(s)}{s^2} = +\infty;$

(f_2^∞) $\quad \liminf_{s\to\infty} \frac{f(s)}{s} < 0.$

Unlike in Theorem 10.2 where no further assumption is needed at infinity, we have to control here the growth of f. We assume that f has an *asymptotically critical* growth at infinity, namely,

(f_3^∞) $\quad \sup_{s\in\mathbb{R}} \frac{|f(s)|}{1+|s|^{2^*-1}} < \infty$, where $2^* = \frac{2d}{d-2}$.

Theorem 10.3 *Let $d \geq 5$ and $f : \mathbb{R} \to \mathbb{R}$ be an odd, continuous function which verifies (f_1^∞), (f_2^∞) and (f_3^∞). Then there exist at least $s_d = [d/2] + (-1)^{d+1} - 1$ sequences $\{\tilde{u}_k^i\}_k \subset H_1^2(\mathbb{S}^d)$, $i \in \{1, ..., s_d\}$, of sign-changing weak solutions of* (P) *distinguished by their symmetry properties. In addition,*

$$\lim_{k\to\infty} \|\tilde{u}_k^i\|_{L^\infty} = \lim_{k\to\infty} \|\tilde{u}_k^i\|_{H_1^2} = \infty \ \ \textit{for every} \ \ i \in \{1, ..., s_d\}.$$

Example 10.2 Let $d \geq 5$ and $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\frac{d+2}{d-2} \geq \alpha > 1$, $|\alpha - \beta| < 1$, and $\gamma \in (0, 1)$. Then, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(s) = |s|^{\alpha-1}s(\gamma + \sin |s|^\beta)$ verifies the hypotheses (f_1^∞), (f_2^∞) and (f_3^∞), respectively.

Certain parts of the proof of Theorem 10.3 are similar to that of Theorem 10.2; so, we present only the differences. We assume throughout of this section that the hypotheses of Theorem 10.3 are fulfilled. Due to (f_2^∞), one can fix $c_\infty > 0$ such that

$$\liminf_{s\to\infty} \frac{f(s)}{s} < -c_\infty < 0.$$

Let $\{\overline{s}_k\} \subset (0,\infty)$ be a sequence converging (increasingly) to $+\infty$, such that $f(\overline{s}_k) < -c_\infty \overline{s}_k$. We define the functions

$$\psi(s) = f(s) + c_\infty s \quad \text{and} \quad \Psi(s) = \int_0^s \psi(t)dt = F(s) + \frac{c_\infty}{2}s^2, \quad s \in \mathbb{R}. \tag{10.18}$$

By construction, $\psi(\overline{s}_k) < 0$; consequently, there are two sequences $\{a_k\}_k$, $\{b_k\}_k \subset (0,\infty)$, both converging to ∞, such that $a_k < \overline{s}_k < b_k < a_{k+1}$ for every $k \in \mathbb{N}$ and

$$\psi(s) \leq 0 \text{ for every } s \in [a_k, b_k]. \tag{10.19}$$

Since $c_\infty > 0$, the norm $\|\cdot\|_{c_\infty}$ defined in the same way as (10.11) with c_∞ instead of c_0, is equivalent to the standard norm $\|\cdot\|_{H_1^2}$. Now, we define the energy functional $\mathcal{E} : H_1^2(\mathbb{S}^d) \to \mathbb{R}$ associated with (P) by

$$\mathcal{E}(u) = \frac{1}{2}\|u\|_{c_\infty}^2 - \int_{\mathbb{S}^d} \Psi(u(\sigma))d\sigma_h.$$

Since $H_1^2(\mathbb{S}^d)$ is continuously embedded into $L^p(\mathbb{S}^d)$, $1 \leq p \leq 2^*$, see [136, Corollary 2.1, p. 33]), using hypothesis (f_3^∞), the functional $\mathcal{E}$ is well-defined, and it belongs to $C^1(H_1^2(\mathbb{S}^d))$. Moreover, since f is odd on the whole $\mathbb{R}$, the functional $\mathcal{E}$ is even.

We fix $i \in \{1, ..., s_d\}$ and the subspace $H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$ of $H_1^2(\mathbb{S}^d)$. Let $\mathcal{E}_i$ be the restriction of the functional $\mathcal{E}$ to $H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$ and for every $k \in \mathbb{N}$, define the set

$$Z_k^i = \{u \in H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d) : \ \|u\|_{L^\infty} \leq b_k\},$$

where b_k is from (10.19).

Proposition 10.7 *The functional $\mathcal{E}_i$ is bounded from below on Z_k^i and its infimum $\tilde{m}_k^i$ on Z_k^i is attained at $\tilde{u}_k^i \in Z_k^i$. Moreover, $\lim_{k\to\infty} \tilde{m}_k^i = -\infty$.*

Proof. It is easy to check that $\mathcal{E}_i$ is bounded from below on Z_k^i. In order to see that it attains its infimum on Z_k^i we show that the function $u \mapsto \int_{\mathbb{S}^d} \Psi(u(\sigma))d\sigma_h$, $u \in H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$ is sequentially weak continuous; in such case, $\mathcal{E}_i$ is sequentially weak lower semicontinuous and we may proceed in the standard way. On one hand, due to (f_3^∞), (10.18) and the oddness of ψ, one can find $c_1 > 0$ such that

$$|\psi(s)| \leq c_1(1 + |s|^{2^*-1}), \quad s \in \mathbb{R}. \tag{10.20}$$

On the other hand, the definition of $G_{d,i}$ shows that the $G_{d,i}$-orbit of every point $\sigma \in \mathbb{S}^d$ has at least dimension 1, i.e., $\dim(G_{d,i}\sigma) \geq 1$ for every $\sigma \in \mathbb{S}^d$. Thus

$$d_G = \min\{\dim(G_{d,i}\sigma) : \sigma \in \mathbb{S}^d\} \geq 1.$$

Applying [30, Lemma 3.2], we conclude in particular that $H_{G_{d,i}}(\mathbb{S}^d)$ is compactly embedded into $L^q(\mathbb{S}^d)$, whenever $q \in \left[1, \frac{2d-2}{d-3}\right)$. Since $\frac{2d-2}{d-3} > 2^*$, the embedding $H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d) \subset H_{G_{d,i}}(\mathbb{S}^d) \hookrightarrow L^{2^*}(\mathbb{S}^d)$ is compact. Combining (10.20) with the above compactness property, we conclude the sequentially weak continuity of the function $u \mapsto \int_{\mathbb{S}^d} \Psi(u(\sigma))d\sigma_h$, $u \in H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$. Consequently, we may assert that the infimum $\tilde{m}_k^i$ on Z_k^i is attained at the point $\tilde{u}_k^i \in Z_k^i$.

We will prove $\lim_{k\to\infty} \tilde{m}_k^i = -\infty$. First, due to (10.18) and (f_1^∞), we have

$$-\infty < \liminf_{s\to\infty} \frac{\Psi(s)}{s^2} \leq \limsup_{s\to\infty} \frac{\Psi(s)}{s^2} = +\infty. \tag{10.21}$$

The left inequality of (10.21) and the evenness of Ψ implies the existence of $\underline{l}, \varrho > 0$ such that

$$\Psi(s) \geq -\underline{l}s^2 \text{ for every } |s| > \varrho. \tag{10.22}$$

Let $D_i \subset \mathbb{S}^d$ and $C_i > 0$ be from Proposition 10.2 (which depend only on $G_{d,i}$ and τ_i), and fix a number $\bar{l} > 0$ large enough such that

$$\bar{l}\mathrm{Vol}_h(D_i) > \left(\underline{l} + \frac{c_\infty}{2}\right)\mathrm{Vol}_h(\mathbb{S}^d) + \frac{C_i^2}{2}. \tag{10.23}$$

Taking into account the right-hand side of (10.21), there is a sequence $\{\tilde{s}_k\}_k \subset (0,\infty)$ such that $\lim_{k\to\infty} \tilde{s}_k = \infty$ and $\Psi(\tilde{s}_k) = \Psi(-\tilde{s}_k) > \bar{l}\tilde{s}_k^2$ for every $k \in \mathbb{N}$.

Let $\{b_{n_k}\}_k$ be an increasing subsequence of $\{b_k\}_k$ such that $\tilde{s}_k \leq b_{n_k}$ for every $k \in \mathbb{N}$. Let $\tilde{w}_k := w_{\tilde{s}_k} \in H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$ be the function from Proposition 10.2 corresponding to the value $\tilde{s}_k > 0$. Then $\tilde{w}_k \in Z_{n_k}^i$ and one has

$$\begin{aligned}
\mathcal{E}_i(\tilde{w}_k) &= \frac{1}{2}\|\tilde{w}_k\|_{c_\infty}^2 - \int_{\mathbb{S}^d} \Psi(\tilde{w}_k(\sigma))d\sigma_h \\
&\leq \frac{1}{2}\left(C_i^2 + c_\infty \mathrm{Vol}_h(\mathbb{S}^d)\right)\tilde{s}_k^2 - \int_{D_i} \Psi(\tilde{w}_k(\sigma))d\sigma_h - \int_{\mathbb{S}^d\setminus D_i} \Psi(\tilde{w}_k(\sigma))d\sigma_h.
\end{aligned}$$

On account of Proposition 10.2 iii), we have

$$\int_{D_i} \Psi(\tilde{w}_k(\sigma))d\sigma_h = \Psi(\tilde{s}_k)\mathrm{Vol}_h(D_i) > \bar{l}\mathrm{Vol}_h(D_i)\tilde{s}_k^2.$$

Due to Proposition 10.2 i) and (10.22), we have

$$\begin{aligned}\int_{\mathbb{S}^d\setminus D_i} \Psi(\tilde{w}_k(\sigma))d\sigma_h &= \int_{(\mathbb{S}^d\setminus D_i)\cap\{|\tilde{w}_k|\le\varrho\}} \Psi(\tilde{w}_k(\sigma))d\sigma_h \\ &\quad + \int_{(\mathbb{S}^d\setminus D_i)\cap\{|\tilde{w}_k|>\varrho\}} \Psi(\tilde{w}_k(\sigma))d\sigma_h \\ &\ge -\left(\max_{[-\varrho,\varrho]}|\Psi| + \underline{l}\tilde{s}_k^2\right)\mathrm{Vol}_h(\mathbb{S}^d).\end{aligned}$$

Combining these estimates, we obtain that

$$\mathcal{E}_i(\tilde{w}_k) \le \tilde{s}_k^2\left(-\bar{l}\mathrm{Vol}_h(D_i) + \left(\underline{l} + \frac{c_\infty}{2}\right)\mathrm{Vol}_h(\mathbb{S}^d) + \frac{C_i^2}{2}\right) + \max_{[-\varrho,\varrho]}|\Psi|\mathrm{Vol}_h(\mathbb{S}^d).$$

Taking into account (10.23) and that $\lim_{k\to\infty}\tilde{s}_k = \infty$, we obtain $\lim_{k\to\infty}\mathcal{E}_i(\tilde{w}_k) = -\infty$. Since $\tilde{m}^i_{n_k} = \mathcal{E}_i(\tilde{u}^i_{n_k}) = \inf_{Z^i_{n_k}}\mathcal{E}_i \le \mathcal{E}_i(\tilde{w}_k)$, then $\lim_{k\to\infty}\tilde{m}^i_{n_k} = -\infty$. Since the sequence $\{\tilde{m}^i_k\}_k$ is non-increasing, the claim follows. □

Proposition 10.8 $\lim_{k\to\infty}\|\tilde{u}^i_k\|_{L^\infty} = \lim_{k\to\infty}\|\tilde{u}^i_k\|_{H^2_1} = \infty$.

Proof. Assume first by contradiction that there exists a subsequence $\{\tilde{u}^i_{n_k}\}_k$ of $\{\tilde{u}^i_k\}_k$ such that $\|\tilde{u}^i_{n_k}\|_{L^\infty} \le M$ for some $M > 0$. In particular, $\{\tilde{u}^i_{n_k}\} \subset Z^i_l$ for some $l \in \mathbb{N}$. Therefore, for every $n_k \ge l$, we have

$$\tilde{m}^i_l \ge \tilde{m}^i_{n_k} = \inf_{Z^i_{n_k}}\mathcal{E}_i = \mathcal{E}_i(\tilde{u}^i_{n_k}) \ge \inf_{Z^i_l}\mathcal{E}_i = \tilde{m}^i_l.$$

Consequently, $\tilde{m}_{n_k} = \tilde{m}_l$ for every $n_k \ge l$, and since the sequence $\{\tilde{m}^i_k\}_k$ is non-increasing, there exists $k_0 \in \mathbb{N}$ such that for every $k \ge k_0$ we have $\tilde{m}^i_k = \tilde{m}^i_l$, contradicting Proposition 10.7.

It remains to prove that $\lim_{k\to\infty}\|\tilde{u}^i_k\|_{H^2_1} = \infty$. Note that (10.20) and the continuity of the embedding $H^2_1(\mathbb{S}^d)$ into $L^{2^*}(\mathbb{S}^d)$ implies that for come $C > 0$ we have

$$\left|\int_{\mathbb{S}^d}\Psi(u(\sigma))d\sigma_h\right| \le C(\|u\|_{H^2_1} + \|u\|^{2^*}_{H^2_1}), \quad \forall u \in H^2_1(\mathbb{S}^d).$$

Similarly as above, we assume that there exists a subsequence $\{\tilde{u}^i_{n_k}\}_k$ of $\{\tilde{u}^i_k\}_k$ such that for some $M > 0$, we have $\|\tilde{u}^i_{n_k}\|_{H^2_1} \le M$. Since $\|\cdot\|_{c_\infty}$ is equivalent with $\|\cdot\|_{H^2_1}$, due to the above inequality, the sequence $\{\mathcal{E}_i(\tilde{u}^i_{n_k})\}_k$ is bounded. But $\tilde{m}^i_{n_k} = \mathcal{E}_i(\tilde{u}^i_{n_k})$, thus, the sequence $\{\tilde{m}^i_{n_k}\}$

is also bounded. This fact contradicts Proposition 10.7. □

Proof of Theorem 10.3. Due to Proposition 10.7, we can find infinitely many distinct elements $\tilde{u}_k^i$; similar reasoning as in Propositions 10.4 and 10.6 show that $\tilde{u}_k^i$ are weak solutions of (P) for every $k \in \mathbb{N}$. The L^∞– and H_1^2–asymptotic behaviour of the sequences of solutions are described in Proposition 10.8. The rest is similar as in Theorem 10.2. □

Remark 10.2 Theorems 10.2 and 10.3 can be successfully applied to treat Emden-Fowler equations of the form

$$-\Delta v = |x|^{\alpha-2} f(|x|^{-\alpha} v), \quad x \in \mathbb{R}^{d+1} \setminus \{0\} \;\; (\alpha < 0) \tag{10.24}$$

whenever $f : \mathbb{R} \to \mathbb{R}$ is enough smooth and oscillates either at zero or at infinity having an asymptotically critical growth. Finding solutions of (10.24) in the form $v(x) = v(r, \sigma) = r^\alpha u(\sigma)$, $(r, \sigma) = (|x|, x/|x|) \in (0, \infty) \times \mathbb{S}^d$ being the spherical coordinates, we obtain

$$-\Delta_h u + \alpha(1 - d - \alpha) u = f(u) \;\; \text{on} \;\; \mathbb{S}^d, \tag{10.25}$$

see also $(9.1)_\lambda$. Assuming (f_1^L) and $\liminf_{s \to L} \frac{f(s)}{s} < \alpha(1 - d - \alpha)$ with $L \in \{0^+, +\infty\}$, and (f_3^∞) whenever $L = +\infty$, we may formulate multiplicity results for (10.25), so for (10.24). Note that the obtained solutions of (10.24) are sign-changing and *non*-radial.

10.5 Historical notes, comments and perspectives

A. Historical notes. One of the most famous problems in Differential Geometry is the so-called Yamabe problem: given a smooth compact Riemannian manifold (M, g) of dimension d, $d \geq 3$, there exists a metric $\tilde{g}$ from the conformal class of g of constant scalar curvature. This problem has a PDE formulation which has been extensively studied by many authors; the reader id referred to the monographs of Aubin [19] and Hebey [136].

Another class of elliptic problems defined on compact manifolds arises from the Emden-Fowler equation, studied for superlinear nonlinearities via minimization or minimax methods by Cotsiolis-Iliopoulos [79], Vázquez-Véron [288], Bidaut-Véron-Véron [39], etc.

In Sections 9.2 and 10 we presented some of our contributions related to Emden-Fowler equations when the nonlinearity has either a sublinear growth at infinity or it has an oscillatory behavior near zero or at infinity.

Elliptic problems involving oscillatory nonlinearities have been studied in Omari-Zanolin [229], Ricceri [258], Saint Raymond [265], subjected to standard Neumann or Dirichlet boundary value conditions on bounded open domains of $\mathbb{R}^n$, or even on unbounded domains, see Faraci-Kristály [111], Kristály [166]. Results in finding sign-changing solutions for semilinear problems can be found in Li-Wang [187], Zou [299] and references therein. The strategy in these last papers is to construct suitable closed convex sets which contain all the positive and negative solutions in the interior, and are invariant with respect to some vector fields. Our approach is rather different than those of [187], [299] and is related to the works of Bartsch-Schneider-Weth [30] and Bartsch-Willem [32], where the existence of non-radial and sign-changing solutions are studied for Schrödinger and polyharmonic equations defined on $\mathbb{R}^n$.

B. Comments. Section 9.2 is based on the paper of Kristály-Rădulescu [175], while the main results of Section 10 are mainly contained in Kristály [169] and Kristály-Marzantowicz [171].

As we mentioned at the beginning of Section 9.2, when the nonlinearity f is a uniformly Lipschitz function (with Lipschitz constant $L > 0$), we have extra information on the eigenvalues:

(a) problem (P_λ) has only the trivial solution whenever $\lambda \in (0, \lambda_L)$;

(b) problem (P_λ) has at least two nontrivial solutions whenever $\lambda > \lambda^*$. Clearly, we have $\lambda^* \geq \lambda_L$, and usually these two numbers do not coincide. (For instance, if $f(s) = \ln(1+s^2)$, $K(\sigma) = \alpha(\sigma) = \text{const.}$, then $\lambda_L = 1$ while $\lambda^* \approx 1.32$.) Unfortunately, we have no any precise result when the parameter belongs to the 'gap-interval' $[\lambda_L, \lambda^*]$.

In Section 10, the minimal number of those sequences of solutions for (P) which contain mutually symmetrically distinct elements is $s_d = [d/2] + (-1)^{d+1} - 1$. Note that $s_d \sim d/2$ as $d \to \infty$. However, in lower dimensions, Theorems 10.2 and 10.3 are not spectacular. For instance, $s_4 = 0$; therefore, on $\mathbb{S}^4$ we have no analogous results as Theorems 10.2 and 10.3. Note that $s_3 = 1$; in fact, for $G_{3,1} = O(2)\times O(2)$ we may apply our arguments. Hence, on $\mathbb{S}^3$ one can find a sequence of solutions of (P) with the described properties in our theorems. We may compare these results with that of Bartsch-Willem [32]; they studied the lower bound of those sequences of solutions for a Schrödinger equation on $\mathbb{R}^{d+1}$ which contain elements in different $O(d+1)$-orbits. Due to [32, Proposition 4.1, p. 457], we deduce that their lower bound is $s'_d = \left[\log_2 \frac{d+3}{3}\right]$ whenever $d \geq 3$ and $d \neq 4$.

Let $\alpha, \beta \in L^\infty(\mathbb{S}^d)$ be two $G_{d,i}$-invariant functions such that $\mathrm{essinf}_{\mathbb{S}^d}\beta >$

0 and consider the problem

$$-\Delta_h u + \alpha(\sigma)u = \beta(\sigma)f(u) \ \text{ on } \ \mathbb{S}^d. \tag{10.26}$$

If $f : \mathbb{R} \to \mathbb{R}$ has an asymptotically critical growth fulfilling (f_1^L) and

$$\liminf_{s\to L} \frac{f(s)}{s} < \operatorname{essinf}_{\mathbb{S}^d} \frac{\alpha}{\beta},$$

problem (10.26) admits a sequence of $G_{d,i}$-invariant (perhaps not sign-changing) weak solutions in both cases, i.e. $L \in \{0^+, \infty\}$. The proofs can be carried out following Theorems 10.2 and 10.3, respectively, considering instead of $H_{G_{d,i}^{\tau_i}}(\mathbb{S}^d)$ the space $H_{G_{d,i}}(\mathbb{S}^d)$. Note that $\alpha : \mathbb{S}^d \to \mathbb{R}$ may change its sign. In particular, this type of result complements the paper of Cotsiolis-Iliopoulos [78].

C. Further perspectives. The symmetry and compactness of the sphere $\mathbb{S}^d$ have been deeply exploited in Section 10. We intend to study a challenging problem related to (P) which is formulated on *non-compact Riemannian symmetric spaces* (for instance, on the hyperbolic space $\mathbb{H}^d = SO_0(d,1)/SO(d)$ which is the dual companion of $\mathbb{S}^d = SO(d+1)/SO(d)$). In order to handle this kind of problem, the action of the isometry group of the symmetric space seems to be essential, as shown by Hebey [136, Chapter 9], Hebey-Vaugon [138].

11
Equations with Critical Exponent

Numbers are the highest degree of knowledge. It is knowledge itself.

Plato (429–347 B.C.)

11.1 Introduction

In the last years many books and papers were dedicated to study Sobolev spaces and equations with critical exponent on compact or non-compact Riemannian manifold with or without boundary. For recent developments, we refer to the books by Ambrosetti and Malchiodi [6], Druet, Hebey, and Robert [96], and Hebey [135], [136]. In this chapter we present some elementary existence results concerning equations with critical exponent. Here we follow the paper of Hebey [137].

First we formulate the Yamabe problem to give the geometric motivation of these type problems. Let (M, g) be a Riemannian space and ∇ the Riemannian connection. Let (x_i) is a local system of coordinates on M, then

$$\nabla_{\frac{\partial}{\partial x_i}} \left(\frac{\partial}{\partial x_j} \right) = \Gamma_{ij}^k \, \frac{\partial}{\partial x_k} \, .$$

The functions Γ_{ij}^k are called the Christoffel symbols of the connection ∇. In the local system of coordinates (x_i) the components of metric tensor g we denote by (g_{ij}) with the inverse matrix (g^{ij}), and let $|g| = \det(g_{ij})$. The divergence operator div_g on the C^1 vector field $X = (X^i)$ is defined by

$$\operatorname{div}_g X = \frac{1}{\sqrt{|g|}} \sum_i \frac{\partial}{\partial x_i} (\sqrt{|g|} X^i),$$

and the Laplace–Beltrami operator by $\Delta_g u = \operatorname{div}_g(\nabla u)$. Here

$$\nabla u = \sum_i g^{ij} \frac{\partial u}{\partial x_i}$$

In the local system of coordinates (x_i) the Laplace-Beltrami operator Δ_g, has the following expression:

$$\Delta_g u = -g^{ij} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial u}{\partial x_k} \right). \tag{11.1}$$

The *curvature* associated to the connection ∇ is defined by:

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$

where $[\cdot,\cdot]$ denote the Poisson bracket of the vector fields X and Y. Consider the 4- covariant tensor $\mathrm{R}(X,Y,Z,T) = g(X, R(Z,T)Y)$ with components $\mathrm{R}_{lkij} = g_{lm} R_{kij}^m$. The Ricci tensor is obtained from curvature tensor by contraction and it is only one nonzero tensor or its negative. Its components are $R_{ij} = R_{ikj}^k$. The Ricci tensor is symmetric and its contraction $S_g = R_{ij} g^{ij}$ is called scalar curvature. A metric of the form $\tilde{g} = e^u g$ is said to be conformal metric to g. We denote by $[g]$ the conformal class of the reference metric g. By definition

$$[g] = \{e^u g \mid u \in C^\infty(M)\}. \tag{11.2}$$

If S_g and $S_{\tilde{g}}$ are the scalar curvatures of g and $\tilde{g}$, one easily gets that

$$e^{2u} S_{\tilde{g}} = S_g + 2(n-1)\Delta_g u - (n-1)(n-2)|\nabla u|_g^2. \tag{11.3}$$

Let us now write $\tilde{g}$ under the form $\tilde{g} = u^{\frac{4}{n-2}} g$, for $u : M \to \mathbb{R}$ some smooth positive function. The above relation becomes

$$\Delta_g u + \frac{n-2}{4(n-1)} S_g u = \frac{n-2}{4(n-1)} S_{\tilde{g}} u^{\frac{n+2}{n-2}}. \tag{11.4}$$

The **Yamabe problem** can be formulated in the following way:

Geometric formulation. *For any smooth compact Riemannian manifold (M,g) of dimension n, $n \geq 3$, there exists $\tilde{g} \in [g]$ of constant scalar* *curvature.*

PDE formulation. *For any smooth compact Riemannian manifold (M, g) of dimension n, $n \geq 3$, there exists $u \in C^\infty(M), u > 0$ and there exists $\lambda \in \mathbb{R}$ such that*

$$\Delta_g u + \frac{n-2}{4(n-1)} S_g u = \lambda u^{\frac{n+2}{n-2}} \tag{11.5}$$

where Δ_g is the Laplacian with the respect to g, and S_g is the scalar curvature of g.

If u and λ satisfies equation (11.5), and if $\tilde{g} = u^{\frac{4}{n-4}} g$, then we gets

$$S_{\tilde{g}} = \frac{4(n-1)}{n-2} \lambda.$$

In particular, this gives a conformal metric to g of constant scalar curvature. The left hand side in this equation (11.5) is referred to as the conformal Laplacian. We denoted by

$$L_g = \Delta_g u + \frac{n-2}{4(n-1)} S_g u.$$

Note that L_g is conformally invariant in the following sense: If $\tilde{g} = \varphi^{\frac{4}{n-2}} g$ is a conformal metric to g, then, for all $u \in C^\infty(M)$,

$$L_{\tilde{g}}(u) = \varphi^{-\frac{n+2}{n-2}} L_g(u\varphi).$$

In this chapter we study the problem like (11.4). For this let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, and $h : M \to \mathbb{R}$ be a smooth function. In the next we study equations of the following form:

$$(CE) \qquad \begin{cases} \Delta_g u + hu = \lambda u^{2^\star - 1}, & \text{in} \quad M \\ u > 0, & \text{in} \quad M, \end{cases}$$

where $\lambda \in \mathbb{R}$ and $2^\star = \frac{2n}{n-2}$ is the critical exponent of the embedding $W^{1,2}(M) \hookrightarrow L^{2^\star}$.

Before to discuss this problem we remember

Theorem 11.1 *Let $(X, \|\cdot\|)$ be a real Banach space, $\Omega \subset X$ be an open subset, $f : \Omega \to \mathbb{R}$ be a differentiable function, and $\Phi : \Omega \to \mathbb{R}^n$ be of* *class* C^1. *Let also* $a \in \mathbb{R}^n$ *be such that* $\mathcal{H} = \Phi^{-1}(a)$ *is not empty. If* $x_0 \in \mathcal{H}$ *is a solution of the minimization problem*

$$f(x_0) = \min_{x \in \mathcal{H}} f(x)$$

and if $D\Phi(x_0)$ *is surjective, then there exists* $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ *such that*

$$Df(x_0) = \sum_{i=1}^{n} \lambda_i D\Phi_i(x_0),$$

where $\Phi = (\Phi_1, \ldots, \Phi_n)$.

The next result is the Hopf's maximum principle.

Theorem 11.2 *Let* $\Omega \subset \mathbb{R}^n$ *be an open connected set and let*

$$L(u) = \sum_{i.j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b_i(x) \frac{}{\partial u} \partial x^i + h(x)u$$

be a linear uniformly elliptic differential operator with bounded coefficients and $h \leq 0$. *Suppose that* $u \in C^2(M)$ *satisfies* $L(u) \geq 0$. *If* u *attains its maximum* $M \geq 0$ *in* Ω, *then* u *is constant equal to* M *on* Ω. *Otherwise if at* $x_0 \in \partial\Omega$, u *is continuous and* $u(x_0) = M \geq 0$ *then the outer normal derivative at* x_0, *if it exists, satisfies* $\frac{\partial u(x_0)}{\partial n} > 0$, *provided* x_0 *belongs to the boundary of a ball included in* Ω. *Moreover, if* $h = 0$, *the same conclusion hold for a maximum* $M < 0$.

An immediate consequence of the Hopf's maximum principle is the following.

Proposition 11.1 *Let* $f : M \times \mathbb{R} \to \mathbb{R}$ *be a continuous function and* $u \in C^2(M)$ *such that*

$$\Delta_g u \geq u(x) f(x, u(x)),$$

then either $u > 0$ *or* $u = 0$.

As usual, $C^\infty(M)$ and $C_0^\infty(M)$ denote the spaces of smooth functions and smooth compactly supported function on M respectively.

Definition 11.1 The Sobolev space $\overset{\circ}{H}{}_k^p(M)$ is the closure of $C_0^\infty(M)$ in $H_k^p(M)$.

If (M,g) is a complete Riemannian manifold, then for any $p \geq 1$, we have $\overset{\circ}{H}{}^p_k(M) = H^p_k(M)$.

We finish this section with the Sobolev embedding theorem and the Rellich- Kondrachov result for compact manifold without and with boundary.

Theorem 11.3 *(Sobolev embedding theorems for compact manifolds) Let M be a cpmpact Riemannian manifold of dimension n.*

a) If $\frac{1}{r} \geq \frac{1}{p} - \frac{k}{n}$, then the embedding $H^p_k(M) \hookrightarrow L^r(M)$ is continuous.

b) (Rellich-Kondrakov theorem) Suppose that the inequality in a) i s strict, then the embedding $H^p_k(M) \hookrightarrow L^r(M)$ is compact.

c) Suppose $0 < \alpha < 1$ and $\frac{1}{p} \leq \frac{k-\alpha}{n}$, then the embedding $H^p_k(M) \hookrightarrow C^\alpha(M)$ is continuous.

Theorem 11.4 *Let (M,g) be a compact n-dimensional Riemannian manifold with boundary ∂M.*

a) The embedding $H^p_1(M) \hookrightarrow L^q(M)$ is continuous, if $p \leq q \leq \frac{np}{n-p}$ and compact for $p \leq q < \frac{np}{n-p}$.

b) If $\partial M \neq \emptyset$, then the embedding $H^p_1(M) hookrightarrow L^q(\partial M)$ is continuous, if $p \leq q \leq \frac{p(n-1)}{n-p}$ and compact for $p \leq q < \frac{p(n-1)}{n-p}$.

Theorem 11.5 *For any smooth compact Riemannian manifold (M,g) of dimension $n \geq 3$, there exists $B > 0$ such that for any $u \in H^2_1(M)$,*

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \leq K_n^2 \int_M |\nabla u|^2 dv_g + B \int_M u^2 dv_g \tag{11.6}$$

and the inequality is sharp.

Theorem 11.6 *(Global elliptic regularity) Let M be a compact Riemann manifold, and suppose that $u \in L^1_{loc}(M)$ is a weak solution to $\Delta_g u = f$.*

a) If $f \in H^p_k(M)$, then $u \in H^p_{k+2}(M)$, and

$$\|u\|_{H^p_{k+2}} \leq C(\|\Delta_g u\|_{H^p_k} + \|u\|_{L^p}).$$

b) If $f \in C^{k,\alpha}(M)$, then $f \in C^{k+2,\alpha}(M)$, and

$$\|u\|_{C^p_{k+2}} \leq C(\|\Delta_g u\|_{C^p_k} + \|u\|_{C^\alpha}).$$

11.2 Subcritical case

In this section we study the following subcritical equation, i.e

$$\begin{cases} \Delta_g u + hu = \lambda u^{q-1}, & \text{in} \quad M \\ \\ u > 0, & \text{in} \quad M, \end{cases} \tag{11.7}$$

where (M, g) b a smooth compact manifold of dimension $n \geq 3$, $h : M \to \mathbb{R}$ a smooth function and $q \in (2, 2^\star)$ be a fixed number.

Definition 11.2 We say that $u \in H_1^2(M)$ is a weak solution of the equation (11.7) if for every $\varphi \in H_1^2(M)$ we have

$$\int_M \langle \nabla u, \nabla \varphi \rangle_g dv_g + \int_M hu\varphi dv_g = \alpha \int_M u^{q-1} \varphi dv_g.$$

We have the following regularity result.

Theorem 11.7 *If $u \in H_1^2(M)$ is a weak solution of (11.7), then $u \in C^\infty(M)$.*

Proof First we prove that the weak solution u is smooth, i.e. $u \in C^\infty(M)$. For this let $f = \mu_q u^{q-1}$, and $p_1 = 2^\star$. Since $u \in H_1^2(M)$, from Theorem 11.3 follows that $u \in L^{p_1}(M)$. Hence $f \in L^{\frac{p_1}{q-1}}(M)$, and it follows from Theorem 11.6 that $u \in H_2^{\frac{p_1}{q-1}}(M)$. Using again Theorem 11.3 embedding theorem, we obtain

1) $u \in L^{p_2}(M)$, where $p_2 = \frac{np_1}{n(q-1)-2p_1}$ if $n(q-1) > 2p_1$
   or
2) $u \in L^s(M)$ for all s if $n(q-1) \leq 2p_1$.

If we repeat this process, we get by finite induction that $u \in L^s(M)$ for all s. Indeed, let $p_0 = \frac{n(q-2)}{2}$. Then $p_1 > p_0$. We define p_i by induction letting

$$\begin{cases} p_{i+1} = \frac{np_i}{n(q-1)-2p_i}, & \text{if} \quad n(q-1) > 2p_i \\ \\ p_{i+1} = +\infty, & \text{if} \quad n(q-1) \leq 2p_i. \end{cases} \tag{11.8}$$

For every $i \in \mathbb{N}^\star$ we have $p_i > p_0$. It follows that $p_{i+1} > p_i$. Moreover, $u \in L^{p_i+1}(M)$ if $n(q-1) > 2p_i$, and $u \in L^s(M)$ for all s if $n(q-1) \leq 2p_i$. Now, either exists $i \in \mathbb{N}^\star$ such that $p_i > \frac{n(q-1)}{2}$, or $p_i \leq \frac{n(q-1)}{2}$ for all i. In the first case, $p_{i+1} = +\infty$ and we get that $u \in L^s(M)$ for all s.

In the second case, (p_i) is an increasing sequence bounded from above. Thus (p_i) converges, and if p is the limit of the p_i's, then

$$p = \frac{np}{n(q-1)-2p}$$

so that $p = \frac{n(q-2)}{2}$, which is impossible. This prove that $u \in L^s(M)$ for all s. By Theorem 11.6 we get $u \in H_2^s(M)$ for all s. From c) Theorem 11.3 follows that $u \in C^1(M)$. Then, since $q > 2, u^{q-1} \in C^1(M)$, and, in particular $u^{q-1} \in H_1^s(M)$ for all s. Using again Theorem 11.6, it follows that $u \in H_3^s(M)$ for all s, and c) Theorem 11.3 implies $u \in C^2(M)$. Since $u \neq 0$, applying the maximum principle, i.e. Theorem 11.1 we get than $u > 0$. From Theorem 11.6 follows that $u \in C^\infty(M)$. Thus the assertion of theorem is proved.

□

We define

$$\mu_q = \inf_{u \in \mathcal{H}_q} \int_M (|\nabla u|^2 + hu^2) dv_g, \tag{11.9}$$

where

$$\mathcal{H}_q = \{u \in H_1^2 \mid \text{such that} \int_M |u|^q dv_g = 1\}. \tag{11.10}$$

We have the following result

Theorem 11.8 *Let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. Given $q \in (2, 2^\star)$, there exists $u \in C^\infty(M), u > 0$ in M such that*

$$\Delta_g u + hu = \mu_q u^{q-1},$$

where $\int_M u^q dv_g = 1$, where μ_q is as above.

Proof Let $(u_i) \subset \mathcal{H}_q$ a minimizing sequence for μ_q. Taking into account Proposition 3.49 [18] we have $|\nabla |u|| = |\nabla u|$ a.e., up to replacing u_i by $|u_i|$, we can assume that $u_i \geq 0$ for all i. Since $(u_i) \subset \mathcal{H}_q$ and $q > 2$, (u_i) using Hölder inequality we get

$$\int_M |u_i|^2 dv_g \leq V_g(M)^{\frac{q-2}{q}} (\int_M |u_i|^q dv_g)^{\frac{2}{q}}.$$

From this inequality follows that (u_i) is bounded in $L^2(M)$. In particular, μ_q is finite, and (u_i) is bounded in $H_1^2(M)$. Since $H_1^2(M)$ is a Hilbert space, follows that is reflexive. Taking into account that the embedding $H_1^2(M) \hookrightarrow L^q(M)$ is compact, there exists $u \in H_1^2(M)$ and a subsequence of (u_i) which will denote in the sameway, such that:

a) $u_i \rightharpoonup u$ weakly in $H_1^2(M)$

b) $u_i \to u$ strongly in $L^q(M)$

c) $u_i(x) \to u(x)$ a.e. $x \in M$.

From c) follows that $u \geq 0$, and by b), $u \in \mathcal{H}_q$. Because the norm is weakly lower semicontinuous and $u_i \rightharpoonup u$ weakly in $H_1^2(M)$ follows that

$$\|u\|_{H_1^2} \leq \liminf_{i\to+\infty} \|u_i\|_{H_1^2}.$$

Because $u_i \to u$ strongly in $L^q(M)$ and since $L^q(M) \subset L^2(M)$, then we obtain

$$\mu_q = \int_M (|\nabla u|^2 + hu^2) dv_g.$$

In particular, u is minimizer for μ_q. By Theorem 11.1 follows the existence of $\alpha \in \mathbb{R}$, such that for any $\varphi \in H_1^2(M)$,

$$\int_M \langle \nabla u, \nabla\varphi\rangle_g dv_g + \int_M hu\varphi dv_g = \alpha \int_M u^{q-1}\varphi dv_g.$$

Taking $\varphi = u$, we get that $\alpha = \mu_q$. Thus $u \in \mathcal{H}_q, u \geq 0$, a weak solutions of (11.7).

$\square$

11.3 Critical case

In this section we study the critical case, i.e. the problem

$$(CE) \qquad \begin{cases} \Delta_g u + hu = \lambda u^{2^\star-1}, & \text{in} \quad M \\ \\ u > 0, & \text{in} \quad M, \end{cases}$$

where, (M, g) be a smooth compact manifold of dimension $n \geq 3$, $h : M \to \mathbb{R}$ a smooth function, where $\lambda \in \mathbb{R}$ and $2^\star = \frac{2n}{n-2}$ is the critical exponent.

We define

$$\mu = \inf_{u\in\mathcal{H}} \int_M (|\nabla u|^2 + hu^2) dv_g,$$

where

$$\mathcal{H} = \{u \in H_1^2 \mid \text{such that} \int_M |u|^{2^\star} dv_g = 1\}.$$

Because $L^{2^\star}(M) \subset L^2(M)$, follows that μ is finite. First we state the following regularity result, which essentially is due to Trudinger [284]. Here we follows the Hebey work [137].

We have the following regularity result.

Theorem 11.9 *Let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, and let $h : M \to \mathbb{R}$ be a smooth function. If $u \in H_1^2(M), u \geq 0$ is a weak solution of the equation*

$$\Delta_g u + hu = \lambda u^{2^\star - 1}$$

where $\lambda \in \mathbb{R}$, then $u \in C^\infty(M)$ and either $u \equiv 0$, or $u > 0$ everywhere.

Proof It is enough to prove that $u \in L^s(M)$ for some $s > 2^\star$, because using the argument from Theorem 11.8 follows that $u \in C^\infty(M)$. For a fixed number $L > 0$ we consider the following functions $F_L, G_L : \mathbb{R} \to \mathbb{R}$ given by

$$F_L(t) = \begin{cases} |t|^{2^\star/2}, & \text{if} \quad |t| \leq L \\ \frac{2^\star}{2} L^{(2^\star-2)/2}|t| - \frac{2^\star-2}{2} L^{2^\star/2}, & \text{if} \quad |t| > L, \end{cases}$$

and

$$G_L(t) = \begin{cases} |t|^{2^\star-1}, & \text{if} \quad |t| \leq L \\ \frac{2^\star}{2} L^{(2^\star-2)}|t| - \frac{2^\star-2}{2} L^{2^\star-1}, & \text{if} \quad |t| > L. \end{cases}$$

From the definitions of the functions F_L and G_L follows

$$F_L \leq t^{2^\star/2}, \quad G_L \leq t^{2^\star/2} \text{ and } (F_L(t))^2 \geq tG_L(t).$$

From these, follows easily $(F_L'(t))^2 \leq \frac{2^\star}{2} G_L'(t)$ for $t \neq L$. Taking into account that F_L and G_L are locally Lipschitz and if we denote $\tilde{F}_L = F_L(u)$ and $\tilde{G}_L = G_L(u)$ follows that $\tilde{F}_L, \tilde{G}_L \in H_1^2(M)$. Since $u \in H_1^2(M)$ is a weak solutions the equation (CE), follows that

$$\int_M \langle \nabla u, \nabla \tilde{G}_L \rangle_g dv_g + \int_M hu\tilde{G}_L dv_g = \alpha \int_M u^{2^\star-1} \tilde{G}_L dv_g.$$

Since $\tilde{G}_L(u) \leq u^{2^\star-1}$, and $u \in L^{2^\star}(M)$, it follows that there exists $C_1, C_2 > 0$, independent of L, such that

$$\int_M G_L'(u)|\nabla u|^2 dv_g \leq C_1 + C_2 \int_M u^{2^\star-1} \tilde{G}_L dv_g$$

and since $(F'_L(t))^2 \leq \frac{2^\star}{2} G'_L(t)$ and $tG_L(t) \leq (F_L(t))^2$, follows that

$$\frac{2^\star}{2} \int_M |\nabla \tilde{F}_L| dv_g \leq C_1 + C_2 \int_M u^{2^\star-2} \tilde{F}_L^2 dv_g.$$

Given $K > 0$, let

$$K^+ = \{x \in M \ : \ u(x) \leq K\},$$

and

$$K^- = \{x \in M \ : \ u(x) \geq K\}.$$

Because $H_1^2(M) \hookrightarrow L^{2^\star}(M)$ the embedding is continuous, from the Hölder's inequalities we get

$$\begin{aligned}
\int_M u^{2^\star-2} \tilde{F}_L^2 dv_g &= \int_{K^-} u^{2^\star-2} \tilde{F}_L^2 dv_g + \int_{K^+} u^{2^\star-2} \tilde{F}_L^2 dv_g \\
&\leq \int_{K^-} u^{2^\star-2} \tilde{F}_L^2 dv_g + \left(\int_{K^+} u^{2^\star}\right)^{2/n} \left(\int_{K^+} \tilde{F}_L^{2^\star}\right)^{2/2^\star} \\
&\leq \int_{K^-} u^{2^\star-2} \tilde{F}_L^2 dv_g + \varepsilon(K) \left(\int_M \tilde{F}_L^{2^\star}\right)^{2/2^\star} \\
&\leq \int_{K^-} u^{2^\star-2} \tilde{F}_L^2 dv_g + C_3 \varepsilon(K) \int_M (|\nabla \tilde{F}_L|^2 + \tilde{F}_L^2) dv_g,
\end{aligned}$$

where $\varepsilon(K) = (\int_{K^+} u^{2^\star} dv_g)^{2/n}$, and C_3 does not depend on K and L.Since $u \in L^{2^\star}(M)$, we have $\lim\limits_{K \to +\infty} \varepsilon(K) = 0$. We fix $K > 0$ such that $C_2 C_3 \varepsilon(K) < \frac{2}{2^\star}$. We suppose that $L > K$, then we have

$$\int_{K^-} u^{2^\star-2} \tilde{F}_L^2 dv_g \leq K^{2(2^\star-1)} V_g,$$

where V_g denotes the volume of M with respect the Riemannian metric g. Since $F_L(t) \leq t^{2^\star/2}$ and $u \in L^{2^\star}(M)$ follows the existence of a real number $C_4 > 0$, which does not depend on L such that

$$\int_M \tilde{F}_L^2 dv_g \leq C_4.$$

In consequence, then there exist $C_5, C_6 > 0$, independent of L, with $C_6 < 1$, such that

$$\int_M |\nabla \tilde{F}_L|^2 dv_g \leq C_5 + C_6 \int_M |\nabla \tilde{F}_L|^2 dv_g.$$

From this inequality follows that

$$\int_M |\nabla \tilde{F}_L|^2 dv_g \leq \frac{C_5}{1 - C_6}.$$

Taking into account the Sobolev inequality for the embedding $H_1^2(M) \hookrightarrow L^{2^\star}(M)$ and the above inequality we get

$$\int_M \tilde{F}^{2^\star} dv_g \leq C_7,$$

where C_7 does not depend on L. If $L \to +\infty$, follows that $u \in L^{(2^\star)^2/2}(M)$. But $(2^\star)^2/2 > 2$ and we get the existence of some $s > 2^\star$ such that $u \in L^s(M)$. Now we can use the procedure from Theorem 11.8 to prove that $u \in C^\infty(M)$. Now from Theorem 11.1 it follows that either $u \equiv 0$ or $u > 0$.

□

Theorem 11.10 *Let (M, g) be a smooth compact manifold of dimension $n \geq 3$, and let $h : M \to \mathbb{R}$ be a smooth function. If $\mu \leq 0$, then the problem (CE) has a smooth positive solution.*

Proof

In the following we distinguish two cases.

First case: $\mu < 0$. Fix $q \in (2, 2^\star)$ and we consider the equation

$$\Delta_g u + hu = \mu_q u^{q-1},$$

where μ_q is defined in (11.8).

From Theorem 11.8, there exists $u_q \in C^\infty(M), u_q > 0$, such that

$$\Delta_g u_q + hu_q = \mu_q u_q^{q-1},$$

and $\int_M u_q^q dv_g = 1$. Hence there exists $u \in \mathcal{H}$ such that $I(u) < 0$, where

$$I(u) = \int_M (|\nabla u|^2 + hu^2) dv_g.$$

We have

$$\mu_q \leq I\left(\frac{u}{(\int_M |u|^q)^{1/q}}\right)$$

and that $\int_M |u|^q dv_g \leq V^{1-\frac{q}{2^\star}}$, where V_g is the volume of M with respect to g, we easily get that there exists $\varepsilon_0 > 0$ such that $\mu_q \leq \varepsilon_0$ for every $q \in (2, 2^\star)$. In similar way, we easily get that there exists $K > 0$ such that $\mu_q \geq -K$ for all $q \in (2, 2^\star)$. Hence, there exists $\varepsilon_0 > 0$ such that

$$-\frac{1}{\varepsilon_0} \leq \mu_q \leq -\varepsilon_0$$

for all $q \in (2, 2^\star)$. Let x_q be a point where u_q is maximum. Then $\Delta_g u_q(x_q) \geq 0$. It follows from the equation satisfied by u_q that

$$h(x_q)u_q \leq \mu_q u_q^{q-1}(x_q).$$

In particular, $h(x_q) < 0$, and

$$u_q^{q-2}(x_q) \leq \frac{1}{\varepsilon_0} \max_{x \in M} |h(x)|,$$

therefore the u_q's are uniformly bounded. From Theorem 11.6 follows that the sequence u_q's are bounded in H_2^p for all p. In particular, a subsequence of the u_q's converge to some u in $C^1(M)$ as $q \to 2^\star$. Assuming that the μ_q's converge to some λ as $q \to 2^\star$, we get that u is a weak solution of

$$\Delta_g u + hu = \lambda u^{2^\star - 1}.$$

Because $\displaystyle\int_M u_q^q dv_g = 1$, follows that u is nonzero and $u_q \to u$ uniformly as $q \to 2^\star$.

From Theorem 11.9 follows that u is smooth and Proposition 11.1 implies that u is everywhere positive. In particular, u is a strong solution of the above equation. With similar arguments as above, we get that $\limsup\limits_{q \to 2^\star} \mu_q \leq \mu$. Independently, it is straightforward that $\mu \leq I\left((\int_M u_q^{2^\star} dv_g)^{-\frac{1}{2^\star}} u_q \right)$, therefore

$$\left(\int_M u_q^{2^\star} dv_g \right)^{\frac{2}{2^\star}} \mu \leq \mu_q$$

for all q. Since $u_q \to u$ uniformly, we have that $\displaystyle\int_M u_q^{2^\star} dv_g \to \int_M u^{2^\star} dv_g$. Hence, we also have that $\liminf\limits_{q \to 2^\star} \mu_q \geq \mu$. It follows that $\mu_q \to \mu$ as $q \to 2^\star$, so that $\lambda = \mu$. Summarizing, we proved that if $\mu < 0$, then there exists $u \in C^\infty(M), u > 0$, such that

$$\Delta_g u + hu = \mu u^{2^\star - 1}$$

and $\displaystyle\int_M u^{2^\star} dv_g = 1$. In particular, u is a minimizing solution of the equation. Moreover, u is obtained as the uniform limit of a subsequence of the u_q's.

The null case: $\mu = 0$.

For the fixed number $q \in (2, 2^\star)$ we consider the equation

$$\Delta_g u + hu = \mu_q u^{q-1},$$

where μ_q is defined in (11.9).

From Theorem 11.8 follows the existence of an element $u_q \in C^\infty(M), u_q > 0$ be such that

$$\Delta_g u_q + hu_q = \mu_q u_q^{q-1}$$

and $\int_M u_q^q dv_g = 1$. First we claim that if $\mu = 0$, then $\mu_q = 0$ for all q. Given $\varepsilon > 0$, we let $u_\varepsilon \in \mathcal{H}$ be such that $I(u_\varepsilon) \leq \varepsilon$. Thanks to Sobolev inequality, there exists $A > 0$ such that for any $u \in H_1^2(M,$

$$\|u\|_{2^\star}^2 \leq A(\|u\|_2^2 + \|u\|_2^2).$$

Taking $u = u_\varepsilon$ in the above inequality, we get that for any $\varepsilon > 0$,

$$1 \leq A(\varepsilon + B\|u_\varepsilon\|_2^2),$$

where $B = 1 + \max_{x \in M} |h(x)|$. Hence, there exists $C > 0$ such that $\|u_\varepsilon\|_2 \geq C$ for all $\varepsilon > 0$ sufficiently small. In particular, for $q > 2$, there exists $C_q > 0$ such that

$$\int_M |u_\varepsilon|^q dv_g \geq C_q.$$

Independently, it is clear that $\mu_q \leq I(\|u_\varepsilon\|_q^{-1} u_\varepsilon)$, so that $\|u_\varepsilon\|_q^2 \mu_q \leq \varepsilon$.

Fixing $q > 2$, and letting $\varepsilon \to 0$, it follows that $\mu_q \leq 0$. On the other hand,

$$\mu_q = I(u_q) = \|u_q\|_{2^\star}^2 I(\|u_q\|_{2^\star}^{-1} u_q) \geq \|u_q\|_{2^\star}^2 \mu,$$

so that $\mu_q \geq 0$. This prove the above claim if $\mu = 0$, then $\mu_q = 0$ for all q. Letting $u = \|u_q\|_{2^\star}^{-1} u_q$ for some q, we get that u is a positive smooth solution of the equation

$$\Delta_g u + hu = \mu u^{2^\star - 1}$$

such that $\int_M u^{2^\star} dv_g = 1$. □

The positive case: $\mu > 0$.
In this case first we prove that the operator $\Delta_g + h$ is coercive in the sense that, there exists $\lambda > 0$ such that for any $u \in H_1^2(M)$,

$$\int_M (|\nabla u|^2 + hu^2) dv_g \geq \lambda \|u\|_{H_1^2}^2.$$

From condition $\mu > 0$ and from Hölder's inequalities follows the existence of $\tilde{\mu} > 0$ such that for any $u \in H_1^2(M)$,

$$\int_M (|\nabla u|^2 + hu^2)dv_g \geq \tilde{\mu} \int_M u^2 dv_g.$$

We let $\varepsilon \in (0, \frac{\tilde{\mu}}{2})$ be such that $(1-\varepsilon)\tilde{\mu} + \varepsilon h \geq \frac{\tilde{\mu}}{2}$. Then

$$\begin{aligned}
\int_M (|\nabla u|^2 + hu^2)dv_g &\geq \varepsilon \int_M (|\nabla u|^2 + hu^2)dv_g + (1-\varepsilon) \int_M u^2 dv_g \\
&\geq \int_M |\nabla u|^2 dv_g + \frac{\tilde{\mu}}{2} \int_M u^2 dv_g \\
&\geq \varepsilon \int_M (|\nabla u|^2 + u^2)dv_g.
\end{aligned}$$

In consequence, the operator $\Delta_g + h$ is coercive.

The main result in the case $\mu > 0$ is the following:

Theorem 11.11 *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, and $h : M \to \mathbb{R}$ be a smooth function. If*

$$\inf_{u \in \mathcal{H}} \int_M (|\nabla u|^2 + hu^2)dv_g < \frac{1}{K_n^2}.$$

Then there exists $u \in C^\infty(M), u > 0$ such that

$$\Delta_g u + hu = \mu u^{2^\star - 1}$$

and $\int_M u^{2^\star} dv_g = 1$. In particular, u is a minimizing solution of the equation.

Proof Let $(u_i) \subset \mathcal{H}$ be a minimizing sequence for μ. Using again Proposition 3.49 [18], we can replace u_i by $|u_i|$, so we can assume that the u_i's are nonnegative. We have that (u_i) is bounded in $H_1^2(M)$. Taking into account that $H_1^2(M)$ is a Hilbert space, we may thus assume that there exists $u \in H_1^2(M)$, such that

1) $u_i \rightharpoonup u$ weakly in $H_1^2(M)$,
2) $u_i \to u$ strongly in $L^2(M)$,
3) $u_i \to u$ almost everywhere as $i \to +\infty$.

In particular, u is nonnegative. From the weak convergence $u_i \rightharpoonup u$ follows that

$$\|u_i\|_2^2 = \|\nabla(u_i - u)\|_2^2 + \|u\|_2^2 + o(1),$$

for all i, where $o(1) \to 0$ as $i \to +\infty$. From Brézis-Lieb Lemma (see Theorem B.2), it follows that

$$\|u_i\|_{2^\star}^{2^\star} = \|u_i - u\|_{2^\star}^{2^\star} + \|u\|_{2^\star}^{2^\star} + o(1),$$

for all i, where, as above, $o(1) \to 0$ as $i \to +\infty$. From Theorem 11.5 follows that, there exists $B > 0$ such that for any i,

$$\|u_i - u\|_{2^\star}^2 \leq K_n^2 \|\nabla(u_i - u)\|_2^2 + B\|u_i - u\|_2^2.$$

Since $u_i \in \mathcal{H}$, it follows that

$$(1 - \|u\|_{2^\star}^{2^\star})^{2/2^\star} \leq K_n^2 \left(\|\nabla u_i\|_2^2 - \|\nabla u\|_2^2\right) + o(1).$$

Since $I(u_i) \to \mu$ and $u_i \to u$ strongly in $L^2(M)$, we also have that

$$\begin{aligned} K_n^2(\|\nabla u_i\|_2^2 - \|\nabla u\|_2^2) &= K_n^2\mu - K_n^2\left(\int_M |\nabla u|^2 dv_g + \int_M hu^2 dv_g\right) + o(1) \\ &\leq K_n^2\mu - K_n^2\mu\|u\|_{2^\star}^2 + o(1). \end{aligned}$$

Hence,

$$(1 - \|u\|_{2^\star}^{2^\star})^{2/2^\star} \leq K_n^2(1 - \|u\|_{2^\star}^2).$$

From the assumption $\mu K_n^2 < 1$ follows that

$$1 - \|u\|_{2^\star}^2 \leq (1 - \|u\|_{2^\star}^{2^\star})^{2/2^\star}.$$

From this we have $\|u\|_{2^\star} = 1$. Then, $\|u_i\|_2 \to \|u\|_2$ as $i \to +\infty$, and since

$$\|\nabla u_i\|_2^2 = \|\nabla(u_i - u)\|_2^2 + \|\nabla u\|_2^2 + o(1),$$

we get that $u_i \to u$ strongly in $H_1^2(M)$ as $i \to +\infty$. In particular, u is a minimizer for μ, and u is a weak nonnegative solution of the equation

$$\Delta_g u + hu = \mu u^{2^\star - 1}.$$

From Theorem 11.9 and Proposition 11.1 follows that u is smooth and positive. □

11.4 Comments and historical notes

In 1960, Yamabe [294] attempted to solve the problem *Given a compact Riemannian manifold (M, g) of dimension $n \geq 3$, then there exists a conformal metric with constant scalar curvature.* Unfortunately, his proof contained an error, discovered by Trudinger [284] and repair the proof, but only with a rather restrictive assumption on the manifold M. In 1976, Aubin [17] and Schoen [267] trait the remaining case. In 1984, Cherrier [68] extend the Yamabe problem to the compact Riemann manifold with boundary. In 1987 appear a very nice paper due to Lee and Parker [185] about the Yamabe problem, where heres gave a proof of the Yamabe problem unifying Aubin's and Schoen's arguments. The Yamabe problem on compact Riemannian manifold with boundary was studied in many papers by Escobar [104], [105], [106]. In 1993, Hebey and Vaugon [138] studied the equivariant Yamabe problem. In the last years many papers are dedicated to study different aspects of Yamabe or Yamabe like problems. For this see the books of Ambrosetti and Malchiodi [6], Druet, Hebey, and Robert [96], and Hebey [135], [136] and also the papers of Li and Li [188] and Il'yasov and Runst [142] and there references. Also many papers is dedicated to study the Yamabe invariant, see for example a very nice paper of Ammann, Dahl, and Humbert [10]. These problems suggest to study elliptic or semilinear elliptic problems on a compact Riemannian manifold with or without boundary. In the present exists a very reach literature which are dedicated to study these problem.

12
Problems to Part II

> In mathematics the art of proposing a question must be held of higher value than solving it.
>
> Georg Cantor (1845–1918)

Problem 12.1 (Hopf's lemma) Let (M, g) be a connected, compact Riemannian manifold and $f : M \to \mathbb{R}$ a smooth function such that

$$\Delta_g f \geq 0.$$

Then f is constant.

Problem 12.2 Consider $\mathbb{S}^n$ the n-dimensional unit sphere endowed with the Riemannian metric induced by the inclusion $\iota : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$. Show that for any smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ we have

$$(\Delta_{\mathbb{R}^{n+1}} f)|_{\mathbb{S}^n} = \Delta_{\mathbb{S}^n}(f|_{\mathbb{S}^n}) - \frac{\partial^2 f}{\partial r^2}|_{\mathbb{S}^n} - n\frac{\partial f}{\partial r}|_{\mathbb{S}^n},$$

where $\Delta_{\mathbb{R}^{n+1}}$, $\Delta_{\mathbb{S}^n}$ and $\frac{\partial}{\partial r}$ are the Laplace operator on $\mathbb{R}^{n+1}$, $\mathbb{S}^n$, and the radial derivative, respectively.

Problem 12.3 Let $\mathbb{S}^n$ be the unit sphere endowed with the usual Riemannian structure from $\mathbb{R}^{n+1}$. Denote by $\mathcal{H}_k$ the vector space of the harmonic polynomial of degree $k \geq 0$ defined on $\mathbb{R}^{n+1}$. Let $\tilde{\mathcal{H}}_k = \{f|_{\mathbb{S}^n} : f \in \mathcal{H}_k\}$.

a) Show that $\Delta_{\mathbb{S}^n} f = k(n + k - 1)f$, for all $f \in \tilde{\mathcal{H}}_k$ and hence $k(n + k - 1)$ is an eigenvalue of the Laplace operator $\Delta_{\mathbb{S}^n}$.

b) $\tilde{\mathcal{H}}_k$ is the eigenspace corresponding to the eigenvalue $\lambda_k = k(n+k-1)$.

c) The set $\{k(n+k-1) \;:\; k \in \mathbb{N}\}$ is the set of eigenvalues of $\Delta_{\mathbb{S}^n}$.

Problem 12.4 (M. Obata [227]) If g is a metric on $\mathbb{S}^n$ that is conformal to the standard metric $\overline{g}$ and has constant scalar curvature, then up to a constant scale factor, g is obtained from $\overline{g}$ by a conformal diffeomorphism of the sphere.

Problem 12.5 Let g_0 be the standard Riemannian metric on $\mathbb{R}^n$. Let $h = (h_{ij})$ be a symmetric bilinear form with compact support and consider the metric $g_\varepsilon = g_0 + \varepsilon h$ on $\mathbb{R}^n$. If R_{g_ε} denotes the scalar curvature of g_ε prove that

$$R_{g_\varepsilon}(x) = \varepsilon R_1(x) + \varepsilon^2 R_2(x) + o(\varepsilon^2),$$

where

$$R_1(x) = \sum_{i,j} \frac{\partial^2 h_{ij}}{\partial x_i \partial x_j} - \Delta \mathrm{tr} h,$$

and

$$\begin{aligned} R_2 \;=\; & -2 \sum_{k,j,l} h_{kj} \frac{\partial^2 h_{ij}}{\partial^2 x_l} + \sum_{k,j,l} h_{kj} \frac{\partial^2 h_{ll}}{\partial x_j \partial x_k} + \frac{3}{4} \sum_{k,j,l} \frac{\partial h_{jl}}{\partial x_k} \frac{\partial h_{jl}}{\partial x_k} \\ & - \sum_{k,j,l} \frac{\partial h_{jl}}{\partial x_l} \frac{\partial h_{jk}}{\partial x_k} + \sum_{k,j,l} \frac{\partial h_{jl}}{\partial x_l} \frac{\partial h_{kk}}{\partial x_j} - \frac{1}{4} \sum_{k,j,l} \frac{\partial h_{ll}}{\partial x_j} \frac{\partial h_{kk}}{\partial x_j} - \frac{1}{2} \sum_{k,j,l} \frac{\partial h_{kl}}{\partial x_j} \frac{\partial h_{jk}}{\partial x_l}. \end{aligned}$$

Also the following formula holds.

$$|g_\varepsilon|^{\frac{1}{2}} = 1 + \frac{\varepsilon}{2} \mathrm{tr} h + \varepsilon^2 \left(\frac{1}{8} (\mathrm{tr} h)^2 - \frac{1}{4} \mathrm{tr}(h^2) \right) + o(\varepsilon^2).$$

Problem 12.6 Let $(M, \overset{\circ}{g})$ be a compact Riemannian manifold with boundary $n \geq 3$. Let $0 \in \partial M$ be a point. We assume that $1 \leq i, j, k, l \leq n-1$. Let $(x_1, x_2, \ldots, x_{n-1})$ be normal coordinate on ∂M at the point 0. Let $\gamma(t)$ be the geodesics leaving from $(x_1, x_2, \ldots, x_{n-1})$ in the orthogonal direction to ∂M and parameterized by the arc length. In this case $(x_1, x_2, \ldots, x_{n-1})$ will be called Fermi coordinates at $0 \in \partial M$. In these coordinates the arc length is written as

$$ds^2 = dt^2 + g_{ij}(x,t) dx_i dx_j.$$

If $g = det(\overset{\circ}{g})$ then $g = det(g_{ij})$. If x_i and t are sufficiently small, then the following formula holds

$$\sqrt{g} = 1 - Ht + \frac{1}{2}(H^2 - \|\pi\|^2 - \mathrm{Ric}(\eta))t^2 - H_i t x_i - \frac{1}{6}\overline{R}_{ij} x_i x_j + O(|(x,t)|^3),$$

where π is the second fundamental form, H is its trace and $\overline{R}_{ij}$ are the coefficients of the Ricci tensor of ∂M.

Problem 12.7 Let (M, g) be a connected compact n-dimensional Riemannian manifold without boundary. An eigenvalue of the p-Laplace operator $\Delta_p u := \mathrm{div}_g(|\nabla u|^{p-2}\nabla u)$ is a real number λ such that there exists $u \in W_1^p(M) \setminus \{0\}$ such that

$$-\Delta_p u = \lambda |u|^{p-2} u.$$

We denote by $\sigma_p(M, g)$ the set of all nonzero eigenvalues. The set $\sigma_p(M, g)$ is not empty, unbounded and included in $(0, +\infty)$. If $\lambda_1 = \inf \sigma_p(M, g)$ prove that $\lambda_1 \in \sigma_p(M, g)$ and

$$\lambda_1 = \min \left\{ \int_M |\nabla\varphi|^p dv_g \ : \ \varphi \in \Sigma_0 \right\}$$

where

$$\Sigma_0 = \left\{ \varphi \in W_1^p(M) \ : \ \int_M |\varphi|^p dv_g = 1, \ \int_M |\varphi|^{p-2}\varphi dv_g = 0 \right\}.$$

If $\omega \subset M$ is a non-empty open subset, define

$$\mu(\omega) = \min \left\{ \int_\omega |\nabla\varphi|^p \ : \varphi \in W_0^{1,p}(\omega), \ \int_\omega |\varphi|^p dv_g = 1 \right\}.$$

Prove that

$$\lambda_1 = \min_{(\omega,\tilde{\omega})} \max\{\mu(\omega), \mu(\tilde{\omega})\},$$

where $(\omega, \tilde{\omega})$ runs over the set of couples of non-empty disjoint open subset of M and if u is any nonzero eigenfunction associated to λ_1, then $M \setminus u^{-1}(0)$ has two connected open components.

Problem 12.8 Let (M, g) be an n-dimensional compact Riemannian manifold without boundary, $K : M \to \mathbb{R}$ a continuous function ($K \neq 0$) and $\lambda \in \mathbb{R}$. We consider the following equation

$$\Delta_g u + \lambda u + K(x) u^q = 0.$$

a) If $\lambda = 0$ and $1 < q < \frac{n+2}{n-2}$ then the equation admits a positive solutions if and only if the following conditions hold:

   i) $\int_M K dv_g < 0$;

   ii) there exists $x_0 \in M$ such that $K(x_0) > 0$.

b) If $\lambda < 0$ and $1 < q < \frac{n+2}{n-2}$ then the equation admits positive solutions if and only if the condition ii) holds.

Problem 12.9 Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and let λ_1 denote the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$, $f'(0) < \lambda_1$, and

$$\lambda_1 < \lim_{u \longrightarrow +\infty} \frac{f(u)}{u} < \infty\,.$$

Prove that the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a solution.

Problem 12.10 Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and assume that p is a real number such that $1 < p < (N+2)/(N-2)$ if $N \geq 3$ and $1 < p < \infty$ if $N \in \{1, 2\}$.

Prove that there exists $\lambda^* > 0$ such that the problem

$$\begin{cases} -\Delta u = \lambda(1+u)^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least two solutions for any $\lambda \in (0, \lambda^*)$.

Problem 12.11 Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and assume that p is a real number such that $1 < p < (N+2)/(N-2)$ if $N \geq 3$ and $1 < p < \infty$ if $N \in \{1, 2\}$. Consider the problem

$$\begin{cases} -\Delta u = u^p + f(x) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{12.1}$$

where f is a smooth function.

Prove that there exists $\delta > 0$ such that problem (12.1) has a solution, provided that $\|f\|_{L^\infty} \leq \delta$.

Problem 12.12 Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and assume that p is a real number such that $1 < p < (N+2)/(N-2)$ if $N \geq 3$ and $1 < p < \infty$ if $N \in \{1, 2\}$. Consider the problem

$$\begin{cases} -\Delta u = |u|^p + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{12.2}$$

where f is a smooth function.

Denote

$$K = \{f \in C^{0,\alpha}(\overline{\Omega});\ \text{problem (12.2) has a solution}\}$$

and

$$P = \{h \in C^{0,\alpha}(\overline{\Omega});\ h \geq 0 \text{ in } \Omega\}\,.$$

a) Prove that K is a convex set and $K \setminus P \subset K$.
b) Prove that for every $f \in \operatorname{Int} K$, problem (12.2) has at least two solutions. In particular, if $f \leq 0$ in Ω, then problem (12.2) has at least two solutions.

13
Mathematical Preliminaries

Knowledge is power. (Ipsa Scientia Potestas Est)

Sir Francis Bacon (1561–1626), *Meditationes Sacræ. De Hæresibus*

13.1 Metrics, geodesics, flag curvature

Let M be a connected m-dimensional C^∞ manifold and let $TM = \bigcup_{p\in M} T_pM$ be its tangent bundle. If the continuous function $F : TM \to [0,\infty)$ satisfies the conditions that it is C^∞ on $TM \setminus \{0\}$; $F(tu) = tF(u)$ for all $t \geq 0$ and $u \in TM$, i.e., F is positively homogeneous of degree one; and the matrix $g_{ij}(u) := [\frac{1}{2}F^2]_{y^iy^j}(u)$ is positive definite for all $u \in TM \setminus \{0\}$, then we say that (M, F) is a *Finsler manifold.* If F is absolutely homogeneous, then (M, F) is said to be *reversible.* Let π^*TM be the pull-back of the tangent bundle TM by $\pi : TM \setminus \{0\} \to M$; then

$$g_{(p,y)} := g_{ij(p,y)}dp^i \otimes dp^j := [\frac{1}{2}F^2]_{y^iy^j}dp^i \otimes dp^j, \quad p \in M,\ y \in T_pM, \tag{13.1}$$

is the natural Riemannian metric on the pulled-back bundle π^*TM.

Unlike the Levi-Civita connection in Riemann geometry, there is no unique natural connection in the Finsler case. Among these connections on π^*TM, we choose the *Chern connection* whose coefficients are denoted by Γ^k_{ij}; see [22, p.38]). This connection induces the *curvature tensor*, denoted by R; see [22, Chapter 3]. The Chern connection defines the *covariant derivative* D_VU of a vector field U in the direction $V \in T_pM$. Since, in general, the Chern connection coefficients Γ^i_{jk} in natural coordinates have a directional dependence, we must say explicitly that $D_V U$ is defined with a fixed reference vector. In particular, let $\sigma : [0, r] \to M$ be a smooth curve with velocity field $T = T(t) = \dot{\sigma}(t)$. Suppose that U and W are vector fields defined along σ. We define $D_T U$ with *reference vector* W as

$$D_T U = \left[\frac{dU^i}{dt} + U^j T^k (\Gamma^i_{jk})_{(\sigma,W)} \right] \frac{\partial}{\partial p^i}_{|\sigma(t)},$$

where $\left\{ \frac{\partial}{\partial p^i}_{|\sigma(t)} \right\}_{i=1,m}$ is a basis of $T_{\sigma(t)}M$. A C^∞ curve $\sigma : [0, r] \to M$, with velocity $T = \dot{\sigma}$ is a (Finslerian) *geodesic* if

$$D_T \left[\frac{T}{F(T)} \right] = 0 \text{ with reference vector } T. \tag{13.2}$$

If the Finslerian velocity of the geodesic σ is constant, then (13.2) becomes

$$\frac{d^2\sigma^i}{dt^2} + \frac{d\sigma^j}{dt}\frac{d\sigma^k}{dt}(\Gamma^i_{jk})_{(\sigma,T)} = 0, \quad i = 1, ..., m = \dim M. \tag{13.3}$$

For any $p \in M$ and $y \in T_pM$ we may define the *exponential map* $\exp_p : T_pM \to M$, $\exp_p(y) = \sigma(1, p, y)$, where $\sigma(t, p, y)$ is the unique solution (geodesic) of the second order differential equation (13.2) (or, (13.3)) which passes through p at $t = 0$ with velocity y.

If U, V and W are vector fields along a curve σ, which has velocity $T = \dot{\sigma}$, we have the *derivative rule*

$$\frac{d}{dt} g_{(\sigma,W)}(U, V) = g_{(\sigma,W)}(D_T U, V) + g_{(\sigma,W)}(U, D_T V) \tag{13.4}$$

whenever $D_T U$ and $D_T V$ are with reference vector W and *one* of the following conditions holds:

- U or V is proportional to W, or
- $W = T$ and σ is a geodesic.

Let $\gamma : [0, 1] \to M$ be a smooth regular curve and $\Sigma : [0, 1] \times [-\varepsilon, \varepsilon] \to M$ be a smooth regular variation of γ (i.e. $\Sigma(t, 0) = \gamma(t)$ for all $t \in [0, 1]$) with variation vector field $U = U(t, u) = \frac{\partial \Sigma}{\partial u}$. Then

$$\frac{\partial}{\partial u} g_{(\sigma,T)}(T, T) = 2g_{(\sigma,T)}(T, D_U T), \tag{13.5}$$

where $T = \frac{\partial \Sigma}{\partial t}$ and the covariant derivative $D_U T$ is with reference vector T.

A vector field J along a geodesic $\sigma : [0, r] \to M$ (with velocity field T) is said to be a *Jacobi field* if it satisfies the equation

$$D_T D_T J + R(J, T)T = 0, \tag{13.6}$$

where R is the curvature tensor. Here, the covariant derivative D_T is defined with reference vector T.

We say that q *is conjugate to* p along the geodesic σ if there exists a nonzero Jacobi field J along σ which vanishes at p and q.

Let $\gamma : [0, r] \to M$ be a piecewise C^∞ curve. Its *integral length* is defined as

$$L_F(\gamma) = \int_0^r F(\gamma(t), \dot\gamma(t))\, dt.$$

Let $\Sigma : [0, r] \times [-\varepsilon, \varepsilon] \to M$ ($\varepsilon > 0$) be a piecewise C^∞ variation of a geodesic $\gamma : [0, r] \to M$ with $\Sigma(\cdot, 0) = \gamma$. Let $T = T(t, u) = \frac{\partial \Sigma}{\partial t}$, $U = U(t, u) = \frac{\partial \Sigma}{\partial u}$ the velocities of the *t-curves* and *u-curves*, respectively. The formula for the *first variation of arc length*, see Bao-Chern-Shen [22, Exercise 5.1.4], gives us

$$L'_F(\Sigma(\cdot, 0)) := \frac{d}{du} L_F(\Sigma(\cdot, u))_{|u=0} = \left[g_T\left(U, \frac{T}{F(T)}\right)_{|u=0} \right]\Bigg|_{t=0}^{t=r}. \tag{13.7}$$

For $p, q \in M$, denote by $\Gamma(p, q)$ the set of all piecewise C^∞ curves $\gamma : [0, r] \to M$ such that $\gamma(0) = p$ and $\gamma(r) = q$. Define the map $d_F : M \times M \to [0, \infty)$ by

$$d_F(p, q) = \inf_{\gamma \in \Gamma(p,q)} L_F(\gamma). \tag{13.8}$$

Of course, we have $d_F(p, q) \geq 0$, where equality holds if and only if $p = q$, and the *triangle inequality* holds, i.e., $d_F(p_0, p_2) \leq d_F(p_0, p_1) + d_F(p_1, p_2)$ for every $p_0, p_1, p_2 \in M$. In general, since F is only a positive homogeneous function, $d_F(p, q) \neq d_F(q, p)$; thus, (M, d_F) is only a quasi-metric space. If (M, g) is a Riemannian manifold, we will use the notation d_g instead of d_F which becomes a usual metric function.

For $p \in M$, $r > 0$, we define the *forward* and *backward Finsler-metric balls*, respectively, with center $p \in M$ and radius $r > 0$, by

$$\mathcal{B}_p^+(r) = \{q \in M : d_F(p, q) < r\} \text{ and } \mathcal{B}_p^-(r) = \{q \in M : d_F(q, p) < r\}.$$

We denote by $B_p(r) := \{y \in T_pM : F(p, y) < r\}$ the open *tangent ball* at $p \in M$ with radius $r > 0$. It is well-known that the topology generated by the forward (resp. backward) metric balls coincide with the underlying manifold topology, respectively.

By Whitehead's theorem (see [292] or [22, Exercise 6.4.3, p. 164]) and [22, Lemma 6.2.1, p. 146] we can conclude the following useful local result (see also [179]).

Proposition 13.1 *Let (M, F) be a Finsler manifold, where F is positively (but perhaps not absolutely) homogeneous of degree one. For every point $p \in M$ there exist a small $\rho_p > 0$ and $c_p > 1$ (depending only on p) such that for every pair of points q_0, q_1 in $\mathcal{B}_p^+(\rho_p)$ we have*

$$\frac{1}{c_p} d_F(q_1, q_0) \leq d_F(q_0, q_1) \leq c_p d_F(q_1, q_0). \tag{13.9}$$

Moreover, for every real number $k \geq 1$ and $q \in \mathcal{B}_p^+(\rho_p/k)$ the mapping $\exp_q$ is C^1-diffeomorphism from $B_q(2\rho_p/k)$ onto $\mathcal{B}_q^+(2\rho_p/k)$ and every pair of points q_0, q_1 in $\mathcal{B}_p^+(\rho_p/k)$ can be joined by a unique minimal geodesic from q_0 to q_1 lying entirely in $\mathcal{B}_p^+(\rho_p/k)$.

A set $M_0 \subseteq M$ is *forward bounded* if there exist $p \in M$ and $r > 0$ such that $M_0 \subseteq \mathcal{B}_p^+(r)$. Similarly, $M_0 \subseteq M$ is *backward bounded* if there exist $p \in M$ and $r > 0$ such that $M_0 \subseteq \mathcal{B}_p^-(r)$.

A set $M_0 \subseteq M$ is *geodesic convex* if for any two points of M_0 there exists a unique geodesic joining them which belongs entirely to M_0.

Let $(p, y) \in TM \setminus 0$ and let V be a section of the pulled-back bundle π^*TM. Then,

$$K(y, V) = \frac{g_{(p,y)}(R(V, y)y, V)}{g_{(p,y)}(y, y)g_{(p,y)}(V, V) - [g_{(p,y)}(y, V)]^2} \tag{13.10}$$

is the *flag curvature* with flag y and transverse edge V. In particular, when the Finsler structure F arises from a Riemannian metric g (i.e., the fundamental tensor $g_{ij} = [\frac{1}{2}F^2]_{y_i y_j}$ does not depend on the direction y), the flag curvature coincides with the usual *sectional curvature*.

If $K(V, W) \leq 0$ for every $0 \neq V, W \in T_pM$, and $p \in M$, with V and W not collinear, we say that the flag curvature of (M, F) is *non-positive*.

A Finsler manifold (M, F) is said to be *forward (resp. backward) geodesically complete* if every geodesic $\sigma : [0, 1] \to M$ parameterized to have constant Finslerian speed, can be extended to a geodesic defined on $[0, \infty)$ (resp. $(-\infty, 1]$). (M, F) is *geodesically complete* if every geodesic $\sigma : [0, 1] \to M$ can be extended to a geodesic defined on $(-\infty, \infty)$. In the Riemannian case, instead of geodesically complete we simply say *complete Riemannian manifold*.

Theorem 13.1 (Theorem of Hopf-Rinow, [22, p. 168]) *Let (M, F) be a connected Finsler manifold, where F is positively (but perhaps not absolutely) homogeneous of degree one. The following two criteria are equivalent:*

(a) (M, F) is forward (backward) geodesically complete;

(b) Every closed and forward (backward) bounded subset of (M, d_F) is compact.

Moreover, if any of the above holds, then every pair of points in M can be joined by a minimizing geodesic.

Theorem 13.2 (Theorem of Cartan-Hadamard, [22, p. 238]) *Let (M, F) be a forward/backward geodesically complete, simply connected Finsler manifold of non-positive flag curvature. Then:*

(a) Geodesics in (M, F) do not contain conjugate points.

(b) The exponential map $\exp_p : T_pM \to M$ is a C^1 diffeomorphism from the tangent space T_pM onto the manifold M.

A Finsler manifold (M, F) is a *Minkowski space* if M is a vector space and F is a Minkowski norm inducing a Finsler structure on M by translation; its flag curvature is identically zero, the geodesics are straight lines, and for any two points $p, q \in M$, we have $F(q - p) = d_F(p, q)$, see Bao-Chern-Shen [22, Chapter 14]. In particular, (M, F) is both forward and backward geodesically complete. The *fundamental inequality* for Minkowski norms implies

$$|g_y(y, w)| \leq \sqrt{g_y(y, y)} \cdot \sqrt{g_w(w, w)} = F(y) \cdot F(w) \quad \text{for all } y \neq 0 \neq w, \tag{13.11}$$

which is the *generalized Cauchy-Schwarz inequality*, see Bao-Chern-Shen [22, p. 6-10].

A Finsler manifold is of *Berwald type* if the Chern connection coefficients Γ_{ij}^k in natural coordinates depend only on the base point. Special Berwald spaces are the (*locally*) *Minkowski space*s and the *Riemannian manifold*s. In the latter case, the Chern connection coefficients Γ_{ij}^k coincide the usual Christofel symbols

$$\overline{\Gamma}_{ij}^k(p) = \frac{1}{2}\left[\left(\frac{\partial g_{mj}}{\partial p_j}\right)_p + \left(\frac{\partial g_{mi}}{\partial p_j}\right)_p - \left(\frac{\partial g_{ij}}{\partial p_m}\right)_p\right] g^{mk}(p)$$

where the g^{ij}'s are such that $g_{im}g^{mj} = \delta_{ij}$.

A Riemannian manifold is said to be of *Hadamard-type*, if it is simply connected, complete, having non-positive sectional curvature.

Theorem 13.3 (Cosine inequality, see [91, Lemma 3.1]) *Let (M,g) be a Hadamard-type Riemannian manifold. Consider the geodesic triangle determined by vertices $a,b,c \in M$. If $\widehat{c}$ is the angle belonging to vertex c and if $A = d_g(b,c)$, $B = d_g(a,c)$, $C = d_g(a,b)$, then*

$$A^2 + B^2 - 2AB\cos\widehat{c} \leq C^2.$$

The following result is probably know, but since we have not found an explicit reference, we give its proof.

Proposition 13.2 *Let (M,g) be a complete, finite-dimensional Riemannian manifold. Then any geodesic convex set $K \subset M$ is contractible.*

Proof Let us fix $p \in K$ arbitrarily. Since K is geodesic convex, every point $q \in K$ can be connected to p uniquely by the geodesic segment $\gamma_q : [0,1] \to K$, i.e., $\gamma_q(0) = p$, $\gamma_q(1) = q$. Moreover, the map $K \ni q \mapsto \exp_p^{-1}(q) \in T_pM$ is well-defined and continuous. Note actually that $\gamma_q(t) = \exp_p(t\exp_p^{-1}(q))$. We define the map $F : [0,1] \times K \to K$ by $F(t,q) = \gamma_q(t)$. It is clear that F is continuous, $F(1,q) = q$ and $F(0,q) = p$ for all $q \in K$, i.e., the identity map id_K is homotopic to the constant map p. □

13.2 Busemann-type inequalities on Finsler manifolds

In the forties, Busemann developed a synthetic geometry on metric spaces. In particular, he axiomatically elaborated a whole theory of non-positively curved metric spaces which have no differential structure a priori and they possess the essential qualitative geometric properties of Finsler manifolds. These spaces are the so-called *G-spaces*, see Busemann [54, p. 37]. This notion of non-positive curvature requires that in small geodesic triangles the length of a side is at least the twice of the geodesic distance of the mid-points of the other two sides, see Busemann [54, p. 237].

To formulate in a precise way this notion, let (M,d) be a quasi-metric space and for every $p \in M$ and radius $r > 0$, we introduce the *forward* and *backward metric ball*s

$$B_p^+(r) = \{q \in M : d(p,q) < r\} \quad \text{and} \quad B_p^-(r) = \{q \in M : d(q,p) < r\}.$$

A continuous curve $\gamma : [a,b] \to M$ with $\gamma(a) = x$, $\gamma(b) = y$ is a *shortest* *geodesic*, if $l(\gamma) = d(x, y)$, where $l(\gamma)$ denotes the *generalized length* of γ and it is defined by

$$l(\gamma) = \sup\{\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : a = t_0 < t_1 < ... < t_n = b,\ n \in \mathbb{N}\}.$$

In the sequel, we always assume that the shortest geodesics are parametrized proportionally to arclength, i.e., $l(\gamma|_{[0,t]}) = tl(\gamma)$.

Remark 13.1 A famous result of Busemann-Meyer (see [55, Theorem 2, p. 186]) from Calculus of Variations shows that the generalized length $l(\gamma)$ and the integral length $L_F(\gamma)$ of any (piecewise) C^∞ curves coincide for Finsler manifolds. Therefore, the minimal Finsler geodesic and shortest geodesic notions coincide.

We say that (M, d) is a *locally geodesic* (*length*) *space* if for every point $p \in M$ there is a $\rho_p > 0$ such that for every two points $x, y \in B_p^+(\rho_p)$ there exists a shortest geodesic joining them.

Definition 13.1 A locally geodesic space (M, d) is said to be a Busemann non-positive curvature space (shortly, Busemann NPC space), if for every $p \in M$ there exists $\rho_p > 0$ such that for any two shortest geodesics $\gamma_1, \gamma_2 : [0, 1] \to M$ with $\gamma_1(0) = \gamma_2(0) = x \in B_p^+(\rho_p)$ and with endpoints $\gamma_1(1), \gamma_2(1) \in B_p^+(\rho_p)$ we have

$$2d(\gamma_1(\frac{1}{2}), \gamma_2(\frac{1}{2})) \leq d(\gamma_1(1), \gamma_2(1)).$$

(We shall say that γ_1 and γ_2 satisfy the Busemann NPC inequality).

Let (M, g) be a Riemannian manifold and (M, d_g) the metric space induced by itself. In this context, the Busemann NPC inequality is well-known. Namely, we have

Proposition 13.3 [54, Theorem (41.6)] *(M, d_g) is a Busemann non-positive curvature space if and only if the sectional curvature of (M, g) is non-positive.*

However, the picture for Finsler spaces is not so nice as in Proposition 13.3 for Riemannian manifolds. To see this, we consider the Hilbert metric of the interior of a simple, closed and convex curve C in the Euclidean plane. In order to describe this metric, let $M_C \subset \mathbb{R}^2$ be the region defined by the interior of the curve C and fix $x_1, x_2 \in \text{Int}(M_C)$. Assume first that $x_1 \neq x_2$. Since C is a convex curve, the straight line passing to the points x_1, x_2 intersects the curve C in two point; denote them by $u_1, u_2 \in C$. Then, there are $\tau_1, \tau_2 \in (0,1)$ such that $x_i = \tau_i u_1 + (1-\tau_i)u_2$ $(i = 1,2)$.

The *Hilbert distance* between x_1 and x_2 is

$$d_H(x_1, x_2) = \left| \log \left(\frac{1-\tau_1}{1-\tau_2} \cdot \frac{\tau_2}{\tau_1} \right) \right|.$$

We complete this definition by $d_H(x,x) = 0$ for every $x \in \mathrm{Int}(M_C)$. One can easily prove that $(\mathrm{Int}(M_C), d_H)$ is a metric space and it is a projective Finsler metric with constant flag curvature -1. However, due to Kelly-Straus, we have

Proposition 13.4 (see [156]) *The metric space* $(\mathrm{Int}(M_C), d_H)$ *is a Busemann non-positive curvature space if and only if the curve* $C \subset \mathbb{R}^2$ *is an ellipse.*

This means that, although for Riemannian spaces the non-positivity of the sectional curvature and Busemann's curvature conditions are mutually equivalent, the non-positivity of the flag curvature of a *generic* Finsler manifold is not enough to guarantee Busemann's property.

Therefore, in order to obtain a characterization of Busemann's curvature condition for Finsler spaces, we have two possibilities:

(I) To find a *new* notion of curvature in Finsler geometry such that for an arbitrary Finsler manifold the non-positivity of this curvature is equivalent with the Busemann non-positive curvature condition, as it was proposed by Z. Shen, see [276, Open Problem 41]; or,

(II) To keep the flag curvature, but put some restrictive condition on the Finsler metric.

In spite of the fact that (reversible) Finsler manifolds are included in G-spaces, only few results are known which establish a link between the *differential invariants* of a Finsler manifold and the *metric properties* of the induced metric space. The main result of this section is due Kristály-Kozma [170] (see also Kristály-Varga-Kozma [179]), which makes a strong connection between an analytical property and a synthetic concept of non-positively curved metric spaces. Namely, we have

Theorem 13.4 *Let* (M, F) *be a Berwald space with non-positive flag curvature, where* F *is positively* (*but perhaps not absolutely*) *homogeneous of degree one. Then* (M, d_F) *is a Busemann NPC space.*

Proof Let us fix $p \in M$ and consider $\rho_p > 0$, $c_p > 1$ from Proposition 13.1. We will prove that $\rho'_p = \frac{\rho_p}{c_p}$ is a good choice in Definition 13.1. To do this, let $\gamma_1, \gamma_2 : [0,1] \to M$ be two (minimal) geodesics with $\gamma_1(0) = \gamma_2(0) = x \in \mathcal{B}_p^+(\rho'_p)$ and $\gamma_1(1), \gamma_2(1) \in \mathcal{B}_p^+(\rho'_p)$. By Proposition 13.1, we can construct a unique geodesic $\gamma : [0,1] \to M$ joining $\gamma_1(1)$ with $\gamma_2(1)$ and $d_F(\gamma_1(1), \gamma_2(1)) = L(\gamma)$. Clearly, $\gamma(s) \in \mathcal{B}_p^+(\rho'_p)$ for all $s \in [0,1]$ (we applied Proposition 13.1 for $k = c_p$). Moreover, $x \in \mathcal{B}^+_{\gamma(s)}(2\rho_p)$. Indeed, by (13.9), we obtain

$$d_F(\gamma(s), x) \leq d_F(\gamma(s), p) + d_F(p, x) \leq c_p d_F(p, \gamma(s)) + \rho'_p \leq (c_p+1)\rho'_p < 2\rho_p.$$

Therefore, we can define $\Sigma : [0,1] \times [0,1] \to M$ by

$$\Sigma(t,s) = \exp_{\gamma(s)}((1-t) \cdot \exp^{-1}_{\gamma(s)}(x)).$$

The curve $t \mapsto \Sigma(1-t, s)$ is a radial geodesic which joins $\gamma(s)$ with x. Taking into account that (M, F) is of Berwald type, the reverse of $t \mapsto \Sigma(1-t, s)$, i.e. $t \mapsto \Sigma(t,s)$ is a geodesic too (see [22, Exercise 5.3.3, p. 128]) for all $s \in [0,1]$. Moreover, $\Sigma(0,0) = x = \gamma_1(0)$, $\Sigma(1,0) = \gamma(0) = \gamma_1(1)$. From the uniqueness of the geodesic between x and $\gamma_1(1)$, we have $\Sigma(\cdot\,, 0) = \gamma_1$. Analogously, we have $\Sigma(\cdot\,, 1) = \gamma_2$. Since Σ is a geodesic variation (of the curves γ_1 and γ_2), the vector field J_s, defined by

$$J_s(t) = \frac{\partial}{\partial s}\Sigma(t,s) \in T_{\Sigma(t,s)}M$$

is a Jacobi field along $\Sigma(\cdot\,, s)$, $s \in [0,1]$ (see [22, p. 130]). In particular, we have $\Sigma(1,s) = \gamma(s)$, $J_s(0) = 0$, $J_s(1) = \frac{\partial}{\partial s}\Sigma(1,s) = \frac{d\gamma}{ds}$ and $J_s(\frac{1}{2}) = \frac{\partial}{\partial s}\Sigma(\frac{1}{2}, s)$.

Now, we fix $s \in [0,1]$. Since $J_s(0) = 0$ and the flag curvature in non-positive, then the geodesic $\Sigma(\cdot, s)$ has no conjugated points, see Theorem 13.2. Therefore,

$$J_s(t) \neq 0 \quad \text{for all} \;\; t \in (0,1].$$

Hence $g_{J_s}(J_s, J_s)(t)$ is well defined for every $t \in (0,1]$. Moreover,

$$F(J_s)(t) := F(\Sigma(t,s), J_s(t)) = [g_{J_s}(J_s, J_s)]^{\frac{1}{2}}(t) \neq 0 \qquad \forall\, t \in (0,1]. \tag{13.12}$$

Let T_s the velocity field of $\Sigma(\cdot, s)$. Applying twice formula (13.4), we obtain

$$\frac{d^2}{dt^2}[g_{J_s}(J_s,J_s)]^{\frac{1}{2}}(t) = \frac{d^2}{dt^2}F(J_s)(t) = \frac{d}{dt}\left[\frac{g_{J_s}(D_{T_s}J_s,J_s)}{F(J_s)}\right](t) =$$

$$\frac{[g_{J_s}(D_{T_s}D_{T_s}J_s,J_s)+g_{J_s}(D_{T_s}J_s,D_{T_s}J_s)]\cdot F(J_s) - g_{J_s}^2(D_{T_s}J_s,J_s)\cdot F(J_s)^{-1}}{F^2(J_s)}(t) =$$

$$\frac{g_{J_s}(D_{T_s}D_{T_s}J_s,J_s)\cdot F^2(J_s) + g_{J_s}(D_{T_s}J_s,D_{T_s}J_s)\cdot F^2(J_s) - g_{J_s}^2(D_{T_s}J_s,J_s)}{F^3(J_s)}(t),$$

where the covariant derivatives (for generic Finsler manifolds) are with reference vector J_s. Since (M, F) is a Berwald space, the Chern connection coefficients do not depend on the direction, i.e., the notion of reference vector becomes irrelevant. Therefore, we can use the Jacobi equation (13.6), concluding that

$$g_{J_s}(D_{T_s}D_{T_s}J_s,J_s) = -g_{J_s}(R(J_s,T_s)T_s,J_s).$$

Using the symmetry property of the curvature tensor, the formula of the flag curvature, and the Schwarz inequality we have

$$\begin{aligned} -g_{J_s}(R(J_s,T_s)T_s,J_s) &= -g_{J_s}(R(T_s,J_s)J_s,T_s) \\ &= -K(J_s,T_s)\cdot[g_{J_s}(J_s,J_s)g_{J_s}(T_s,T_s) - g_{J_s}^2(J_s,T_s)] \geq 0. \end{aligned}$$

For the last two terms of the numerator we apply again the Schwarz inequality and we conclude that

$$\frac{d^2}{dt^2}F(J_s)(t) \geq 0 \quad \text{for all } t \in (0,1].$$

Since $J_s(t) \neq 0$ for $t \in (0,1]$, the mapping $t \mapsto F(J_s)(t)$ is C^∞ on $(0,1]$. From the above inequality and the second order Taylor expansion about $v \in (0,1]$, we obtain

$$F(J_s)(v) + (t-v)\frac{d}{dt}F(J_s)(v) \leq F(J_s)(t) \quad \text{for all } t \in (0,1]. \tag{13.13}$$

Letting $t \to 0$ and $v = 1/2$ in (13.13), by the continuity of F, we obtain

$$F(J_s)(\frac{1}{2}) - \frac{1}{2}\frac{d}{dt}F(J_s)(\frac{1}{2}) \leq 0.$$

Let $v = 1/2$ and $t = 1$ in (13.13), and adding the obtained inequality with the above one, we conclude that

$$2F(\Sigma(\frac{1}{2},s),\frac{\partial}{\partial s}\Sigma(\frac{1}{2},s)) = 2F(J_s)(\frac{1}{2}) \leq F(J_s)(1) = F(\gamma(s),\frac{d\gamma}{ds}).$$

Integrating the last inequality with respect to s from 0 to 1, we obtain

$$\begin{aligned} 2L_F(\Sigma(\frac{1}{2},\cdot)) &= 2\int_0^1 F(\Sigma(\frac{1}{2},s),\frac{\partial}{\partial s}\Sigma(\frac{1}{2},s))\,ds \\ &\leq \int_0^1 F(\gamma(s),\frac{d\gamma}{ds})\,ds \\ &= L_F(\gamma) \\ &= d_F(\gamma_1(1),\gamma_2(1)). \end{aligned}$$

Since $\Sigma(\frac{1}{2},0)=\gamma_1(\frac{1}{2}),\Sigma(\frac{1}{2},1)=\gamma_2(\frac{1}{2})$ and $\Sigma(\frac{1}{2},\cdot)$ is a C^∞ curve, by the definition of the metric function d_F, we conclude that γ_1 and γ_2 satisfy the Busemann NPC inequality. This concludes the proof of Theorem 13.4. □

In view of Theorem 13.4, Berwald spaces seem to be the first class of Finsler metrics that are non-positively curved in the sense of Busemann and which are neither flat nor Riemannian. Moreover, the above result suggests a full characterization of the Busemann curvature notion for Berwald spaces. Indeed, we refer the reader to Kristály-Kozma [170] where the converse of Theorem 13.4 is also proved; here we omit this technical part since only the above result is applied for Economical problems.

Note that Theorem 13.4 includes a partial answer to the question of Busemann [54] (see also [235, p. 87]), i.e., every reversible Berwald space of non-positive flag curvature has convex capsules (i.e., the loci equidistant to geodesic segments).

In the fifties, Aleksandrov introduced independently an other notion of curvature in metric spaces, based on the convexity of the distance function. It is well-known that the condition of Busemann curvature is weaker than the Aleksandrov one, see [150, Corollary 2.3.1]. Nevertheless, in Riemannian spaces the Aleksandrov curvature condition holds if and only if the sectional curvature is non-positive (see [52, Theorem 1A.6]), but in the Finsler case the picture is quite rigid. Namely, if on a reversible Finsler manifold (M,F) the Aleksandrov curvature condition holds (on the induced metric space by (M,F)) then (M,F) it must be Riemannian, see [52, Proposition 1.14].

A direct consequence of Theorem 13.4 is

Corollary 13.1 *Let (M,F) be a forward/backward geodesically complete, simply connected Berwald space with non-positive flag curvature,* *where F is positively (but perhaps not absolutely) homogeneous of degree one. Then (M, d_F) is a global Busemann NPC space, i.e., the Busemann NPC inequality holds for any pair of geodesics.*

The following result is crucial in Chapters 14 and 15.

Proposition 13.5 *Let (M, F) be a forward/backward geodesically complete, simply connected Berwald space with non-positive flag curvature, where F is positively (but perhaps not absolutely) homogeneous of degree one. Fix two geodesics $\gamma_1, \gamma_2 : [0,1] \to M$. Then, the function $t \mapsto d_F(\gamma_1(t), \gamma_2(t))$ is convex.*

Proof Due to Hopf-Rinow theorem (see Theorem 13.1), there exists a geodesic $\gamma_3 : [0,1] \to M$ joining $\gamma_1(0)$ and $\gamma_2(1)$. Moreover, due to Cartan-Hadamard theorem (see Theorem 13.2), γ_3 is unique. Applying Corollary 13.1 first to the pair γ_1, γ_3 and then to the pair γ_3, γ_2 (with opposite orientation), we obtain

$$d_F(\gamma_1(\frac{1}{2}), \gamma_3(\frac{1}{2})) \leq \frac{1}{2} d_F(\gamma_1(1), \gamma_3(1));$$

$$d_F(\gamma_3(\frac{1}{2}), \gamma_2(\frac{1}{2})) \leq \frac{1}{2} d_F(\gamma_3(0), \gamma_2(0)).$$

Note that the opposite of γ_3 and γ_2 are also geodesics, since (M, F) is of Berwald type, see [22, Example 5.3.3]. Now, using the triangle inequality, we obtain

$$d_F(\gamma_1(\frac{1}{2}), \gamma_2(\frac{1}{2})) \leq \frac{1}{2} d_F(\gamma_1(1), \gamma_2(1)) + \frac{1}{2} d_F(\gamma_1(0), \gamma_2(0)),$$

which means actually the $\frac{1}{2}$-convexity of the function $t \mapsto d_F(\gamma_1(t), \gamma_2(t))$. Continuing in this way and taking a limit if necessarily, we conclude the convexity of the above function. □

13.3 Variational inequalities

Existence results for Nash equilibria are often derived from intersection theorems (KKM theorems) or fixed point theorems. For instance, the original proof of Nash concerning equilibrium point is based on the Brouwer fixed point theorem. These theorems are actually equivalent to minimax theorems or variational inequalities, as Ky Fan minimax theorem, etc. In this section we recall a few results which will be used in Chapter 16.

A nonempty set X is *acyclic* if it is connected and its Čech homology (coefficients in a fixed field) is zero in dimensions greater than zero. Note that every contractible set is acyclic (but the converse need not holds in general). The following result is a Ky Fan type minimax theorem, proved by McClendon:

Theorem 13.5 [207, Theorem 3.1] *Suppose that X is a compact acyclic finite-dimensional ANR. Suppose $h : X \times X \to \mathbb{R}$ is a function such that $\{(x, y) : h(y, y) > h(x, y)\}$ is open and $\{x : h(y, y) > h(x, y)\}$ is contractible or empty for all $y \in X$. Then there is a $y_0 \in X$ with $h(y_0, y_0) \leq h(x, y_0)$ for all $x \in X$.*

In the sequel, we state some well-known results from the theory of variational inequalities. To do this, we consider a Gâteaux differentiable function $f : K \to \mathbb{R}$ where K is a closed convex subset of the topological vector space X.

Lemma 13.1 *Let $x_0 \in K$ be a relative minimum point of f to K, i.e., $f(x) \geq f(x_0)$ for every $x \in K$. Then,*

$$f'(x_0)(x - x_0) \geq 0, \quad \forall x \in K. \tag{13.14}$$

Furthermore, if f is convex, then the converse also holds.

Lemma 13.2 *Let $x_0 \in K$ such that*

$$f'(x)(x - x_0) \geq 0, \quad \forall x \in K. \tag{13.15}$$

Then, $x_0 \in K$ is a relative minimum point of f to K. Furthermore, if f is convex, then the converse also holds.

Note that if we replace f' by an operator $A : X \to X^*$, then (13.14) appearing in Lemma 13.1 is called a Stampacchia-type variational inequality, while (13.15) from Lemma 13.2 is a Minty-type variational inequality.

14
Minimization of Cost-functions on Manifolds

> Geography has made us neighbors. History has made us friends. Economics has made us partners, and necessity has made us allies. Those whom God has so joined together, let no man put asunder.
>
> John F. Kennedy (1917–1963)

14.1 Introduction

Let us consider three markets P_1, P_2, P_3 placed on an inclined plane (slope) with an angle α to the horizontal plane, denoted by (S_α). Assume that three cars transport products from (resp. to) deposit $P \in (S_\alpha)$ to (resp. from) markets $P_1, P_2, P_3 \in (S_\alpha)$ such that

- they move always in (S_α) along straight roads;
- the Earth gravity acts on them (we omit other physical perturbations such as friction, air resistance, etc.);
- the transport costs coincide with the *distance* (measuring actually the *time* elapsed to arrive) from (resp. to) deposit P to (resp. from) markets P_i $(i = 1, 2, 3)$.

We emphasize that usually the two distances, i.e., from the deposit to the markets and conversely, are *not* the same. The point here is that the travel speed depends heavily on both the slope of the terrain and the direction of travel. More precisely, if a car moves with a constant speed v $[m/s]$ on a horizontal plane, it goes $l_t = vt + \frac{g}{2}t^2 \sin\alpha \cos\theta$ meters in

t seconds on (S_α), where θ is the angle between the straight road and the direct downhill road (θ is measured in clockwise direction). The law of the above phenomenon can be described relatively to the horizontal plane by means of the parametrized function

$$F_\alpha(y_1, y_2) = \frac{y_1^2 + y_2^2}{v\sqrt{y_1^2 + y_2^2} + \frac{g}{2} y_1 \sin\alpha}, \qquad (y_1, y_2) \in \mathbb{R}^2 \setminus \{(0,0)\}. \quad (14.1)$$

Here, $g \approx 9.81 m/s^2$. The *distance* (measuring the *time* to arrive) from $P = (P^1, P^2)$ to $P_i = (P_i^1, P_i^2)$ is

$$d_\alpha(P, P_i) = F_\alpha(P_i^1 - P^1, P_i^2 - P^2),$$

and for the converse it is

$$d_\alpha(P_i, P) = F_\alpha(P^1 - P_i^1, P^2 - P_i^2).$$

Consequently, we have to minimize the functions

$$C_f(P) = \sum_{i=1}^{3} d_\alpha(P, P_i) \quad \text{and} \quad C_b(P) = \sum_{i=1}^{3} d_\alpha(P_i, P), \qquad (14.2)$$

when P moves on (S_α). The function C_f (resp. C_b) denotes the *total forward* (*resp. backward*) *cost* between the deposit $P \in (S_\alpha)$ and markets $P_1, P_2, P_3 \in (S_\alpha)$. The minimum points of C_f and C_b, respectively, may be far from each other (see Figure 14.1), due to the fact that F_α (and d_α) is not symmetric unless $\alpha = 0$, i.e., $F_\alpha(-y_1, -y_2) \neq F_\alpha(y_1, y_2)$ for each $(y_1, y_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

We will use in general T_f (resp. T_b) to denote a minimum point of C_f (resp. C_b), which corresponds to the position of a deposit when we measure costs in forward (resp. backward) manner, see (14.2).

In the case $\alpha = 0$ (when (S_α) is a horizontal plane), the functions C_f and C_b coincide (the same is true for T_f and T_b). The minimum point $T = T_f = T_b$ is the well-known *Torricelli point* corresponding to the triangle $P_1P_2P_{3\Delta}$. Note that $F_0(y_1, y_2) = \sqrt{y_1^2 + y_2^2}/v$ corresponds to the standard Euclidean metric; indeed,

$$d_0(P, P_i) = d_0(P_i, P) = \sqrt{(P_i^1 - P^1)^2 + (P_i^2 - P^2)^2}/v$$

measures the time, which is needed to arrive from P to P_i (and vice-versa) with constant velocity v.

Unfortunately, finding critical points as possible minima does not yield any result: either the minimization function is not smooth enough (usually, it is only a locally Lipschitz function) or the system, which would

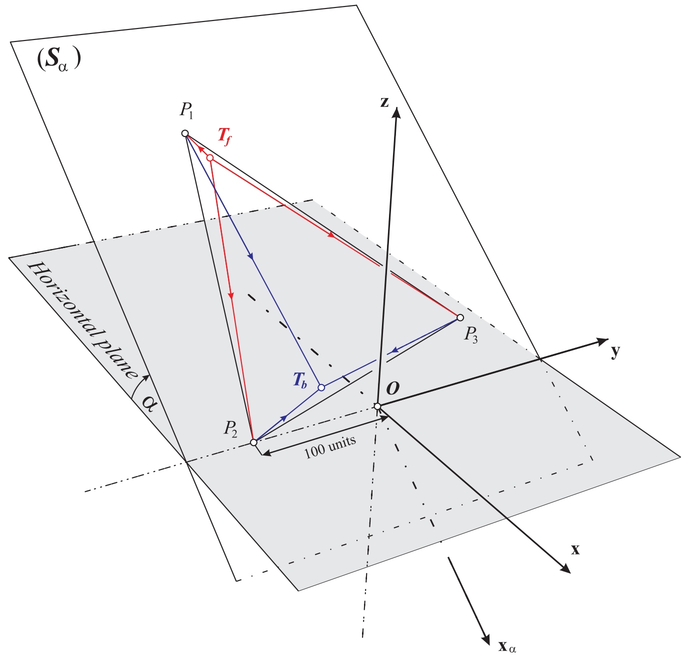

Fig. 14.1. We fix $P_1 = (-250, -50)$, $P_2 = (0, -100)$ and $P_3 = (-50, 100)$ on the slope (S_α) with angle $\alpha = 35^o$. If $v = 10$, the minimum of the total forward cost on the slope is $C_f \approx 40.3265$; the corresponding deposit is located at $T_f \approx (-226.11, -39.4995) \in (S_\alpha)$. However, the minimum of the total backward cost on the slope is $C_b \approx 38.4143$; the corresponding deposit has the coordinates $T_b \approx (-25.1332, -35.097) \in (S_\alpha)$.

give the critical points, becomes very complicated even in quite simple cases (see (14.5) below). Consequently, the main purpose of the present chapter is to study the set of these minima (existence, location) in various geometrical settings.

Note, that the function appearing in (14.1) is a typically Finsler metric on $\mathbb{R}^2$, introduced and studied first by Matsumoto [204]. In this way, elements from Riemann-Finsler geometry are needed in order to handle the question formulated above. In the next sections we prove some necessarily, existence, uniqueness and multiplicity results for the economical problem on non-positively curved Berwald space which model various real life phenomena. Simultaneously, relevant numerical examples and counterexamples are constructed by means of evolutionary methods and computational geometry tools, emphasizing the applicability and sharpness of our results.

14.2 A necessary condition

Let (M, F) be an m-dimensional connected Finsler manifold, where F is positively (but perhaps not absolutely) homogeneous of degree one.

In this section we prove some results concerning the set of minima for functions

$$C_f(P_i, n, s)(P) = \sum_{i=1}^{n} d_F^s(P, P_i) \quad \text{and} \quad C_b(P_i, n, s)(P) = \sum_{i=1}^{n} d_F^s(P_i, P),$$

where $s \geq 1$ and $P_i \in M$, $i = 1, \ldots, n$, correspond to $n \in \mathbb{N}$ markets. The value $C_f(P_i, n, s)(P)$ (resp. $C_b(P_i, n, s)(P)$) denotes the *total s-forward* (*resp. s-backward*) *cost* between the deposit $P \in M$ and the markets $P_i \in M$, $i = 1, ..., n$. When $s = 1$, we simply say *total forward* (*resp. backward*) *cost.*

By using the triangle inequality, for every $x_0, x_1, x_2 \in M$ we have

$$|d_F(x_1, x_0) - d_F(x_2, x_0)| \leq \max\{d_F(x_1, x_2), d_F(x_2, x_1)\}. \tag{14.3}$$

Given any point $P \in M$, there exists a coordinate map φ_P defined on the closure of some precompact open subset U containing P such that φ_P maps the set U diffeomorphically onto the open Euclidean ball $B^m(r)$, $r > 0$, with $\varphi_P(P) = 0_{\mathbb{R}^m}$. Moreover, there is a constant $c > 1$, depending only on P and U such that

$$c^{-1}\|\varphi_P(x_1) - \varphi_P(x_2)\| \leq d_F(x_1, x_2) \leq c\|\varphi_P(x_1) - \varphi_P(x_2)\| \tag{14.4}$$

for every $x_1, x_2 \in U$; see [22, p. 149]. Here, $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^m$. We claim that for every $Q \in M$, the function $d_F(\varphi_P^{-1}(\cdot), Q)$ is a Lipschitz function on $\varphi_P(U) = B^m(r)$. Indeed, for every $y_i = \varphi_P(x_i) \in \varphi_P(U)$, $i = 1, 2$, due to (14.3) and (14.4), one has

$$|d_F(\varphi_P^{-1}(y_1), Q) - d_F(\varphi_P^{-1}(y_2), Q)| = |d_F(x_1, Q) - d_F(x_2, Q)| \leq$$

$$\leq \max\{d_F(x_1, x_2), d_F(x_2, x_1)\} \leq c\|y_1 - y_2\|.$$

Consequently, for every $Q \in M$, there exists the *generalized gradient* of the locally Lipschitz function $d_F(\varphi_P^{-1}(\cdot), Q)$ on $\varphi_P(U) = B^m(r)$, see Clarke [71, p. 27], i.e., for every $y \in \varphi_P(U) = B^m(r)$ we have

$$\partial d_F(\varphi_P^{-1}(\cdot), Q)(y) = \{\xi \in \mathbb{R}^m : d_F^0(\varphi_P^{-1}(\cdot), Q)(y; h) \geq \langle \xi, h \rangle \ \text{ for all } h \in \mathbb{R}^m\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^m$ and

$$d_F^0(\varphi_P^{-1}(\cdot), Q)(y; h) = \limsup_{z \to y,\ t \to 0^+} \frac{d_F(\varphi_P^{-1}(z + th), Q) - d_F(\varphi_P^{-1}(z), Q)}{t}$$

is the *generalized directional derivative.*

Theorem 14.1 (Necessary Condition) *Assume that $T_f \in M$ is a minimum point for $C_f(P_i, n, s)$ and φ_{T_f} is a map as above. Then*

$$0_{\mathbb{R}^m} \in \sum_{i=1}^{n} d_F^{s-1}(T_f, P_i)\partial d_F(\varphi_{T_f}^{-1}(\cdot), P_i)(\varphi_{T_f}(T_f)). \tag{14.5}$$

Proof Since $T_f \in M$ is a minimum point of the locally Lipschitz function $C_f(P_i, n, s)$, then

$$0_{\mathbb{R}^m} \in \partial\left(\sum_{i=1}^{n} d_F^s(\varphi_{T_f}^{-1}(\cdot), P_i)\right)(\varphi_{T_f}(T_f)),$$

see [71, Proposition 2.3.2]. Now, using the basic properties of the generalized gradient, see [71, Proposition 2.3.3] and [71, Theorem 2.3.10], we conclude the proof. □

Remark 14.1 A result similar to Theorem 14.1 can also be obtained for $C_b(P_i, n, s)$.

Example 14.1 Let $M = \mathbb{R}^m$, $m \geq 2$, be endowed with the natural Euclidean metric. Taking into account (14.5), a simple computation shows that the unique minimum point $T_f = T_b$ (i.e., the place of the deposit) for $C_f(P_i, n, 2) = C_b(P_i, n, 2)$ is the centre of gravity of markets $\{P_1, ..., P_n\}$, i.e., $\frac{1}{n}\sum_{i=1}^{n} P_i$. In this case, φ_{T_f} can be the identity map on $\mathbb{R}^m$.

Remark 14.2 The system (14.5) may become very complicate even for simple cases; it is enough to consider the Matsumoto metric given by (14.1). In such cases, we are not able to give an explicit formula for minimal points.

14.3 Existence and uniqueness results

The next result gives an alternative concerning the number of minimum points of the function $C_f(P_i, n, s)$ in a general geometrical framework. (Similar result can be obtained for $C_b(P_i, n, s)$.) Namely, we have the following theorem.

Theorem 14.2 *Let (M, F) be a simply connected, geodesically complete Berwald manifold of nonpositive flag curvature, where F is positively (but perhaps not absolutely) homogeneous of degree one. Then*

(a) *there exists either a unique or infinitely many minimum points for $C_f(P_i, n, 1)$;*
(b) *there exists a unique minimum point for $C_f(P_i, n, s)$ whenever $s > 1$.*

Proof First of all, we observe that M is not a backward bounded set. Indeed, if we assume that it is, then M is compact due to Hopf-Rinow theorem, see Theorem 13.1. On the other hand, due Cartan-Hadamard theorem, see Theorem 13.2, the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism for every $p \in M$. Thus, the tangent space $T_pM = \exp_p^{-1}(M)$ is compact, a contradiction. Since M is not backward bounded, in particular, for every $i = 1, ..., n$, we have that

$$\sup_{P\in M} d_F(P, P_i) = \infty.$$

Consequently, outside of a large backward bounded subset of M, denoted by M_0, the value of $C_f(P_i, n, s)$ is large. But, M_0 being compact, the continuous function $C_f(P_i, n, s)$ attains its infimum, i.e., the set of the minima for $C_f(P_i, n, s)$ is always nonempty.

On the other hand, due to Proposition 13.5 for every nonconstant geodesic $\sigma : [0, 1] \to M$ and $p \in M$, the function $t \mapsto d_F(\sigma(t), p)$ is convex and $t \mapsto d_F^s(\sigma(t), p)$ is strictly convex, whenever $s > 1$ (see also [150, Corollary 2.2.6]).

(a) Let us assume that there are at least two minimum points for $C_f(P_i, n, 1)$, denoting them by T_f^0 and T_f^1. Let $\sigma : [0, 1] \to M$ be a geodesic with constant Finslerian speed such that $\sigma(0) = T_f^0$ and $\sigma(1) = T_f^1$. Then, for every $t \in (0, 1)$ we have

$$\begin{aligned} C_f(P_i, n, 1)(\sigma(t)) &= \sum_{i=1}^n d_F(\sigma(t), P_i) \\ &\leq (1-t)\sum_{i=1}^n d_F(\sigma(0), P_i) + t\sum_{i=1}^n d_F(\sigma(1), P_i) \qquad (14.6) \\ &= (1-t)\min C_f(P_i, n, 1) + t\min C_f(P_i, n, 1) \\ &= \min C_f(P_i, n, 1). \end{aligned}$$

Consequently, for every $t \in [0, 1]$, $\sigma(t) \in M$ is a minimum point for $C_f(P_i, n, 1)$.

(b) It follows directly from the strict convexity of the function $t \mapsto d_F^s(\sigma(t), p)$, whenever $s > 1$; indeed, in (14.6) we have $<$ instead of $\leq$ which shows we cannot have more then one minimum point for $C_f(P_i, n, s)$. □

Example 14.2 Let F be the Finsler metric introduced in (14.1). One can see that $(\mathbb{R}^2, F)$ is a typically nonsymmetric Finsler manifold. Actually, it is a (locally) Minkowski space, so a Berwald space as well; its Chern connection vanishes, see [22, p. 384]. According to (13.3) and (13.10), the geodesics are straight lines (hence $(\mathbb{R}^2, F)$ is geodesically complete in both sense) and the flag curvature is identically 0. Thus, we can apply Theorem 14.2. For instance, if we consider the points $P_1 = (a, -b) \in \mathbb{R}^2$ and $P_2 = (a, b) \in \mathbb{R}^2$ with $b \neq 0$, the minimum points of the function $C_f(P_i, 2, 1)$ form the segment $[P_1, P_2]$, *independently* of the value of α. The same is true for $C_b(P_i, 2, 1)$. However, considering more complicated constellations, the situation changes dramatically, see Figure 14.2.

It would be interesting to study in similar cases the precise orbit of the (Torricelli) points T_f^α and T_b^α when α varies from 0 to $\pi/2$. Several numerical experiments show that T_f^α tends to a top point of the convex polygon (as in the Figure 14.2).

In the sequel, we want to study our problem in a special constellation: we assume the markets are situated on a common "straight line", i.e., on a geodesic which is in a Riemannian manifold. Note that, in the Riemannian context, the forward and backward costs coincide, i.e.,

$$C_f(P_i, n, 1) = C_b(P_i, n, 1).$$

We denote this common value by $C(P_i, n, 1)$. We have

Theorem 14.3 *Let (M, g) be a Hadamard-type Riemannian manifold. Assume the points $P_i \in M$, $i = 1, ..., n$, $(n \geq 2)$, belong to a geodesic $\sigma : [0, 1] \to M$ such that $P_i = \sigma(t_i)$ with $0 \leq t_1 < ... < t_n \leq 1$. Then*

(a) *the unique minimum point for $C(P_i, n, 1)$ is $P_{[n/2]}$ whenever n is odd;*

(b) *the minimum points for $C(P_i, n, 1)$ is the whole geodesic segment situated on σ between $P_{n/2}$ and $P_{n/2+1}$ whenever n is even.*

Proof Since (M, g) is complete, we extend σ to $(-\infty, \infty)$, keeping the same notation. First, we prove that the minimum point(s) for $C(P_i, n, 1)$

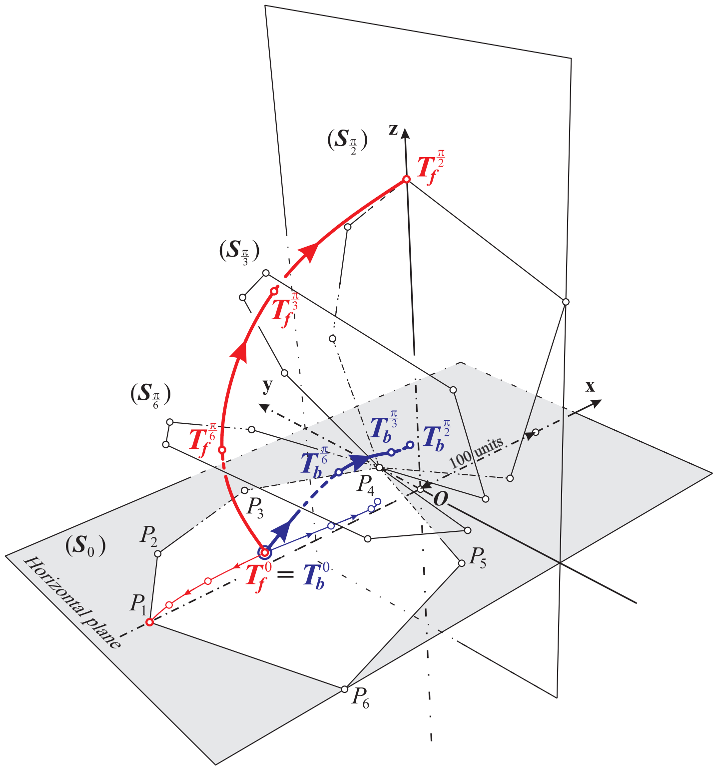

Fig. 14.2. A hexagon with vertices $P_1, P_2, ..., P_6$ in the Matsumoto space. Increasing the slope's angle α from 0 to $\pi/2$, points T_f^α and T_b^α are wandering in the presented directions. Orbits of points T_f^α and T_b^α were generated by natural cubic spline curve interpolation.

belong to the geodesic σ. We assume the contrary, i.e., let $T \in M \setminus \mathrm{Image}(\sigma)$ be a minimum point of $C(P_i, n, 1)$. Let $T_\perp \in \mathrm{Image}(\sigma)$ be the projection of T on the geodesic σ, i.e.

$$d_g(T, T_\perp) = \min_{t \in \mathbb{R}} d_g(T, \sigma(t)).$$

It is clear that the (unique) geodesic lying between T and $T_\perp$ is perpendicular to σ with respect to the Riemannian metric g.

Let $i_0 \in \{1, ..., n\}$ such that $P_{i_0} \neq T_\perp$. Applying the cosine inequality, see Theorem 13.3 (a), for the triangle with vertices P_{i_0}, T and $T_\perp$ (so, $\widehat{T_\perp} = \pi/2$), we have

$$d_g^2(T_\perp, T) + d_g^2(T_\perp, P_{i_0}) \leq d_g^2(T, P_{i_0}).$$

Since

$$d_g(T_\perp, T) > 0,$$

we have

$$d_g(T_\perp, P_{i_0}) < d_g(T, P_{i_0}).$$

Consequently,

$$C(P_i, n, 1)(T_\perp) = \sum_{i=1}^{n} d_g(T_\perp, P_i) < \sum_{i=1}^{n} d_g(P, P_i) = \min C(P_i, n, 1),$$

a contradiction. Now, conclusions (a) and (b) follow easily by using simple arithmetical reasons. □

14.4 Examples on the Finslerian-Poincaré disc

We emphasize that Theorem 14.3 is sharp in the following sense: neither the nonpositivity of the sectional curvature (see Example 14.3) nor the Riemannian structure (see Example 14.4) can be omitted.

Example 14.3 (Sphere) Let us consider the 2-dimensional unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ endowed with its natural Riemannian metric h inherited by $\mathbb{R}^3$. We know that it has constant curvature 1. Let us fix $P_1, P_2 \in \mathbb{S}^2$ ($P_1 \neq P_2$) and their antipodals $P_3 = -P_1$, $P_4 = -P_2$. There exists a unique great circle (geodesic) connecting P_i, $i = 1, ..., 4$. However, we observe that the function $C(P_i, 4, 1)$ is *constant* on $\mathbb{S}^2$; its value is 2π. Consequently, *every* point on $\mathbb{S}^2$ is a minimum for the function $C(P_i, 4, 1)$.

Example 14.4 (Finslerian-Poincaré disc) Let us consider the disc

$$M = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}.$$

Introducing the polar coordinates (r, θ) on M, i.e., $x = r\cos\theta$, $y = r\sin\theta$, we define the non-reversible Finsler metric on M by

$$F((r, \theta), V) = \frac{1}{1 - \frac{r^2}{4}} \sqrt{p^2 + r^2 q^2} + \frac{pr}{1 - \frac{r^4}{16}},$$

where

$$V = p\frac{\partial}{\partial r} + q\frac{\partial}{\partial \theta} \in T_{(r,\theta)}M.$$

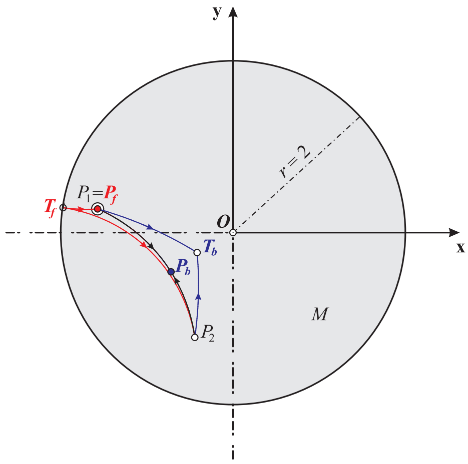

Fig. 14.3.
Step 1: The minimum of the total backward (resp. forward) cost function $C_b(P_i, 2, 1)$ (resp. $C_f(P_i, 2, 1)$) is *restricted to the geodesic* determined by $P_1(1.6, 170°)$ and $P_2(1.3, 250°)$. The point which minimizes $C_b(P_i, 2, 1)$ (resp. $C_f(P_i, 2, 1)$) is approximated by $P_b(0.8541, 212.2545°)$ (resp. $P_f = P_1$); in this case $C_b(P_i, 2, 1)(P_b) \approx 1.26$ (resp. $C_f(P_i, 2, 1)(P_f) \approx 2.32507$).
Step 2: The minimum of the total backward (resp. forward) cost function $C_b(P_i, 2, 1)$ (resp. $C_f(P_i, 2, 1)$) is *on the whole Randers space* M. The minimum point of total backward (resp. forward) cost function is approximated by $T_b(0.4472, 212.5589°)$ (resp. $T_f(1.9999, 171.5237°)$), which gives $C_b(P_i, 2, 1)(T_b) \approx 0.950825 < C_b(P_i, 2, 1)(P_b)$ (resp. $C_f(P_i, 2, 1)(T_f) \approx 2.32079 < C_f(P_i, 2, 1)(P_f)$).

The pair (M, F) is the so-called *Finslerian-Poincaré disc.* Within the classification of Finsler manifolds, (M, F) is a *Randers space,* see [22, Section 12.6], which has the following properties:

- (p1) it has constant negative flag curvature $-1/4$;
- (p2) the geodesics have the following trajectories: Euclidean circular arcs that intersect the boundary ∂M of M at Euclidean right angles; Euclidean straight rays that emanate from the origin; and Euclidean straight rays that aim to the origin;
- (p3) $\mathrm{dist}_F((0,0), \partial M) = \infty$, while $\mathrm{dist}_F(\partial M, (0,0)) = \log 2$.

Although (M, F) is forward geodesically complete (but *not* backward geodesically complete), it has constant negative flag curvature $-\frac{1}{4}$ and it is contractible (thus, simply connected), the conclusion of Theorem 14.3 may be false. Indeed, one can find points in M (belonging to the same geodesic) such that the minimum point for the total forward (resp. backward) cost function is *not* situated on the geodesic, see Figure 14.3.

Remark 14.3 Note that Example 14.4 (Finslerian-Poincaré disc) may give a model of a *gravitational field* whose centre of gravity is located at the origin $O = (0, 0)$, while the boundary ∂M means the "infinity". Suppose that in this gravitational field, we have several spaceships, which are delivering some cargo to certain bases or to another spacecraft. Also, assume that these spaceships are of the same type and they consume k liter/second fuel ($k > 0$). Note that the expression $F(d\sigma)$ denotes the physical *time* elapsed to traverse a short portion $d\sigma$ of the spaceship orbit. Consequently, traversing a short path $d\sigma$, a spaceship consumes $kF(d\sigma)$ liter of fuel. In this way, the number $k\int_0^1 F(\sigma(t), d\sigma(t))dt$ expresses the quantity of fuel used up by a spaceship traversing an orbit $\sigma : [0, 1] \to M$.

Suppose that two spaceships have to meet each other (for logistical reasons) starting their trip from bases P_1 and P_2, respectively. Consuming as low total quantity of fuel as possible, they will choose T_b as a meeting point and *not* P_b on the geodesic determined by P_1 and P_2. Thus, the point T_b could be a position for an optimal deposit-base.

Now, suppose that we have two damaged spacecraft (e.g., without fuel) at positions P_1 and P_2. Two rescue spaceships consuming as low total quantity of fuel as possible, will blastoff from base T_f and not from $P_f = P_1$ on the geodesic determined by P_1 and P_2. In this case, the point T_f is the position for an optimal rescue-base. If the spaceships in trouble are close to the center of the gravitational field M, then any rescue-base located closely also to the center O, implies the consumption of a great amount of energy (fuel) by the rescue spaceships in order to reach their destinations (namely, P_1 and P_2). Indeed, they have to overcome the strong gravitational force near the center O. Consequently, this is the reason why the point T_f is so far from O, as Figure 14.3 shows. Note that further numerical experiments support this observation. However, there are certain special cases when the position of the optimal rescue-base is either P_1 or P_2: from these two points, the farthest one from the gravitational center O will be the position of the rescue-base. In such case, the orbit of the (single) rescue spaceship is exactly the geodesic determined by points P_1 and P_2.

14.5 Comments and further perspectives

A. Comments. The results of this chapter are based on the paper of Kristály-Kozma [170] and Kristály-Moroşanu-Róth[172]. In this chapter we studied variational problems arising from Economical contexts via Riemann-Finsler geometry. Real life phenomena may be well modelized involving Finsler metrics, which represent external force as current or gravitation. A special class of Finsler manifolds, called Berwald space, played a central role in our investigation. Indeed, a recent result of Kristály-Kozma [170] concerning metric relations on non-positively curved Berwald spaces has been exploited which includes several real life applications (as the slope metric of a hillside, described by Matsumoto [204]). Beside of these applications, we also presented a completely new material on non-positively curved Berwald spaces as well as a conjecture regarding the rigidity of Finsler manifolds under the Busemann curvature condition which could be of interest for the community of Geometers.

B. Further perspectives. We propose an optimization problems which arises in a real life situation.

Problem 14.1 *There are given n ships moving on different paths (we know all the details on their speed, direction, etc.). Determine the position of the optimum point(s) of the aircraft-carrier (mother ship) from where other n ships can reach the first n ships within a given time interval using the lowest amount of (total) combustible.*

We give some hints concerning Problem 14.1. We consider it in a particular case as follows. Two pleasure boats are moving on given paths within an estuary ending in a waterfall (see Figure 14.4), and a mother ship M_S is positioned in the same area for safety reasons transporting two lifeboats. The problem is to determine the optimal position of the mother ship M_S at every moment such that the two lifeboats reach the two pleasure boats within a T period consuming the minimal total combustible (for a lifeboat it is allowed to wait the pleasure boat but not conversely). On Figure 14.4, the points $x_1(0)$ and $x_2(0)$ correspond to the alerting moment (when the lifeboats start their trips) while $x_1(T)$ and $x_2(T)$ are the last possible points where the lifeboats and the corresponding pleasure boats may meet each other. In this case, the external force is the *current flow* towards the waterfall; the law describing this force – up to some constants – can be given as a *sub*manifold of the Finslerian-Poincaré disc, see Example 14.4.

We now formulate the first problem within a general mathematical framework. We consider the quasi-metric space (M, d_F) associated with a Finsler manifold (M, F) which is not necessarily symmetric, the paths

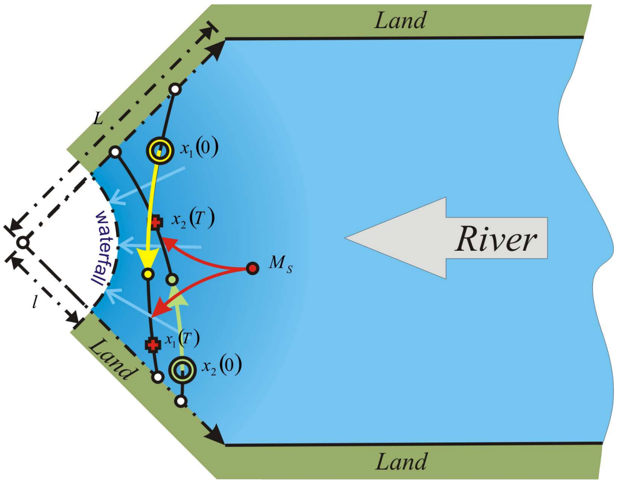

Fig. 14.4. The force of the current flow towards the waterfall may be described by means of the Finslerian-Poincaré disc model. The points $x_1(0)$ and $x_2(0)$ correspond to the alerting moment when the lifeboats from the mother-ship M_S start their trips, while $x_1(T)$ and $x_2(T)$ are the last possible points where the lifeboats and the corresponding pleasure boats may meet each other.

$x_i : [0,T] \to M$ $(i = 1, ..., n)$ and some numbers $v_i > 0$ $(i = 1, ..., n)$ representing the speeds of the last n ships moving towards the paths x_i in order to meet the i^{th} ship from the first group. Let us consider the function

$$f_i(P) = \min_{t\in[0,T]} \{d_F(P, x_i(t)) : d_F(P, x_i(t)) \leq v_i t\}$$

and the convex functions $\psi_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, ..., n)$. The variational problem we are dealing with is

$$\min\{f(P) : P \in M\} \tag{P_{optim}}$$

where

$$f(P) = \sum_{i=1}^{n} \psi_i(f_i(P)).$$

We assume that

$$S = \bigcap_{i=1}^{n} \{P \in M : d_F(P, x_i(T)) \leq v_i T\} \neq \emptyset,$$

which is a sufficient condition for the existence of a solution to the variational problem (P_{optim}).

15
Best Approximation Problems on Manifolds

> If that enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by the laws.
>
> Henri Poincaré (1854–1912)

15.1 Introduction

One of the most famous questions of functional and numerical analysis is the best approximation problem: find the nearest point (called also projection) from a given point to a nonempty set, both situated in an ambient space endowed with a certain metric structure by means of which one can measure metric distances. In the classical theory of best approximations the ambient space has a vector space structure, the distance function is symmetric which comes from a norm or, specially, from an inner product. However, *non*-symmetry is abundant in important real life situations; indeed, it is enough to consider the swimming (with/against a current) in a river, or the walking (up/down) on a mountain slope. These kinds of non-symmetric phenomena lead us to spaces with possible no vector space structure while the distance function is not necessarily symmetric, see also Chapter 14. Thus, the most appropriate framework is to consider not necessarily reversible Finsler manifolds. This nonlinear context throws completely new light upon the problem of best approximations where well-tried methods usually fail.

The purpose of this chapter is to initiate a systematic study of best approximation problems on Finsler manifolds by exploiting notions from Finsler geometry as geodesics, forward/backward geodesically completeness, flag curvature, fundamental inequality of Finsler geometry, and first variation of arc length. Beyond of this new approach, we premise a sharp contrast between the classical results of best approximation theory and those obtained in the present paper.

Throughout this chapter, (M, F) is a connected finite-dimensional Finsler manifold, F is positively (but perhaps not absolutely) homogeneous of degree one. Let $d_F : M \times M \to [0, \infty)$ be the quasi-metric associated with F, see (13.8). Let $q \in M$ be a point and $S \subset M$ a nonempty set. Since d_F is not necessarily symmetric, let

$$\mathrm{dist}_F(q, S) = \inf\{d_F(q, s) : s \in S\} \quad \text{and} \quad \mathrm{dist}_F(S, q) = \inf\{d_F(s, q) : s \in S\}$$

the *forward* and *backward distance*s between the point q and the set S, respectively. The *forward* (resp. *backward*) *best approximation problem* can be formulated as follows: find $s_f \in S$ (resp. $s_b \in S$) such that

$$d_F(q, s_f) = \mathrm{dist}_F(q, S) \quad (\text{resp.} \quad d_F(s_b, q) = \mathrm{dist}_F(S, q)).$$

We define the set of *forward* (resp. *backward*) *projections* of q to S by

$$P_S^+(q) = \{s_f \in S : d_F(q, s_f) = \mathrm{dist}_F(q, S)\}$$

$$(\text{resp.} \quad P_S^-(q) = \{s_b \in S : d_F(s_b, q) = \mathrm{dist}_F(S, q)\}).$$

If $P_S^+(q)$ and $P_S^-(q)$ coincide (for instance, when the Finsler metric F is absolutely homogeneous), we simple write $P_S(q)$ instead of the above sets. If any of the above sets is a singleton, we do not make any difference between the set and its unique point.

In the following sections we are dealing in detail with the existence, characterization and non-expansiveness of projections, geodesic convexity and Chebyshevity of closed sets as well as with nearest points between two sets.

15.2 Existence of projections

Let (M, F) be a Finsler manifold. A set $S \subset M$ is *forward* (resp. *backward*) *proximinal* if $P_S^+(q) \neq \emptyset$ (resp. $P_S^-(q) \neq \emptyset$) for every $q \in M$.

Theorem 15.1 *Let (M, F) be a forward (resp. backward) geodesically complete Finsler manifold, $S \subset M$ a nonempty closed set. Then, S is a forward (resp. backward) proximinal set.*

Proof We consider the statement only for the "forward" case, the "backward" case works similarly. Let us fix $q \in M$ arbitrarily and let $\{s_n\}_{n\in\mathbb{N}} \subset S$ be a forward minimizing sequence, i.e., $\lim_{n\to\infty} d_F(q, s_n) = \mathrm{dist}_F(q, S) =: d_0$. In order to conclude the proof, it is enough to show that the sequence $\{s_n\}_{n\in\mathbb{N}}$ contains a convergent subsequence. Indeed, let $\{s_{k_n}\}_n \subset S$ be a convergent subsequence of $\{s_n\}_n$. Since S is closed, then $\lim_{n\to\infty} s_{k_n} = s_f \in S$. Moreover, since $d_F(q, \cdot)$ is continuous, we have $d_F(q, s_f) = \lim_{n\to\infty} d_F(q, s_{k_n}) = \mathrm{dist}_F(q, S)$, which proves that $s_f \in P_S^+(q)$, i.e., S is a forward proximinal set.

Let $A = \{s_n : n \in \mathbb{N}\}$ and assume that A has no any convergent subsequence. Then, for every $a \in \overline{A}$, there exists $r_a > 0$ such that

$$\mathcal{B}_a^+(r_a) \cap A \subseteq \{a\}. \tag{15.1}$$

On the other hand, for n large enough, we have $d_F(q, s_n) < d_0 + 1$; thus, we may assume that $A \subset \mathcal{B}_q^+(d_0+1)$. Since the set $\overline{A}$ is forward bounded and closed, due to Hopf-Rinow's theorem, it is also compact. Since $\overline{A} \subseteq \cup_{a\in\overline{A}} \mathcal{B}_a^+(r_a)$ and $\mathcal{B}_a^+(r_a)$ are open balls in the manifold topology, there exist $a_1, ..., a_l \in \overline{A}$ such that $\overline{A} \subseteq \cup_{i=1}^l \mathcal{B}_{a_i}^+(r_{a_i})$. In particular, for every $n \in \mathbb{N}$, there exists $i_0 \in \{1, ..., l\}$ such that $s_n \in \mathcal{B}_{a_{i_0}}^+(r_{a_{i_0}})$. Due to (15.1),

$$s_n \in \mathcal{B}_{a_{i_0}}^+(r_{a_{i_0}}) \cap A \subseteq \{a_{i_0}\},$$

i.e., $s_n = a_{i_0}$. Consequently, the set A contains finitely many elements, which contradicts our assumption. $\square$

The geodesically completeness cannot be dropped in Theorem 15.1. We describe here a concrete example on the Finslerian-Poincaré disc, see Example 14.4. The idea to construct a closed, not backward proximinal set in (M, F) comes from the property (p3) and from the fact that (M, F) is *not* geodesically backward complete, see Bao-Chern-Shen [22, p. 342]. We consider the point $q = (1, 0) \in M$ and the set $S = \{(t, -\sqrt{2}) : t \in [1, \sqrt{2})\} \subset M$, see Figure 1. Clearly, S is closed in the topology of (M, F). However, a numerical calculation based on property (p2) and on Maple codes shows that the function $t \mapsto d_F((t, -\sqrt{2}), q)$, $t \in [1, \sqrt{2})$, is strictly *decreasing*, thus the value $\mathrm{dist}_F(S, q) = \lim_{t\to\sqrt{2}} d_F((t, -\sqrt{2}), q) \approx 0.8808$ is *not* achieved by any point of S, i.e., $P_S^-(q) = \emptyset$.

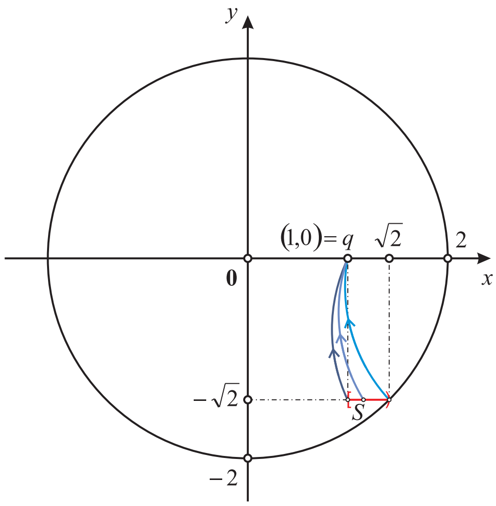

Fig. 15.1. $S = \{(t, -\sqrt{2}) : t \in [1, \sqrt{2})\}$ is a closed, but not backward proximinal set on the Finslerian-Poincaré disc.

15.3 Geometric properties of projections

Let $(X, \langle\cdot,\cdot\rangle)$ be a real prehilbert space, $S \subset X$ be a nonempty convex set, $x \in X$, $y \in S$. Due to Moskovitz-Dines [220] one has:

$$y \in P_S(x) \Leftrightarrow \langle x - y, z - y\rangle \leq 0 \ \text{ for all } z \in S. \tag{15.2}$$

The geometrical meaning of this characterization is that the vectors $x-y$ and $z - y$ (for every $z \in S$) form an obtuse-angle with respect to the inner product $\langle\cdot,\cdot\rangle$.

For a Finsler manifold (M, F), the function F is not necessarily induced by an inner product, thus angle-measuring looses its original meaning. The 'inner product' on a Finsler manifold (M, F) is the Riemannian metric on the pulled-back bundle π^*TM depending not only on the points of M but also on the *directions* in TM, defined by $g_{(p,y)}$ from (13.1), $y \in T_pM$.

Let $S \subset M$ be a nonempty set and $q \in M$. We consider the following statements:

(MD_1^+) : $s \in P_S^+(q)$;
(MD_2^+) : If $\gamma : [0, 1] \to M$ is the unique minimal geodesic from $\gamma(0) = q$ to

$\gamma(1) = s \in S$, then every geodesic $\sigma : [0, \delta] \to S$ $(\delta \geq 0)$ emanating from the point s fulfills $g_{(s,\dot\gamma(1))}(\dot\gamma(1), \dot\sigma(0)) \geq 0$.

Theorem 15.2 *Let (M, F) be a forward geodesically complete Finsler manifold, $S \subset M$ be a nonempty closed set. Then, $(MD_1^+) \Rightarrow (MD_2^+)$.*

Proof One may assume that $\delta > 0$; otherwise, the statement is trivial since the curve σ shrinks to a point. Standard ODE theory shows that the geodesic segment $\sigma : [0, \delta] \to M$ has a C^∞ extension $\tilde\sigma : (-\varepsilon, \delta] \to M$, where $\varepsilon > 0$, see Bao-Chern-Shen [22, Exercise 5.4.2]. Since γ is the unique minimal geodesic lying the points q and s, there exists $v_0 \in T_qM$ such that $\gamma(t) = \exp_q(tv_0)$ and $\exp_{q*}$ is nonsingular at the point v_0. On account of Bao-Chern-Shen [22, p. 205], one may fix $0 < \varepsilon_0 < \min\{\varepsilon, \delta\}$ and a C^∞ vector field $V : [-\varepsilon_0, \varepsilon_0] \to T_qM$ such that $V(0) = v_0$, $\exp_q(V(u)) = \tilde\sigma(u)$ and $\exp_{q*}$ is not singular at $V(u)$ for every $u \in [-\varepsilon_0, \varepsilon_0]$. Let us introduce the map $\Sigma : [0, 1] \times [-\varepsilon_0, \varepsilon_0] \to M$ by $\Sigma(t, u) = \exp_q(tV(u))$. Note that Σ is well-defined and Σ is a variation of the geodesic γ with $\Sigma(\cdot, 0) = \gamma$. By definition, we have

$$\begin{aligned} L'_F(\Sigma(\cdot,0)) &= \frac{d}{du}L_F(\Sigma(\cdot,u))_{|u=0} = \lim_{u\to 0^+}\frac{L_F(\Sigma(\cdot,u)) - L_F(\Sigma(\cdot,0))}{u} \\ &\geq \lim_{u\to 0^+}\frac{d_F(q,\tilde\sigma(u)) - L_F(\gamma)}{u} = \lim_{u\to 0^+}\frac{d_F(q,\sigma(u)) - d_F(q,s)}{u}. \end{aligned}$$

Since $s \in P_S^+(q)$ and $\sigma(u) \in S$ for small values of $u \geq 0$, the latter limit is non-negative; thus, $L'_F(\Sigma(\cdot, 0)) \geq 0$. On the other hand, combining this relation with the first variation formula, see (13.7), we obtain

$$0 \leq L'_F(\Sigma(\cdot,0)) = \frac{g_{(s,\dot\gamma(1))}(\dot\gamma(1),\dot\sigma(0))}{F(s,\dot\gamma(1))}.$$

□

We now consider the statements for the "backward" case:

(MD_1^-) : $s \in P_S^-(q)$;

(MD_2^-) : If $\gamma : [0, 1] \to M$ is the unique minimal geodesic from $\gamma(0) = s \in S$ to $\gamma(1) = q$, then every geodesic $\sigma : [0, \delta] \to S$ $(\delta \geq 0)$ emanating from the point s fulfills $g_{(s,\dot\gamma(0))}(\dot\gamma(0), \dot\sigma(0)) \leq 0$.

The proof of the following result works similarly to that of Theorem 15.2.

Theorem 15.3 *Let (M, F) be a backward geodesically complete Finsler manifold, $S \subset M$ be a nonempty, closed set. Then, $(MD_1^-) \Rightarrow (MD_2^-)$.*

Remark 15.1 Let (M, g) be a Riemannian manifold. In this special case, since the fundamental tensor g is independent on the directions, the statements (MD_1^+) and (MD_2^+) (as well as (MD_1^-) and (MD_2^-)) reduce to:

(MD_1) : $s \in P_S(q)$;

(MD_2) : If $\gamma : [0, 1] \to M$ is the unique minimal geodesic from $\gamma(0) = s \in S$ to $\gamma(1) = q$, then every geodesic $\sigma : [0, \delta] \to S$ $(\delta \geq 0)$ emanating from the point s fulfills $g(\dot{\gamma}(0), \dot{\sigma}(0)) \leq 0$.

Remark 15.2 The implications in Theorems 15.2 and 15.3 *cannot* be reversed in general, not even for Riemannian manifolds. Indeed, we consider the m-dimensional unit sphere $(\mathbb{S}^m, g_0)$ $(m \geq 2)$, and we fix the set S as the equator of $\mathbb{S}^m$. Let also $q \in \mathbb{S}^m \setminus (S \cup \{N, -N\})$ where N is the North pole. One can see that hypothesis (MD_2) holds exactly for *two* points $s_1 \in S$ and $s_2 = -s_1 \in S$; these points are the intersection points of the equator S and the plane throughout the points q, N and $-N$. Let us fix an order of the points on the great circle: $N, q, s_1, -N, s_2$. Then, we have $\text{dist}_{g_0}(q, S) = d_{g_0}(q, s_1) < \pi/2 < d_{g_0}(q, s_2)$, thus $s_1 \in P_S(q)$ but $s_2 \notin P_S(q)$.

In spite of Remark 15.2 we have two Moskovitz-Dines type *characterizations*. The first is due to Walter [290] for Hadamard-type Riemannian manifolds.

Theorem 15.4 *Let (M, g) be a Hadamard-type Riemannian manifold and let $S \subset M$ be a nonempty, closed, geodesic convex set. Then $(MD_1) \Leftrightarrow (MD_2)$.*

For not necessarily reversible Minkowski spaces, we may prove the following.

Theorem 15.5 *Let $(M, F) = (\mathbb{R}^m, F)$ be a Minkowski space and $S \subset M$ be a nonempty, closed, geodesic convex set. Then*

(i) $(MD_1^+) \Leftrightarrow (MD_2^+)$;

(ii) $(MD_1^-) \Leftrightarrow (MD_2^-)$.

Proof On account of Theorems 15.2 and 15.3 we have $(MD_1^+) \Rightarrow (MD_2^+)$

and $(MD_1^-) \Rightarrow (MD_2^-)$, respectively. We now prove that $(MD_2^+) \Rightarrow (MD_1^+)$; the implication $(MD_2^-) \Rightarrow (MD_1^-)$ works in a similar way.

We may assume that $s \neq q$. The unique minimal geodesic $\gamma : [0,1] \to M$ from $\gamma(0) = q$ to $\gamma(1) = s \in S$ is $\gamma(t) = q + t(s-q)$. Fix $z \in S$, and define the geodesic segment $\sigma : [0,1] \to M$ by $\sigma(t) = s + t(z-s)$. Since S is geodesic convex, then $\sigma([0,1]) \subseteq S$. Consequently, the inequality from (MD_2^+) reduces to

$$g_{s-q}(s-q, z-s) \geq 0. \tag{15.3}$$

Relations (15.3) and (13.11) yield

$$\begin{aligned} d_F^2(q,s) &= F^2(s-q) \\ &= g_{s-q}(s-q, s-q) \\ &\leq g_{s-q}(s-q, z-q) \\ &\leq F(s-q) \cdot F(z-q) \\ &= d_F(q,s) \cdot d_F(q,z), \end{aligned}$$

i.e., $d_F(q,s) \leq d_F(q,z)$. Since $z \in S$ is arbitrarily fixed, we have $s \in P_S^+(q)$. □

15.4 Geodesic convexity and Chebyshev sets

If $(X, \langle \cdot, \cdot \rangle)$ is a finite-dimensional Hilbert space and $S \subset X$ is a nonempty closed set, the equivalence of the following statements is well-known (see Borwein [43] and Phelps [239]):

- S is convex;
- S is Chebyshev, i.e., $\mathrm{card} P_S(x) = 1$ for every $x \in X$;
- P_S is non-expansive, i.e., for every $x_1, x_2 \in X$,

$$\|P_S(x_1) - P_S(x_2)\| \leq \|x_1 - x_2\|.$$

On manifolds, as might be expected, uniqueness of the projection is influenced not only by the geodesical convexity of the set but also by the *curvature* of the space. The latter is well emphasized by considering the standard m-dimensional unit sphere $\mathbb{S}^m$ $(m \geq 2)$ with its natural Riemannian metric g_0, having constant sectional curvature 1. The projection of the North pole $N \in \mathbb{S}^m$ to a geodesic ball centered at the South pole and radius $r < 1/2$ is the *whole* boundary of this geodesic ball. As far as we know, there is only one class guaranteeing the uniqueness of projections on non-positively curved spaces from any point to any; namely, the geodesic convex sets on global Busemann NPC spaces, see Jost [150, Section 3.3]. Note that all Hadamard-type Riemannian manifolds belong to this class, see also Udrişte [286].

However, uniqueness of projections to generic geodesic convex sets are not known either for reversible or for non-reversible Finsler manifolds with non-positive curvature. In the sequel, we delimit a class of not necessarily reversible Finsler manifolds (so, in particular, non-Riemannian manifolds) for which the forward/backward projections are singletons to any geodesic convex sets.

Let (M, F) be a Finsler manifold. A set $S \subset M$ is a *forward* (resp. *backward*) *Chebyshev* set if $\text{card}P_S^+(q) = 1$ (resp. $\text{card}P_S^-(q) = 1$) for each $q \in M$.

Theorem 15.6 *Let (M, F) be a forward (resp. backward) geodesically complete, simply connected Berwald space with non-positive flag curvature, and $S \subset M$ a nonempty, closed, geodesic convex set. Then, S is a forward (resp. backward) Chebyshev set.*

Proof By Theorem 15.1, we know that $\text{card}P_S^+(q) \geq 1$ (resp. $\text{card}P_S^-(q) \geq 1$) for every $q \in M$. On the other hand, Proposition 13.5 guarantees that $t \mapsto d_F(\gamma_1(t), \gamma_2(t))$ is convex for any two minimal geodesics $\gamma_1, \gamma_2 : [0, 1] \to M$ (with not necessarily a common starting point). In particular, for every $q \in M$ and any non-constant geodesic $\gamma : [0, 1] \to M$, the function $t \mapsto d_F^2(q, \gamma(t))$ is strictly convex. Now, assume that $\text{card}P_S^+(q) > 1$, i.e., there exists $s_1, s_2 \in P_S^+(q)$, $s_1 \neq s_2$. Let $\gamma : [0, 1] \to M$ be the minimal geodesic which joins these points. Consequently, $\text{Im}\gamma \subset S$ and for every $0 < t < 1$, we have

$$d_F^2(q, \gamma(t)) < t d_F^2(q, \gamma(1)) + (1 - t) d_F^2(q, \gamma(0)) = \text{dist}_F^2(q, S),$$

a contradiction. □

Remark 15.3 Both Hadamard-type Riemannian manifolds and Minkowski spaces fulfill the hypotheses of Theorem 15.6. Now, we present a typical class of Berwald spaces where Theorem 15.6 applies. Let (N, h) be an arbitrarily closed hyperbolic Riemannian manifold of dimension at least 2, and $\varepsilon > 0$. Let us define the Finsler metric $F_\varepsilon : T(\mathbb{R} \times N) \to [0, \infty)$ by

$$F_\varepsilon(t, p; \tau, w) = \sqrt{h_p(w, w) + \tau^2 + \varepsilon \sqrt{h_p^2(w, w) + \tau^4}},$$

where $(t, p) \in \mathbb{R} \times N$ and $(\tau, w) \in T_{(t,p)}(\mathbb{R} \times N)$. Shen [273] pointed out that the pair $(\mathbb{R} \times N, F_\varepsilon)$ is a reversible Berwald space with non-positive flag curvature which is neither a Riemannian manifold nor a Minkowski space.

Remark 15.4 There are Finsler manifolds with non-positive flag curvature which are not curved in the sense of Busemann and the function $t \mapsto d_F(\gamma_1(t), \gamma_2(t))$ is *not* convex; simple examples of these types can be given theoretically on Hilbert geometries (see Socié-Méthou [277]), or numerically on certain Randers spaces (see Shen [271], [272]) by using evolutive programming in the spirit of Kristály-Moroşanu-Róth [172]. Thus, no convexity property of the function $t \mapsto d_F(\gamma_1(t), \gamma_2(t))$ can be guaranteed in general which raises doubts on the validity of Theorem 15.6 for arbitrary Finsler manifolds with non-positive flag curvature.

Remark 15.5 The Chebyshevity of a closed set does not imply always its geodesic convexity in arbitrarily Hadamard-type Riemannian manifolds. Indeed, consider the Poincaré upper half-plane model $\mathbb{H}^2$ (with sectional curvature -1) and the closed set $S = \mathbb{H}^2 \setminus B_{(0,1)}(2)$, where $B_{(0,1)}(2)$ is the standard euclidean 2-dimensional open ball with center $(0,1) \in \mathbb{R}^2$ and radius 2. S is a Chebyshev set since every horocycle which is below of the boundary ∂S and tangent to the same set ∂S, is situated in the interior of $B_{(0,1)}(2)$. However, it is clear that S is not geodesic convex. For further examples and comments, see Grognet [130, Section 2]. These examples show the converse of Theorem 15.6 does not hold in general. Moreover, the following rigidity result does hold.

Theorem 15.7 (Busemann *[54, p. 152]*) *Let (M, F) be a simply-connected, complete, reversible Finsler manifold with non-positive flag curvature. If any Chebyshev set in M is geodesic convex, then (M, F) is a Minkowski space.*

The following characterization of geodesic convexity is due to Grognet [130] (see also Udrişte [286] for an alternative proof via the second variation formula of the arc lenght):

Theorem 15.8 *Let (M, g) be a Hadamard-type Riemannian manifold and $S \subset M$ be a nonempty closed set. The following two statements are equivalent:*

(i) *S is geodesic convex;*

(ii) *P_S is non-expansive, i.e., $d_g(P_S(q_1), P_S(q_2)) \leq d_g(q_1, q_2)$ for any $q_i \in M$, $i = 1, 2$.*

Proof (i)⇒(ii) This part is contained in Udrişte [286]; for completeness we give its outline. First, on account of Theorem 15.6, S is a Chebyshev set. Fix $q_1, q_2 \in M$ and let $\gamma_i : [-1, 1] \to M$ be the unique geodesic with constant speed from $\gamma_i(-1) = q_i$ to $\gamma_i(1) = s_i = P_S(q_i)$, $i = 1, 2$. Define the variation $\Sigma : [0, 1] \times [-1, 1] \to M$ by $\Sigma(t, u) = \exp_{\gamma_1(u)}(t \exp^{-1}_{\gamma_1(u)}(\gamma_2(u)))$. Note that

$$\begin{aligned} L_g^2(\Sigma(\cdot, u)) &= \left(\int_0^1 \sqrt{g(\frac{\partial \Sigma}{\partial t}(t, u), \frac{\partial \Sigma}{\partial t}(t, u))} dt \right)^2 \\ &= \int_0^1 g(\frac{\partial \Sigma}{\partial t}(t, u), \frac{\partial \Sigma}{\partial t}(t, u)) dt \,. \end{aligned}$$

Throughout the first variational formula, we obtain that

$$(L_g^2)'(\Sigma(\cdot, 1)) = 2 \left[g(\gamma_2'(1), \frac{\partial \Sigma}{\partial t}(1, 1)) - g(\gamma_1'(1), \frac{\partial \Sigma}{\partial t}(0, 1)) \right].$$

Since S is geodesic convex, then the geodesic $\Sigma(\cdot, 1)$ belongs to S. Applying Theorem 15.4, we obtain that

$$g(-\gamma_1'(1), \frac{\partial \Sigma}{\partial t}(0, 1)) \leq 0 \quad \text{and} \quad g(-\gamma_2'(1), -\frac{\partial \Sigma}{\partial t}(1, 1)) \leq 0 \,.$$

Consequently, $(L_g^2)'(\Sigma(\cdot, 1)) \leq 0$. Now, using the second variational formula, and taking into account that γ_1, γ_2 are geodesics, we deduce that

$$(L_g^2)''(\Sigma(\cdot, u)) = 2 \int_0^1 \left[g\left(\frac{D}{\partial t} \frac{\partial \Sigma}{\partial t}, \frac{D}{\partial t} \frac{\partial \Sigma}{\partial t} \right) - R\left(\frac{\partial \Sigma}{\partial t}, \frac{\partial \Sigma}{\partial u}, \frac{\partial \Sigma}{\partial t}, \frac{\partial \Sigma}{\partial u} \right) \right] dt,$$

where R is the Riemannian curvature tensor. Since the sectional curvature is non-positive, we obtain $(L_g^2)''(\Sigma(\cdot, u)) \geq 0$. Now, the Taylor expansion yields $L_g(\Sigma(\cdot, 1)) \leq L_g(\Sigma(\cdot, -1))$, i.e., $d_g(s_1, s_2) \leq d_g(q_1, q_2)$.

(ii)⇒(i) Assume that S is not geodesic convex, i.e., there exists two distinct points $s_1, s_2 \in S$ such that the unique geodesic with constant speed $\gamma : [0, 1] \to M$, with $\gamma(0) = s_1$ and $\gamma(1) = s_2$ has the property that $\gamma(t) \notin S$ for every $t \in (0, 1)$. Since S is proximinal, cf. Theorem 15.1, we fix an element $\tilde{s} \in P_S(\gamma(1/2))$.

We claim that either $d_g(s_1, \tilde{s})$ or $d_g(\tilde{s}, s_2)$ is strictly greater than $d_g(s_1, s_2)/2$. If $\tilde{s} = s_1$ or $\tilde{s} = s_2$, the claim is true. Now, if $s_1 \neq \tilde{s} \neq s_2$, we assume that $\max\{d_g(s_1, \tilde{s}), d_g(\tilde{s}, s_2)\} \leq d_g(s_1, s_2)/2$. Therefore, $d_g(s_1, s_2) \leq d_g(s_1, \tilde{s}) + d_g(\tilde{s}, s_2) \leq d_g(s_1, s_2)$, which implies that $\tilde{s} \in S$

belongs to the geodesic γ, contradicting our initial assumption. Thus, the claim is true. Suppose that $d_g(s_1, \tilde{s}) > d_g(s_1, s_2)/2$; the other case works similarly. Since $P_S(s_1) = s_1$ and $d_g(s_1, s_2)/2 = d_g(s_1, \gamma(1/2))$, relation $d_g(s_1, \tilde{s}) > d_g(s_1, \gamma(1/2))$ contradicts (ii). □

Remark 15.6 In spite of Theorem 15.8 there is no way to give a similar characterization for generic Finsler manifolds with non-positive flag curvature. In the sequel, we give such a simple counterexample on Matsumoto mountain slope $(\mathbb{R}^2, F_\alpha)$, see (14.1). More precisely, let $v = 10$ and $\alpha = 60^o$ in $(\mathbb{R}^2, F_\alpha)$, and

$$S = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 = 1\}.$$

It is clear that S is closed and geodesic convex. Let $q_1 = (0, 0)$ and $q_2 = (0, 1)$. Since $q_2 \in S$, then $P_S^+(q_2) = q_2$. Due to Theorem 15.6, card$P_S^+(q_1)$=1, and from Theorem 15.5, $s_f \in P_S^+(q_1)$ if and only if $g_{s_f}(s_f, e) = 0$ where $e = (1, -1)$ is the direction vector of the straight line S. Solving the equation $g_{s_f}(s_f, e) = 0$, we obtain the unique solution $s_f = (0.58988935, 0.41011065) \in S$. Moreover,

$$\begin{aligned} d_{F_\alpha}(P_S^+(q_1), P_S^+(q_2)) &= d_{F_\alpha}(s_f, q_2) = F_\alpha(q_2 - s_f) \\ &= 0.58988935 \cdot F_\alpha(-1, 1) = 0.11923844 \\ &> 0.1 = F_\alpha(q_2 - q_1) \\ &= d_{F_\alpha}(q_1, q_2). \end{aligned}$$

Consequently, P_S^+ is *not* a non-expansive map in general.

15.5 Optimal connection of two submanifolds

Let us assume that two disjoint closed sets M_1 and M_2 are fixed in a (not necessarily reversible) Finsler manifold (M, F). Roughly speaking, we are interested in the number of those Finslerian geodesics which connect M_1 and M_2 in an optimal way, i.e., satisfying certain boundary conditions. In order to handle this problem, following Mercuri [209], Caponio, Javaloyes and Masiello [56], we first describe the structure of a special Riemann-Hilbert manifold.

Let (M, F) be a forward or backward complete Finsler manifold and let us endow in the same time M with any complete Riemannian metric h. Let N be a smooth submanifold of $M \times M$. We consider the collection $\Lambda_N(M)$ of curves $c : [0, 1] \to M$ with $(c(0), c(1)) \in N$ and having H^1 regularity, that is, c is absolutely continuous and the integral

$\int_0^1 h(\dot{c},\dot{c})ds$ is finite. It is well known that $\Lambda_N(M)$ is a Hilbert manifold modeled on any of the equivalent Hilbert spaces of H^1 sections, with endpoints in TN, of the pulled back bundle c^*TM, c being any regular curve in $\Lambda_N(M)$. For every H^1 sections X and Y of c^*TM the scalar product is given by

$$\langle X, Y\rangle_1 = \int_0^1 h(X,Y)ds + \int_0^1 h(\nabla_c^h X, \nabla_c^h Y)ds \tag{15.4}$$

where ∇_c^h is the usual covariant derivative along c associated to the Levi-Civita connection of the Riemannian metric h.

Let $J : \Lambda_N(M) \to \mathbb{R}$ be the energy functional defined by

$$J(c) = \frac{1}{2}\int_0^1 F^2(c,\dot{c})ds. \tag{15.5}$$

For further use, we denote the function F^2 by G. Note that the functional J is of class C^{2-} on the space $\Lambda_N(M)$, i.e., it is of class C^1 with locally Lipschitz differential.

Proposition 15.1 *A curve $\gamma \in \Lambda_N(M)$ is a constant (non zero) speed geodesic for the Finsler manifold (M,F) satisfying the boundary condition*

$$g_{(\gamma(0),\dot{\gamma}(0))}(V,\dot{\gamma}(0)) = g_{(\gamma(1),\dot{\gamma}(1))}(W,\dot{\gamma}(1)), \quad \forall (V,W) \in T_{(\gamma(0),\gamma(1))}N, \tag{15.6}$$

if and only if it is a (non constant) critical point of J.

Proof Assume that $\gamma \in \Lambda_N(M)$ is a critical point of J. One can prove that γ is a smooth regular curve which argument is based on local coordinates. Now let $Z \in T_\gamma\Lambda_N(M)$ be a smooth vector field along γ and let $\Sigma : [0,1]\times[-\varepsilon,\varepsilon] \to M$ be a smooth regular variation of γ with variation vector field $U = \frac{\partial\Sigma}{\partial u}$ having endpoints in TN and such that $U(t,0) = Z(t)$ for all $t \in [0,1]$. Let also $T = \frac{\partial\Sigma}{\partial t}$. Since $G(x,y) = g_{(x,y)}(y,y)$ for any $(x,y) \in TM \setminus 0$, from (13.5) we get

$$\frac{d}{du}J(\Sigma) = \frac{1}{2}\int_0^1 \frac{\partial}{\partial u} g_{(\Sigma,T)}(T,T)dt = \int_0^1 g_{(\Sigma,T)}(T, D_U T)dt. \tag{15.7}$$

Since Σ is smooth, we have that $D_U T = D_T U$ both considered with the reference vector T; therefore, using this equality in (15.7) and evaluating at $u = 0$ we obtain

$$dJ(\gamma)[Z] = \int_0^1 g_{(\gamma,\dot\gamma)}(\dot\gamma, D_{\dot\gamma}Z)dt, \tag{15.8}$$

where $D_{\dot\gamma}Z$ has reference vector $\dot\gamma$. Now, applying relation (13.4), we have

$$\frac{d}{dt}g_{(\gamma,\dot\gamma)}(\dot\gamma, Z) = g_{(\gamma,\dot\gamma)}(D_{\dot\gamma}\dot\gamma, Z) + g_{(\gamma,\dot\gamma)}(\dot\gamma, D_{\dot\gamma}Z),$$

which, when applied to (15.8), gives us

$$0 = dJ(\gamma)[Z] = -\int_0^1 g_{(\gamma,\dot\gamma)}(D_{\dot\gamma}\dot\gamma, Z)dt + g_{(\gamma(1),\dot\gamma(1))}(\dot\gamma(1), Z(1)) - g_{(\gamma(0),\dot\gamma(0))}(\dot\gamma(0), Z(0)). \tag{15.9}$$

Now, by choosing an endpoints vanishing vector field Z one can see that γ should verify the equation $D_{\dot\gamma}\dot\gamma = 0$, i.e., γ is a constant speed geodesic. Consequently, the remaining part of the above relation gives precisely the boundary conditions (15.6).

For the converse, we observe that if γ is a constant non-zero speed geodesic satisfying the boundary conditions (15.6) then (15.9) holds and hence γ is a critical point of J. □

Remark 15.7 Two particular cases is presented concerning the form of N.

(i) Let Δ be the diagonal in $M \times M$ and $N = \Delta$. By using the Euler theorem for homogeneous functions, we know that $\partial_y G(x, y) = 2g_{(x,y)}(\cdot, y)$ for any $(x, y) \in TM$. Hence, from $\gamma(0) = \gamma(1)$ and (15.6) we clearly have

$$\partial_y G(\gamma(0), \dot\gamma(0)) = \partial_y G(\gamma(0), \dot\gamma(1)).$$

Since the map $y \mapsto \partial_y G(x, y)$ is an injective map, we necessarily have that $\dot\gamma(0) = \dot\gamma(1)$, i.e., the curve γ in Proposition 15.1 is a closed geodesic. See Mercuri [209].

(ii) Let M_1 and M_2 be two submanifolds of M and $N = M_1 \times M_2$. In (15.6) put $W = 0$. Then, for any $V \in T_{\gamma(0)}M_1$ we get $g_{(\gamma(0),\dot\gamma(0))}(V, \dot\gamma(0)) = 0$. Analogously, taking $V = 0$, we have $g_{(\gamma(1),\dot\gamma(1))}(W, \dot\gamma(0)) = 0$ for any $W \in T_{\gamma(1)}M_2$. When (M, F) is a Riemannian manifold, these conditions are actually the well-known perpendicularity conditions to M_1 and M_2, respectively. See Grove [133].

Based on the papers of Caponio, Javaloyes and Masiello [56], Kozma, Kristály and Varga [160], and Mercuri [209], we sketch the proof of the fact that J satisfies the (PS)-condition under natural assumptions. This result allows us to give several multiplicity results concerning the number of geodesics joining two different submanifolds in a Finsler manifold.

Proposition 15.2 *Let (M,F) be forward (resp. backward) complete and N be a closed submanifold on $M \times M$ such that the first projection (resp. the second projection) of N to M is compact. Then J satisfies the (PS)-condition on $\Lambda_N(M)$.*

Proof (Sketch) We sketch the proof in the forward complete case, the backward being similar.

Step 1. Differentiable structure on $\Lambda_N(M)$. The manifold $\Lambda_N(M)$ is a closed submanifold of the complete Hilbert manifold $\Lambda(M)$; the last one being the set of all the H^1 curves in M parameterized on $[0,1]$ with scalar product from (15.4). The differentiable manifold structure on $\Lambda(M)$ is given by the charts $\{(O_\omega, \exp_\omega^{-1})\}_{\omega\in C^\infty(M)}$, where $\exp_\omega^{-1}$ is the inverse of the map $\exp_\omega(\xi) = \exp_{\omega(t)} \xi(t)$, for all $\xi \in H^1(O_\omega)$, being O_ω a neighborhood of the zero section in $\omega^* TM$.

Step 2. Uniform convergence of a (PS)-sequence. We consider the sequence $\{c_n\}_{n\in\mathbb{N}}$ contained in $\Lambda_N(M)$ which verifies the assumptions from the (PS)-condition. In particular, $\{J(c_n)\}_n$ is bounded. We shall prove that the sequence $\{c_n\}_n$ converges uniformly. To do this, we fix a point $\overline{p} \in p_1(N)$, where p_1 is the first projection on $M \times M$. Then

$$\begin{aligned} d_F(\overline{p}, c_n(s)) &\leq d_F(\overline{p}, c_n(0)) + d_F(c_n(0), c_n(s)) \\ &\leq d_F(\overline{p}, c_n(0)) + \int_0^1 F(c_n, \dot{c}_n)ds, \end{aligned}$$

for all $s \in [0,1]$, $n \in \mathbb{N}$. Since $p_1(N)$ is compact, there exists a constant K such that $d_F(\overline{p}, c_n(0)) \leq K$. By the Hölder inequality, for every $s \in [0,1]$ we have

$$d_F(\overline{p}, c_n(s)) \leq K + \left(\int_0^1 G(c_n, \dot{c}_n)ds\right)^{\frac{1}{2}} = K + \sqrt{J(c_n)} \leq K_1.$$

Consequently, the set $S = \{c_n(s) : s \in [0,1], n \in \mathbb{N}\}$ is a forward bounded set; thus, Hopf-Rinow theorem, see Theorem 13.1, shows that there exists a compact subset C of M which contains S. Hence there exist $k_1, k_2 > 0$ such that

$$k_1|v|^2 \leq G(x,v) \leq k_2|v|^2, \quad \forall x \in C,\ v \in T_xM.$$

Here, we denoted by $|\cdot|$ the norm associated to the metric h. Moreover, let d_h be the distance associated to the Riemannian metric h, then using the last and Hölder's inequalities, we get

$$\begin{aligned} d_h(c_n(s_1), c_n(s_2)) &\leq \int_{s_1}^{s_2} |\dot{c}_n| ds \leq \sqrt{s_2 - s_1} \left(\int_0^1 |\dot{c}_n|^2 ds \right)^{\frac{1}{2}} \\ &\leq \frac{1}{k_1} \sqrt{s_2 - s_1} \left(\int_0^1 G(c_n, \dot{c}_n) ds \right)^{\frac{1}{2}} \leq K_2 \sqrt{s_2 - s_1}, \end{aligned}$$

with $s_1 < s_2$ in $[0,1]$ and $K_2 > 0$, so that $\{c_n(t)\}$ is relatively compact for every $t \in [0,1]$ and uniformly Hölder. Thus we can use the symmetric distance d_h and the Ascoli-Arzelà theorem to obtain a subsequence, that will be denoted again by $\{c_n\}$, converging uniformly to a curve $\overline{c}$ of class C^0 parameterized in $[0,1]$ and having endpoints in N.

Step 3. Reduction of the strong convergence of $\{c_n\}_n$ to an appropriate convergence in H^1-topology. For any $\eta > 0$ small enough the set $\mathcal{C} = \{\exp_{\overline{c}(s)} v : \ s \in [0,1]; \ v \in \overline{B}(\overline{c}(s), \eta)\}$ is compact in M. Let $\mathrm{inj}(p)$ be the injectivity radius of p in (M,h) and $\rho = \inf\{\mathrm{inj}(p) : \ p \in \mathcal{C}\}$. Since the injectivity radius is continuous, we have that $\rho > 0$; therefore, there exists a curve ω of class C^∞ such that $\|\overline{c} - \omega\|_\infty < \min\{\eta, \rho/2\}$. Let $[0,1] \ni t \to \mathbf{E}(t) = (E_1(t), \dots, E_m(t))$ be a parallel orthonormal frame along ω with $m = \dim M$, and let $P_t : \mathbb{R}^m \to T_{\omega(t)}M$ defined as $P_t(v_1, \dots, v_m) = v_1 E(t) + \cdots + v_m E_m(t)$. Consider the Euclidean open ball of radius ρ, denoted in the sequel by U, and the map $\varphi(t,v) = \exp_{\omega(t)} P_t(v)$. Since $\rho \leq \mathrm{inj}(\omega(t))$, the map $\varphi_t : U \to M$, defined as $\varphi_t(v) = \varphi(t,v)$, is locally invertible and injective with invertible differential $d\varphi_t(v)$, for every $t \in [0,1]$ and $v \in U$. By taking a smaller open set in U that contains the closed ball of radius $\rho/2$ and contained in the closed ball of radius $2\rho/3$, we can assume that all the continuous functions involved in the rest of the proof are uniformly bounded in $[0,1] \times U$ or in $\bigcup_{t \in [0,1]} \{t\} \times \varphi(\{t\} \times U)$, as for example the norms of $d\varphi(t,v)$ and $d\varphi(t,x)$, where $\varphi(t,x) = \varphi_t^{-1}(x)$. Let $\mathcal{O}_\omega$ be a neighborhood of ω in $H^1([0,1],M)$ such that the map $\varphi_*^{-1} : \mathcal{O}_\omega \to H^1([0,1],U)$, defined as $\varphi_*^{-1}(x)(t) = \varphi_t^{-1}(x(t))$ is the map of coordinate system centered at ω. Observe that the inverse of φ_*^{-1} is the map φ_*, defined by $\varphi_*(\xi)(t) = \varphi(t, \xi(t))$. Clearly if n is big enough, $c_n \in \varphi_*(H^1([0,1],U))$, so that we call $\xi_n = \varphi_*^{-1}(c_n)$. Consequently, taking into account the above reduction argument inspired by [1, Appendix A.1], the strong convergence of $\{c_n\}_n$ is equivalent with the strong convergence of $\{\xi_n\}_n$ in $H^1([0,1],U)$.

Step 4. Splitting argument. Taking the vector space of dimension $2m$

defined by $V = \{\zeta \in C^\infty([0,1], \mathbb{R}^m) : \zeta'' - \zeta = 0\}$, we have the following orthogonal splitting

$$H^1([0,1], \mathbb{R}^m) = H_0^1([0,1], \mathbb{R}^m) \oplus V.$$

Therefore, if $n \in \mathbb{N}$ is big enough there exist $\xi_n^0 \in H_0^1([0,1], U)$ and $\zeta_n \in V$ such that $\xi_n = \xi_n^0 + \zeta_n$. Note that $\{c_n\}$ is a sequence which verifies the assumptions of the (PS)-condition. Moreover, the norm of $d\varphi_*$ is bounded in $H^1([0,1], U)$, $\{\zeta_n\}_n$ is a converging sequence in the C^1 norm (which follows from the C^0 convergence of $\{\xi_n\}_n$ and the smooth dependence of the solutions of the differential equation defining V from boundary data) and $\{\xi_n\}_n$ is a bounded sequence in $H^1([0,1], U)$ (ξ_n^0 is a bounded sequence in $H_0^1([0,1], U)$). Due to the above facts and considering J defined on the manifold $\Lambda(M)$, we have

$$d(J \circ \varphi_*)(\xi_n)[\xi_n - \xi_k]$$

$$= d(J \circ \varphi_*)(\xi_n)[\xi_n^0 - \xi_k^0] + d(J \circ \varphi_*)(\xi_n)[\zeta_n - \zeta_k]$$

$$= dJ(c_n)[d\varphi_*(\xi_n)[\xi_n^0 - \xi_k^0]] + d(J \circ \varphi_*)(\xi_n)[\zeta_n - \zeta_k] \to 0. \tag{15.10}$$

Step 5. Boundedness of $\{\xi_n\}_n$ *in* $H^1([0,1], U)$. To see this, it is enough to observe that

$$\int_0^1 |\dot{\xi}_n|^2 ds = \int_0^1 |d\varphi(s, c_n)[(1, \dot{c}_n)]|^2 ds$$

$$\leq K_3 \int_0^1 (1 + h(\dot{c}_n, \dot{c}_n)) ds \leq K_3 + K_4 J(c_n) < K_5 < +\infty, \tag{15.11}$$

where $\varphi(s, x) = \varphi_s^{-1}(x)$, for every $s \in [0,1]$ and $x \in \varphi_s(U)$, and K_3, K_4, K_5 are positive constants.

Step 6. (Final) The sequence $\{\xi_n\}_n$ *is fundamental in* $H^1([0,1], U)$. Technical estimations show that

$$\int_0^1 |\dot{\xi}_n - \dot{\xi}_k|^2 ds \to 0$$

as $n, k \to \infty$. This concludes the proof. □

Now, we are in the position to establish an existence and some multiplicity results concerning the number of geodesics joining two submanifolds in a Finsler manifold.

Theorem 15.9 *Let (M, F) be a forward or backward complete Finsler manifold and let M_1 and M_2 be two closed submanifolds of M such that M_1 or M_2 is compact. Then in any homotopy class of curves from M_1 to M_2 there exists a geodesic with energy smaller than that of any other curve in this class satisfying (15.6). Furthermore, there are at least* $\mathrm{cat}\Lambda_{M_1\times M_2}(M)$ *geodesics joining M_1 and M_2 with the property (15.6).*

Proof Since $\Lambda_{M_1\times M_2}(M)$ is a complete Hilbert-Riemann manifold and the energy functional satisfies the (PS)-condition, see Proposition 15.2, it follows that the energy integral attains its infimum on any component of $\Lambda_{M_1\times M_2}(M)$ and its lower bound, see Theorem 1.4. The infimum points are critical points of J, while any critical point c for J is a geodesic with the property (15.6), see Proposition 15.1. Therefore, the first part is proved. Now, applying Theorem 1.9, we obtain our last claim. □

Theorem 15.10 *Let (M, F) be a compact, connected and simply connected Finsler manifold, and let M_1 and M_2 be two closed disjoint submanifolds of M with M_1 contractible. Then there are infinitely many geodesics joining M_1 and M_2 with the property (15.6).*

Proof Due to Theorem C.3, we have that $\mathrm{cuplong}\,\Lambda_{M_1\times M_2}(M) = \infty$. Using the inequality $\mathrm{cat}\,\Lambda_{M_1\times M_2}(M) \geq 1 + \mathrm{cuplong}\,\Lambda_{M_1\times M_2}(M)$, from Theorem 15.9 the statement follows. □

Theorem 15.11 *Let (M, F) be a complete, non-contractible Finsler manifold, and let M_1 and M_2 be two closed, disjoint and contractible submanifolds of M such that M_1 or M_2 is compact. Then there are infinitely many geodesics joining M_1 and M_2 with the property (15.6).*

Proof Since $M_1\times M_2$ is a submanifold of $M\times M$, the inclusion $\Lambda_{M_1\times M_2}M \hookrightarrow C^0_{M_1\times M_2}(M) = \{\sigma \in C^0([0,1], M) : \sigma(0) \in M_1, \sigma(1) \in M_2\}$ is a homotopy equivalence, see Grove [133, Theorem 1.3]. Since M_1 and M_2 are contractible subsets of M, the sets $C^0_{M_1\times M_2}(M)$ and $M_1 \times M_2 \times \Omega(M)$ are homotopically equivalent, see Fadell-Husseini [109, Proposition 3.2]. Since M is non contractible, we have $\mathrm{cat}\,\Omega(M) = \infty$, see Theorem C.1. Therefore, $\mathrm{cat}\,\Lambda_{M_1\times M_2}(M) = \infty$ and we apply again Theorem 15.9 to obtain the desired conclusion. □

15.6 Final remarks and perspectives

Further questions arise concerning the best approximation problems on Finsler manifolds. We formulate only a few of them:

(i) The Moskovitz-Dines type characterization of projections has been proved for Minkowski spaces (Theorem 15.5) and for Hadamard-type Riemannian manifolds (Theorem 15.4). We feel quite certain we face a rigidity result: if $(MD_1^+) \Leftrightarrow (MD_2^+)$ (or, $(MD_1^-) \Leftrightarrow (MD_2^-)$) holds on a Finsler manifold (M, F) with non-positive flag curvature, than (M, F) is either a Riemannian manifold or a Minkowski space.

(ii) We were able to prove the uniqueness result (Theorem 15.6) only on Berwald spaces. Try to extend Theorem 15.6 to any (not necessarily reversible) Finsler manifold with non-positive flag curvature; see also Remark 15.4.

(iii) Determine those classes of Finsler manifolds with non-positive flag curvature whose (forward and backward) projections are non-expansive for every nonempty, closed, geodesic convex set. We believe the only class fulfilling this property is the class of Hadamard-type Riemannian manifolds. There are three supporting reasons for this fact: (i) non-expansiveness of the projection map holds on Hadamard-type Riemannian manifolds, see Theorem 15.8; (ii) in Remark 15.6 we gave a counterexample for non-expansivity of the projection map on a slightly general case than Riemannian manifolds; (iii) the non-expansiveness of the projection map holds on a normed vector space if and only if the norm comes from an inner product, see Phelps [239].

16

A Variational Approach of the Nash Equilibria

I did have strange ideas during certain periods of time.

John F. Nash (b. 1928)

16.1 Introduction

Nash equilibrium plays a central role in game theory; it is a concept strategy of a game involving n-players ($n \geq 2$) in which every player know the equilibrium strategies of the other players, and changing his/her own strategy alone, a player has nothing to gain.

Let $K_1, \ldots, K_n$ ($n \geq 2$) be the nonempty sets of strategies of the players and $f_i : K_1 \times \ldots \times K_n \to \mathbb{R}$ ($i \in \{1, \ldots, n\}$) be the payoff functions. A point $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{K}$ is a *Nash equilibrium point for* $(\mathbf{f}, \mathbf{K})$ if

$$f_i(\mathbf{p}; q_i) \geq f_i(\mathbf{p}) \text{ for all } q_i \in K_i, \ i \in \{1, \ldots, n\}.$$

Here and in the sequel, the following notations are used: $\mathbf{K} = \Pi_{i=1}^n K_i$; $\mathbf{p} = (p_1, \ldots, p_n)$; $(\mathbf{f}, \mathbf{K}) = (f_1, \ldots, f_n; K_1, \ldots, K_n)$; $(\mathbf{p}; q_i) = (p_1, \ldots, q_i, \ldots, p_n)$.

The most well-known existence result is due to Nash (see [225], [224]) which works for compact and convex subsets K_i of Hausdorff topological vector spaces, and the continuous payoff functions f_i are (quasi)convex in the i^{th}-variable, $i \in \{1, \ldots, n\}$. A natural question that arises at this point is:

How is it possible to guarantee the existence of Nash equilibrium points for a family of payoff functions (perhaps set-valued) without any convexity, or even more, when their domains are not convex in the usual sense?

In this chapter we focus our attention to the existence and location of Nash equilibrium points in various context. First, in Section 16.2 we give several formulations of Nash equilibria via variational inequalities of Stampacchia and Minty type following the paper of Cavazzuti, Pappalardo and Passacantando [62], in Section 16.3 we are dealing with Nash equilibrium points for set-valued maps on vector spaces. In Section 16.4, we treat a case when the domain of the strategy sets are not necessarily convex in the usual sense by applying elements from Riemannian geometry.

16.2 Nash equilibria and variational inequalities

We first recall the well-known result of Nash (see [225], [224]), concerning the existence of at least one Nash equilibrium point for a system $(\mathbf{f}, \mathbf{K})$.

Theorem 16.1 *Let $K_1, \ldots, K_n$ be nonempty compact convex subsets of Hausdorff topological vector spaces, and let $f_i : \mathbf{K} \to \mathbb{R}$ $(i \in \{1, \ldots, n\})$ be continuous functions such that $q_i \in K_i \mapsto f_i(\mathbf{p}; q_i)$ is quasiconvex for all fixed $p_j \in K_j$ $(j \neq i)$. Then there exists a Nash equilibrium point for $(\mathbf{f}, \mathbf{K})$.*

In the sequel, assume that the sets of strategies of the players, i.e., $K_1, \ldots, K_n$ $(n \geq 2)$, are closed convex subsets of a common vector space X. Moreover, we also assume that the payoff functions f_i are continuous on $\mathbf{K}$ and each f_i is of class C^1 in its i^{th} variable on an open set $D_i \supseteq K_i$. Let $\mathbf{p} \in \mathbf{K}$; we introduce the notations

$$\nabla \mathbf{f}(\mathbf{p}) = (\frac{\partial f_1}{\partial p_1}(\mathbf{p}), ..., \frac{\partial f_n}{\partial p_n}(\mathbf{p})) \quad \text{and} \quad \langle \nabla \mathbf{f}(\mathbf{p}), \mathbf{q} \rangle = \sum_{i=1}^{n} \langle \frac{\partial f_i}{\partial p_i}(\mathbf{p}), q_i \rangle,$$

where $\langle \cdot, \cdot \rangle$ inside the sum denotes the duality pair between X and X^*.

A point $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{K}$ is a *Nash-Stampacchia point for* $(\mathbf{f}, \mathbf{K})$ if

$$\langle \nabla \mathbf{f}(\mathbf{p}), \mathbf{q} - \mathbf{p} \rangle \geq 0 \quad \text{for all } \mathbf{q} \in \mathbf{K}.$$

In a similar way, a point $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{K}$ is a *Nash-Minty point for* $(\mathbf{f}, \mathbf{K})$ if

$$\langle \nabla \mathbf{f}(\mathbf{q}), \mathbf{q} - \mathbf{p} \rangle \geq 0 \quad \text{for all } \mathbf{q} \in \mathbf{K}.$$

Theorem 16.2 *(a) Every Nash equilibrium point for $(\mathbf{f}, \mathbf{K})$ is a Nash-Stampacchia point for $(\mathbf{f}, \mathbf{K})$.*

(b) Every Nash-Minty equilibrium point for $(\mathbf{f}, \mathbf{K})$ *is a Nash equilibrium point for* $(\mathbf{f}, \mathbf{K})$.

(c) If each f_i *is convex in its* i^{th} *variable, then the converses also hold in (a) and (b).*

Proof The proof easily follows from Lemmas 13.1 and 13.2. □

Theorem 16.3 *Assume that* $\nabla \mathbf{f}$ *is a strictly monotone operator. Then, the following assertions hold true:*

(a) $(\mathbf{f}, \mathbf{K})$ *has at most one Nash-Stampacchia point.*

(b) $(\mathbf{f}, \mathbf{K})$ *has at most one Nash equilibrium point.*

Proof (a) Assume that $\mathbf{p_1} \neq \mathbf{p_2} \in \mathbf{K}$ are Nash-Stampacchia points for $(\mathbf{f}, \mathbf{K})$. In particular, we clearly have that

$$\langle \nabla \mathbf{f}(\mathbf{p_1}), \mathbf{p_2} - \mathbf{p_1} \rangle \geq 0,$$

$$\langle \nabla \mathbf{f}(\mathbf{p_2}), \mathbf{p_1} - \mathbf{p_2} \rangle \geq 0.$$

Therefore, by the strict monotonicity assumptions we have

$$0 \leq \langle \nabla \mathbf{f}(\mathbf{p_1}) - \nabla \mathbf{f}(\mathbf{p_2}), \mathbf{p_2} - \mathbf{p_1} \rangle < 0,$$

contradiction.

(b) It follows from Theorem 16.2 (a). □

Usually, the set of Nash-Stampacchia points and the set of Nash equilibrium points are different. A simple example which support this statement is presented in what follows, see also [153].

Example 16.1 Let $K_1 = K_2 = [-1, 1]$, $D_1 = D_2 = \mathbb{R}$, and $f_1, f_2 : D_1 \times D_2 \to \mathbb{R}$ defined by

$$f_1(x_1, x_2) = x_1^2 x_2 + x_1, \quad f_2(x_1, x_2) = -x_1^2 x_2^2 + x_2.$$

One can prove that the set of Nash-Stampacchia points for $(\mathbf{f}, \mathbf{K})$ is

$$\{(1/2, -1), (-1, 1/2), (-1, -1), (1, -1)\}.$$

However, the unique Nash equilibrium point for $(\mathbf{f}, \mathbf{K})$ is $(-1, -1)$.

In the rest of this section we assume that X is a prehilbert space. Consequently, the inequality from the definition of Nash-Stampacchia point for $(\mathbf{f}, \mathbf{K})$ can be written equivalently as

$$\langle (\mathbf{p} - \alpha \nabla \mathbf{f}(\mathbf{p})) - \mathbf{p}, \mathbf{q} - \mathbf{p} \rangle \leq 0 \text{ for all } \mathbf{q} \in \mathbf{K}, \ \alpha > 0.$$

Furthermore, in terms of projections, the Moskovitz-Dines type relation, see relation 15.2, gives actually that

$$\mathbf{p} \in P_{\mathbf{K}}(\mathbf{p} - \alpha\nabla\mathbf{f}(\mathbf{p})) \text{ for all } \alpha > 0.$$

Note that since $\mathbf{K}$ is convex, it is also a Chebyshev set, i.e., the above relation has the form

$$0 = P_{\mathbf{K}}(\mathbf{p} - \alpha\nabla\mathbf{f}(\mathbf{p})) - \mathbf{p} \text{ for all } \alpha > 0.$$

The last relation motivates the study of the following continuous dynamical system

$$\dot{\mathbf{x}}(t) = P_{\mathbf{K}}(\mathbf{x}(t) - \alpha\nabla\mathbf{f}(\mathbf{x}(t))) - \mathbf{x}(t), \tag{16.1}$$

where $\alpha > 0$ is fixed. Following Cavazzuti, Pappalardo and Passacantando [62], we prove

Theorem 16.4 *Assume that* $\nabla\mathbf{f}(\mathbf{p}) = 0$. *If* $\nabla\mathbf{f}$ *is strongly monotone on* $\mathbf{K}$ *with constant* $\eta > 0$ *and Lipschitz continuous on* $\mathbf{K}$ *with constant* $L > 0$*, then there exists* $\alpha_0 > 0$ *such that for every* $\alpha \in (0, \alpha_0)$ *there exists a constant* $C > 0$ *such that for each solution* $\mathbf{x}(t)$ *of (16.1), with* $\mathbf{x}(0) \in \mathbf{K}$*, we have*

$$\|\mathbf{x}(t) - \mathbf{p}\| \le \|\mathbf{x}(0) - \mathbf{p}\| \exp(-Ct), \quad \forall t \ge 0,$$

i.e., $\mathbf{x}(t)$ *converges exponentially to* $\mathbf{p} \in \mathbf{K}$.

Proof First of all, due to the non-expansiveness of the projection operator $P_{\mathbf{K}}$, we have that

$$\|P_{\mathbf{K}}(\mathbf{x}(t) - \alpha\nabla\mathbf{f}(\mathbf{x}(t))) - \mathbf{p}\|^2 =$$

$$= \|P_{\mathbf{K}}(\mathbf{x}(t) - \alpha\nabla\mathbf{f}(\mathbf{x}(t))) - P_{\mathbf{K}}(\mathbf{p} - \alpha\nabla\mathbf{f}(\mathbf{p}))\|^2 \le$$

$$\le \|\mathbf{x}(t) - \mathbf{p} - \alpha(\nabla\mathbf{f}(\mathbf{x}(t)) - \nabla\mathbf{f}(\mathbf{p}))\|^2$$

$$= \|\mathbf{x}(t) - \mathbf{p}\|^2 - 2\alpha\langle\nabla\mathbf{f}(\mathbf{x}(t)) - \nabla\mathbf{f}(\mathbf{p}), \mathbf{x}(t) - \mathbf{p}\rangle + \alpha^2\|\nabla\mathbf{f}(\mathbf{x}(t)) - \nabla\mathbf{f}(\mathbf{p})\|^2$$

$$\le (1 - 2\alpha\eta + \alpha^2 L^2)\|\mathbf{x}(t) - \mathbf{p}\|^2.$$

Let $\alpha_0 = 2\eta/L^2$ and $\alpha \in (0, \alpha_0)$. Note that the whole orbit of $\mathbf{x}(t)$ remains in $\mathbf{K}$. We consider the function

$$h(t) = \frac{1}{2}\|\mathbf{x}(t) - \mathbf{p}\|^2.$$

Then, we have

$$\begin{aligned}
\dot{h}(t) &= \langle \mathbf{x}(t)-\mathbf{p}, P_{\mathbf{K}}(\mathbf{x}(t)-\alpha\nabla\mathbf{f}(\mathbf{x}(t)))-\mathbf{x}(t)\rangle \\
&= \langle \mathbf{x}(t)-\mathbf{p}, P_{\mathbf{K}}(\mathbf{x}(t)-\alpha\nabla\mathbf{f}(\mathbf{x}(t)))-\mathbf{p}\rangle - \|\mathbf{x}(t)-\mathbf{p}\|^2 \\
&\le \|\mathbf{x}(t)-\mathbf{p}\|\cdot\|P_{\mathbf{K}}(\mathbf{x}(t)-\alpha\nabla\mathbf{f}(\mathbf{x}(t)))-\mathbf{p}\| - \|\mathbf{x}(t)-\mathbf{p}\|^2 \\
&\le (\sqrt{1-2\alpha\eta+\alpha^2L^2}-1)\|\mathbf{x}(t)-\mathbf{p}\|^2 \\
&= -2Ch(t),
\end{aligned}$$

where $C = 1-\sqrt{1-2\alpha\eta+\alpha^2L^2} > 0$. Consequently, we have $\dot{h}(t) \le -2Ch(t)$, with $h(0) = \frac{1}{2}\|\mathbf{x}(0)-\mathbf{p}\|^2$. After an integration, we have that

$$\|\mathbf{x}(t)-\mathbf{p}\| \le \|\mathbf{x}(0)-\mathbf{p}\|\exp(-Ct), \quad \forall t \ge 0.$$

The proof is complete. $\square$

16.3 Nash equilibria for set-valued maps

In some cases the objective/payoff functions are not single-valued, i.e., the values of a payoff functions are not real numbers but some sets. Set-valued versions of the Nash equilibrium problem can be founded in the literature, see for example Guillerme [134] and Luo [196]. In this section we propose a new approach in the study of the existence of Nash equilibria for set-valued maps which is based on the contingent derivative.

Let $K_1,\ldots,K_n$ be nonempty subsets of a real normed space X, and let $F_i : \mathbf{K} \rightsquigarrow \mathbb{R}$ ($i \in \{1,\ldots,n\}$) be set-valued maps with nonempty compact values. A point $\mathbf{p} = (p_1,\ldots,p_n) \in \mathbf{K}$ is a *Nash equilibrium point for* $(\mathbf{F},\mathbf{K}) = (F_1,\ldots,F_n;K_1,\ldots,K_n)$ if $\mathbf{p}$ is a Nash equilibrium point for $(\min\mathbf{F},\mathbf{K})$.

The following result is an easy consequence of Theorem 16.1.

Theorem 16.5 *Let $K_1,\ldots,K_n$ be nonempty compact convex subsets of a real normed space X, and let $F_i : \mathbf{K} \rightsquigarrow \mathbb{R}$ ($i \in \{1,\ldots,n\}$) be continuous set-valued maps on $\mathbf{K}$ with nonempty compact values such that $q_i \in K_i \rightsquigarrow F_i(\mathbf{p};q_i)$ is convex on K_i for all fixed $p_j \in K_j$ ($j \neq i$). Then there exists a Nash equilibrium point for $(\mathbf{F},\mathbf{K})$.*

Proof Let $f_i = \min F_i$ ($i \in \{1,\ldots,n\}$). It is easy to prove that the functions f_i are continuous and convex in the i^{th} variable. We apply Theorem 16.1. $\square$

Let $K = K_1 = K_2 = [-1, 1]$, $X = \mathbb{R}$, and $F_1, F_2 : K \times K \rightsquigarrow \mathbb{R}$ be defined by

$$F_1(x_1, x_2) = [\max\{|x_1|, |x_2|\} - 1, 0]; \quad (16.2)$$
$$F_2(x_1, x_2) = [1 - \max\{|x_1|, |x_2|\}, 2 - x_1^2 - x_2^2]. \quad (16.3)$$

Clearly, F_1 and F_2 are continuous on $K \times K$, but $x_2 \in K \rightsquigarrow F_2(0, x_2)$ is not convex on K. Therefore, Theorem 16.5 can not be applied. However, we will see in Example 16.2 that there are Nash equilibrium points for $(\mathbf{F}, \mathbf{K}) = (F_1, F_2; K, K)$. To determine these points we elaborate a method by using the contingent derivative of set-valued maps.

Let $K_1, \ldots, K_n$ be nonempty convex subsets of a real normed space X, and let $F_i : K_1 \times \ldots \times X \times \ldots \times K_n \rightsquigarrow \mathbb{R}$ $(i \in \{1, \ldots, n\})$ be set-valued maps with compact, nonempty values. Here, X is in the i^{th} position. Let be a fixed element. We can define the (partial) contingent derivative of F_i in the i^{th} variable at the point $(\mathbf{p}, \min F_i(\mathbf{p}))$, i.e., the contingent derivative of $F_i(p_1, \ldots, p_{i-1}, \cdot, p_{i+1}, \ldots, p_n)$ at the point $(p_i, \min F_i(\mathbf{p}))$, see Appendix, Definition E.4.

Definition 16.1 A point $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{K}$ is said to be a Nash contingent point for $(\mathbf{F}, \mathbf{K})$ if

$$D_i F_i(\mathbf{p}, \min F_i(\mathbf{p}))(q_i - p_i) \subseteq \mathbb{R}_+$$

for all $q_i \in K_i$ and all $i \in \{1, \ldots, n\}$.

Proposition 16.1 *Let* $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{K}$ *be a Nash equilibrium point for* $(\mathbf{F}, \mathbf{K})$). *If each map* $q_i \rightsquigarrow F_i(\mathbf{p}; q_i)$ $(i \in \{1, \ldots, n\})$ *is* K_i*-locally Lipschitz, then* $\mathbf{p}$ *is a Nash contingent point for* $(\mathbf{F}, \mathbf{K})$.

Proof Since $\mathbf{p}$ is a Nash equilibrium point for $(\mathbf{F}, \mathbf{K})$, we have

$$F_i(\mathbf{p}; q_i) - \min F_i(\mathbf{p}) \subseteq \mathbb{R}_+ \quad (16.4)$$

for all $q_i \in K_i$ and all $i \in \{1, \ldots, n\}$.

Let $i \in \{1, \ldots, n\}$, $q_i \in K_i$ be fixed elements and we fix arbitrarily $c_i \in D_i F_i(\mathbf{p}, \min F_i(\mathbf{p}))(q_i - p_i)$. We prove that $c_i \geq 0$. From (E.2) we have that

$$\liminf_{t \to 0^+} \operatorname{dist}\left(c_i, \frac{F_i(\mathbf{p}; p_i + t(q_i - p_i)) - \min F_i(\mathbf{p})}{t}\right) = 0, \quad (16.5)$$

since the map $v_i \rightsquigarrow F_i(\mathbf{p}; v_i)$ is K_i-locally Lipschitz. Since K_i is convex, for $t > 0$ small enough, $y_i^t = p_i + t(q_i - p_i) \in K_i$. Using (16.4), we have that

$$\frac{F_i(\mathbf{p}; y_i^t) - \min F_i(\mathbf{p})}{t} \subseteq \mathbb{R}_+.$$

Suppose that $c_i < 0$. Then

$$0 < |c_i| = \mathrm{dist}(c_i, \mathbb{R}_+) \leq \mathrm{dist}\left(c_i, \frac{F_i(\mathbf{p}; y_i^t) - \min F_i(\mathbf{p})}{t}\right),$$

which is in contradiction with (16.5). □

The above proposition allows to select the Nash equilibrium points from the set of Nash contingent points. In most of the cases the determination of contingent Nash points is much more easier than those of the Nash equilibrium points. We will present two examples in this direction. We will use the notation

$$\lfloor a, b \rfloor = [\min\{a, b\}, \max\{a, b\}], \text{ where } a, b \in \mathbb{R}.$$

Example 16.2 Let us consider the set-valued maps from (16.2) and (16.3) in extended forms, i.e. $K = K_1 = K_2 = [-1, 1]$, $X = \mathbb{R}$ and let $F_1 : X \times K \rightsquigarrow \mathbb{R}$, $F_2 : K \times X \rightsquigarrow \mathbb{R}$ be defined by

$$F_1(x_1, x_2) = \lfloor \max\{|x_1|, |x_2|\} - 1, 0 \rfloor;$$

$$F_2(x_1, x_2) = \lfloor 1 - \max\{|x_1|, |x_2|\}, 2 - x_1^2 - x_2^2 \rfloor.$$

It is easy to verify that $u \in K \rightsquigarrow F_1(u, x_2)$ and $v \in K \rightsquigarrow F_2(x_1, v)$ are K-locally Lipschitz maps ($x_1, x_2 \in K$ being fixed points). We are first looking for those points $(x_1, x_2) \in K \times K$ which fulfill

$$D_1F_1((x_1, x_2), \min F_1(x_1, x_2))(v_1 - x_1) \subseteq \mathbb{R}_+ \text{ for all } v_1 \in K; \quad (16.6)$$
$$D_2F_2((x_1, x_2), \min F_2(x_1, x_2))(v_2 - x_2) \subseteq \mathbb{R}_+ \text{ for all } v_2 \in K. \quad (16.7)$$

Using the geometric meaning of the contingent derivative (see relation (E.1)), after an elementary discussion and computation, the points which satisfy the inclusions (16.6) and (16.7) respectively, are

$$CNP_1 = \{(x_1, x_2) \in K \times K : |x_1| \leq |x_2|\};$$

$$CNP_2 = \{(x_1, x_2) \in K \times K : |x_1| > |x_2|\} \cup \{(x_1, x_2) \in K \times K : |x_2| = 1\}.$$

Therefore, the contingent Nash points for $(\mathbf{F}, \mathbf{K}) = (F_1, F_2; K, K)$ are

$$CNP = CNP_1 \cap CNP_2 = \{(x_1, x_2) \in K \times K : |x_2| = 1\}.$$

A direct computation shows that the set of Nash equilibrium points for $(\mathbf{F}, \mathbf{K})$ coincides the set CNP.

Example 16.3 Let $K = [-1, 1]$, and let $F_1 : \mathbb{R} \times K \rightsquigarrow \mathbb{R}$ and $F_2 : K \times \mathbb{R} \rightsquigarrow \mathbb{R}$ be defined by

$$F_1(x_1, x_2) = \begin{cases} \lfloor 2, |x_1 - 1| \rfloor, & \text{if } x_2 \geq 0, \\ \lfloor 2, |x_1 + 1| \rfloor, & \text{if } x_2 < 0, \end{cases}$$

and

$$F_2(x_1, x_2) = \begin{cases} \lfloor 2, |x_2 + 1| \rfloor, & \text{if } x_1 \geq 0, \\ \lfloor 2, |x_2 - 1| \rfloor, & \text{if } x_1 < 0. \end{cases}$$

The points which satisfy the corresponding inclusions from (16.6) and (16.7) respectively, are

$$CNP_1 = \{(1, x_2) : x_2 \in [0, 1]\} \cup \{(-1, x_2) : x_2 \in [-1, 0)\};$$

$$CNP_2 = \{(x_1, -1) : x_1 \in [0, 1]\} \cup \{(x_1, 1) : x_1 \in [-1, 0)\}.$$

The maps $u \in K \rightsquigarrow F_1(u, x_2)$ and $v \in K \rightsquigarrow F_2(x_1, v)$ are K-locally Lipschitz ($x_1, x_2 \in K$ being fixed points), however the set of Nash contingent points for $(\mathbf{F}, \mathbf{K}) = (F_1, F_2; K, K)$ is $CNP = CNP_1 \cap CNP_2$, which is empty. Hence the set of Nash equilibrium points is empty too, due to Proposition 16.1.

Note that the set-valued maps from Examples 16.2 and 16.3 are not continuous on $K \times K$. In the sequel, we give sufficient conditions to obtain Nash contingent points, assuming some continuity hypotheses on the set-valued maps. Before to do this, we can state the converse of Proposition 16.1 by taking a convexity assumption for the corresponding set-valued maps. Let X be a normed space, and let K be a nonempty subset of X.

Definition 16.2 The set-valued map $F : X \rightsquigarrow \mathbb{R}$ is called K-pseudo-convex at $(u, c) \in \text{Graph}(F)$ if

$$F(u') \subseteq c + DF(u, c)(u' - u) \text{ } for \text{ } all \text{ } u' \in K.$$

In the case when $K = \text{Dom}(F)$, the above definition reduces to [16, Definition 5.1.1]. The following result can be easily proved.

Proposition 16.2 *Let $K_1, \ldots, K_n$ be nonempty subsets of a real normed* *space X, and let $F_i : K_1 \times \ldots \times X \times \ldots \times K_n \rightsquigarrow \mathbb{R}$ $(i \in \{1, \ldots, n\})$ be set-valued maps with nonempty compact values. Let $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{K}$ be a Nash contingent point for $(\mathbf{F}, \mathbf{K})$. If $q_i \rightsquigarrow F_i(\mathbf{p}; q_i)$ $(i \in \{1, \ldots, n\})$ is K_i-pseudo-convex at $(p_i, \min F_i(\mathbf{p}))$, then $\mathbf{p}$ is a Nash equilibrium point for $(\mathbf{F}, \mathbf{K})$.*

We give and existence theorem concerning the Nash contingent points for a system $(\mathbf{F}, \mathbf{K})$ by using a Ky Fan type result proved by Kristály and Varga [176]:

Theorem 16.6 *Let $\tilde{X}$ be a real normed space, let K be a nonempty convex compact subset of $\tilde{X}$, and let $F : K \times K \rightsquigarrow \mathbb{R}$ be a set-valued map satisfying the following conditions:*
(i) for all $v \in K$, $u \in K \rightsquigarrow F(u, v)$ is lower semicontinuous on K;
(ii) for all $u \in K$, $v \in K \rightsquigarrow F(u, v)$ is convex on K;
(iii) for all $u \in K$, $F(u, u) \subseteq \mathbb{R}_+$.
Then there exists an element $\overline{u} \in K$ such that

$$F(\overline{u}, v) \subseteq \mathbb{R}_+ \text{ for all } v \in K. \tag{16.8}$$

Let K_i and F_i $(i \in \{1, \ldots, n\})$ be as in Proposition 16.2. We denote by $F_i|_{\mathbf{K}}$ the restriction of F_i to $\mathbf{K} = K_1 \times \ldots \times K_i \times \ldots \times K_n$.

Definition 16.3 $F_i|_K$ is called i-lower semicontinuously differentiable if

$$\mathrm{Graph}(F_i|_{\mathbf{K}}) \times X \ni (\mathbf{p}, c, h) \rightsquigarrow D_i F_i(\mathbf{p}, c)(h)$$

is lower semicontinuous.

Now, we are in the position to give an existence result concerning the Nash contingent points.

Theorem 16.7 *Let $K_1, \ldots, K_n$ be nonempty compact convex subsets of a real normed space X and let $F_i : K_1 \times \ldots \times X \times \ldots \times K_n \rightsquigarrow \mathbb{R}$ $(i \in \{1, \ldots, n\})$ be set-valued maps with nonempty compact values and with closed graph. Suppose that for each $i \in \{1, \ldots, n\}$ the map F_i is K_i-locally Lipschitz in the i^{th} variable, $F_i|_{\mathbf{K}}$ is continuous on $\mathbf{K}$ and i-lower semicontinuously differentiable. Then there exists a Nash contingent point for $(\mathbf{F}, \mathbf{K})$.*

Proof Let $\tilde{X} = X \times \ldots \times X$. Clearly, $\mathbf{K}$ is a compact, convex subset of

$\tilde{X}$. We define the map $G : \mathbf{K} \times \mathbf{K} \rightsquigarrow \mathbb{R}$ by

$$G(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{n} D_i F_i(\mathbf{p}, \min F_i(\mathbf{p}))(q_i - p_i),$$

where $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$. We will verify the hypotheses from Theorem 16.6.

(i) Let us fix $\mathbf{q} \in \mathbf{K}$. Since the sum of finite lower semicontinuous maps is lower semicontinuous, it is enough to prove that $\mathbf{p} \in \mathbf{K} \rightsquigarrow D_i F_i(\mathbf{p}, \min F_i(\mathbf{p}))(q_i - p_i)$ is lower semicontinuous on $\mathbf{K}$ for each $i \in \{1, \ldots, n\}$. To do this, let us fix an $\mathbf{p} \in \mathbf{K}$. Now, let $c_i \in D_i F_i(\mathbf{p}, \min F_i(\mathbf{p}))(q_i - p_i)$ and $\{\mathbf{p}^m\}$ be a sequence from $\mathbf{K}$ which converges to $\mathbf{p}$. Since $F_i|_{\mathbf{K}}$ is continuous on $\mathbf{K}$, the function $\min(F_i|_{\mathbf{K}})$ is also continuous. Therefore $\min F_i(\mathbf{p}^m) \to \min F_i(\mathbf{p})$ as $m \to \infty$. Using the fact that $F_i|_{\mathbf{K}}$ is i-lower semicontinuously differentiable, there exists a sequence $c_i^m \in D_i F_i(\mathbf{p}^m, \min F_i(\mathbf{p}^m))(q_i - p_i^m)$ such that $c_i^m \to c_i$ as $m \to \infty$.

(ii) Let us consider $\mathbf{p} \in \mathbf{K}$. fixed. We will prove that $\mathbf{q} \in \mathbf{K} \rightsquigarrow G(\mathbf{p}, \mathbf{q})$ is convex on $\mathbf{K}$. Since the sum of convex maps is also convex, it is enough to prove that $\mathbf{q} \in \mathbf{K} \rightsquigarrow D_i F_i(\mathbf{p}, \min F_i(\mathbf{p}))(q_i - p_i)$ is convex. Using the fact that $F_i|_{\mathbf{K}}$ is i-lower semicontinuously differentiable (in particular $q_i \rightsquigarrow F_i(\mathbf{p}; q_i)$ is sleek at $(p_i, \min F_i(\mathbf{p})))$ and Remark E.2, the above set-valued map is a closed convex process.

(iii) Let $\mathbf{p} \in \mathbf{K}$ and $i \in \{1, \ldots, n\}$ be fixed. We prove that every element c_i from $D_i F_i(\mathbf{p}, \min F_i(\mathbf{p}))(0)$ is non-negative. In fact, since $q_i \rightsquigarrow F_i(\mathbf{p}; q_i)$ is K_i-locally Lipschitz, from (E.2) we have

$$\liminf_{t \to 0^+} \operatorname{dist}\left(c_i, \frac{F_i(\mathbf{p}; p_i + t \cdot 0) - \min F_i(\mathbf{p})}{t}\right) = 0.$$

Since

$$\frac{F_i(\mathbf{p}) - \min F_i(\mathbf{p})}{t} \subseteq \mathbb{R}_+ \text{ for all } t > 0,$$

we obtain that $c_i \geq 0$. Therefore, $G(\mathbf{p}, \mathbf{p}) = \sum_{i=1}^{n} D_i F_i(\mathbf{p}, \min F_i(\mathbf{p}))(0) \subseteq \mathbb{R}_+$.

By Theorem 16.6 we obtain an element $\overline{\mathbf{p}} = (\overline{p}_1, \ldots, \overline{p}_n) \in \mathbf{K}$ such that

$$\sum_{i=1}^{n} D_i F_i(\overline{\mathbf{p}}, \min F_i(\overline{\mathbf{p}}))(q_i - \overline{p}_i) \subseteq \mathbb{R}_+ \text{ for all } \mathbf{q} \in \mathbf{K}. \tag{16.9}$$

Let $i \in \{1, \ldots, n\}$ be fixed. We may choose $q_j = \overline{p}_j,\ j \neq i$. Since the (partial) contingent derivatives are closed convex process, we clearly have that $0 \in D_j F_j(\overline{\mathbf{p}}, \min F_j(\overline{\mathbf{p}}))(0),\ j \neq i$. From (16.9), we obtain

$$D_i F_i(\overline{\mathbf{p}}, \min F_i(\overline{\mathbf{p}}))(q_i - \overline{p}_i) \subseteq \mathbb{R}_+ \text{ for all } q_i \in K_i.$$

The proof is complete. $\square$

Using Remark E.1, we immediately obtain a consequence of the above theorem. A similar result was obtained by Kassay, Kolumbán and Páles [153].

Corollary 16.1 *Let $K_1, \ldots, K_n$ be nonempty compact convex subsets of a real normed space X and let $f_i : K_1 \times \ldots \times X \times \ldots \times K_n \to \mathbb{R}$ ($i \in \{1, \ldots, n\}$) be continuous functions. Suppose that there exist open convex sets $D_i \subseteq X$ such that $K_i \subseteq D_i$ and f_i is continuously differentiable in the i^{th} variable on D_i and $\partial_i f_i$ is continuous on $K_1 \times \ldots \times D_i \times \ldots \times K_n$. Then there exists an element $\mathbf{p} \in \mathbf{K}$ such that*

$$\langle \partial_i f_i(\mathbf{p}), q_i - p_i \rangle \geq 0$$

for all $q_i \in K_i$ and all $i \in \{1, \ldots, n\}$.

Example 16.4 Let $K = K_1 = K_2 = [-1, 1]$, $X = D_1 = D_2 = \mathbb{R}$, and let $f_1, f_2 : D_1 \times D_1 \to \mathbb{R}$ be defined by

$$f_1(x_1, x_2) = x_1 x_2 + x_1^2, \quad f_2(x_1, x_2) = x_2 - 3x_1 x_2^2.$$

Clearly, the above sets and functions satisfy the assumptions from Corollary 16.1, hence a solution for the above system is guaranteed. It is easy to observe that, for $i \in \{1, 2\}$, we have $\partial_i f_i(x_1, x_2)(v_i - x_i) \geq 0$ for all $v_i \in K$ if and only if

- $\partial_i f_i(x_1, x_2) = 0$ if $-1 < x_i < 1$;
- $\partial_i f_i(x_1, x_2) \geq 0$, if $x_i = -1$ and
- $\partial_i f_i(x_1, x_2) \leq 0$ if $x_i = 1$.

Discussing all the cases, we establish that the set of Nash contingent points for $(\mathbf{f}, \mathbf{K}) = (f_1, f_2; K, K)$ is $CNP = \{(\frac{1}{2}, -1)\}$. This point will be also a Nash equilibrium point for $(\mathbf{f}, \mathbf{K})$.

16.4 Lack of convexity: a Riemannian approach

In this section we are going to treat a case when the strategy sets K_i are *not* necessarily convex in the usual sense. In order to handle this problem a geometrical method will be exploited: we assume that one can find suitable Riemannian manifolds (M_i, g_i) such that the set K_i becomes a geodesic convex set in M_i. Note that the choice of such Riemannian structures does not influence the Nash equilibrium points for $(\mathbf{f}, \mathbf{K})$. Moreover, after fixing these manifolds if the payoff functions f_i become *convex on* K_i (i.e., $f_i \circ \gamma_i : [0,1] \to \mathbb{R}$ is convex in the usual sense for every geodesic $\gamma_i : [0,1] \to K_i$), the following existence result may be stated.

Theorem 16.8 *Let* (M_i, g_i) *be finite-dimensional Riemannian manifolds,* $K_i \subset M_i$ *be nonempty, compact, geodesic convex sets, and* $f_i : \mathbf{K} \to \mathbb{R}$ *be continuous functions such that* $K_i \ni q_i \mapsto f_i(\mathbf{p}; q_i)$ *is convex on* K_i *for every* $\mathbf{p} \in \mathbf{K}$, $i \in \{1, \ldots, n\}$. *Then, there exists a Nash equilibrium point for* $(\mathbf{f}, \mathbf{K})$.

Proof We apply Theorem 13.5 by choosing $X = \mathbf{K} = \Pi_{i=1}^n K_i$ and $h : X \times X \to \mathbb{R}$ defined by $h(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^n [f_i(\mathbf{p}; q_i) - f_i(\mathbf{p})]$. First of all, note that the sets K_i are ANRs, being closed subsets of finite-dimensional manifolds (thus, locally contractible spaces). Moreover, since a product of a finite family of ANRs is an ANR (see Bessage and Pelczyński [37, p. 69]), it follows that X is an ANR. Due to Proposition 13.2, X is contractible, thus acyclic.

Note that the function h is continuous, and $h(\mathbf{p}, \mathbf{p}) = 0$ for every $\mathbf{p} \in X$. Consequently, the set $\{(\mathbf{q}, \mathbf{p}) \in X \times X : 0 > h(\mathbf{q}, \mathbf{p})\}$ is open.

It remains to prove that $S_{\mathbf{p}} = \{\mathbf{q} \in X : 0 > h(\mathbf{q}, \mathbf{p})\}$ is contractible or empty for all $\mathbf{p} \in X$. Assume that $S_{\mathbf{p}} \neq \emptyset$ for some $\mathbf{p} \in X$. Then, there exists $i_0 \in \{1, ..., n\}$ such that $f_{i_0}(\mathbf{p}; q_{i_0}) - f_{i_0}(\mathbf{p}) < 0$ for some $q_{i_0} \in K_{i_0}$. Therefore, $\mathbf{q} = (\mathbf{p}; q_{i_0}) \in S_{\mathbf{p}}$, i.e., $\mathrm{pr}_i S_{\mathbf{p}} \neq \emptyset$ for every $i \in \{1, ..., n\}$. Now, we fix $\mathbf{q}^j = (q_1^j, ..., q_n^j) \in S_{\mathbf{p}}$, $j \in \{1, 2\}$ and let $\gamma_i : [0,1] \to K_i$ be the unique geodesic joining the points $q_i^1 \in K_i$ and $q_i^2 \in K_i$ (note that K_i is geodesic convex), $i \in \{1, ..., n\}$. Let $\gamma : [0,1] \to \mathbf{K}$ defined by $\gamma(t) = (\gamma_1(t), ..., \gamma_n(t))$. Due to the convexity of the function $K_i \ni q_i \mapsto f_i(\mathbf{p}; q_i)$, for every $t \in [0,1]$, we have

$$\begin{aligned}
h(\gamma(t), \mathbf{p}) &= \sum_{i=1}^n [f_i(\mathbf{p}; \gamma_i(t)) - f_i(\mathbf{p})] \\
&\leq \sum_{i=1}^n [t f_i(\mathbf{p}; \gamma_i(1)) + (1-t) f_i(\mathbf{p}; \gamma_i(0)) - f_i(\mathbf{p})] \\
&= t h(\mathbf{q}^2, \mathbf{p}) + (1-t) h(\mathbf{q}^1, \mathbf{p}) \\
&< 0.
\end{aligned}$$

Consequently, $\gamma(t) \in S_{\mathbf{p}}$ for every $t \in [0,1]$, i.e., $S_{\mathbf{p}}$ is a geodesic convex set in the product manifold $\mathbf{M} = \Pi_{i=1}^n M_i$ endowed with its natural (warped-)product metric (with the constant weight functions 1), see O'Neill [230, p. 208]. Now, Proposition 13.2 implies that $S_{\mathbf{p}}$ is contractible. Alternatively, we may exploit the fact that the projections $\mathrm{pr}_i S_{\mathbf{p}}$ are geodesic convex, thus contractible sets, $i \in \{1, ..., n\}$.

On account of Theorem 13.5, there exists $\mathbf{p} \in \mathbf{K}$ such that $0 = h(\mathbf{p},\mathbf{p}) \le h(\mathbf{q},\mathbf{p})$ for every $\mathbf{q} \in \mathbf{K}$. In particular, putting $\mathbf{q} = (\mathbf{p}; q_i)$, $q_i \in K_i$ fixed, we obtain that $f_i(\mathbf{p}; q_i) - f_i(\mathbf{p}) \ge 0$ for every $i \in \{1, ..., n\}$, i.e., $\mathbf{p}$ is a Nash equilibrium point for $(\mathbf{f},\mathbf{K})$. □

Our further investigation is motivated by the following two questions:

- What about the case when some payoff functions f_i are *not* convex on K_i in spite of the geodesic convexity of K_i on (M_i, g_i)?
- Even for convex payoff functions f_i on K_i, how can we *localize* the Nash equilibrium points for $(\mathbf{f},\mathbf{K})$?

The following concept is destined for simultaneously handling the above questions.

Let (M_i, g_i) be complete finite-dimensional Riemannian manifolds, $K_i \subset M_i$ be nonempty, geodesic convex sets, and $f_i : (\mathbf{K}; D_i) \to \mathbb{R}$ functions such that $D_i \ni q_i \mapsto f_i(\mathbf{p}; q_i)$ is of class C^1 for every $\mathbf{p} \in \mathbf{K}$, where $(\mathbf{K}; D_i) = K_1 \times ... \times D_i \times ... \times K_n$, with D_i open and geodesic convex, and $K_i \subseteq D_i \subseteq M_i$, $i \in \{1, \ldots, n\}$.

A point $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{K}$ is a *Nash critical point for* $(\mathbf{f},\mathbf{K})$ if

$$g_i(\partial_i f_i(\mathbf{p}), \exp_{p_i}^{-1}(q_i)) \ge 0 \;\text{ for all } q_i \in K_i,\; i \in \{1, \ldots, n\}.$$

Here, $\partial_i f_i(\mathbf{p})$ denotes the i^{th} partial derivative of f_i at the point $p_i \in K_i$.

A useful relation between Nash equilibrium points and Nash critical points is established by the following result.

Proposition 16.3 *Any Nash equilibrium point for* $(\mathbf{f},\mathbf{K})$ *is a Nash critical point for* $(\mathbf{f},\mathbf{K})$*. In addition, if* $D_i \ni q_i \mapsto f_i(\mathbf{p}; q_i)$ *is convex for every* $\mathbf{p} \in \mathbf{K}$*,* $i \in \{1, \ldots, n\}$*, the converse also holds.*

Proof Let $\mathbf{p} \in \mathbf{K}$ be a Nash equilibrium point for $(\mathbf{f},\mathbf{K})$. In particular, for every $q_i \in K_i$ with $i \in \{1, \ldots, n\}$ fixed, the geodesic segment $t \mapsto \exp_{p_i}(t \exp_{p_i}^{-1}(q_i))$, $t \in [0,1]$ joining the points p_i and q_i, belongs entirely to K_i; thus,

$$f_i(\mathbf{p}; \exp_{p_i}(t \exp_{p_i}^{-1}(q_i))) - f_i(\mathbf{p}) \ge 0 \;\text{ for all } t \in [0,1]. \tag{16.10}$$

Combining relation (16.10) with

$$g_i(\partial_i f_i(\mathbf{p}), \exp_{p_i}^{-1}(q_i)) = \lim_{t\to 0^+} \frac{f_i(\mathbf{p}; \exp_{p_i}(t \exp_{p_i}^{-1}(q_i))) - f_i(\mathbf{p})}{t},$$

we indeed have that $\mathbf{p} \in \mathbf{K}$ is a Nash critical point for $(\mathbf{f}, \mathbf{K})$.

Now, let us assume that $\mathbf{p} \in \mathbf{K}$ is a Nash critical point for $(\mathbf{f}, \mathbf{K})$, i.e.,

$$0 \le g_i(\partial_i f_i(\mathbf{p}), \exp_{p_i}^{-1}(q_i)) = \lim_{t\to 0^+} \frac{f_i(\mathbf{p}; \exp_{p_i}(t \exp_{p_i}^{-1}(q_i))) - f_i(\mathbf{p})}{t}. \tag{16.11}$$

The function

$$g(t) = \frac{f_i(\mathbf{p}; \exp_{p_i}(t \exp_{p_i}^{-1}(q_i))) - f_i(\mathbf{p})}{t}$$

is well-defined on the whole interval $(0, 1]$; indeed, $t \mapsto \exp_{p_i}(t \exp_{p_i}^{-1}(q_i))$ is the minimal geodesic joining the points $p_i \in K_i$ and $q_i \in K_i$ which belongs to $K_i \subset D_i$. Moreover, a standard computation shows that $t \mapsto g(t)$ is non-decreasing on $(0, 1]$ due to the convexity of $D_i \ni q_i \mapsto f_i(\mathbf{p}; q_i)$. Consequently, (16.11) implies that

$$0 \le \lim_{t\to 0^+} g(t) \le g(1) = f_i(\mathbf{p}; \exp_{p_i}(\exp_{p_i}^{-1}(q_i))) - f_i(\mathbf{p}) = f_i(\mathbf{p}; q_i) - f_i(\mathbf{p}),$$

which completes the proof. $\square$

In order to state our existence result concerning Nash critical points, we consider the hypothesis

(H) $K_i \ni q_i \mapsto g_i(\partial_i f_i(\mathbf{p}), \exp_{p_i}^{-1}(q_i))$ *is convex for every* $\mathbf{p} \in \mathbf{K}$ *and* $i \in \{1, ..., n\}$.

Remark 16.1 Let $I_1, I_2 \subseteq \{1, ..., n\}$ be such that $I_1 \cup I_2 = \{1, ..., n\}$. Hypothesis (H) holds, for instance, when

- (M_i, g_i) is Euclidean, $i \in I_1$;
- $K_i = \mathrm{Im}\gamma_i$ where $\gamma_i : [0, 1] \to M_i$ is a minimal geodesic, $i \in I_2$.

(a) In the first case, we have $\exp_{p_i} = \mathrm{id}_{\mathbb{R}^{\dim M_i}}$; in this case $K_i \ni q_i \mapsto \langle \partial_i f_i(\mathbf{p}), q_i - p_i \rangle$ becomes affine, so convex.

(b) In the second case, if $\sigma_i : [0, 1] \to M_i$ is a geodesic segment joining the points $\sigma_i(0) = \gamma_i(\tilde{t}_0)$ with $\sigma_i(1) = \gamma_i(\tilde{t}_1)$ $(0 \le \tilde{t}_0 < \tilde{t}_1 \le 1)$, then $\mathrm{Im}\sigma_i \subseteq \mathrm{Im}\gamma_i = K_i$. Fix $p_i = \gamma_i(t_i) \in K_i$ $(0 \le \tilde{t}_i \le 1)$. Let $a_0, a_1 \in \mathbb{R}$ $(a_0 < a_1)$ such that $\exp_{p_i}(a_0 \gamma_i'(t_i)) = \gamma_i(\tilde{t}_0)$ and $\exp_{p_i}(a_1 \gamma_i'(t_i)) = \gamma_i(\tilde{t}_1)$. Then, $\sigma_i(t) = \exp_{p_i}((a_0 + (a_1 - a_0)t)\gamma_i'(t_i))$. Now, the claim easily follows since the convexity of function from (H) reduces to the affinity of $t \mapsto g_i(\partial_i f_i(\mathbf{p}), (a_0 + (a_1 - a_0)t)\gamma_i'(t_i))$.

Theorem 16.9 *Let (M_i, g_i) be complete finite-dimensional Riemannian manifolds, $K_i \subset M_i$ be nonempty, compact, geodesic convex sets and $f_i : (\mathbf{K}; D_i) \to \mathbb{R}$ continuous functions such that $D_i \ni q_i \mapsto f_i(\mathbf{p}; q_i)$ is of class C^1 for every $\mathbf{p} \in \mathbf{K}$, $i \in \{1, \ldots, n\}$. If* (H) *holds, there exists a Nash critical point for* $(\mathbf{f}, \mathbf{K})$.

Proof The proof is similar to that of Theorem 16.8; we show only the differences. Let $X = \mathbf{K} = \Pi_{i=1}^n K_i$ and $h : X \times X \to \mathbb{R}$ defined by $h(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^n g_i(\partial_i f_i(\mathbf{p}), \exp_{p_i}^{-1}(q_i))$. It is clear that $h(\mathbf{p}, \mathbf{p}) = 0$ for every $\mathbf{p} \in X$.

First of all, the (upper-semi)continuity of $h(\cdot, \cdot)$ on $X \times X$ implies the fact that the set $\{(\mathbf{q}, \mathbf{p}) \in X \times X : 0 > h(\mathbf{q}, \mathbf{p})\}$ is open.

Now, let $\mathbf{p} \in X$ such that $S_\mathbf{p} = \{\mathbf{q} \in X : 0 > h(\mathbf{q}, \mathbf{p})\}$ is not empty. Then, there exists $i_0 \in \{1, ..., n\}$ such that $g_{i_0}(\partial_{i_0} f_{i_0}(\mathbf{p}), \exp_{p_{i_0}}^{-1}(q_{i_0})) < 0$ for some $q_{i_0} \in K_{i_0}$. Consequently, $\mathbf{q} = (\mathbf{p}; q_{i_0}) \in S_\mathbf{p}$, i.e., $\mathrm{pr}_i S_\mathbf{p} \neq \emptyset$ for every $i \in \{1, ..., n\}$. Now, we fix $\mathbf{q}^j = (q_1^j, ..., q_n^j) \in S_\mathbf{p}$, $j \in \{1, 2\}$, and let $\gamma_i : [0, 1] \to K_i$ be the unique geodesic joining the points $q_i^1 \in K_i$ and $q_i^2 \in K_i$. Let also $\gamma : [0, 1] \to \mathbf{K}$ defined by $\gamma(t) = (\gamma_1(t), ..., \gamma_n(t))$. Due to hypotheses (H), the convexity of the function $[0, 1] \ni t \mapsto h(\gamma(t), \mathbf{p})$, $t \in [0, 1]$ easily follows. Therefore, $\gamma(t) \in S_\mathbf{p}$ for every $t \in [0, 1]$, i.e., $S_\mathbf{p}$ is a geodesic convex set, thus contractible.

Theorem 13.5 implies the existence of $\mathbf{p} \in \mathbf{K}$ such that $0 = h(\mathbf{p}, \mathbf{p}) \leq h(\mathbf{q}, \mathbf{p})$ for every $\mathbf{q} \in \mathbf{K}$. In particular, if $\mathbf{q} = (\mathbf{p}; q_i)$, $q_i \in K_i$ fixed, we obtain that $g_i(\partial_i f_i(\mathbf{p}), \exp_{p_i}^{-1}(q_i)) \geq 0$ for every $i \in \{1, ..., n\}$, i.e., $\mathbf{p}$ is a Nash critical point for $(\mathbf{f}, \mathbf{K})$. The proof is complete. $\square$

Remark 16.2 Proposition 16.3 and Theorem 16.9 give together a possible answer to the location of Nash equilibrium points. Indeed, on account of Theorem 16.9 we are able to find explicitly the Nash critical points for $(\mathbf{f}, \mathbf{K})$; then, due to Proposition 16.3, among Nash critical points for $(\mathbf{f}, \mathbf{K})$ we may choose the Nash equilibrium points for $(\mathbf{f}, \mathbf{K})$.

Remark 16.3 The set of Nash critical points and the set of Nash equilibrium points are usually different. Indeed, let $f_1, f_2 : [-1, 1]^2 \to \mathbb{R}$ be defined by $f_1(x_1, x_2) = f_2(x_1, x_2) = x_1^3 + x_2^3$. Note that $(0, 0)$, $(0, -1)$, $(-1, 0)$ and $(-1, -1)$ are Nash critical points, but only $(-1, -1)$ is a Nash equilibrium point for $(f_1, f_2; [-1, 1], [-1, 1])$.

The following examples show the applicability of our results.

Example 16.5 Let $K_1 = [-1,1]$, $K_2 = \{(\cos t, \sin t) : t \in [\pi/4, 3\pi/4]\}$, and $f_1, f_2 : K_1 \times K_2 \to \mathbb{R}$ defined for every $x \in K_1$, $(y_1, y_2) \in K_2$ by

$$f_1(x,(y_1,y_2)) = |x|y_1^2 - y_2, \quad f_2(x,(y_1,y_2)) = (1-|x|)(y_1^2 - y_2^2).$$

Note that $K_1 \subset \mathbb{R}$ is convex in the usual sense, but $K_2 \subset \mathbb{R}^2$ is not. However, if we consider the Poincaré upper-plane model $(\mathbb{H}^2, g_{\mathbb{H}})$, the set $K_2 \subset \mathbb{H}^2$ is geodesic convex with respect to the metric $g_{\mathbb{H}}$, being the image of a geodesic segment from $(\mathbb{H}^2, g_{\mathbb{H}})$. It is clear that $f_1(\cdot,(y_1,y_2))$ is a convex function on K_1 in the usual sense for every $(y_1,y_2) \in K_2$. Moreover, $f_2(x,\cdot)$ is also a convex function on $K_2 \subset \mathbb{H}^2$ for every $x \in K_1$. Indeed, the latter fact reduces to the convexity of the function $t \mapsto (1-|x|)\cos(2t)$, $t \in [\pi/4, 3\pi/4]$. Therefore, Theorem 16.8 guarantees the existence of at least one Nash equilibrium point for $(f_1, f_2; K_1, K_2)$. Using Proposition 16.3, a simple calculation shows that the set of Nash equilibrium (=critical) points for $(f_1, f_2; K_1, K_2)$ is $K_1 \times \{(0,1)\}$.

Example 16.6 Let $K_1 = [-1,1]^2$, $K_2 = \{(y_1,y_2) : y_2 = y_1^2,\ y_1 \in [0,1]\}$, and $f_1, f_2 : K_1 \times K_2 \to \mathbb{R}$ defined for every $(x_1,x_2) \in K_1$, $(y_1,y_2) \in K_2$ by

$$f_1((x_1,x_2),(y_1,y_2)) = -x_1^2 y_2 + x_2 y_1, \quad f_2((x_1,x_2),(y_1,y_2)) = x_1 y_2^2 + x_2 y_1^2.$$

The set $K_1 \subset \mathbb{R}^2$ is convex, but $K_2 \subset \mathbb{R}^2$ is not in the usual sense. However, K_2 may be considered as the image of a geodesic segment on the paraboloid of revolution $p_{\rm rev}(u,v) = (v\cos u, v\sin u, v^2)$. More precisely, K_2 becomes geodesic convex on $\mathrm{Im}p_{\rm rev}$, being actually a compact part of a meridian on $\mathrm{Im}p_{\rm rev}$. Note that neither $f_1(\cdot,(y_1,y_2))$ nor $f_2((x_1,x_2),\cdot)$ is convex (the convexity of the latter function being considered on $K_2 \subset \mathrm{Im}p_{\rm rev}$); thus, Theorem 16.8 is not applicable. In view of Remark 16.1 (c), Theorem 16.9 can be applied in order to determine the set of Nash critical points. This set is nothing but the set of solutions in the form $((\tilde{x}_1,\tilde{x}_2),(\tilde{y},\tilde{y}^2)) \in K_1 \times K_2$ of the system

$$\begin{cases} -2\tilde{x}_1\tilde{y}^2(x_1-\tilde{x}_1) + \tilde{y}(x_2-\tilde{x}_2) \ge 0, & \forall (x_1,x_2) \in K_1, \\ \tilde{y}(2\tilde{y}^2\tilde{x}_1 + \tilde{x}_2)(y-\tilde{y}) \ge 0, & \forall y \in [0,1]. \end{cases} \tag{System$_{\rm NCP}$}$$

We distinguish three cases: (a) $\tilde{y} = 0$; (b) $\tilde{y} = 1$; and (c) $0 < \tilde{y} < 1$.

(a) $\tilde{y} = 0$. Then, any $((\tilde{x}_1,\tilde{x}_2),(0,0)) \in K_1 \times K_2$ solves (System$_{\rm NCP}$).

(b) $\tilde{y} = 1$. After an easy computation, we obtain that $((-1,-1),(1,1)) \in K_1 \times K_2$ and $((0,-1),(1,1)) \in K_1 \times K_2$ solve (System$_{\rm NCP}$).

(c) $0 < \tilde{y} < 1$. The unique situation when ($\text{System}_{\text{NCP}}$) is solvable is $\tilde{y} = \sqrt{2}/2$. In this case, ($\text{System}_{\text{NCP}}$) has a unique solution $((1,-1),(\sqrt{2}/2,1/2)) \in K_1 \times K_2$.

Consequently, the set of Nash critical points for $(f_1, f_2; K_1, K_2)$, denoted in the sequel by S_{NCP}, is the union of the points from (a), (b) and (c), respectively.

Let us denote by S_{NEP} the set of Nash equilibrium points for $(f_1, f_2; K_1, K_2)$. Due to Proposition 16.3, we may select the elements of S_{NEP} from S_{NCP}. Therefore, the elements of S_{NEP} are the solutions $((\tilde{x}_1,\tilde{x}_2),(\tilde{y},\tilde{y}^2)) \in S_{\text{NCP}}$ of the system

$$\begin{cases} -x_1^2\tilde{y}^2 + x_2\tilde{y} \geq -\tilde{x}_1^2\tilde{y}^2 + \tilde{x}_2\tilde{y}, & \forall (x_1,x_2) \in K_1, \\ \tilde{x}_1 y^4 + \tilde{x}_2 y^2 \geq \tilde{x}_1\tilde{y}^4 + \tilde{x}_2\tilde{y}^2, & \forall y \in [0,1]. \end{cases} \qquad (\text{System}_{\text{NEP}})$$

We consider again the above three cases.

(a) $\tilde{y} = 0$. Among the elements $((\tilde{x}_1,\tilde{x}_2),(0,0)) \in K_1 \times K_2$ which solve ($\text{System}_{\text{NCP}}$), only those are solutions for ($\text{System}_{\text{NEP}}$) which fulfill the condition $\tilde{x}_2 \geq \max\{-\tilde{x}_1, 0\}$.

(b) $\tilde{y} = 1$. We have $((-1,-1),(1,1)) \in S_{\text{NEP}}$, but $((0,-1),(1,1)) \notin S_{\text{NEP}}$.

(c) $0 < \tilde{y} < 1$. We have $((1,-1),(\sqrt{2}/2,1/2)) \in S_{\text{NEP}}$.

16.5 Historical comments and perspectives

A. Historical comments. Although many extensions and applications can be found of Nash's result (see, for instance, Chang [66], Georgiev [125], Guillerme [134], Kulpa and Szymanski [181], Luo [196], Morgan and Scalzo [214], Yu and Zhang [295], and references therein), to the best of our knowledge, only a few works can be found in the literature dealing with the question addressed in the Introduction, i.e., the location of Nash equilibrium points for a functions with non-usual properties (for instance, set-valued maps) which are defined on sets with non-standard structures. Nessah and Kerstens [226] characterize the existence of Nash equilibrium points on non-convex strategy sets via a modified diagonal transfer convexity notion; Tala and Marchi [283] also treat games with non-convex strategies reducing the problem to the convex case via certain homeomorphism; Kassay, Kolumbán and Páles [153] and Ziad [298] considered Nash equilibrium points on convex domains for non-convex payoff functions having suitable regularity instead of their convexity.

In this chapter we treated the existence and location of Nash equilibrium points in various context. Via variational inequalities and dynamical systems, we established stability results concerning Nash critical points following the paper of Cavazzuti, Pappalardo and Passacantando [62]. In Section 16.3 we studied Nash equilibrium points for set-valued maps on vector spaces. The purpose of the last section was to initiate a new approach concerning the study of Nash equilibria, where the payoff functions are defined on sets which are not necessarily convex in the usual sense, by embedding these sets into suitable Riemannian manifolds.

B. Perspectives. The main challenging question is to study the (exponential) stability of Nash critical points for functions which are defined on sets embedded into certain Riemannian manifolds. Since the non-expansiveness and the validity of Moskovitz-Dines property of the projection operator are indispensable in such an argument (see Theorem 16.4), we believe that this fact is possible only for those cases where the strategy sets can be embedded into Hadamard-type Riemannian manifoldsm (see Theorems 15.4 and 15.8).

17
Problems to Part III

> God exists since mathematics is consistent, and the Devil exists since we cannot prove it.
>
> André Weil (1906–1998)

Problem 17.1 Prove that $(\text{Int}(M_C), d_H)$ is a metric space, where C is a simple, closed and convex curves and d_H is the Hilbert distance. [Hint: See Section 13.2]

Problem 17.2 Let C be an ellipse with semi-axes $a > b > 0$, and consider three points $P_1 = (a/2, 0)$, $P_2 = (-a/2, 0)$ and $P_3 = (a/2, -b/2)$ in the interior of the ellipse $\text{Int}(M_C)$. Determine the minimum points of the total cost function $C(P_i, 3, s) = \sum_{i=1}^{3} d_H^s(\cdot, P_i)$ on $\text{Int}(M_C)$ for $s = 1$ and $s = 2$, respectively.

Problem 17.3 Let $M = \mathbb{R}^m$, $m \geq 2$, be endowed with the natural Euclidean metric, and $P_i \in M$, $i = 1, ..., n$ given points. Prove that the unique minimum point $T_f = T_b$ (i.e., the place of the deposit) for $C_f(P_i, n, s) = C_b(P_i, n, s)$ is the center of gravity of points $\{P_1, ..., P_n\}$ if and only if $s = 2$. [Hint: See Example 14.1]

Problem 17.4 There is a ship which moves on a closed curve and there are n fixed ambulance stations. Suppose that at a certain moment, the ship's crew asks for first-aid continuing their trip. Determine the position of ambulance station from where the team assistance reaches the ship within minimal time. [Hint: Follow Problem 14.1]

Problem 17.5 If M_0 is a submanifold of the Riemannian manifold (M, g) and $\gamma : [0, 1] \to M$ is a geodesic of M with the properties that $\gamma(1) \in M_0$ and $d_g(\gamma(0), y) \geq L_g(\gamma)$ for every $y \in M_0$, then $g(\dot\gamma(1), X) = 0$ for every $X \in T_{\gamma(1)}M_0$, i.e., γ is perpendicular to M_0 at the point of contact. [Hint: Use the first variational formula and follow Proposition 15.1]

Problem 17.6 If M_1 and M_2 are submanifolds of the Riemannian manifold (M, g) and $\gamma : [0, 1] \to M$ is a geodesic of M with the properties that $x_0 = \gamma(0) \in M_1$, $y_0 = \gamma(1) \in M_2$, and $d_g(x, y) \geq d_g(x_0, y_0) = L_g(\gamma)$ for every $(x, y) \in M_1 \times M_2$, then $g(\dot\gamma(0), X) = 0$ and $g(\dot\gamma(1), Y) = 0$ for every $(X, Y) \in T_{\gamma(0)}M_1 \times T_{\gamma(1)}M_2$, i.e., γ is a common perpendicular of M_1 and M_2. [Hint: Use the first variational formula and follow Proposition 15.1]

Problem 17.7 Let (M, g) be a simply connected Riemannian manifold with non-positive sectional curvature and let $\alpha, \beta : \mathbb{R} \to M$ be two geodesics. We allow that $\beta(t)$ =constant, and $\|\dot\alpha\| = 1$. The distance function $f(t) = d_g(\alpha(t), \beta(t))$ has the following properties:

1) the equation $f(t) = 0$ has at most one solution;

2) f is a function of class C^∞ for all $t \in \mathbb{R}$ with $f(t) \neq 0$.

[Hint: Use the first and second variational formula.]

Problem 17.8 Let $K_1 = [-5, 3]$, $K_2 = \{(\cos t, \sin t) : t \in [\pi/4, 3\pi/4]\}$, and $f_1, f_2 : K_1 \times K_2 \to \mathbb{R}$ defined for every $x \in K_1$, $(y_1, y_2) \in K_2$ by $f_1(x, (y_1, y_2)) = x^2 y_1^3 - y_2^2$, $f_2(x, (y_1, y_2)) = (16 - x^2)(y_1^2 - y_2^2)$. Determine the set of Nash equilibrium points for $(\mathbf{f}, \mathbf{K})$.

Problem 17.9 Let $K_1 = \{(\sin t, \cos t) : t \in [\pi/8, 3\pi/7]\}$, $K_2 = \{(y_1, y_2) : y_2 = y_1^2 + 2,\ y_1 \in [0, 2]\}$, and $f_1, f_2 : K_1 \times K_2 \to \mathbb{R}$ defined for every $(x_1, x_2) \in K_1$, $(y_1, y_2) \in K_2$ by $f_1((x_1, x_2), (y_1, y_2)) = 3x_1 y_2^2 + 2x_2^3 y_1^3$, $f_2((x_1, x_2), (y_1, y_2)) = 4x_1^2 y_2^3 + 5x_2^2 y_1$. Determine the set of Nash equilibrium points for $(\mathbf{f}, \mathbf{K})$.

Problem 17.10 Let $K = K_1 = K_2 = [-2, 3]$, $X = \mathbb{R}$, and $F_1, F_2 : K \times K \rightsquigarrow \mathbb{R}$ be defined by $F_1(x_1, x_2) = [\max\{x_1^2, x_2^3\} - 1, 0]$ and $F_2(x_1, x_2) = [1 - \max\{|x_1|, x_2^2\}, 6 - x_1 - x_2^2]$. Determine the set of Nash equilibrium points for $(\mathbf{F}, \mathbf{K}) = (F_1, F_2; K, K)$.

Problem 17.11 (Location of Post Office) Let $P_1 = (1, 0) \in \mathbb{R}^2$ and

$P_2 = (0, 1) \in \mathbb{R}^2$ be two Post Offices, and let two players moving on the set $\mathbf{K} = K \times K$ where $K = (-\infty, 0]$. The purpose of the ith player is to minimize the distance between $(x_1, x_2) \in \mathbf{K}$ and his favorite goal P_i. What about the case when the players moving area is subject to the social constraint $S = \{(x_1, x_2) \in \mathbf{K} : x_1 + x_2 \leq -2\}$? [Hint: Use Nash point notions from Section 16.2.]

Appendices

A
Elements of convex analysis

By plucking her petals, you do not gather the beauty of the flower.

Rabindranath Tagore
(1861–1941)

A.1 Convex sets and convex functions

Mathematically, the notion of *convex set* can be made precise by defining the *segment* joining any two points x and y in a vector space to be the set of of all points having the form $\lambda x + (1-\lambda)y$ for $0 \le \lambda \le 1$. A set is convex if and only if it contains all the segments joining any two of its points. The geometric nature of this definition is depicted in Figures A.1 and A.2.

Given a collection of convex sets, it is always possible to create other convex sets by the operations of *dilation*, *sum* or *intersection*. More precisely, if A and B are convex sets in a linear space then the sets

$$\alpha A := \{\alpha x;\ x \in A\} \qquad \text{where } \alpha \in \mathbb{R}$$

and

$$A + B := \{a + b;\ a \in A,\, b \in B\}$$

are convex, too. Moreover, any intersection of convex sets is convex.

The notion of convex function was introduced in the first part of the twentieth century, though it was implicitly used earlier by Gibbs and Maxwell in order to describe relationships between thermodynamic variables.

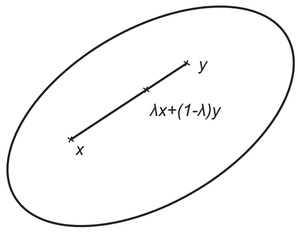

Fig. A.1. Geometric illustration of a convex set.

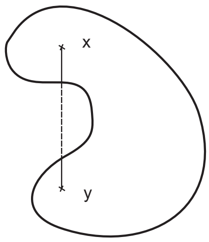

Fig. A.2. Shape of a nonconvex set.

Definition A.1 Let C be a convex set in a linear space. A function $f : C \to \mathbb{R}$ is convex if and only if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \qquad \text{for any } \lambda \in [0,1].$$

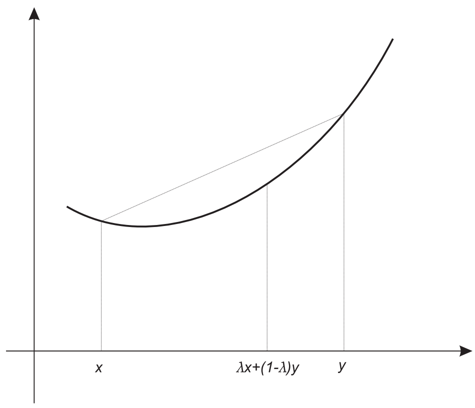

Fig. A.3. Geometric illustration of a convex function.

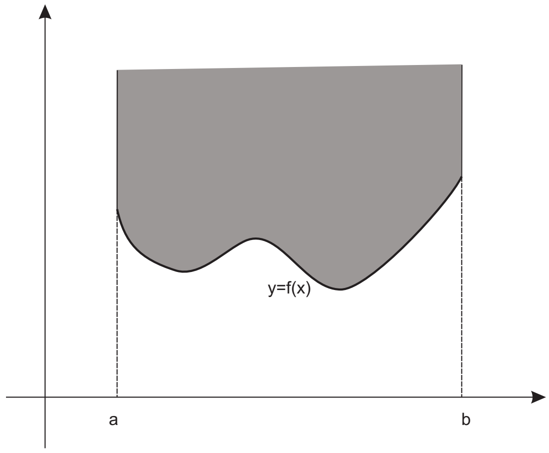

Fig. A.4. Geometric illustration of the epigraph of a function $f : [a, b] \to \mathbb{R}$.

Thus, a functional is convex if the segment connecting two points on its graph lies above the graph of the function, as shown in Figure A.3. Now, this definition is often expressed concisely by defining the set of all points that lie above the graph of a functional to be the *epigraph* (see Figure A.4). This set plays an important role in optimization theory and is usually referred in economics as an *upper contour set.*

Definition A.2 The epigraph of a functional f acting on a vector space X is the subset

$$\mathrm{epi}\,(f) := \{(x, a) \in X \times \mathbb{R};\; f(x) \leq a\}\,.$$

The following result makes clear how the notions of convexity of a set and convexity of a functional are related.

Proposition A.1 *Suppose f is a functional defined on a convex set. Then f is a convex functional if and only if the set* epi (f) *is convex.*

We have seen in this volume that convexity plays an important role in many optimization problems. In addition, some classes of inequality constraints couched in terms of *convex cones* are amenable to Lagrange multiplier methods for abstract spaces. Convex cones are also used to define orderings on vector spaces, which in turn facilitate the introduction of inequality constraints.

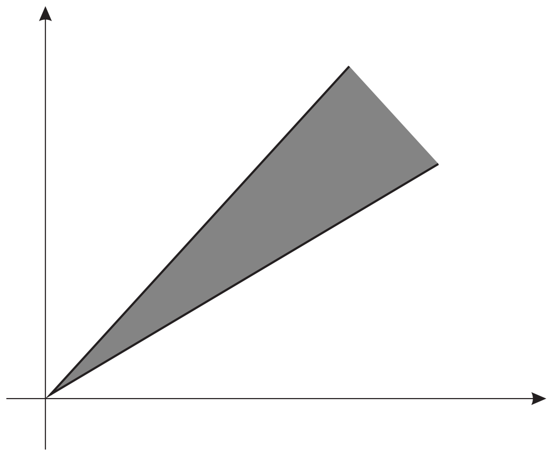

Fig. A.5. A convex closed cone with vertex at the origin in $\mathbb{R}^2$.

Definition A.3 Let X be a vector space. A set $C_0 \subset X$ is a cone with vertex at the origin if

$$x \in X \qquad \text{implies} \qquad \alpha x \in C_0\,,$$

for all $\alpha \geq 0$. A cone with vertex at a point $v \in X$ is defined to be the translation $v + C$ of a cone C_0 with vertex at the origin:

$$C = \{v + x;\ x \in C_0\}\,.$$

A positive cone $C_0^+ \subset X$ is defined by

$$C_0^+ := \{x;\ x \in C_0,\, x \geq 0\}\,.$$

A cone with vertex at the origin need not be convex. We also point out that a cone need not be closed, in general. Examples of cones (either convex or not convex) are depicted in Figures A.5, A.6, and A.7.

The next theorem is a result due to Efimov and Stechkin in the theory of best approximation. Roughly speaking, this property assures that in a certain class of Banach spaces, every sequentially weakly closed Chebyshev set is convex.

Theorem A.1 (Lemma 1 of [99], Theorem 2 of [285]) *Let X be a uniformly convex Banach space with strictly convex topological dual. Assume that M is a sequentially weakly closed non–convex subset of X.*

Then, for any convex dense subset S of X, there exists $x_0 \in S$ such

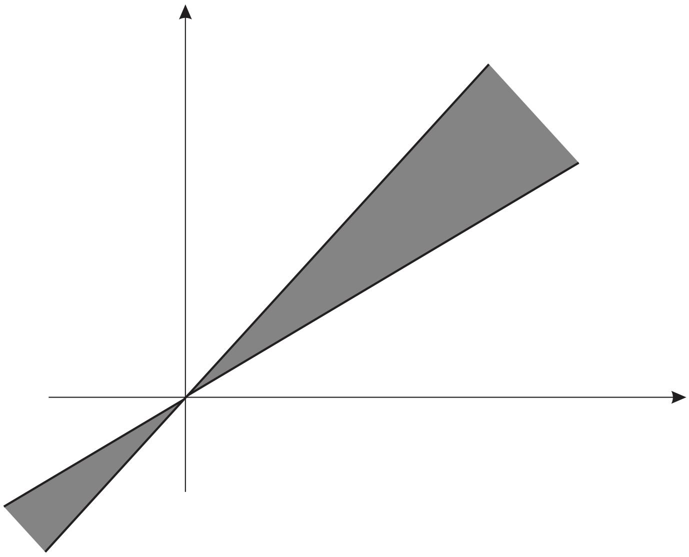

Fig. A.6. A nonconvex closed cone with vertex at the origin in $\mathbb{R}^2$.

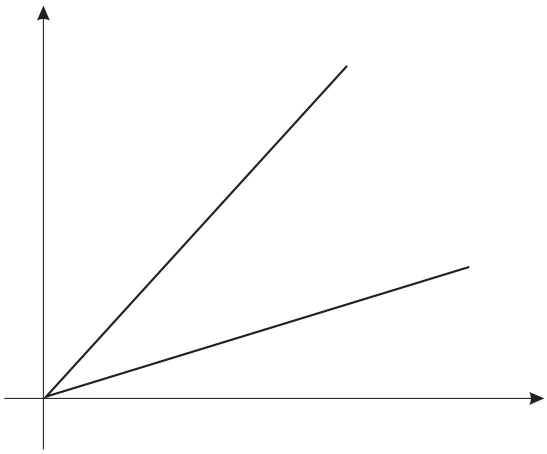

Fig. A.7. A nonconvex open cone with vertex at the origin in $\mathbb{R}^2$.

that the set

$$\{y \in M \ : \ \|y - x_0\| = d(x_0, M)\}$$

contains at least two points.

A.2 Convex analysis in Banach spaces

Throughout this section we assume that $(X, ||\cdot||)$ is a Banach space.

Definition A.4 Let $\alpha : X \to (-\infty, +\infty]$ be a function and let $x \in X$ be a point where α is finite. The one-sided directional derivative of α at x with respect to a vector $y \in X$ is

$$\alpha'(x; y) = \lim_{\lambda \searrow 0} \frac{\alpha(x + \lambda y) - \alpha(x)}{\lambda},$$

if it exists.

Lemma A.1 *Let be $\alpha : X \to (-\infty, +\infty]$ be a convex function, and let $x \in X$ be a point where alpha(x) is finite. For each $y \in X$, the difference quotient in the definition of $\alpha'(x; y)$ is a non-decreasing function of $\lambda > 0$, so that $\alpha'(x; y)$ exists and*

$$\alpha'(x; y) = \lim_{\lambda \searrow 0} \frac{\alpha(x + \lambda y) - \alpha(x)}{\lambda}.$$

Moreover, $\alpha'(x; y)$ is a positively homogeneous convex of $y \in X$.

Definition A.5 Let $\alpha : X \to (-\infty, +\infty]$ be a convex function. An element $x^\star \in X$ is said to be a subgradient of the convex function α at a point $x \in X$ provided that for any $z \in X$,

$$\alpha(z) \geq \alpha(x) + \langle x^\star, z - x \rangle.$$

The set of all subgradient of α at x is called the subdifferential of α at x and is denoted by $\partial\alpha(x)$.

Lemma A.2 *Let $\alpha : X \to (-\infty, +\infty]$ be a convex function and let $x \in X$ be a point where $\alpha(x)$ is finite. Then $x^\star \in \partial\alpha(x)$ if and only if for all $y \in X$,*

$$\alpha'(x; y) \geq \langle x^\star, y \rangle.$$

Now, let $S \subset X$ be a non-empty closed convex set and denote by $\alpha_S : X \to (-\infty, +\infty]$ the indicator function associated to S, that is,

$$j_S(x) = \begin{cases} 0, & \text{if } x \in S \\ +\infty, & \text{if } x \in X \setminus S. \end{cases} \tag{A.1}$$

In this case $\partial\alpha_S(x) = \{x^\star \in X^\star \mid \langle x^\star, z - x \rangle \leq 0, \quad \forall\ z \in S\}$, that is, $\partial\alpha_S(x)$ coincide with the normal cone associated to S at the point $x \in S$. Thus, $\partial\alpha_S(x) = N_S(x)$.

Let $S \subset X$ be a nonempty set. As usual, we denote the *distance function* $\mathrm{dist}_S : X \to \mathbb{R}$ defined by

$$\mathrm{dist}_S(x) = \inf\{\|x - y\| : y \in S\}.$$

It is clear that dist_S is a globally Lipschitz function. Consequently, one may define its *directional derivative* in the sense of Clarke at the point $x \in S$ with direction $v \in X$, denoted by $\mathrm{dist}_S^0(x; v)$.

Definition A.6 Let $x \in S$. A vector $v \in X$ is tangent to S at x if $\mathrm{dist}_S^0(x; v) = 0$. The set of tangent vectors of S at x is denoted by $T_S(x)$.

Note that $T_S(x)$ is a closed convex cone in X; in particular, $0 \in T_S(x)$.

Definition A.7 Let $x \in S$. The normal cone to S at x is

$$N_S(x) = \{x^* \in X^* : \langle x^*, x\rangle \leq 0 \text{ for all } v \in T_S(x)\}.$$

Proposition A.2 *Let $X = X_1 \times X_2$, where X_1, X_2 are Banach spaces, and let $x = (x_1, x_2) \in S_1 \times S_2$, where S_1, S_2 are subsets of X_1, X_2, respectively. Then*

$$T_{S_1 \times S_2}(x) = T_{S_1}(x_1) \times T_{S_2}(x_2);$$

$$N_{S_1 \times S_2}(x) = N_{S_1}(x_1) \times N_{S_2}(x_2).$$

B
Function spaces

All the effects of Nature are
only the mathematical
consequences of a small
number of immutable laws.

Pierre-Simon Laplace
(1749–1827)

In this part we recall some basic facts on Lebesgue and Sobolev spaces. A central place is dedicated to the main theorems in these basic function spaces.

B.1 Lebesgue spaces

Let $\Omega \subset \mathbb{R}^N$ be an open set. We recall that the Lebesgue spaces are defined by

$$L^p(\Omega) := \left\{ u : \Omega \to \mathbb{R};\ u \text{ is measurable and } \int_\Omega |u(x)|^p \, dx < \infty \right\}$$

if $1 \leq p < \infty$, and

$$L^\infty(\Omega) := \{ u : \Omega \to \mathbb{R};\ u \text{ is measurable and } \operatorname{ess\,sup}_{x \in \Omega} |u(x)| < \infty \} \, .$$

For any $1 \leq p \leq \infty$ we define the space

$$L^p_{\mathrm{loc}}(\Omega) := \{ u : \Omega \to \mathbb{R};\ u \in L^p(\omega) \text{ for each } \omega \subset\subset \Omega \} \, .$$

We have used in this volume the following basic results on Lebesgue spaces.

Theorem B.1 *(Fatou's Lemma). Assume that for any $n \geq 1$, $u_n \in L^1(\Omega)$ and $u_n \geq 0$ a.e. on Ω. Then*

$$\int_\Omega \liminf_{n\longrightarrow\infty} u_n \, dx \leq \liminf_{n\longrightarrow\infty} \int_\Omega u_n \, dx \, .$$

The next theorem is due to Brezis and Lieb [48] provides us with a correction term that changes Fatou's lemma from an inequality to an equality.

Theorem B.2 *(Brézis–Lieb Lemma). Let (u_n) be a sequence of functions defined on Ω that converges pointwise a.e. to a function u. Assume that for some $1 \leq p < \infty$, there exists a positive constant C such that for all $x \in \Omega$,*

$$\sup_{n\geq 1} \int_\Omega |u_n(x)|^p \, dx < C \, .$$

Then

$$\lim_{n\longrightarrow\infty} \int_\Omega \left[|u_n(x)|^p - |u_n(x) - u(x)|^p\right] \, dx = \int_\Omega |u(x)|^p \, dx \, .$$

Theorem B.3 *(Lebesgue's Dominated Convergence Theorem). Assume that for any $n \geq 1$, $u_n \in L^1(\Omega)$. Suppose that $u_n \rightarrow u$ a.e. on Ω and $|u_n| \leq v$ a.e., for some function $v \in L^1(\Omega)$. Then*

$$\lim_{n\longrightarrow\infty} \int_\Omega u_n(x) \, dx = \int_\Omega u(x) \, dx \, .$$

B.2 Sobolev spaces

Let $u \in L^1_{\mathrm{loc}}(\Omega)$ be a function. A function $v_\alpha \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega u(x) D_\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_\Omega v_\alpha(x) \varphi(x) dx$$

for any $\varphi \in C_0^\infty(\Omega)$ is called the weak distributional derivative of u and is denoted by $D^\alpha u$. Here $\alpha = (\alpha_1, ..., \alpha_n)$, where α_i $(i = 1, ..., N)$ are nonnegative integers and $D^\alpha u = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}$, with $|\alpha| = \alpha_1 + ... + \alpha_N$. It is clear that if such a v_α exists, it is unique up to sets of measure zero.

To define a Sobolev space we introduce a functional $|| \cdot ||_{m,p}$, where m

is a nonnegative integer and $1 \le p \le \infty$, as follows

$$||u||_{m,p} = \left(\sum_{0\le|\alpha|\le m} \int_\Omega |D^\alpha u(x)|^p dx \right)^{1/p},$$

for $1 \le p < \infty$ and

$$||u||_{m,\infty} = \max_{0\le|\alpha|\le m} \sup_\Omega |D^\alpha u(x)|$$

if $p = \infty$. It is obvious that $||\cdot||_{m,p}$ and $||\cdot||_{m,\infty}$ define norms on any vector space of functions for which values of these functionals are finite, provided functions are identified in the space if they are equal almost everywhere.

For any integer $m \ge 1$ and any $1 \le p \le \infty$ we can define the following spaces

$$H^{m,p}(\Omega) = \text{the completion of } \{u \in C^m(\Omega) \ : \ ||u_{m,p}|| < \infty\}$$

with respect the norm $||\cdot||_{m,p}$;

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for } 0 \le |\alpha| \le m\},$$

where D^α denotes the weak partial derivative of u;

$$W_0^{m,p}(\Omega) = \text{the closure of } C_0^\infty(\Omega) \text{ in the space } W^{m,p}(\Omega).$$

The above spaces equipped with the norms $||\cdot||_{m,p}$ are called Sobolev spaces over Ω.

It is clear that $W^{0,p} = W_0^{0.p} = L^p$. It is also known that $H^{m,p}(\Omega) = W^{1,p}(\Omega)$ for every domain $\Omega \subset \mathbb{R}^N$. The spaces $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are Banach spaces and these spaces are reflexive if and only if $1 < p < \infty$. Moreover, $W^{k,2}(\Omega)$ and $W_0^{k,2}(\Omega)$ are Hilbert spaces with scalar product

$$\langle u, v\rangle = \sum_{|\alpha|\le k} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx,$$

where $D^0 u = u$. Also, $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are separable for $1 \le p < \infty$. For any $1 \le p < \infty$ we denote $p' = p/(p-1)$ (if $p = 1, p' = \infty$) and we denote by $W^{-m,p'}(\Omega)$ the dual space of $W_0^{m,p}(\Omega)$ and we write

$$W^{-m,p'}(\Omega) = \left(W^{m,p}(\Omega)\right)^\star.$$

Then every element $T \in W^{-m,p'}(\Omega)$ has the form

$$T(u) = \sum_{|\alpha|\le k} \int_\Omega D^\alpha u(x) v_\alpha(x) dx,$$

where v_α are suitable elements in $L^{p'}(\Omega)$.

B.3 Compact embedding results

Let $(X, ||\cdot||_X), (Y, ||\cdot||_Y)$ be Banach spaces. We say that X is continuously embedded into Y if there exists an injective linear map $i : X \to Y$ and a constant C such that $||i(x)||_Y \leq C||x||_X$ for all $x \in X$. We identify X with the image $i(X)$. We say that X is compactly embedded into Y if i is a compact map, that is, i maps bounded subsets of X into relatively compact subset of Y.

The following theorem has a particular importance in the variational and qualitative analysis of differential and partial differential equations, due to the control over the nonlinear terms.

Theorem B.4 (Sobolev embedding theorem) *Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary. Then*

(i) If $kp < N$, then $W^{k,p}(\Omega)$ is continuously embedded into $L^q(\Omega)$ for each $1 \leq q \leq \dfrac{Np}{N-kp}$; this embedding is compact, if $q < \dfrac{Np}{N-kp}$.

(ii) If $0 \leq m < k - \dfrac{N}{p} < m+1$, then $W^{k,p}(\Omega)$ is continuously embedded into $C^{m,\alpha}(\Omega)$, for $0 \leq \alpha \leq k - m - \dfrac{m}{p}$, this embedding is compact if $\alpha < k - m - \dfrac{m}{p}$.

Let $C^{m,\beta}(\Omega)$ denote the set of all functions u belonging to $C^m(\Omega)$ and whose partial derivatives $D^\alpha u$, with $|\alpha| = m$, are Hölder continuous with exponent $0 < \beta < 1$.

For a bounded domain Ω, $C^{m,\beta}(\Omega)$ becomes a Banach space with a norm

$$||u||_{C^{m,\beta}} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_\infty + \sum_{|\alpha|=m} \sup_{x \neq y \in \Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\beta}.$$

For an arbitrary domain $\Omega \subset \mathbb{R}^N$ and $u \in W_0^{k,p}(\Omega)$ we have

$$K\|u\|_{p^\star}^p \leq \sum_{|\alpha|=k} \int_\Omega |D^\alpha u(x)|^p dx, \tag{B.1}$$

where $p^\star = \dfrac{Np}{N-kp}$ and $K = K(k,n,p)$ is the best Sobolev constant. This inequality says that $W_o^{k,p}$ is continuously embedded in $L^{p^\star}(\Omega)$.

If $u \in W^{k,p}(\mathbb{R}^N)$, then the best Sobolev constant S is given by

$$K(k,n,p) = \pi^{-\frac{1}{2}} N^{-\frac{1}{p}} \left(\frac{p-1}{N-p}\right)^{1-\frac{1}{p}} \left\{ \frac{\Gamma(1+\dfrac{N}{2})\Gamma(N)}{\Gamma(\dfrac{N}{p})\Gamma(1+N-\dfrac{N}{p})} \right\}^{\frac{1}{N}},$$

where $\Gamma(\,\cdot\,)$ denotes Euler's Gamma function. Moreover, the equality holds in (B.1) if and only if u has the form

$$u(x) = \left[a + b|x|^{\frac{p}{p-1}}\right]^{1-\frac{N}{p}}, \tag{B.2}$$

where a and b are positive constants.

If $k = 1, p = 2$, then the sharp constant will be denoted by K_n and we have

$$K_n = \sqrt{\frac{4}{n(n-2)\omega_n^{\frac{2}{n}}}}, \tag{B.3}$$

where ω_n is the volume of the unit sphere.

We denote by $D^{k,p}(\mathbb{R}^N)$ completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{D^{k,p}}^p = \sum_{|\alpha|=k} \int_\Omega |D^\alpha u(x)|^p dx\,.$$

According to the inequality (B.1), the function space $D^{k,p}(\mathbb{R}^N)$ is continuously embedded in $L^{p^\star}(\mathbb{R}^N)$.

Let Ω be an unbounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and let G be a subgroup of $O(N)$ whose elements leave Ω invariant, that is, $g(\Omega) = \Omega$ for all $g \in G$. We assume that Ω is compatible with G, that is, for some $r > 0$

$$m(y,r,G) \to \infty, \quad \text{as } \operatorname{dist}(y,\Omega) \le r, \quad |y| \to \infty \tag{B.4}$$

where

$$m(y,r,G) = \sup\left\{ n \in \mathbb{N} \;:\; \begin{array}{l} \exists \;\; g_1, g_2, ..., g_n \in G \text{s.t.} \\ B(g_j y, r) \cap B(g_k y, r) = \emptyset \text{ if } j \neq k \end{array} \right\}.$$

Let $X = W_0^{1,p}(\Omega)$ and define a representation of G over X as follows:

$$(\pi(g)u)(x) = u(g^{-1}x), \quad g \in G, \quad u \in X \quad x \in \Omega.$$

As usual we shall write gu in place of $\pi(g)u$.

A function u defined on Ω is said to be G-invariant if

$$u(gx) = u(x), \quad \forall \;\; g \in G, \;\; \text{a.e. } x \in \Omega.$$

Then $u \in X$ is G-invariant if and only if

$$u \in X^G = W^{1,p}_{0,G}(\Omega) := \{u \in X : gu = u, \;\forall \; g \in G \;\}.$$

On X we take the norm

$$\|u\| = \left\{ \int_\Omega (|\nabla u|^p + |u|^p) \right\}^{1/p}.$$

The next result is essentially given by Willem [292] and in this form was proved by Kobayashi-Ôtani [158].

Theorem B.5 *If Ω is compatible with G, then the embeddings*

$$X^G = W^{1,p}_{0,G}(\Omega) \hookrightarrow L^q(\Omega), \quad p < q < p^\star = \frac{Np}{N-p}$$

are compact.

Now, we consider the space

$$H^1(\mathbb{R}^N) := \left\{ \; u \in L^2(\mathbb{R}^N) \;\; : \;\; \nabla u \in L^2(\mathbb{R}^N) \right\}$$

with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} [\nabla u \nabla v + uv].$$

The space $(H^1(\mathbb{R}^N), \langle \cdot, \cdot \rangle)$ becomes a Hilbert space.

Let G be a subgroup of $O(N)$. We define

$$H^1_G(\mathbb{R}^N) = \{u \in H^1 \;\; : gu = u, \;\forall \; g \in G\},$$

where $gu(x) := u(g^{-1}x)$.

The following result establishes a compact embedding in the case of lack of compactness.

Theorem B.6 (Lions [190]) *Let $N_j \geq 2, j = 1, ..., k, \displaystyle\sum_{j=1}^k N_j = N$ and*

$$G := O(N_1) \times O(N_2) \times ... \times O(N_k).$$

Then the following embedding is compact:

$$H^1_G(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N), \quad 2 < p < 2^\star.$$

In the following we consider $\Omega = \tilde{\Omega} \times \mathbb{R}^N, N - m \geq 2, \tilde{\Omega} \subset \mathbb{R}^m (m \geq 1)$ is open bounded and $1 \leq p \leq N$. We consider the space $W_0^{1,p}(\Omega)$ with the norm $||u|| = \left(\int_\Omega |\nabla u|^p \right)^{\frac{1}{p}}$. Let G be a subgroup of $O(N)$ defined by $G = id^m \times O(N - m)$. The action of G on $W_0^{1,p}(\Omega)$ is given by $gu(x_1, x_2) = u(x_1, g_1 x_2)$ for every $(x_1, x_2) \in \tilde{\Omega} \times \mathbb{R}^{N-m}$ and $g = id^m \times g_1 \in G$. The subspace of invariant functions is defined by

$$W_{0,G}^{1,p} = \{u \in X : gu = u, \ \forall g \in G\}.$$

The action of G on $W_0^{1,p}(\Omega)$ is isometric, that is

$$||gu|| = ||u||, \ \forall g \in G.$$

We have the following result.

Theorem B.7 (Lions [190]) *If $2 \leq p \leq N$, then the embedding*

$$W_{0,G}^{1,p} \hookrightarrow L^s(\Omega), \quad p < s < p^\star$$

is compact.

We end this section with a result given by Esteban and Lions [108]. For this we consider $\tilde{\Omega} \subset \mathbb{R}^m$ a bounded open set $\Omega = \tilde{\Omega} \times \mathbb{R}$ and let

$$H_0^1(\Omega) = \{u \in L^2(\Omega) \ : \ \nabla u \in L^2(\Omega), u = 0 \text{ on } \partial\Omega = \partial\tilde{\Omega} \times \mathbb{R}\}.$$

We denote by K the cone of $H_0^1(\Omega)$ defined by

$$\begin{aligned}
\mathcal{K} = \{u \quad \in \quad & H_0^1(\omega \times \mathbb{R}) : u \text{ is nonnegative,} \\
& y \mapsto u(x, y) \text{ is nonincreasing for } x \in \omega, \ y \geq 0, \text{ and} \\
& y \mapsto u(x, y) \text{ is nondecreasing for } x \in \omega, \ y \leq 0\},
\end{aligned}$$

Theorem B.8 (Esteban and Lions [108]) *The embeddings $K \hookrightarrow L^q(\Omega)$ are compact for $q \in (2, 2_m^*)$, where $2_m^* = \frac{2(m+1)}{m-1}$ if $m > 1$, and $2_m^* = \infty$ if $m = 1$.*

B.4 Sobolev spaces on Riemann manifolds

Let (M, g) be a Riemannian manifold of dimension n. For k an integer and k an integer and $u \in C^\infty(M), \nabla^k u$ denotes the k-th covariant derivative of u (with the convection $\nabla^0 u = u$.) The component of ∇u in the local coordinates $(x^1, \cdots, x^n)$ are given by

$$(\nabla^2)_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial u}{\partial x^k}.$$

By definition on has

$$|\nabla^k u|^2 = g^{i_1 j_1} \cdots g^{i_k j_k} (\nabla^k u)_{i_1 \cdots i_k} (\nabla^k u)_{j_1 \cdots j_k}.$$

For $k \in \mathbb{N}$ and $p \geq 1$ real, we denote by $\mathcal{C}_k^m(M)$ the space of smooth functions $u \in C^\infty(M)$ such that $|\nabla^j u| \in L^p(M)$ for any $j = 0, \cdots, k$. Hence,

$$\mathcal{C}_k^p = \{u \in C^\infty(M) : \ \forall \ j = 0, ..., k, \quad \int_M |\nabla^j u|^p dv(g) < \infty \ \}$$

where, in local coordinates, $dv(g) = \sqrt{det(g_{ij})}dx$, and where dx stands for the Lebesque's volume element of $\mathbb{R}^n$. If M is compact, on has that $\mathcal{C}_k^p(M) = C^\infty(M)$ for all k and $p \geq 1$.

Definition B.1 The Sobolev space $H_k^p(M)$ is the completion of $\mathcal{C}_k^p(M)$ with respect the norm

$$||u||_{H_k^p} = \sum_{j=0}^k \left(\int_M |\nabla^j u|^p dv(g) \right)^{\frac{1}{p}}.$$

More precisely, one can look at $H_k^p(M)$ as the space of functions $u \in L^p(M)$ which are limit in $L^p(M)$ of a Cauchy sequence $(u_m) \subset \mathcal{C}_k$, and define the norm $||u||_{H_k^p}$ as above where $|\nabla^j u|, 0 \leq j \leq k$, is now the limit in $L^p(M)$ of $|\nabla^j u_m|$. These space are Banach space, and if $p > 1$, then H_k^p is reflexive. We note that, if M is compact, $H_k^p(M)$ does not depend on the Riemannian metric. If $p = 2$, $H_k^2(M)$ is a Hilbert space when equipped with the equivalent norm

$$||u|| = \sqrt{\sum_{j=0}^k \int_M |\nabla^j u|^2 dv(g)}. \tag{B.5}$$

The scalar product $\langle \cdot, \cdot \rangle$ associated to $|| \cdot ||$ is defined by

$$\langle u, v \rangle = \sum_{m=0}^k \int_M \left(g^{i_1 j_1} \cdots g^{i_m j_m} (\nabla^m u)_{i_1 \ldots i_m} (\nabla^m v)_{j_1 \ldots j_m} \right) dv(g). \tag{B.6}$$

We denote by $C^k(M)$ the set of k times continuously differentiable functions, for which the norm

$$\|u\|_{C^k} = \sum_{i=1}^{n} \sup_M |\nabla^i u|$$

is finite. The Hölder space $C^{k,\alpha}(M)$ is defined for $0 < \alpha < 1$ as the set of $u \in C^k(M)$ for which the norm

$$\|u\|_{C^{k,\alpha}} = \|u\|_{C^k} + \sup_{x,y} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{|x-y|^\alpha}$$

is finite, where the supremum is over all $x \neq y$ such that y is contained in a normal neighborhood of x, and $\nabla^k u(y)$ is taken to mean the tensor at x obtained by parallel transport along the radial geodesics from x to y.

As usual, $C^\infty(M)$ and $C_0^\infty(M)$ denote the spaces of smooth functions and smooth compactly supported function on M respectively.

Definition B.2 The Sobolev space $\overset{\circ}{H}{}^p_k(M)$ is the closure of $C_0^\infty(M)$ in $H_k^p(M)$.

If (M,g) is a complete Riemannian manifold, then for any $p \geq 1$, we have $\overset{\circ}{H}{}^p_k(M) = H_k^p(M)$.

We finish this section with the Sobolev embedding theorem and the Rellich–Kondrachov result for compact manifold without and with boundary.

Theorem B.9 *(Sobolev embedding theorems for compact manifolds) Let M be a compact Riemannian manifold of dimension n.*

a) If $\frac{1}{r} \geq \frac{1}{p} - \frac{k}{n}$, then the embedding $H_k^p(M) \hookrightarrow L^r(M)$ is continuous.

b) (Rellich-Kondrachov theorem) Suppose that the inequality in a) i s strict, then the embedding $H_k^p(M) \hookrightarrow L^r(M)$ is compact.

c) Suppose $0 < \alpha < 1$ and $\frac{1}{p} \leq \frac{k-\alpha}{n}$, then the embedding $H_k^p(M) \hookrightarrow C^\alpha(M)$ is continuous.

Theorem B.10 *Let (M,g) be a compact n-dimensional Riemannian manifold with boundary ∂M.*

a) The embedding $H_1^p(M) \hookrightarrow L^q(M)$ is continuous, if $p \leq q \leq \frac{np}{n-p}$ and compact for $p \leq q < \frac{np}{n-p}$.

b) If $\partial M \neq \emptyset$*, then the embedding* $H_1^p(M) \hookrightarrow L^q(\partial M)$ *is continuous, if* $p \leq q \leq \frac{p(n-1)}{n-p}$ *and compact for* $p \leq q < \frac{p(n-1)}{n-p}$.

Theorem B.11 *For any smooth compact Riemannian manifold* (M, g) *of dimension* $n \geq 3$*, there exists* $B > 0$ *such that for any* $u \in H_1^2(M)$,

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \leq K_n^2 \int_M |\nabla u|^2 dv_g + B \int_M u^2 dv_g \tag{B.7}$$

and the inequality is sharp.

Theorem B.12 *(Global elliptic regularity) Let* M *be a compact Riemann manifold, and suppose that* $u \in L^1_{\mathrm{loc}}(M)$ *is a weak solution to* $\Delta_g u = f$.

a) If $f \in H_k^p(M)$*, then* $u \in H_{k+2}^p(M)$*, and*

$$\|u\|_{H_{k+2}^p} \leq C(\|\Delta_g u\|_{H_k^p} + \|u\|_{L^p}).$$

b) If $f \in C^{k,\alpha}(M)$*, then* $f \in C^{k+2,\alpha}(M)$*, and*

$$\|u\|_{C_{k+2}^p} \leq C(\|\Delta_g u\|_{C_k^p} + \|u\|_{C^\alpha}).$$

C

Category and genus

Every human activity, good or
bad, except mathematics,
must come to an end.

Paul Erdös (1913–1996)

Topological tools play a central role in the study of variational problems. Though this approach was foreshadowed in the works of H. Poincaré and G. Birkhoff, the force of these ideas was realized in the first decades of the 20th century, in the pioneering works of Ljusternik and Schnirelmann [194] and Morse [215, 216]. In this section we recall the notions of Ljusternik-Schnirelmann category and Krasnoselski genus as well as some basic properties of them.

Definition C.1 Let M be a topological space and $A \subset M$ a subset. The continuous map $\eta : A \times [0,1] \to M$ is called a deformation of A in M if $\eta(u,0) = u$ for every $u \in A$. The set A is said be contractible in M if there exists a deformation $\eta : A \times [0,1] \to M$ with $\eta(A,1) = \{p\}$ for some $p \in M$.

Definition C.2 Let M be a topological space. A set $A \subset M$ is said to be of Ljusternik-Schnirelmann category k in M (denoted $\mathrm{cat}_M(A) = k$) if it can be covered by k but not by $k-1$ closed sets which are contractible to a point in M. If such k does not exist, then $\mathrm{cat}_M(A) = +\infty$. We define $\mathrm{cat}(M) = \mathrm{cat}_M(M)$.

The Ljusternik-Schnirelmann category has the following basic properties.

Proposition C.1 *If A and B are subsets of the topological space M, then the following properties hold true.*

(a) *If $A \subseteq B$, then $\mathrm{cat}_M(A) \leq \mathrm{cat}_M(B)$;*
(b) $\mathrm{cat}_M(A \cup B) \leq \mathrm{cat}_M(A) + \mathrm{cat}_M(B)$.
(c) *If A is closed in X and $\eta : A \times [0,1] \to M$ a deformation of A in X, then $\mathrm{cat}_M(A) \leq \mathrm{cat}_M(\eta(A,1))$.*
(d) *If M is an ANR (in particular, it is a Banach-Finsler manifold of class C^1) and $A \subset M$, then there exist a neighborhood of A in M, such that $\mathrm{cat}_M(\overline{U}) = \mathrm{cat}_M(A)$.*

Theorem C.1 [109] *Let M be a non-contractible topological space and let $\Omega(M)$ denote the space of based loops in M. Then $\mathrm{cat}(\Omega(M)) = \infty$.*

Definition C.1 Let M be an arcwise connected topological space. Then $\mathrm{cuplong}(M)$ is the greatest non-vanishing iterated cup-product of singular cocycles of M such that each component is of dimension greater than zero.

Theorem C.2 [268] *If M is an arcwise connected metric space, then* $\mathrm{cat}(M) \geq \mathrm{cuplong}(M) + 1$.

Theorem C.3 [269] *Let M be a compact, connected, simply connected and non-contractible Riemannian manifold. If $\Omega(M)$ denotes the space of the free loops on M, then* $\mathrm{cuplong}(\Omega) = \infty$.

Let X be a real Banach space and denote

$$\mathcal{A} = \{A \subset X \setminus \{0\} : A = -A,\ A \text{ is closed}\}.$$

Definition C.3 We say that the positive integer k is the Krasnoselski genus of $A \in \mathcal{A}$, if there exists an odd map $\varphi \in C(A, \mathbb{R}^k \setminus \{0\})$ and k is the smallest integer with this property. The genus of the set A is denoted by $\gamma(A) = k$. When there does not exist a finite such k, set $\gamma(A) = \infty$. Finally, set $\gamma(\emptyset) = 0$.

Example C.1 Suppose $A \subset X \setminus \{0\}$ is closed and $A \cap (-A) = \emptyset$. Let $\tilde{A} = A \cup (-A)$. Then $\gamma(\tilde{A}) = 1$ since the function $\varphi(x) = 1$ for $x \in A$ and $\varphi(x) = -1$ for $x \in -A$ is odd and lies in $C(\tilde{A}, \mathbb{R} \setminus \{0\})$.

For $A \in \mathcal{A}$ and $\delta > 0$ we denote by $N_\delta(A)$ the uniform δ-neighborhood of A, that is, $N_\delta(A) = \{x \in X : \mathrm{dist}_A(x) \leq \delta\}$.

Proposition C.2 *Let $A, B \in \mathcal{A}$. Then the following properties hold true.*

1° *Normalization: If $x \neq 0$, $\gamma(\{x\} \cup \{-x\}) = 1$;*
2° *Mapping property: If there exists an odd map $f \in C(A, B)$, then $\gamma(A) \leq \gamma(B)$;*
3° *Monotonicity property: If $A \subset B$, then $\gamma(A) \leq \gamma(B)$;*
4° *Subadditivity: $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$;*
5° *Continuity property: If A is compact then $\gamma(A) < \infty$ and there is $\delta > 0$ such that $N_\delta(A) \in \mathcal{A}$. In such a case, $\gamma(N_\delta(A)) = \gamma(A)$.*

Example C.2 (a) If S^{k-1} is the $(k-1)$-dimensional unit sphere in $\mathbb{R}^k$, $k \in \mathbb{N}$, then $\gamma(S^{k-1}) = k$.

(b) If S is the unit sphere in an infinite dimensional and separable Banach space, then $\gamma(S) = +\infty$.

D
Clarke and Degiovanni gradients

> Profound study of nature is
> the most fertile source of
> mathematical discoveries.
>
> Baron Jean Baptiste Joseph
> Fourier (1768–1830)

In many time independent problems arising in applications, the solutions of a given problem are critical points of an appropriate energy functional f, which is usually supposed to be real and of class C^1 (or even differentiable) on a real Banach space. One may ask what happens if f, which often is associated to the original equation in a canonical way, fails to be C^1 or differentiable. In this case the gradient of f must be replaced by a generalized one, in a sense which is described in this Appendix, either for locally Lipschitz or for continuous/lower semi-continuous functionals. In the latter case we use in the general framework of *metric spaces.*

D.1 Locally Lipschitz functionals

Let X will denote a real Banach space. Let X^* be its dual and, for every $x \in X$ and $x^* \in X^*$, let $\langle x^*, x\rangle$ denote the duality pairing between X^* and X.

Definition D.1 A functional $f : X \to \mathbb{R}$ is said to be locally Lipschitz provided that, for every $x \in X$, there exists a neighbourhood V of x and a positive constant $k = k(V)$ depending on V such that for all $y, z \in V$,

$$|f(y) - f(z)| \leq k \,||y - z||\,.$$

Definition D.2 Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional and $x, v \in X$. We call the Clarke generalized directional derivative of f in x with respect to the direction v the number

$$f^0(x,v) = \limsup_{\substack{y \to x \\ \lambda \searrow 0}} \frac{f(y + \lambda v) - f(y)}{\lambda}\,.$$

If f is a locally Lipschitz functional, then $f^0(x,v)$ is a finite number and

$$|f^0(x,v)| \leq k||v||\,.$$

Moreover, if $x \in X$ is fixed, then the mapping $v \longmapsto f^0(x,v)$ is positive homogeneous and subadditive, so convex continuous. By the Hahn-Banach theorem, there exists a linear map $x^* : X \to \mathbb{R}$ such that for every $v \in X$,

$$x^*(v) \leq f^0(x,v)\,.$$

Definition D.3 Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional and $x \in X$. The generalized gradient (Clarke subdifferential) of f at the point x is the nonempty subset $\partial f(x)$ of X^* which is defined by

$$\partial f(x) = \{x^* \in X^*;\ f^0(x,v) \geq \langle x^*, v\rangle, \quad \text{for all} \quad v \in X\}\,.$$

We also point out that if f is convex, then $\partial f(x)$ coincides with the subdifferential of f in x in the sense of the convex analysis, that is,

$$\partial f(x) = \{x^* \in X^*;\ f(y) - f(x) \geq \langle x^*, y - x\rangle, \quad \text{for all} \quad y \in X\}\,.$$

We list in what follows the main properties of the Clarke gradient of a locally Lipschitz functional. We refer to [71], [72] for further details and proofs.

a) For every $x \in X$, $\partial f(x)$ is a convex and $\sigma(X^*, X)$-compact set.

b) For every $x, v \in X$ the following holds

$$f^0(x,v) = \max\{\langle x^*, v\rangle;\ x^* \in \partial f(x)\}\,.$$

c) The multivalued mapping $x \longmapsto \partial f(x)$ is upper semicontinuous, in the sense that for every $x_0 \in X$, $\varepsilon > 0$ and $v \in X$, there exists $\delta > 0$ such that, for any $x^* \in \partial f(x)$ satisfying $||x - x_0|| < \delta$, there is some $x_0^* \in \partial f(x_0)$ satisfying $|\langle x^* - x_0^*, v\rangle| < \varepsilon$.

d) The functional $f^0(\cdot,\cdot)$ is upper semi-continuous.

e) If x is an extremum point of f, then $0 \in \partial f(x)$.

f) The mapping

$$\lambda(x) = \min_{x^* \in \partial f(x)} ||x^*||$$

exists and is lower semicontinuous.

g) $\partial(-f)(x) = -\partial f(x)$.

h) Lebourg's mean value theorem: if x and y are two distinct point in X then there exists a point z situated in the open segment joining x and y such that

$$f(y) - f(x) \in \langle \partial f(z), y - x \rangle$$

i) If f has a Gâteaux derivative f' which is continuous in a neighbourhood of x, then $\partial f(x) = \{f'(x)\}$. If X is finite dimensional, then $\partial f(x)$ reduces at one point if and only if f is Fréchet-differentiable at x.

Definition D.4 A point $x \in X$ is said to be a critical point of the locally Lipschitz functional $f : X \to \mathbb{R}$ if $0 \in \partial f(x)$, that is $f^0(x, v) \geq 0$, for every $v \in X$. A number c is a critical value of f provided that there exists a critical point $x \in X$ such that $f(x) = c$.

We remark that a minimum point is also a critical point. Indeed, if x is a local minimum point, then for every $v \in X$,

$$0 \leq \limsup_{\lambda \searrow 0} \frac{f(x + \lambda v) - f(x)}{\lambda} \leq f^0(x, v)\,.$$

Definition D.5 If $f : X \to \mathbb{R}$ is a locally Lipschitz functional and c is a real number, we say that f satisfies the Palais-Smale condition at the level c (in short, $(\mathrm{PS})_c$) if any sequence (x_n) in X satisfying $f(x_n) \longrightarrow c$ and $\lambda(x_n) \longrightarrow 0$, contains a convergent subsequence. The mapping f satisfies the Palais-Smale condition (in short, (PS)) if every sequence (x_n) which satisfies $(f(x_n))$ is bounded and $\lambda(x_n) \longrightarrow 0$, has a convergent subsequence.

D.2 Continuous or lower semi-continuous functionals

We are now interested in the main properties of the corresponding properties in the case where f is a continuous or even a lower semi-continuous functional defined on a metric space.

Definition D.6 Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a continuous function. Let $u \in X$ be a fixed element. We denote by

$|df|(u)$ the supremum of the $\sigma \in [0, \infty[$ such that there exist $\delta > 0$ and a continuous map $\mathcal{H} : B(u, \delta) \times [0, \delta] \to X$ such that for all $v \in B(u, \delta)$ and $t \in [0, \delta]$ we have

a) $d(\mathcal{H}(v, t), v) \leq t$;
b) $f(\mathcal{H}(v, t)) \leq f(v) - \sigma t$.

The extended real number $|df|(u)$ is called the *weak slope* of f at u.

Definition D.7 Let (X, d) be a metric space and $f : X \to \mathbb{R}$ be a continuous function. A point $u \in X$ is said to be a critical point of f if $|df|(u) = 0$.

Definition D.8 Let (X, d) be a metric space. A continuous function $f : X \to \mathbb{R}$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ (shortly, $(PS)_c$-condition) if every sequence $\{u_n\} \subset X$ such that $\lim_{n\to\infty} f(u_n) = c$ and $\lim_{n\to\infty} |df|(u_n) = 0$, possesses a convergent subsequence.

For every $c \in \mathbb{R}$ we set

$$K_c = \{u \in X : |df|(u) = 0,\ f(u) = c\};$$

$$f^c = \{u \in X : f(u) \leq c\}.$$

Remark D.1 Note that $|df|(u) = \|f'(u)\|$ whenever X is a Finsler manifold of class C^1 and f is of class C^1, see [83]. In particular, when X is a Banach space, Definition D.8 reduces to Definition 1.1 (a).

Theorem D.1 [Deformation lemma; [74, Theorem 2.3]] *Let (X, d) be a complete metric space and let $f : X \to \mathbb{R}$ be a continuous function, and $c \in \mathbb{R}$. Assume that f satisfies the $(PS)_c$-condition. Then, given $\overline{\varepsilon} > 0$, $\mathcal{O}$ a neighborhood of K_c (if $K_c = \emptyset$, we choose $\mathcal{O} = \emptyset$) and $\lambda > 0$, there exist $\varepsilon > 0$ and a continuous map $\eta : X \times [0, 1] \to X$ such that*

(i) $d(\eta(u, t), u)) \leq \lambda t$ *for all* $u \in X$ *and* $t \in [0, 1]$;
(ii) $f(\eta(u, t)) \leq f(u)$ *for all* $u \in X$ *and* $t \in [0, 1]$;
(iii) $\eta(u, t) = u$ *for all* $u \notin f^{-1}(]c - \overline{\varepsilon}, c + \overline{\varepsilon}[)$ *and* $t \in [0, 1]$;
iv) $\eta(f^{c+\varepsilon} \setminus \mathcal{O}, 1) \subseteq f^{c-\varepsilon}$.

These notions can be extended to *arbitrary* functions defined on a metric space (X, d). Let $f : X \to \mathbb{R}$ be a function. We denote by $B_r(u)$ the open ball of center u and radius r and we set

$$\operatorname{epi}(f) = \{(u, \lambda) \in X \times \mathbb{R} :\ f(u) \leq \lambda\}\ .$$

In the following, $X \times \mathbb{R}$ will be endowed with the metric

$$d\left((u,\lambda),(v,\mu)\right) = \left(d(u,v)^2 + (\lambda-\mu)^2\right)^{1/2}$$

and epi (f) with the induced metric.

Definition D.9 For every $u \in X$ with $f(u) \in \mathbb{R}$, we denote by $|df|(u)$ the supremum of the σ's in $[0,+\infty[$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{H} : (B_\delta(u,f(u)) \cap \mathrm{epi}\,(f)) \times [0,\delta] \to X$$

satisfying

$$d(\mathcal{H}((w,\mu),t),w) \le t\,, \qquad f(\mathcal{H}((w,\mu),t)) \le \mu - \sigma t\,,$$

whenever $(w,\mu) \in B_\delta(u,f(u)) \cap \mathrm{epi}\,(f)$ and $t \in [0,\delta]$.

The extended real number $|df|(u)$ is called the weak slope of f at u.

Define a function $\mathcal{G}_f : \mathrm{epi}\,(f) \to \mathbb{R}$ by $\mathcal{G}_f(u,\lambda) = \lambda$. Of course, $\mathcal{G}_f$ is Lipschitz continuous of constant 1.

Proposition D.1 *For every $u \in X$ with $f(u) \in \mathbb{R}$, we have $f(u) = \mathcal{G}_f(u,f(u))$ and*

$$|df|(u) = \begin{cases} \dfrac{|d\mathcal{G}_f|(u,f(u))}{\sqrt{1-|d\mathcal{G}_f|(u,f(u))^2}} & \text{if } |d\mathcal{G}_f|(u,f(u)) < 1\,, \\ +\infty & \text{if } |d\mathcal{G}_f|(u,f(u)) = 1\,. \end{cases}$$

We refer to [59, Proposition 2.3] for a complete proof of this result.

The previous proposition allows us to reduce, at some extent, the study of the general function f to that of the continuous function $\mathcal{G}_f$. Moreover, if f is continuous then Definition D.9 reduces to Definition D.6.

Two important notions which are very well related to the notion of weak slope are those of *generalized directional derivative* and *generalized gradient* in the sense of Degiovanni. For every $u \in X$ with $f(u) \in \mathbb{R}$, $v \in X$ and $\varepsilon > 0$, let $f^0_\varepsilon(u;v)$ be the infimum of $r \in \mathbb{R}$ such that there are $\delta > 0$ and a continuous map

$$\mathcal{V} : (B_\delta(u,f(u)) \cap \mathrm{epi}\,(f)) \times (0,\delta] \to B_\varepsilon(v)$$

satisfying

$$f(w + t\mathcal{V}((w,\mu),t) \le \mu + rt\,,$$

whenever $(w,\mu) \in B_\delta(u,f(u)) \cap \operatorname{epi}(f)$ and $t \in (0,\delta]$. Under these assumptions, we define the *generalized directional derivative* of f at u with respect to v by

$$f^0(u;v) := \sup_{\varepsilon>0} f^0_\varepsilon(u;v)\,.$$

We also define the *generalized gradient* of f at u by

$$\partial f(u) := \{\varphi \in X^*;\ f^0(u;v) \geq \varphi(v) \quad \text{for every } v \in X\}\,.$$

Theorem D.2 *If $u \in X$ and $f(u) \in \mathbb{R}$, the following facts hold:*

(*a*) $|df|(u) < +\infty \iff \partial f(u) \neq \emptyset$;
(*b*) $|df|(u) < +\infty \implies \partial f(u) \geq \min\{\|u^*\| : u^* \in \partial f(u)\}$.

We point out that $f^0(u;v)$ is greater or equal than the directional derivative in the sense of Clarke-Rockafellar, hence $\partial f(u)$ contains the subdifferential of f at u in the sense of Clarke. However, if $f : X \to \mathbb{R}$ is locally Lipschitz, these notions agree with those of Clarke. Thus, in such a case, $f^0(u;\cdot)$ is also Lipschitz continuous and we have for all $u, v \in X$,

$$f^0(u;v) = \limsup_{\substack{z\to u,\, w\to v\\ t\to 0^+}} \frac{f(z+tw)-f(z)}{t}\,, \tag{D.1}$$

$$\big\{(u,v) \mapsto f^0(u;v)\big\} \text{ is upper semicontinuous on } X \times X\,. \tag{D.2}$$

By means of the weak slope, we can now introduce the two main notions of critical point theory.

Definition D.10 We say that $u \in X$ is a (lower) critical point of f, if $f(u) \in \mathbb{R}$ and $|df|(u) = 0$. We say that $c \in \mathbb{R}$ is a (lower) critical value of f, if there exists a (lower) critical point $u \in X$ of f with $f(u) = c$.

Definition D.11 Let $c \in \mathbb{R}$. A sequence (u_n) in X is said to be a Palais-Smale sequence at level c ($(PS)_c-$sequence, for short) for f, if $f(u_n) \to c$ and $|df|(u_n) \to 0$.

We say that f satisfies the Palais-Smale condition at level c ($(PS)_c$, for short), if every $(PS)_c-$sequence (u_n) for f admits a convergent subsequence (u_{n_k}) in X.

The main feature of the weak slope is that it allows to prove natural extensions of the classical critical point theory for general continuous functions defined on complete metric spaces. Moreover, one can try to reduce the study of a lower semicontinuous function f to that of the continuous function $\mathcal{G}_f$. Actually, Proposition D.1 suggests to exploit the bijective correspondence between the set where f is finite and the graph of f.

Assuming that (X, d) is a metric space and $f : X \to \mathbb{R}$ is continuous, for any $a \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ we set

$$f^a := \{u \in X;\ f(u) \leq a\}\,.$$

Definition D.12 Let a, $b \in \overline{\mathbb{R}}$ with $a \leq b$. The pair (f^b, f^a) is said to be trivial if for every neighborhoods $[\alpha', \alpha'']$ of a and $[\beta', \beta'']$ of b in $\overline{\mathbb{R}}$, there exists a continuous function $\mathcal{H} : f^{\beta'} \times [0,1] \to f^{\beta''}$ such that for all $x \in f^{\beta'}$, $\mathcal{H}(x, 0) = x$, $\mathcal{H}(f^{\beta'} \times \{1\}) \subset f^{\alpha''}$, and $\mathcal{H}(f^{\alpha'} \times [0,1]) \subset f^{\alpha''}$.

Definition D.13 A real number c is said to be an essential value of f if for every $\varepsilon > 0$ there exist a, $b \in (c - \varepsilon, c + \varepsilon)$ with $a < b$ such that the pair (f^b, f^a) is not trivial.

Example D.1 Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = e^x - y^2$. Then 0 is an essential value of f, but not a critical value of f. On the other hand, condition $(PS)_0$ is not satisfied for f

Definition D.14 Let X be a normed space and $f : X \to \overline{\mathbb{R}}$ an even function with $f(0) < +\infty$. For every $(0, \lambda) \in \operatorname{epi}(f)$ we denote by $|d_{\mathbb{Z}_2}\mathcal{G}_f|(0, \lambda)$ the supremum of the σ's in $[0, +\infty)$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2) : (B_\delta(0, \lambda) \cap \operatorname{epi}(f)) \times [0, \delta] \to \operatorname{epi}(f)$$

satisfying

$$d\,(\mathcal{H}((w,\mu), t), (w, \mu)) \leq t\,, \qquad \mathcal{H}_2((w,\mu), t) \leq \mu - \sigma t\,,$$

$$\mathcal{H}_1((-w,\mu), t) = -\mathcal{H}_1((w,\mu), t)\,,$$

whenever $(w, \mu) \in B_\delta(0, \lambda) \cap \operatorname{epi}(f)$ and $t \in [0, \delta]$.

The following result is the *saddle point* theorem for nondifferentiable functions.

Theorem D.3 *Let X be a Banach space and assume that $f : X \to \mathbb{R} \cup \{+\infty\}$ is an even lower semicontinuous function. Assume that there exists a strictly increasing sequence (V_h) of finite-dimensional subspaces of X with the following properties:*

(*a*) *there exist a closed subspace Z of X, $\varrho > 0$ and $\alpha > f(0)$ such that $X = V_0 \oplus Z$ and*

$$\forall u \in Z : \|u\| = \varrho \implies f(u) \geq \alpha\,;$$

(*b*) *there exists a sequence (R_h) in $]\varrho, +\infty[$ such that*

$$\forall u \in V_h : \|u\| \geq R_h \implies f(u) \leq f(0)\,;$$

(*c*) *for every $c \geq \alpha$, the function f satisfies $(PS)_c$ and $(epi)_c$;*

(*d*) *we have $|d_{\mathbb{Z}_2}\mathcal{G}_f|(0, \lambda) \neq 0$ whenever $\lambda \geq \alpha$.*

Then there exists a sequence (u_h) of critical points of f with $f(u_h) \to +\infty$.

E
Elements of set–valued analysis

> To myself I am only a child playing on the beach, while vast oceans of truth lie undiscovered before me.
>
> Sir Isaac Newton (1642–1727)

Let X and Y be metric spaces, and let $F : X \rightsquigarrow Y$ be a set-valued map with nonempty values.

Definition E.1 (i) $F : X \rightsquigarrow Y$ is upper semi-continuous at $u \in X$ (usc at u) if for any neighborhood U of $F(u)$ there exists $\eta > 0$ such that for every $u' \in B_X(u, \eta)$ we have $F(u') \subseteq U$.

(ii) $F : X \rightsquigarrow Y$ is lower semi-continuous at $u \in X$ (lsc at u) if for any $y \in F(u)$ and for any sequence of elements $\{u_n\}$ in X converging to u there exists a sequence $\{y_n\}$ converging to y and $y_n \in F(u_n)$.

(iii) $F : X \rightsquigarrow Y$ is upper (resp. lower) semi-continuous on X if F is upper (resp. lower) semi-continuous at every point $u \in X$.

(iv) $F : X \rightsquigarrow Y$ is continuous at $u \in X$ if it is both upper semi-continuous and lower semi-continuous at u, and that it is continuous on X if and only if it is continuous at every point of X.

Let M be a subset of Y. We set

$$F^{-1}(M) = \{x \in X : F(u) \cap M \neq \emptyset\};$$

$$F^{+1}(M) = \{x \in X : F(u) \subseteq M\}.$$

The subset $F^{-1}(M)$ is called the *inverse image* of M by F and $F^{+1}(M)$ is called the *core* of M by F. The following characterization of the upper (resp. lower) semicontinuity of F on X can be given.

Proposition E.1 *Let X and Y be metric spaces. A set-valued map $F : X \rightsquigarrow Y$ with nonempty values is upper semicontinuous on X if and only if the inverse image of any closed subset $M \subseteq Y$ is closed, and it is lower semicontinuous on X if and only if the core of any closed subset $M \subseteq Y$ is closed.*

Let X and Y be vector spaces, let $F : X \rightsquigarrow Y$ be a set-valued map, and let K be a convex subset of X.

Definition E.2 (i) F is a process if $\lambda F(u) = F(\lambda u)$ for all $u \in X$, $\lambda > 0$, and $0 \in F(0)$.

(ii) We say that F is convex on K if

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subseteq F(\lambda x_1 + (1-\lambda)x_2)$$

for all $x_1, x_2 \in K$ and all $\lambda \in [0,1]$.

Let X be a normed space, K be a subset of X, and let $u \in \overline{K}$, where $\overline{K}$ is the closure of K. The *contingent cone* $T_K(u)$ is defined by

$$T_K(u) = \{v \in X : \liminf_{t \to 0^+} \frac{\operatorname{dist}(u+tv, K)}{t} = 0\}.$$

In particular, if K is convex, then $K \subseteq u + T_K(u)$.

Let $F : X \rightsquigarrow \mathbb{R}$ be a set-valued map with nonempty compact values.

Definition E.3 (i) We say that F is Lipschitz around $u \in X$ if there exist a positive constant L and a neighborhood U of u such that

$$F(u_1) \subseteq F(u_2) + L\|u_1 - u_2\| \cdot [-1,1] \ \textit{for all}\ u_1, u_2 \in U.$$

(ii) Let $K \subseteq X$. F is K-locally Lipschitz if it is Lipschitz around all $u \in K$.

Proposition E.2 *Let X be a normed space. If $F : X \rightsquigarrow \mathbb{R}$ is K-locally Lipschitz, then the restriction $F|_K : K \rightsquigarrow \mathbb{R}$ is continuous on K.*

Definition E.4 The contingent derivative $DF(u,c)$ of $F : X \rightsquigarrow \mathbb{R}$ at $(u,c) \in \operatorname{Graph} F$ is the set-valued map from X to $\mathbb{R}$ defined by

$$\operatorname{Graph}(DF(u,c)) = T_{\operatorname{Graph} F}(u,c), \tag{E.1}$$

where $T_{\operatorname{Graph} F}(u,c)$ is the contingent cone at (u,c) to the $\operatorname{Graph} F$.

We can characterize the contingent derivative by a limit of a differential quotient. Let $(u,c) \in \text{Graph}F$ and suppose that F is Lipschitz around u. We have

$$v \in DF(u,c)(h) \Longleftrightarrow \liminf_{t\to 0^+} \text{dist}(v, \frac{F(u+th)-c}{t}) = 0 \qquad \text{(E.2)}$$

(see Aubin and Frankowska [16, Proposition 5.1.4, p.186]). If we introduce the *Kuratowski upper limit*

$$\text{Limsup}_{u'\to u} F(u') = \left\{ c \in \mathbb{R} : \liminf_{u'\to u} \text{dist}(c, F(u')) = 0 \right\},$$

then (E.2) can be written in the following form:

$$DF(u,c)(h) = \text{Limsup}_{t\to 0^+} \frac{F(u+th)-c}{t}.$$

Remark E.1 Let us consider the case when F is single-valued, that is, $F(u) = \{f(u)\}$ for all $u \in X$. Suppose that $f : X \to \mathbb{R}$ is continuously differentiable. From Aubin and Frankowska [16, Proposition 5.1.3, p.184], we have that

$$DF(u, f(u))(h) = f'(u)(h) \text{ for all } h \in X. \qquad \text{(E.3)}$$

Definition E.5 (i) We say that $F : X \rightsquigarrow \mathbb{R}$ is sleek at $(u,c) \in \text{Graph}\, F$ if the map

$$\text{Graph}F \ni (u',c') \rightsquigarrow \text{Graph}\,(DF(u',c'))$$

is lower semicontinuous at (u,c). F is said to be sleek if it is sleek at every point $(u,c) \in \text{Graph}\, F$.

(ii) $F : X \rightsquigarrow \mathbb{R}$ is lower semicontinuously differentiable if the map

$$(u,c,h) \in \text{Graph}\, F \times X \rightsquigarrow DF(u,c)(h)$$

is lower semicontinuous.

The lower semicontinuous differentiability of a set-valued may clearly implies its sleekness.

Remark E.2 If $F : X \rightsquigarrow \mathbb{R}$ is a set-valued map with nonempty compact values and closed graph, and it is sleek at $(u,c) \in \text{Graph}\, F$, then its contingent derivative at (u,c) is a closed convex process (see [16, Theorem 4.1.8, p.130]).

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