Dyson series and short time asymptotics for the Green function of stochastic volatility models in Finance

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Abstract

- The main result: **a method to numericaly approximate the solutions of certain parabolic equations.**
- Ultimate applications: to **Finance** (option pricing).
- Connection through **Probability** and **Stochastic calc.**
- Techniques from **Geometric analysis** (complete metrics) and **Numerical methods** for polyhedral domains (Melrose, Struwe, Dauge, Schwab, ...).
- Error estimates using pseudodifferential operators (Farkas, Schulze, Triebel, ...).

Outline

Motivation

- **General Problem:** Find fast, precise numerical methods to price Financial Derivatives (Contingent Claims).
- Almost 600 Trillion on the market (US 14T).
- Lehman Broth. deriv. (2T) > Stimulus package (800B).
- The pricing problem can be reduced to solving a (backw.) parabolic partial differential equation (Kolmogorov).
- Difficult to solve numerically (evolution equations, curse of dimensionality, non-bounded domains).

Black-Scholes-Merton model

Risky asset X_t modeled by *geometric Brownian* motion (w. drift)

$$dX_t = \mu X_t dt + \sigma X_t dW_t.$$

with σ the *volatility* and μ the *average rate of return* (**constant**).

The **no arbitrage** value of a *European Option* U on X_t with payoff $h(X_T)$ at maturity T is then the discounted expectation

$$U_{BSM}(t,x) = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}h(X_t) \,|\, X_t = x] \\ = \int_0^\infty \mathcal{G}^{BSM}_{T-t}(x,y)h(y)dy,$$

$\mathcal{G}^{BSM}_{T-t}(x,y)$ =risk-neutral transition density kernel = **Green function** and r =*risk free interest rate*. **We approximate $\mathcal{G}_t$ for general models.**

Black-Scholes-Merton formula

For *Call Options* (the right to buy the asset X at time T for the price K)

$$h(X_T) = \max\{0, X_T - K\}$$

and explicit evaluation of the above integral is possible:

$$U_{BSM}(t, x) = x\mathcal{N}(d_-) + Ke^{-r(T-t)}\mathcal{N}(d_+),$$

where $\mathcal{N}(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ and

$$d_\pm = \frac{\ln(x/K) + (r \pm \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}.$$

Very few other explicit formulas are known.

Black-Scholes PDE

Let us change $\tau = T - t$ and denote

$$LU(\tau, x) = \frac{\sigma^2 x^2}{2} \partial_x^2 U(\tau, x) + rx\partial_x U(\tau, x) - rU(\tau, x)$$

(Black-Scholes). Then the option price $U = U_{BSM}$ satisfies

$$\begin{cases} (\partial_\tau - L)U(\tau, x) = 0, \\ U(0, x) = h(x), \end{cases}$$

Therefore $\mathcal{G}_\tau^{BSM}(x, y)$ is the **Green function** (fundamental solution) of $\partial_\tau - L$, namely,

$$U_{BSM}(\tau, x) = \int_0^\infty \mathcal{G}_\tau^{BSM}(x, y) h(y) dy.$$

Other equations

The **Black-Scholes Partial Differential Equation** (PDE) will be a test case for the results to follow.

Problems in practice:

- The prices of stocks are not log-normal (fat tails, driven by a Levy process: Schwab-Farkas).
- The implied volatility is not constant (volatility smile).

Other models were also proposed

$$dX_t = \mu X_t dt + \sigma(t) X_t^{\beta} dW_t.$$

time-dependent CEV model, with PDE $\partial_t - L$

$$LU(\tau, x) = \frac{\sigma(t)^2 x^{2\beta}}{2} \partial_x^2 U(\tau, x) + rx\partial_x U(\tau, x) - rU(\tau, x).$$

SABR Model

Lesniewsky and all.: the volatility is not only non-constant, it is even **stochastic** (μ, ν const., often $C(x) = x^\beta$)

$$\begin{cases} dX_t = \mu X_t dt + \sigma_t C(X_t) dW_t \\ d\sigma_t = \nu \sigma_t dZ_t. \end{cases}$$

For two (correlated) Gaussian processes W_t and Z_t,

$$d[W_t, Z_t] = \rho dt,$$

the PDE is $\partial_t - L$ with ($y =$volatility)

$$2L = y^2\big(C(x)^2 \partial_x^2 + 2\rho\nu C(x)\partial_x\partial_y + \nu^2 \partial_y^2\big).$$

SABR Solutions

- No exact (closed form) solutions are known.
- Approximate solutions using the natural metric defined by the SABR PDE (**Varadhan metric**).
- **Varadhan metric** = hyperbolic metric for $C(X) = X$.
- The Laplace operator L_0 associated to the hyperbolic metric on $\mathbb{R}^2_+$ is such that $\partial_t - L_0$ has an explicit Green function
- $L = L_0 + V$, with V of order one. Use Dyson series (below).

Green's Function

- **Our Main Problem:** Solve numerically the parabolic partial differential equation

$$\begin{cases} \partial_t u(t,x) - Lu(t,x) = f(t) \\ u(0,x) = h(x), \quad x \in \mathbb{R}^n, \end{cases}$$

$$L = \sum_{i,j} a_{ij}(x)\partial_i\partial_j + \sum_j b_j(x)\partial_j + c(x),$$

Often (!) $\partial_t u - Lu = 0$ and $u(0) = h$ is given by

$$u(t,x) = \int \mathcal{G}_t^L(x,y)h(y)dy =: e^{tL}u(x).$$

- **Our main result:** explicitly computable approximations of the Green function $\mathcal{G}_t^L(x,y)$ that is accurate to order t^k.

Geometry

Assumptions (and results): in terms of geometry

$[a_{ij}]^{-1} = [a^{ij}]$ **symmetric, positive definite**, and $\sum_{ij} a^{ij} dx^i dx^j$ (Varadhan metric).

Length of a curve $\gamma : [a, b] \to \mathbb{R}^n$ is then

$$\ell(\gamma) = \int_a^b \sqrt{\sum_{ij} a^{ij} \gamma_i'(t) \gamma_j'(t)}\, dt$$

$d(x, y) = \inf \ell(\gamma)$, for $\ell(a) = x$, $\ell(b) = y$.

$$\lim_{t \to 0+} t \ln \mathcal{G}_t(x, y) = -d(x, y)^2/4.$$

Varadhan for time independent case

WKB Heat Kernel short-time asymptotic

Short-time asymptotic expansions well-known in literature:
Assume $L =$ Laplace-Beltrami operator

$$\mathcal{G}_t(x,y) = \frac{e^{-\frac{d(x,y)^2}{4t}}}{(4\pi t)^{N/2}} \left(G_0(x,y) + G_1(x,y)t + G_2(x,y)t^2 + \dots \right),$$

$d(x,y)$ distance in Varadhan metric (McKean-Singer, Atiyah-Singer, Bismut, Avellaneda, ...).

More generally, operators of the form $\nabla^*\nabla + V$ for a potential V, the famous WKB approximation (Henry-Labordere, Kampen, Gatheral and Lawrence, ...).

For the CEV model

$$d(x,y) = \frac{\sqrt{2}}{\sigma}\left|\int_x^y t^{-\beta} dt\right|,$$

so $d(0,x) < \infty$ for $\beta \in (0,1)$, incomplete metric.

For the Black-Scholes-Merton PDE ($\beta = 1$), the metric is $d(x,y) = \sqrt{2}\,\big|\ln(x/y)\big|/\sigma$, complete: $d(0,x) = \infty$, and

$$\mathcal{G}_\tau(x,y) = \frac{1}{\sigma x\sqrt{2\pi\tau}} e^{-\frac{\left(\ln(x/y)+\sigma^2\tau/2\right)^2}{2\sigma^2\tau}}$$

Issues: In general, the distance $d(x,y)$ and the coefficient functions G_j are difficult to compute.

Semi-classical Heat Kernel asymptotics

Let $w = (x - y)/\sqrt{t}$

$$\mathcal{G}_t^L(x,y) \sim t^{-n/2} \sum_{j \geq 0} t^{j/2} p_j(x,w) e^{\frac{-w^T A(x)^{-1} w}{4}}$$

$$= \sum_{j \geq 0} t^{(j-n)/2} p_j\big(x, t^{-1/2}(x-y)\big) e^{\frac{-(x-y)^T A(x)^{-1}(x-y)}{4t}},$$

$p_j(x,w)$ a polynomial of degree j in w (Greiner, Taylor, us, ...).

Issues: Computation of p_j? Approximating the diffusion (covariance) matrix by $A(x)$ seems not to be the best choice.

Our results:

- An algorithm to compute the polynomials p_j,
- error estimates for the remainder and tests,
- x replaced by $z(x, y)$ between x and y in exponential.

Let $w = (x - y)/\sqrt{t}$, $\xi = (x - z)/\sqrt{t}$,

$$\mathcal{G}_t^L(x, y) \sim \sum_{j \geq 0} t^{(j-n)/2} p_j(z, \xi, \partial_x) e^{\frac{-w^T A(z)^{-1} w}{4}}$$

$$= \sum_{j \geq 0} t^{(j-n)/2} \mathfrak{P}_j\big(z, t^{-1/2}(x - z), t^{-1/2}(x - y)\big) e^{\frac{-(x-y)^T A(z)^{-1}(x-y)}{4t}}.$$

$$\mathcal{G}_t^{[\mu,z]}(x, y) = \sum_{j=0}^{\mu} t^{(j-n)/2} \mathfrak{P}_j\big(z, t^{-1/2}\xi, t^{-1/2} w\big) e^{\frac{-w^T A(z)^{-1} w}{4t}}.$$

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Main Result

We can find explicit polynomials $\mathfrak{P}^{\ell}(z, x, y)$ such that the error

$$e^{tL}f(x) = \int_{\mathbb{R}^N} \mathcal{G}_t^{[\mu,z]}(x,y)f(y)dy + t^{(\mu+1)/2}\mathcal{E}_t^{[\mu,z]}f(x).$$

satisfies

$$\|\mathcal{E}_t^{[\mu,z]}f\|_{W_a^{m+k,p}} \leq Ct^{-k/2}\|f\|_{W_a^{m,p}},$$

with C independent of $t \in [0, T]$, $0 < T < \infty$.

Here $W_a^{m,p}$ are **weighted Sobolev spaces** (derivatives up to order m in $e^{a|x|}L^p$).

Suitable for the computation of greeks.

Our expansion

- *parabolic rescaling*, $z =$ dilation center, $s = \sqrt{t}$:

$$u^{s,z}(t,x) := u(s^2 t, z + s(x-z)),$$

$$L^{s,z} := \sum_{i,j=1}^{N} a_{ij}^{s,z}(x)\partial_i\partial_j + s\sum_{i=1}^{N} b_i^{s,z}(x)\partial_i + s^2 c^{s,z}(x),$$

$$(\partial_t - L^{s,z})u^{s,z} = 0.$$

- Compute $e^{L^{s,z}}$ instead of e^{tL}.
- Taylor expansion in s coupled with time-ordered perturbative expansion via Duhamel's principle.
- Eventually, the dilation center z will be allowed to be a function of x, y (improved accuracy).

Assumptions

- We work with $u : [0, \infty) \times \mathbb{R}^N \to \mathbb{C}$.
- The **main assumptions:** $[a_{ij}]$ is symmetric, positive definite, and the functions a_{ij}, b_i, c, and $\det([a_{ij}])^{-1}$ and their derivatives are bounded.
- $\det([a_{ij}])^{-1}$ bounded means uniform ellipticity.
- In particular, the Varadhan metric is complete.
- Extends to other cases (Black-Scholes-Merton), but we need a complete metric.
- $z : \mathbb{R}^{2N} \to \mathbb{R}^N$ satisfies $z(x, x) = x$ and has bounded derivatives (admissible). Ex: $z(x, y) = x,\ y$, or $(x + y)/2$.

Some remarks

- First few terms of expansion agrees with semi-classical approx if $z = x$. Otherwise, more general.

  Choice of $z = x$ not always optimal.
- Error estimates are global on $\mathbb{R}^N$ (generalize to *complete non-compact* manifolds).
- Even when $z = x$, our method is more easily implementable in practice.
- Solution has closed form in term of Error Functions if initial data is piece-wise linear (e.g. option pricing data).
- Very fast implementation is crucial in applications

1D Formulas

Let $L(x) = \frac{1}{2}a(t,x)^2\partial_x^2 + b(t,x)\partial_x + c(t,x)$, second-order approx kernel:

$$G_2(x,y;z) = \Big(\frac{1}{2}L^z_{2,\tau} + L^z_{2,x} + \frac{1}{2}[L^z_0, L^z_{2,x}] + \frac{1}{6}[L^z_0,[L^z_0,L^z_{2,x}]]$$
$$+\frac{1}{2}L^{z,2}_1 + \frac{1}{3}L^z_1[L^z_0,L^z_1] + \frac{1}{6}[L^z_0,L^z_1]L^z_1 + \frac{1}{8}[L^z_0,L^z_1]^2\Big)\, e^{L^z_0}$$
$$= \left(P_0 + \sum_{i=1}^{6} P_i H_i(x-y)\right) e^{L^z_0}(x-y)\,.$$

where H_j are Hermite polynomials and P_j are polynomials in $x-z$ with coefficients given in terms of the values of the functions a, b, and c, and their derivatives, all evaluated at $z = z(x,y)$, as follows

$$
\begin{aligned}
P_0 &= c = c(0,z), \quad P_1 = b'(x-z),\\
P_2 &= \frac{1}{2}\Big[\frac{1}{2}a^3a'' + a^2b' + a^2a'^2/2 + b^2 + a'^2(x-z)^2\\
&\qquad + a\left(ba' + \dot{a} + a''(x-z)^2\right)\Big],\\
P_3 &= a(x-z)(a'b + \frac{1}{2}a^2a'' + \frac{3}{2}aa'^2),\\
P_4 &= \frac{a^2}{3}\Big[\frac{1}{2}a^3a'' + 2a^2a'^2 + \frac{3}{2}aa'b + \frac{3}{2}a'^2(x-z)^2\Big],\\
P_5 &= \frac{1}{2}a^4a'^2(x-z), \quad P_6 = \frac{1}{8}a^6a'^2.
\end{aligned}
\tag{1}
$$

The CEV Model

J.C. Cox, S. A. Ross (skewed “smiles”)

$$L(x) = \frac{1}{2}\sigma(t)^2 x^{2\alpha}\partial_x^2 + rx\partial_x - r, \qquad \alpha > 0.$$

Series solution formulas exist (Cox-Ross, D.Emanuel-J. MacBeth) in terms of Bessel's functions.

Our 1st-order approximate solution for $z = x$:

$$U_{CEV}^{[1]}(\tau, x) = \frac{\sigma x^{\alpha-1}\sqrt{\tau}}{2\sqrt{2\pi}} e^{-\frac{(x-K)^2}{2\sigma^{2\alpha}\tau}} \left((2-\alpha)x - \alpha K\right) + \frac{1}{2}\cdot\left(\mathrm{erf}\left(\frac{x-K}{\sqrt{2\tau}\sigma x^{\alpha}}\right) + 1\right)\left((1+r\tau)x - K\right)$$

Numerical Test

For $\beta = 1$ (Black-Scholes-Merton) our second order approximation is as good as the exact formula for $\sigma^2\tau < .01$. For CEV $\beta = \frac{2}{3}, K = 15, \sigma = 0.3, r = 0.1$, $5 < x < 25$ we get

Figure: $\tau = 0.1$

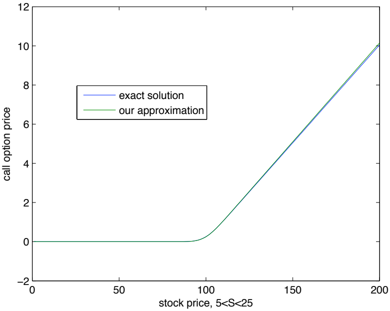
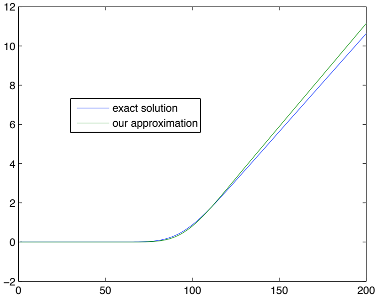


Figure: $\tau = 0.5$

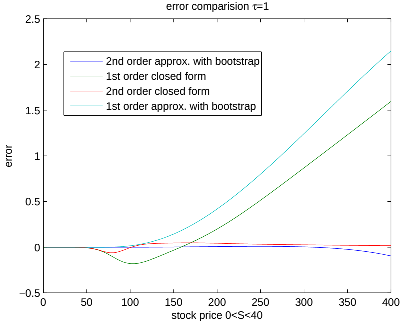


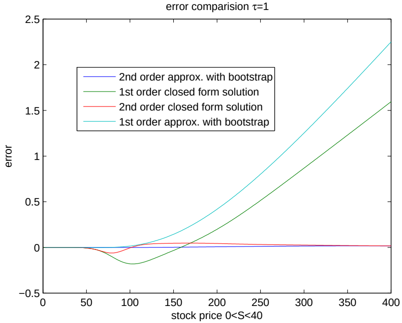
Large time solutions

BSM model: maturity $\tau = 1$, $K = 20$ (100 on picture), $r = 10\%$, and $\sigma = 0.5$.

$n = 10$ bootstrap steps (interm. val. $\tau = k/10$).

Numerical integration on $(0, 200)$ (400).





Duhamel's Formula

Consider the equation

$$\begin{cases} (\partial_t - L)u(t,x) = 0, \\ u(0,x) = h(x), \end{cases}$$

Then $u = e^{tL}h$, with e^{tL} an analytic semigroup. The equation

$$\begin{cases} (\partial_t - L)u(t,x) = f(\tau), \\ u(0,x) = h(x), \end{cases}$$

has solution

$$u(t) = e^{tL}h + \int_0^t e^{(t-s)L} f(s)ds.$$

Time ordered products

Let us write $L = L_0 + V$ and our equation in the form

$$\begin{cases} (\partial_t - L_0)u(t,x) = Vu, \\ u(0,x) = h(x), \end{cases}$$

Since $u = e^{tL}h$, we obtain from Duhamel's formula

$$e^{tL}h = e^{tL_0}h + \int_0^t e^{(t-s)L_0} V e^{sL} u ds.$$

Iterating Duhamel's gives time-ordered (Dyson) expansion:

$$e^{L} = e^{L_0} + \int_0^1 e^{(1-\tau_1)L_0} V e^{\tau_1 L_0} d\tau_1$$

$$+ \int_0^1 \int_0^{\tau_1} e^{(1-\tau_1)L_0} V e^{(\tau_1-\tau_2)L_0} V e^{\tau_2 L_0} d\tau_2 d\tau_1 + \cdots +$$

$$+ \int_0^1 \int_0^{\tau_1} \ldots \int_0^{\tau_{d-1}} e^{(1-\tau_1)L_0} V e^{(\tau_1-\tau_2)L_0} \ldots e^{(\tau_{d-1}-\tau_d)L_0} V e^{\tau_d L_0} d\tau_d \ldots d\tau_1$$

$$+ \int_0^1 \ldots \int_0^{\tau_d} e^{(1-\tau_1)L_0} V e^{(\tau_1-\tau_2)L_0} \ldots e^{(\tau_d-\tau_{d+1})L_0} V e^{\tau_{d+1} L} d\tau_{d+1} \ldots d\tau_1$$

$L_0 = L^{0,z} = \sum a_{ij}\partial_i\partial_j$ from parabolic rescaling,

The idea is that we can compute $e^{\tau L_0}$ and the integrals, except the last term, which will be included in the error.

Dilation and Taylor expansion

Given fixed point z

$$L^{s,z} := \sum_{i,j=1}^{N} a_{ij}^{s,z}(x)\partial_i\partial_j + s\sum_{i=1}^{N} b_i^{s,z}(x)\partial_i + s^2 c^{s,z}(x) \quad \Rightarrow$$

$$\mathcal{G}_t^L(x,y) = s^{-N}\mathcal{G}_1^{L^{s,z}}(z + s^{-1}(x-z), z + s^{-1}(y-z)), \quad t = s^2.$$

So it is enough to compute $e^{L^{s,z}}$, for all s, and then by rescaling back we obtain e^L, where $u(t) = e^{tL}h$.

Taylor expansion

Taylor expand $L^{s,z}$ to order $n = d$ in s at 0:

$$L^{s,z} = \sum_{m=0}^{n} s^m L^z_m + s^{n+1} L^{s,z}_{n+1} = L_0 + V$$

where all the term containing powers of s are contained in V. Then collect the powers of s.

Thus $L_0 = L_0^z = \sum_{ij} a^{ij}(z)\partial_i\partial_j$, constant-coefficient operator:

$$e^{L_0^z}(x,y) = \frac{1}{\sqrt{4\pi \det(A(z))}} e^{-\frac{(x-y)^T A^{-1}(z)(x-y)}{4}}.$$

Collecting the powers of s, Dyson expansion becomes:

$$e^{L^{s,z}} = e^{L_0^z} + \sum_{\ell=1}^{\mu} s^\ell \Lambda_z^\ell + \sum_{\ell=\mu+1}^{\max(\ell,n+1)} s^\ell \Lambda_z^\ell = \sum_{\ell=0}^{\mu} s^\ell \Lambda_z^\ell + s^{\mu+1}\mathbb{E}_\mu^{s,z},$$

$\mathbb{E}_\mu^{s,z}$ = the error, and $\Lambda_z^0 = e^{L_0^z}$

In

$$e^{L^{s,z}} = e^{L_0^z} + \sum_{\ell=1}^{\mu} s^\ell \Lambda_z^\ell + \sum_{\ell=\mu+1}^{\max(\ell,n+1)} s^\ell \Lambda_z^\ell = \sum_{\ell=0}^{\mu} s^\ell \Lambda_z^\ell + s^{\mu+1}\mathbb{E}_\mu^{s,z},$$

we decompose $\Lambda_z^\ell := \sum_\alpha \Lambda_{\alpha,z}$

$$\Lambda_{\alpha,z} := \int_{\Sigma_k} e^{\tau_0 L_0^z} L_{\alpha_1}^z e^{\tau_1 L_0^z} L_{\alpha_2}^z \cdots L_{\alpha_k}^z e^{\tau_k L_0^z} d\tau,$$

for $1 \le k \le n$, while if $k = n+1$,

$$\Lambda_{\alpha,z} := \int_{\Sigma_{n+1}} e^{\tau_0 L_0^z} L_{\alpha_1}^z e^{\tau_1 L_0^z} L_{\alpha_2}^z \cdots L_{\alpha_{n+1}}^z e^{\tau_{n+1} L^{s,z}} d\tau.$$

Commutators

If $\alpha \in \mathfrak{A}_\ell$, $T = L_\alpha^z$ differential operator of order 2 and degree ℓ polynomial coefficients.

Introduce $ad_T(L) = [T, L] = TL - LT$, $ad_T^j(L) = ad_T(ad_T^{j-1}(L))$.

We prove a Campbell-Baker-Hasdorff formula

$$e^{\theta L_0} T = P_\alpha(L_0, T, \theta) e^{\theta L_0},$$

where

$$P_\ell(L, T, \theta) := \sum_{k=0}^{\ell} \frac{\theta^k}{k!} \mathrm{ad}_L^k(T).$$

Expansion revisited

For $\ell \leq n$, then have

$$\Lambda_\alpha^z = \int_{\Sigma_k} \prod_{i=1}^{k} P_{\alpha_i}(L_0^z, L_{\alpha_i}^z; 1-\tau_i) d\tau e^{L_0^z} := \mathcal{P}_\alpha(x, z, \partial) e^{L_0^z},$$

with

$$\mathcal{P}_\alpha(x, z, \partial) = \sum_{|\beta| \leq \ell} \sum_{|\gamma| \leq \ell + 2k} a_{\beta,\gamma}(z)(x-z)^\beta \partial_x^\gamma,$$

$a_{\beta,\gamma}$ smooth, bounded function.

Using explicit formula for $e^{L_0^z}(x, y)$, dilating back, substituting $z = z(x, y)$ gives the final formula for the approximation kernel.

Error Estimates

Let $z = z(x, y)$ admissible, $s = t^{1/2}$.
Two types of error terms in $\mathcal{E}_t^{[\mu,z]}$, operator with kernel:

$$s^{-N}\mathbb{E}_{\mu}^{s,z}(z + s^{-1}(x - z), z + s^{-1}(y - z)).$$

1. For $\mu < \ell \leq n$, operators $\mathcal{L}_{s,\ell}$ with kernel

$$s^{-N}\Lambda_z^{\ell}(z + s^{-1}(x - z), z + s^{-1}(y - z)).$$

2. For $\ell \geq n + 1$, operators $\mathcal{L}_{s,\ell}$ with kernel

$$s^{-N}\Lambda_z^{s,\ell}(z + s^{-1}(x - z), z + s^{-1}(y - z)).$$

1. For $\mu < \ell \leq n$, obtain error bounds *uniformly* in s in $W_a^{r,p}$, for all $r \in \mathbb{R}$ by showing:

$$\mathcal{L}_{s,\ell} = b_s(x, \partial), \qquad b_s(x, \xi) = a_s(x, s\xi),$$

   for some family of symbols a_s bounded in $S^0_{1,0}$.

2. For $\ell \geq n+1$, use Riesz Lemma along with:

$$\partial_x^\beta \partial_z^{\beta'} \partial_y^{\beta''} \Lambda_z^{s,\ell}(x,y) = \langle \partial^\beta \delta_x \, , \partial_z^{\beta'} \Lambda_z^{s,\ell} \, \partial^{\beta''} \delta_y \rangle \Rightarrow$$

$$\|\mathcal{L}_{s,\ell}\|_{W^{r+k,p} \to W^{r,p}} \leq C_T \, t^{-k/2}, \qquad t \in (0, T].$$

   This bound not optimal, but sufficient to prove *sharp* estimate for $\mathcal{E}_t^{[\mu,z]}$ by choosing $n > \mu + N - 1$.