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DOCTORAL THESIS

BOOLEAN SUBALGEBRAS
AND
SPECTRAL AUTOMORPHISMS IN QUANTUM LOGICS

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To the memory of Alecu Ivanov,
my mentor and dear friend

Contents

Introduction

Quantum logics are logic-algebraic structures that arise in the study of the foundations of quantum mechanics.

According to the conventional Hilbert space formulation of quantum mechanics, states and observables are represented by operators in a Hilbert space associated to the quantum system under investigation, the so-called *state space*. Propositions, which represent yes-no experiments concerning the system, form an orthomodular lattice, isomorphic to the partially ordered set of projection operators $\mathcal{P}(H)$ on the state space H. It is called *the logic* associated to the quantum system. Thus, orthomodular lattices, which are sometimes assumed to be complete, atomic and to fulfill the covering property (just like $\mathcal{P}(H)$) bear the name of *quantum logics*. We propose and investigate the following question:

What amount of quantum mechanics is coded into the structure of the propositional system?

In other words, we intend to investigate to what extent some of the fundamental physical facts concerning quantum systems can be described in the more general framework of orthomodular lattices, without the support of Hilbert space–specific tools.

With this question in mind, we attempt to build, in abstract orthomodular lattices, something similar to the spectral theory in Hilbert space. For this purpose, we introduce and study *spectral automorphisms.*

According to the contemporary theory of quantum measurement, yes-no measurements that may be unsharp, called effects, are represented by so-called effect operators, self-adjoint positive operators on the state space H, smaller than identity. As an abstraction of the structure of the set of effect operators, the effect algebra structure is defined.

In the second part of the thesis, we move our investigation to the framework of unsharp quantum logics, represented by effect algebras. We generalize spectral automorphisms to effect algebras and obtain in this framework results that are analogous to the ones obtained in orthomodular lattices.

Finally, as a rather separate undertaking, we study atomic effect algebras endowed with a family of morphisms called *compression base*, analyzing the consequences of atoms being foci of compressions in the compression base. We then apply some of the obtained results to the particular case of effect algebras endowed with a sequential product.

The thesis is divided into two parts. The first part, composed of the first three chapters, is devoted to the study of "sharp" quantum logics, as represented by orthomodular lattices, arising from the conventional Hilbert space formulation of the quantum mechanics. In the second part, consisting of the chapters four to seven, we adopt the framework of "unsharp" quantum logics, represented by effect algebras, emerging from the contemporary theory of quantum measurement. A brief description of the chapters contents follows.

Chapter 1. The first chapter of the thesis is devoted to a presentation of orthomodular structures such as orthomodular posets and lattices and of their basic properties. The physically meaningful relation of compatibility is discussed. Blocks, commutants and center of such structures are covered. The last section of the chapter is dedicated to atomicity, as well as covering and exchange properties. The facts presented in this chapter are covered in various monographs, such as, e.g., [**38, 46, 51, 53, 62**].

Chapter 2. The second chapter contains a discussion of the problem concerning the possibility of embedding quantum logics into classical ones. The origin of this problem can be traced back to a famous paper of Einstein, Podolsky and Rosen, where authors conjectured that a "completion" of quantum mechanical formalism, leading to its embedding" into a larger, classical and deterministic theory is possible.

We give an overview of classical and newer results concerning this matter by Kochen and Specker [**39**], Zierler and Schlessinger [**64**], Calude, Hertling and Svozil [**6**], Harding and Ptak [**35**] in a unitary treatment.

Chapter 3. The last chapter of the first part of the thesis consists of the original results obtained concerning spectral automorphisms in orthomodular lattices. First, we introduce spectral automorphisms. We define the spectrum of a spectral automorphism and study a few examples. Then, we analyze the possibility of constructing such automorphisms in products and horizontal sums of lattices. A factorization of the spectrum of a spectral automorphism is found. We give various characterizations, as well as necessary or sufficient conditions for an automorphism to be spectral or for a Boolean algebra to be its spectrum. Then, we prove that the presence of spectral automorphisms allows us to distinguish between classical and non-classical theories. For finite dimensional quantum logics, we show that for every spectral automorphism there is a basis of invariant atoms. This is an analogue of the spectral theorem for unitary operators having purely point spectrum. An interesting consequence is that, if there are physical motivations for admitting that a finite dimensional theory must have spectral symmetries, it cannot be represented by the lattice of projections of a finite dimensional *real* Hilbert space. The last part of this chapter addresses the problem of the unitary time evolution of a system from the point of view of the spectral automorphisms theory. An analogue of the Stone theorem concerning strongly continuous one-parameter unitary groups is given. The results in this chapter have been published in the articles "Spectral automorphisms in quantum logics", by Ivanov and Caragheorgheopol (2010), and "Characterizations of spectral automorphisms and a Stone-type theorem in orthomodular lattices", by Caragheorgheopol and Tkadlec (2011), both appeared in *International Journal of Theoretical Physics.*

Chapter 4. In the fourth chapter, we present background information on unsharp quantum logics, as represented by effect algebras. We discuss special elements, coexistence relation, which generalizes compatibility from orthomodular posets, various substructures and important classes of effect algebras, as well as automorphisms in effect algebras. The facts presented in this chapter can be found, e.g., in the book of Dvurečenskij and Pulmannová [**14**] which gathers many of the recent results in the field of quantum structures.

Chapter 5. In the fifth chapter, we present sequential, compressible and compression base effect algebras, which will be needed in the sequel. They were introduced by Gudder [**28, 29**] and Gudder and Greechie [**31**].

Sequential product in effect algebras formalizes the case of sequentially performed measurements. The prototypical example of a sequential product is defined on the set $\mathcal{E}(H)$ of effects operators by $A \circ B = A^{1/2}BA^{1/2}$.

The set $\mathcal{E}(H)$ of effect operators can be endowed with a family of morphisms $(J_P)_{P\in\mathcal{P}(H)}$ defined by $J_P(A) = PAP$, called *compressions* and indexed by the projection operators $P \in \mathcal{P}(H)$ which are also called the *foci* of compressions. The family $(J_P)_{P\in\mathcal{P}(H)}$ is said to form a *compression base* of $\mathcal{E}(H)$. Inspired by the main features of the family $(J_P)_{P\in\mathcal{P}(H)}$, the notions of compression, compression base and compressible efect algebra were introduced in abstract effect algebras. As it turns out, compression base effect algebras generalize sequential, as well as compressible effect algebras.

Chapter 6. In this chapter, we generalize spectral automorphisms to compression base effect algebras, which are currently considered as the appropriate mathematical structures for representing physical systems [**21**]. We obtain characterizations of spectral automorphisms in compression base effect algebras and various properties of spectral automorphisms and of their spectra. In order to evaluate how well our theory performs in practice, we apply it to an example of a spectral automorphism on the standard effect algebra of a finite-dimensional Hilbert space and we show the consequences of spectrality of an automorphism for the unitary Hilbert space operator that generates it. In the last section, spectral families of automorphisms are discussed and an effect algebra version of the Stone-type theorem in Chapter 3 is obtained. The results of this chapter are included in the article "Spectral automorphisms in CB-effect algebras", by Caragheorgheopol, which was accepted for publication by *Mathematica Slovaca* and will appear in Volume 62, No. 6 (2012).

Chapter 7. The last chapter of the thesis contains original results concerning atomic compression base effect algebras and the consequences of atoms being foci of compressions. Part of our work generalizes results obtained by Tkadlec [**59**] in atomic sequential effect algebras. The notion of projection-atomicity is introduced and studied and conditions that force a compression base effect algebra or the set of compression foci to be Boolean are given. We apply some of these results to the important particular case of sequential effect algebra and strengthen previous results obtained by Gudder and Greechie [**31**] and Tkadlec [**59**]. The results of this chapter have been published in the article "Atomic effect algebras with compression bases", by Caragheorgheopol and Tkadlec, which appeared in *Journal of Mathematical Physics* (2011).

Part 1

Quantum Logics as Orthomodular Structures

Preliminaries

Finding a mathematical and logical model for quantum mechanics has been a challenge since the first part of the twentieth century. Researchers like von Neumann, Birkhoff, Husimi, Dirac, Mackey , Piron and many others have contributed to this task. As a result of their work, Hilbert space theory has been established as the appropriate mathematical framework for the study of quantum mechanics. At the same time, as it was clear that the facts concerning the measurements of complementary variable in quantum mechanics, such as, e.g., position and momentum, are inconsistent with the distributivity law that works within classical Boolean logic, a new type of logic became necessary. With their historical paper "The Logic of Quantum Mechanics" (1936), Garrett Birkhoff and John von Neumann started the search for a quantum logic.

In what follows, we will try to briefly explain the meaning of the essential elements of the axiomatic model of such a quantum logic. In the meantime, this will provide us with the physical interpretation of the results obtained in the framework of this model. Detailed descriptions of the way that this axiomatic model arises can be found in, e.g., [**2, 45, 51, 62**]. The mathematical modeling in quantum physics often has a speculative character, sometimes involving philosophical problems of physics. After this short introduction, we will devote ourselves to the study of the *mathematical* theory of quantum logics. Nevertheless, we hope that our results remain meaningful from a physical point of view.

The standard Hilbert space formulation of quantum mechanics is based on a few essential notions such as *states*, *observables*, *propositions* pertaining to a quantum system under investigation. Following mainly Mackey's approach [**45**] and its discussion by Beltrametti and Cassinelli in [**2**], let us try to sketch an explanation of their meaning, mathematical models and mutual relations.

By a yes-no experiment, we mean a test performed on a physical system using a measuring apparatus that has only two possible outcomes, which can be labeled "yes" and "no". By preparation procedure of the physical system we understand all the information about the operations performed on the system until the moment when the test is performed.

We consider two preparation procedures of a physical system to be equivalent if we cannot distinguish them by any yes-no experiment (i.e., every yes-no experiment has the same probability of a "yes" outcome for both preparations). We shall call a *state* of the system an equivalence class of preparation procedures. Let us denote by $\mathcal{S}$ the set of states associated to a physical system.

An *observable* of a physical system is generally understood as referring to a measurable physical quantity of the system. It is assumed that such a physical quantity denoted by A takes values on the real axis. If M is a Borel subset of $\mathbb{R}$, then the question whether the measured value of A lies in M is a yes-no experiment that we denote by (A, M). We consider two such yes-no experiments (A, M) and (B, N) to be equivalent if they have the same probability of a "yes" outcome in every possible state of the system. An equivalence class of yes-no experiments of the above type will be called a *proposition* about the system. Let us denote by $\mathcal{L}$ the set of propositions associated to a physical system.

Let us remark that every state $s \in \mathcal{S}$ of the system defines—and can be regarded as being described by—a probability function defined on the set $\mathcal{L}$, taking values in the interval $[0, 1]$ which associates to every proposition $p \in \mathcal{L}$ the probability $s(p)$ of a "yes" outcome when the system is prepared in the state s.

It is now intuitively clear that an observable A of a system in a state s can be completely described by the knowledge of the probability $s(p)$ for all propositions p represented by yes-no experiments (A, M), with M spanning the Borel sets of $\mathbb{R}$ (i.e., by the knowledge, for all Borel sets M, of the probability that the measuring of A in the state s will yield a result in M).

A natural ordering is defined on the set $\mathcal{L}$ of propositions by putting $p \leq q$ if and only if $s(p) \leq s(q)$ for all $s \in \mathcal{S}$. Let us remark that the yes-no experiment $(A, \emptyset)$ is a representative of the trivial proposition that has always (i.e., in all states) the answer "no", which we will denote by $\mathbf{0}$ and the yes-no experiment $(A, \mathbb{R})$ is a representative of the trivial proposition that has always the answer "yes", which we will denote by $\mathbf{1}$. Then $\mathbf{0} \leq p \leq \mathbf{1}$ for all $p \in \mathcal{L}$.

Orthogonality can be defined on $\mathcal{L}$ as follows: we call propositions p and q *orthogonal* (in symbols, $p \perp q$) if $s(p) + s(q) \leq 1$ for all states $s \in \mathcal{S}$.

To every proposition p, represented by (A, M), we can associate its negation, represented by $(A, \mathbb{R} \setminus M)$ and denoted by p'.

According to a crucial axiom of Mackey [**45**], for every sequence of pairwise orthogonal propositions $p_1, p_2, \ldots \in \mathcal{L}$, there exist a proposition $q \in \mathcal{L}$ such that $s(q) + s(p_1) + s(p_2) + \ldots = 1$ for all $s \in \mathcal{S}$.

Anticipating some terminology (for which the reader may consult section 1.1), we can say that, as a consequence of this axiom, the mapping $p \mapsto p'$ becomes an orthocomplementation and $\mathcal{L}$ becomes an orthomodular lattice which is also σ–complete. Moreover, the probability function $s : \mathcal{L} \to [0, 1]$ associated to every state $s \in \mathcal{S}$ becomes a probability measure on $\mathcal{L}$, in the sense of the following definition:

If $\mathcal{L}$ is a σ–complete orthomodular lattice, a mapping $s : \mathcal{L} \to [0, 1]$ is a *probability measure* on $\mathcal{L}$ if $s(\mathbf{1}) = 1$ and for every sequence $(p_n)_{n\in\mathbb{N}}$ of pairwise orthogonal elements in $\mathcal{L}$, $s(\bigvee_{n\in\mathbb{N}} p_n) = \Sigma_{n\in\mathbb{N}} s(p_n)$.

We are led to the idea that the states of a physical system can be identified with probability measures on the orthomodular lattice of propositions of the system. Let us note, as a side remark, that sometimes, the lattice of the propositions of a system is not assumed to be σ–complete and the states (probability measures) are not assumed to be σ–additive as in the above definition, but only additive, as in Definition 2.2.4.

In his celebrated book, “Mathematical Foundations of Quantum Mechanics” [**48**], John von Neumann introduced Hilbert space as the appropriate mathematical framework for quantum mechanics. According to the conventional Hilbert space formulation of quantum mechanics, observables are represented by self-adjoint operators on the Hilbert space H associated to the quantum system, while states are represented by density operators (i.e., trace class operators of trace 1) on H. At this point, one might ask one’s self what is the connection between these mathematical representations and the notions of state and observable previously described here. To answer this question, an additional assumption is necessary, and this assumption is precisely the content of Mackey’s “quantum” axiom VII. This axiom states that *the partially ordered set of all propositions in quantum mechanics is isomorphic to the partially ordered set of projection operators (or, equivalently, of closed subspaces) of a separable, infinite dimensional, complex Hilbert space.*

Let us recall that projection operators on a Hilbert space H are its self-adjoint idempotents. To each projection operator there corresponds a unique closed linear subspace of the Hilbert space H which is the range of the projection. This is a one-to-one correspondence. The set of projection operators on a Hilbert space H, which we will denote henceforward by $\mathcal{P}(H)$ is organized as a complete orthomodular lattice which, by virtue of the above said correspondence, is isomorphical to (and can be identified with) the complete orthomodular lattice of closed subspaces of H (see, e.g., [**34, 38**]). To be precise, if $P_1, P_2 \in \mathcal{P}(H)$ and $\mathcal{M}_1, \mathcal{M}_2$ are their respective ranges, the order relation in $\mathcal{P}(H)$ is defined by putting $P_1 \le P_2$ whenever $\mathcal{M}_1 \subseteq \mathcal{M}_2$. The orthocomplementation is defined on $\mathcal{P}(H)$ by $P' = \mathbf{1} - P$ (where $\mathbf{1}$ denotes the identity of H), i.e., P' is the projection onto the closed subspace that is orthogonal to the range of P.

Let us return to Mackey's axiom VII which we also called the "quantum" axiom. This name is a reference to the fact that it is precisely the assumption of this axiom that makes the difference between quantum systems and other physical systems. We should remark that Mackey himself commented [**45**] that this axiom seems "entirely *ad hoc*" and that "we are far from being forced to accept this axiom as logically inevitable". He adds, however, that "we make it because it 'works', that is, it leads to a theory which explains physical phenomena and successfully predicts the results of experiments".

By virtue of Mackey's axiom VII, we can identify the lattice of propositions $\mathcal{L}$ of a quantum system and the lattice $\mathcal{P}(H)$ of projectors on the Hilbert space associated to the quantum system. A state, represented by a density operator ρ, induces a probability measure on $\mathcal{P}(H)$ by $P \mapsto \mathrm{tr}(\rho P)$ for all $P \in \mathcal{P}(H)$. Conversely, according to a famous and highly nontrivial theorem of Gleason (see, e.g., [**62, 2**]), if $dim(H) \geq 3$, then every probability measure on $\mathcal{P}(H)$ arises from a density operator ρ on H by $P \mapsto \mathrm{tr}(\rho P)$, for all $P \in \mathcal{P}(H)$. Therefore, the set of density operators and the set of probability measures on $\mathcal{P}(H)$ are in a one-to-one correspondence which, moreover, preserves the convex structure of both sets. It should be mentioned here that the extreme points of the convex set of states are called *pure states* and they correspond to density operators which are projection operators on unidimensional subspaces—so-called *rays*— of H. By a slight abuse of language, the unit vectors generating the rays are called sometimes pure states.

We have sketched, up to this point, the explanation of the correlation between states seen as equivalence classes of preparation procedures, probability measures on the orthomodular lattice of propositions (or projection operators) and density operators. Let us try to accomplish the corresponding task concerning observables. This will also allow us to add a final missing step to our explanation on states.

According to von Neumann, an observable A is represented by a self- adjoint operator (which we will denote also by A) on the Hilbert space H associated to the system. The spectral values of the operator are interpreted as the possible outcome of the measurement of the observable. According to the spectral theorem for self-adjoint operators (see, e.g., [**55, 34**]), to A corresponds a spectral projection valued measure (PV-measure), i.e. a mapping $M \mapsto P_A(M)$ which associates to every Borel set M of $\mathbb{R}$ a projection operator $P_A(M)$, such that:

- $P_A(\emptyset) = \mathbf{0}$, $P_A(\mathbb{R}) = \mathbf{1}$;
- $P_A(\bigcup_{n\in\mathbb{N}} M_n) = \Sigma_{n\in\mathbb{N}} P_A(M_n)$ for every sequence of mutually disjoint Borel sets $(M_n)_{n\in\mathbb{N}}$ (the series on the right converges in the strong operator topology);

- $P_A(M_1 \cap M_2) = P_A(M_1)P_A(M_2)$.

By Mackey's axiom VII, as mentioned before, the set $\mathcal{L}$ of propositions can be identified with the set of projection operators $\mathcal{P}(H)$. It is natural to consider that the proposition $p \in \mathcal{L}$ represented by the yes-no experiment (A, M) (i.e., by the question whether the measured value of A lies in the Borel set M) corresponds to the projection operator $P_A(M)$. Moreover, the probability of obtaining for the observable A a measured value within the Borel set M, when the system is prepared in a state s represented by the density operator ρ is given by $\mathrm{tr}(\rho P_A(M))$. It is not difficult to see that the mapping $M \mapsto \mathrm{tr}(\rho P_A(M))$ defined on the Borel sets of $\mathbb{R}$ is a probability measure.

Summarizing, we have justified the correspondence between observables as self-adjoint operators and observables as PV-measures. Moreover, we have specified in what way, for an observable defined as a self-adjoint operator A, we can compute the probability of obtaining a measured value within a Borel set M, when the system is prepared in a state s. As we mentioned before, when we introduced the notion of observable, this intuitively corresponds to a complete description of the observable.

Finally, let us remark, as an argument supporting Mackey's axiom VII, that if the spectrum of an observable represents the possible outcomes of its measurements, the spectrum of projection operators is a subset of $\{0, 1\}$, which suggests the interpretation of observable represented by projection operators as propositions corresponding to yes-no experiments.

In view of the foregoing description of the mathematical model of a quantum system, it should be clear why the partially ordered set of propositions of the system is called the *logic* of the system and hence, why orthomodular lattices (which are sometimes assumed to be complete, or to have the covering property, like the lattice $\mathcal{P}(H)$ which they generalize) are called *quantum logics*.

An important question that one might ask is formulated by Beltrametti and Cassinelli [**2**] as follows:

"The fact that states and physical quantities can be defined in terms of $\mathcal{P}(H)$ suggests the following problem: Take from the outset a partially ordered set $\mathcal{L}$, to be physically interpreted as the set of 'propositions' and having some of the properties of $\mathcal{P}(H)$ but without any notion of Hilbert space. Consider the set $\mathcal{S}$ of all probability measures on $\mathcal{L}$ and the set $\mathcal{O}$ of all functions from $\mathcal{B}(\mathbb{R})$ into $\mathcal{L}$ that have the formal properties of spectral measures. Then, is it possible to determine a Hilbert space H such that $\mathcal{L}$ is identified with $\mathcal{P}(H)$, $\mathcal{S}$ with the set of all density operators on H and $\mathcal{O}$ with the set of all self-adjoint operators on H? Briefly, the question is: to what extent is the Hilbert space description of quantum systems coded into the ordered structure of propositions?"

A great part of our research work presented here, is essentially motivated by another—in a way, complementary—question: is it possible to avoid the use of Hilbert space specific tools and replace them with instruments belonging to the lattice of propositions, in the description of quantum systems?

CHAPTER 1

Basics on Orthomodular Structures

In this introductive chapter we will be concerned with presenting the main orthomodular structures which arise from quantum mechanics—most notably, orthomodular posets and lattices—and their properties. We will also discuss important examples and ways to construct new such structures from given ones. A separate section is devoted to the very important—in view of its physical significance—notion of compatibility and its properties. Here we also cover blocks, commutants and center of such structures. The possibility of including a set of pairwise compatible elements of an orthomodular poset/lattice in a block is discussed. Finally, the last section of this chapter is dedicated to atomicity, covering and exchange property and the introduction of lattices with dimension. The different facts presented in this chapter are covered in various monographs, like, e.g., [**2, 38, 46, 49, 51, 53, 62**].

1.1. Definitions of orthomodular structures

DEFINITION 1.1.1. If P is a set, endowed with a partial relation "$\leq$" which is reflexive, antisymmetric and transitive (i.e., a partial order relation), then $(P, \leq)$ is a *poset*. If there exist the smallest element and the greatest element of P (which will be denoted by $\mathbf{0}$ and $\mathbf{1}$, respectively), then $(P, \leq)$ is a *bounded poset*.

DEFINITION 1.1.2. Let $(P, \leq)$ be a poset and $A \subseteq P$. Then $c \in P$ is the *infimum*, or *greatest lower bound*, or *meet* of A (in P) if $(i) c \leq a$ for every $a \in A$ and (ii) for every $d \in P$ such that $d \leq a$ for every $a \in A$ it follows $d \leq c$. We will denote the infimum of A by $\bigwedge A$. In particular, $a \wedge b$ will denote the infimum of $\{a, b\}$. If it is not clear in what poset the infimum is taken, we will write $\wedge_P$ for the infimum taken in P.

Dually, $e \in P$ is the *supremum*, or *lowest upper bound*, or *join* of A (in P) if $(i) a \leq e$ for every $a \in A$ and (ii) for every $f \in P$ such that $a \leq f$ for every $a \in A$ it follows $e \leq f$. We will denote the supremum of A by $\bigvee A$. In particular, $a \vee b$ will denote the supremum of $\{a, b\}$. If there is a risk of confusion about the poset in which the supremum is taken, we shall write $\vee_P$ for the supremum taken in P.

Let us remark that infimum and supremum need not exist for every subset of a poset $(P, \leq)$, or for every pair of elements $a, b \in P$.

DEFINITION 1.1.3. A poset $(P, \leq)$ such that $a \wedge b$ and $a \vee b$ exist for every $a, b \in P$ is a *lattice*. If $\bigwedge A$ and $\bigvee A$ exist for every subset A of P, it is a *complete lattice*. If $\bigwedge A$ and $\bigvee A$ exist for every countable subset A of P, it is a *σ-complete lattice*

DEFINITION 1.1.4. Let $(P, \leq)$ be a bounded poset. A unary operation ' on P such that, for every $a, b \in P$, the following conditions are fulfilled:

(1) $a \leq b$ implies $b' \leq a'$,
(2) $a'' = a$,
(3) $a \vee a' = \mathbf{1}$ and $a \wedge a' = \mathbf{0}$,

is an *orthocomplementation* on P.

DEFINITION 1.1.5. A bounded poset with an orthocomplementation is an *orthoposet*. An orthoposet which is a lattice is an *ortholattice*.

EXAMPLE 1.1.6. Let P be the power set of a set M, partially ordered by set inclusion. The empty set $\emptyset$ and M are the smallest and greatest elements of P, respectively. An orthocomplementation can be defined by the set-theoretical complement relative to M, i.e., $A' = M \setminus A$ for every $A \in P$. Then $(P, \subseteq, ')$ is an orthoposet, even ortholattice, with set-theoretical intersection as infimum and set-theoretical reunion as supremum.

DEFINITION 1.1.7. A relation *orthogonal*, denoted by "$\perp$" is defined for elements a, b of an orthoposet by

$$a \perp b \iff a \leq b'$$

It is easy to see that relation "$\perp$" is symmetric, since $a \leq b'$ if and only if $b \leq a'$. Let us notice that, in Example 1.1.6, two elements of P are orthogonal if they are disjoint subsets of M.

REMARK 1.1.8. In an orthoposet (ortholattice), de Morgan's laws hold:

(1) $(a \vee b)' = a' \wedge b'$
(2) $(a \wedge b)' = a' \vee b'$

for every elements a, b (in the case of an orthoposet, the relations should be understood in the sense that if one side exists, the other does too, and the equality holds).

DEFINITION 1.1.9. An orthoposet (ortholattice) $(P, \leq, ')$ satisfies the *orthomodular law* if for every $a, b \in P$,

(OM1) $\quad a \leq b$ implies there exists $c \in P, c \perp a$ such that $b = a \vee c$

.

DEFINITION 1.1.10. An orthoposet with the property that every pair of orthogonal elements has supremum and satisfying the orthomodular law is an *orthomodular poset*. If, moreover, the supremum exists for every countable set of pairwise orthogonal elements, it is a *σ–complete orthomodular poset.*

DEFINITION 1.1.11. An ortholattice satisfying the orthomodular law is an *orthomodular lattice.*

PROPOSITION 1.1.12. *Let P be an orthomodular poset and $a, b \in P$ such that $a \leq b$. There exists a unique element $c = a' \wedge b$ of P such that $c \perp a$ and $b = a \vee c$.*

PROOF. The orthomodular law asserts that an element $c \in P$ such that $c \perp a$ and $b = a \vee c$ exists. The element $a' \wedge b$ also exists, since $a \leq b$ implies $b' \leq a'$, hence $b' \perp a$, which in turn entails $a \vee b'$ exists, and, in view of de Morgan's law, $a' \wedge b$ exists as well. Clearly $c \leq a', b$, hence $c \leq a' \wedge b$. It follows that $b = a \vee c \leq a \vee (a' \wedge b) \leq b$, where $a \vee (a' \wedge b)$ exists because $a' \wedge b \leq a'$ (i.e., $a \perp a' \wedge b$) and the last inequality is a consequence of $a, a' \wedge b \leq b$. We conclude that $b = a \vee c = a \vee (a' \wedge b)$.

Let us recall that $c \leq a' \wedge b$. According to the orthomodular law, there exists $d \in P$ such that $d \perp c$ and $c \vee d = a' \wedge b$. It follows that $(a \vee c) \vee d = a \vee (c \vee d) = a \vee (a' \wedge b) = a \vee c$, hence $d \leq a \vee c = b$. However, $d \leq c'$ and $d \leq a' \wedge b \leq a'$, therefore $d \leq a' \wedge c' = (a \vee c)' = b'$. It follows that $d \leq b \wedge b' = \mathbf{0}$, hence $d = \mathbf{0}$ and $c = a' \wedge b$. $\square$

REMARK 1.1.13. In view of Proposition 1.1.12, the orthomodular law can be enounced equivalently as follows:

"An orthoposet (ortholattice) $(P, \leq, ')$ satisfies the orthomodular law if for every $a, b \in P$,

(OM2) $$a \leq b \text{ implies } b = a \vee (a' \wedge b)."$$

EXAMPLE 1.1.14. In the introduction to the first part of the thesis, we have presented the prototypical example of orthomodular lattice— from which the entire discussion about orthomodular structures as models for the logic of quantum mechanics arises— namely the lattice $\mathcal{P}(H)$ of projection operators on a Hilbert space H. Recall that, if $P_1, P_2 \in \mathcal{P}(H)$ and $\mathcal{M}_1, \mathcal{M}_2$ are their respective ranges, the order relation in $\mathcal{P}(H)$ is defined by putting $P_1 \leq P_2$ whenever $\mathcal{M}_1 \subseteq \mathcal{M}_2$. The orthocomplementation is defined on $\mathcal{P}(H)$ by $P' = \mathbf{1} - P$ (where $\mathbf{1}$ denotes the identity of H), i.e., P' is the projection onto the closed subspace that is the orthogonal complement to the range of P. By Sasaki's theorem (see, e.g., [**38**, Section 5]), $\mathcal{P}(H)$ is a complete orthomodular lattice. Let us notice that $P_1 \wedge P_2$ is the projection on the closed subspace $\mathcal{M}_1 \cap \mathcal{M}_2$, while $P_1 \vee P_2$ is the projection on the smallest closed subspace that includes $\mathcal{M}_1 \cup \mathcal{M}_2$. If P_1 and P_2 commute, then their meet and join can be described by the algebraic relations: $P_1 \wedge P_2 = P_1 P_2$ and $P_1 \vee P_2 = P_1 + P_2 - P_1 P_2$.

EXAMPLE 1.1.15. If L is an orthomodular lattice and $a \in L \setminus \{\mathbf{0}\}$, then $([\mathbf{0}, a], \leq|_{[\mathbf{0},a]}, {}^*, \mathbf{0}, a)$ with $[\mathbf{0}, a] = \{b \in L\colon\ b \leq a\}$ and ${}^*\colon\ b \mapsto b' \wedge a$ for every $b \in [\mathbf{0}, a]$ is an orthomodular lattice.

Definition 1.1.16. A lattice satisfies the *distributive laws* if, for every its elements a, b, c, the following relations hold:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), \tag{D}$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c). \tag{D*}$$

In this case, the lattice is *distributive.*

Remark 1.1.17. Let us remark that a lattice is distributive if and only if the property (D) is satisfied for every its elements a, b, c if and only if the property (D*) holds for every its elements a, b, c.

Clearly, distributivity of an ortholattice implies orthomodularity. The converse, however, does not hold, as one can easily see by analyzing the example of the orthomodular lattice of closed subspaces of a bidimensional Hilbert space H, with H as the greatest element of the lattice, denoted by $\mathbf{1}$ and the $\{0\}$ subspace as the least element, denoted by $\mathbf{0}$. Indeed, if we consider three distinct 1-dimensional subspaces of H and denote them as elements a, b, c of the lattice, we find that $a \vee (b \wedge c) = a \vee \mathbf{0} = a$, while $(a \vee b) \wedge (a \vee c) = \mathbf{1} \wedge \mathbf{1} = \mathbf{1}$, hence the distributive law is not satisfied.

Definition 1.1.18. A distributive ortholattice is a *Boolean algebra.*

Let us remark that the above definition, which is suitable for our purposes here, is not the usual one. In fact there are a few alternative ways to define Boolean algebras. Such alternative definitions can be found, e.g., in [**49, 57**]. In [**49**, Proposition 4.10.1] the equivalence of the classical definition and the one presented here is proved.

1.2. Compatibility. Basic properties

In this section, we introduce the notion of compatibility and present some of its properties. The importance of compatibility derives from its physical meaning. Compatible pairs represent simultaneously verifiable events, hence their importance in the axiomatics of quantum theories. Our presentation follows mainly the exposition in [**53**, Chapter 1] although [**46**] was also used.

Definition 1.2.1. Let P be an orthomodular poset. Elements $a, b \in P$ are *compatible (in P)* if there exist mutually orthogonal elements $a_1, b_1, c \in P$ such that $a = a_1 \vee c$ and $b = b_1 \vee c$. In this case we will write $a \leftrightarrow_P b$, or just $a \leftrightarrow b$, when there's no risk of confusion. For M a subset of P, we shall write $a \leftrightarrow M$ when $a \leftrightarrow m$ for every $m \in M$.

Lemma 1.2.2. *Let a and b be elements of an orthomodular poset P. Then:*

(1) *$a \leq b$ implies $a \leftrightarrow b$;*
(2) *$a \perp b$ if and only if $a \leftrightarrow b$ and $a \wedge b = \mathbf{0}$;*

(3) *the following are equivalent:* $a \leftrightarrow b$, $a' \leftrightarrow b$, $a \leftrightarrow b'$, $a' \leftrightarrow b'$.

PROOF. (1) According to the orthomodular law (Definition 1.1.9), if $a \leq b$, there exists $c \in P$, $c \perp a$ such that $b = a \vee c$. It follows that $a, c, \mathbf{0}$ are mutually orthogonal elements such that $a = a \vee \mathbf{0}$ and $b = a \vee c$. Thus, $a \leftrightarrow b$.

(2)"$\Rightarrow$" From $a \perp b$ follows $a \leq b'$. Let $c \leq a, b$. Then, $c \leq b$ and $c \leq a \leq b'$, therefore $c \leq b \wedge b' = \mathbf{0}$. It follows that $a \wedge b = \mathbf{0}$. Since $a, b, \mathbf{0}$ are mutually orthogonal and $a = a \vee \mathbf{0}$, $b = b \vee \mathbf{0}$, we conclude that $a \leftrightarrow b$.

"$\Leftarrow$" Since $a \leftrightarrow b$, there exist a_1, b_1, c mutually orthogonal elements such that $a = a_1 \vee c$ and $b = b_1 \vee c$. Therefore, $c \leq a, b$, hence $c \leq a \wedge b = \mathbf{0}$. It follows that $c = \mathbf{0}$, hence $a = a_1$ and $b = b_1$ and therefore $a \perp b$.

(3) The only thing we need to prove is that $a \leftrightarrow b$ implies $a' \leftrightarrow b$ (and the rest of the statement (3) follows in an obvious way).

Let us assume $a \leftrightarrow b$. There exist then $a_1, b_1, c \in P$ mutually orthogonal such that $a = a_1 \vee c$ and $b = b_1 \vee c$. Clearly $a \vee b = a_1 \vee b_1 \vee c$, and there exist $d = (a \vee b)' = a' \wedge b'$. It follows that d is orthogonal to a_1, b_1, c and $a_1 \vee b_1 \vee c \vee d = \mathbf{1}$. It is then not difficult to see that $a' = b_1 \vee d$, and we know $b = b_1 \vee c$ and b_1, c, d are mutually orthogonal. Therefore $a' \leftrightarrow b$. □

PROPOSITION 1.2.3. [**53**, Proposition 1.3.5] *Let a and b be elements of an orthomodular poset P such that $a \leftrightarrow b$. Then $a \wedge b$ and $a \vee b$ exist in P. If $a_1, b_1, c \in P$ are mutually orthogonal elements such that $a = a_1 \vee c$ and $b = b_1 \vee c$, then $c = a \wedge b$, $a_1 = a \wedge b'$ and $b_1 = b \wedge a'$.*

COROLLARY 1.2.4. *Elements a and b of an orthomodular poset P are compatible if and only if $a = (a \wedge b) \vee (a \wedge b')$ and $b = (b \wedge a) \vee (b \wedge a')$.*

PROPOSITION 1.2.5. [**38**, Ch. 1, Section 3, Theorem 2] *If $(L, \leq, ')$ is an ortholattice, the following statements are equivalent:*

(1) *the orthomodular law holds, i.e., $a \leq b$ implies $b = a \vee (a' \wedge b)$, for every $a, b \in L$;*
(2) *$a \leq b$ and $a' \wedge b = \mathbf{0}$ imply $a = b$ for every $a, b \in L$;*
(3) *$a = (a \wedge b) \vee (a \wedge b')$ implies $b = (b \wedge a) \vee (b \wedge a')$ for every $a, b \in L$.*

COROLLARY 1.2.6. *Let $(L, \leq, ')$ be an orthomodular lattice. Then, for every $a, b \in L$, $a \leftrightarrow b$ if and only if $a = (a \wedge b) \vee (a \wedge b')$ if and only if $b = (b \wedge a) \vee (b \wedge a')$.*

REMARK 1.2.7. In Boolean algebras, every pair of elements is compatible (since conditions in Corollary 1.2.6 are satisfied, due to distributivity). It follows that, according to Lemma 1.2.2 (2), in Boolean algebras, $a \perp b$ if and only if $a \wedge b = \mathbf{0}$, i.e., orthogonality is identical to disjunction. It should be mentioned that this is not the case in orthomodular lattices, where orthogonality is much stronger than disjunction (as one can easily see in the case of the orthomodular lattice of subspaces of a Hilbert space).

THEOREM 1.2.8. [**38**, Ch. 1, Section 3, Proposition 4] *Let* $(L, \leq, ')$ *be an orthomodular lattice and* M *be a subset such that* $\bigvee M$ *exists. If* $b \in L$ *is such that* $b \leftrightarrow M$*, then:*

(1) $b \leftrightarrow \bigvee M$
(2) $b \wedge (\bigvee M) = \bigvee\{b \wedge m\colon\ m \in M\}$

The following corollary will be of practical use.

COROLLARY 1.2.9. *Let* $(L, \leq, ')$ *be an orthomodular lattice and let* $a, b, c \in L$ *be such that* $a \leftrightarrow \{b, c\}$*. Then:*

(1) $a \leftrightarrow (b \vee c)$*;*
(2) $a \leftrightarrow (b \wedge c)$

PROOF. (1) Follows from Theorem 1.2.8 (1), by taking $M = \{b, c\}$.
(2) Since $a \leftrightarrow \{b, c\}$, according to Lemma 1.2.2 (3), it follows that $a \leftrightarrow \{b', c'\}$. According to (1), $a \leftrightarrow b' \vee c'$ and by Lemma 1.2.2 (3) again, $a \leftrightarrow (b' \vee c')'$. According to de Morgan's laws (Remark 1.1.8), $(b' \vee c')' = b \wedge c$, and thus $a \leftrightarrow b \wedge c$. □

DEFINITION 1.2.10. Let a, b, c be elements of a lattice L. We call $\{a, b, c\}$ a *distributive triple* if the distributive laws (D) and (D*) hold for all permutations of the set $\{a, b, c\}$.

PROPOSITION 1.2.11. [**53**, Proposition 1.3.11] *Let* $(L, \leq, ')$ *be an orthomodular lattice and* $a, b, c \in L$*. If* $a \leftrightarrow b$ *and* $a \leftrightarrow c$*, then* $\{a, b, c\}$ *is a distributive triple.*

COROLLARY 1.2.12. *An orthomodular poset is a Boolean algebra if and only if every pair of its elements is compatible.*

PROOF. Let $(P, \leq, ')$ be an orthomodular poset such that every pair of its elements is compatible. According to Proposition 1.2.3, the infimum and supremum exist for every pair of elements of P, hence P is a lattice. According to Proposition 1.2.11, P is a distributive orthomodular lattice, i.e., a Boolean algebra. The converse assertion is trivial. □

DEFINITION 1.2.13. A maximal set of mutually compatible elements of an orthomodular lattice is a *C-class* (compatibility class).

REMARK 1.2.14. (1) It should be noted that C-classes are not proper equivalence classes, since compatibility is *not transitive.* For an example, let us consider a 3-dimensional Hilbert space H and its associated lattice of subspaces, with $\mathbf{1} = H$ and $\mathbf{0} = \{0\}$ and let a, b, c be distinct unidimensional subspaces such that $a \perp b$, $a \perp c$ and b is not orthogonal to c. Then $a \leftrightarrow b$, $a \leftrightarrow c$, but $b \not\leftrightarrow c$, since $(b \wedge c) \vee (b \wedge c') = \mathbf{0} \vee \mathbf{0} = \mathbf{0} \neq b$.

(2) By a simple Zorn lemma argument, C-classes exist. To see this, just consider $\mathcal{M}$ the set of all subsets with mutually compatible elements of the orthomodular lattice, ordered by set inclusion. Let $\mathcal{N}$ be a subset of $\mathcal{M}$ that is a chain. The reunion of elements of $\mathcal{N}$ is an upper bound for $\mathcal{N}$, which is in $\mathcal{M}$. According to Zorn lemma, $\mathcal{M}$ has maximal elements, i.e., C-classes exist.

1.3. Orthomodular substructures

We shall first discuss the substructures of an ortholattice, which can, in particular, be an orthomodular lattice or a Boolean algebra.

Definition 1.3.1. A subset of an ortholattice is a *subalgebra* if it contains the least and greatest elements and it is closed under lattice operations $\vee, \wedge$ and orthocomplementation $'$.

Let us remark that such a subalgebra of an ortholattice (in particular, of an orthomodular lattice, Boolean algebra, respectively) is an ortholattice itself (in particular, an orthomodular lattice, Boolean algebra, respectively) with the induced operations.

Definition 1.3.2. Let L be an ortholattice. A subalgebra of L which is a Boolean algebra with the induced operations from L is a *Boolean subalgebra* of L.

It is a straightforward verification that the intersection of an arbitrary family of subalgebras of an ortholattice (orthomodular lattice, Boolean algebra, respectively) L is a subalgebra of L. Therefore, the following definition makes sense.

Definition 1.3.3. Let L be an ortholattice (orthomodular lattice, Boolean algebra, respectively) and $A \subseteq L$. The smallest subalgebra that includes A is called the *subalgebra generated by* A and is the intersection of all subalgebras containing A, denoted, in what follows, by $[A]$.

Definition 1.3.4. The maximal Boolean subalgebras of an orthomodular lattice are called its *blocks*.

Proposition 1.3.5. *The blocks of an orthomodular lattice are its C-classes.*

Proof. Let $(L, \leq, ')$ be an orthomodular lattice. It suffices to prove that its C-classes are subalgebras of L, i.e., they are closed under $\vee$ and $'$ (closure under $\wedge$ then follows).

Let $B \subseteq L$ be a C-class. If $a \in L$ and $a \leftrightarrow B$, then $a \in B$, due to the maximality of C-classes. If $b \in B$, then $b \leftrightarrow B$, hence $b' \leftrightarrow B$, according to Lemma 1.2.2, and it follows that $b' \in B$. Let now $M \subseteq B$ such that $\bigvee M$ exists in L. Since $m \leftrightarrow B$ for every $m \in M$, according to Theorem 1.2.8, $\bigvee M \leftrightarrow B$, hence $\bigvee M \in B$. $\square$

COROLLARY 1.3.6. *Every subset A with pairwise compatible elements of an orthomodular lattice L is part of a block of L.*

COROLLARY 1.3.7. *If A is a subset of pairwise compatible elements of an orthomodular lattice L, then $[A]$ is a Boolean algebra.*

PROOF. According to Corollary 1.3.6, there exists a block of L which includes A. Then $[A]$ is a Boolean subalgebra of that block, hence a Boolean algebra. □

The following result is from [**62**]:

PROPOSITION 1.3.8. *Let $\{B_i\}_{i\in I}$ an arbitrary family of Boolean subalgebras of an orthomodular lattice L. Then $\left[\bigcup_{i\in I} B_i\right]$ is a Boolean subalgebra of L if and only if $B_i \leftrightarrow B_j$ for any $i, j \in I$.*

REMARK 1.3.9. Every element of an orthomodular lattice L is part of a block, hence L is the set-theoretical union of its blocks. This is especially significant if we take into account that in the context of quantum logics, the blocks correspond to experimental arrangements and their elements are the propositions that can be verified by the arrangement.

We shall now consider the case of substructures in orthomodular posets. As we shall see, their behavior is quite different from the ortholattice case.

DEFINITION 1.3.10. A subset of an orthomodular poset is a *suborthoposet* if it contains the least and greatest elements and it is closed under orthocomplementation and under suprema of orthogonal pairs.

REMARK 1.3.11. It is not difficult to see that a suborthoposet M of an orthomodular poset P is itself an orthomodular poset with the order and orthocomplementation inherited from P (see, e.g., [**53**, Section 1.2]). However, it should be noted that for a nonorthogonal pair of elements of M, the supremum calculated in P (assuming it exists) need not be in M. Moreover, there may exist the supremum calculated in M which is different from the one calculated in P.

Before we can illustrate the above statements by an example, we need the next definition and the proposition that follows.

DEFINITION 1.3.12. Let P be an orthomodular poset and $\Delta \subseteq P$. The *commutant* of Δ in P is the set $\{a \in P\colon\ a \leftrightarrow \Delta\}$. It will be denoted henceforth by $K_P(\Delta)$ or, if there is no possibility of confusion about P, simply by $K(\Delta)$.

REMARK 1.3.13. Let M, N be subsets of an orthomodular poset P. It is then easy to see that $M \subseteq K\left(K\left(M\right)\right)$ and $M \subseteq N \implies K\left(N\right) \subseteq K\left(M\right)$.

PROPOSITION 1.3.14. (see, e.g.,[**53**, Proposition 1.3.16]) *If P is an orthomodular poset and $\Delta \subseteq P$, $\Delta \neq \emptyset$, then $K(\Delta)$ is a suborthoposet of P (which is an orthomodular poset in its own right).*

The following example is from [**53**, Section 1.3], although we use it here for a different purpose.

EXAMPLE 1.3.15. Let Ω be a finite set with an even number of elements. We denote by Ω_{even} the set of its subsets with an even number of elements. Then, Ω_{even} ordered by set inclusion and with the orthocomplement A' defined as the set complement A^C, for every $A \in \Omega_{even}$, is an orthomodular poset.

Let now $\Omega = \{1,2,3,4,5,6,7,8\}$ and $P = \Omega_{even}$. Let us consider $\Delta \subseteq P$, $\Delta = \big\{\{1,2,3,4\},\{1,2,5,6\},\{1,3,5,7\}\big\}$ and denote $M = K(\Delta)$. According to Proposition 1.3.14, M is a suborthoposet of P and an orthomodular poset in its own right. The supremum in P of the nonorthogonal elements $\{1,2,3,4\}$ and $\{1,2,5,6\}$ is $\{1,2,3,4,5,6\}$, which is not an element of M. Indeed, assuming $\{1,2,3,4,5,6\} \in M = K(\Delta)$ it follows that $\{1,2,3,4,5,6\} \leftrightarrow \{1,3,5,7\}$ which is false. Moreover, in M, the same elements $\{1,2,3,4\}$ and $\{1,2,5,6\}$ have Ω as supremum.

The next proposition shows a special type of suborthoposet of an orthomodular poset in which the "pathology" described in Remark 1.3.11 and illustrated by the above example cannot appear.

PROPOSITION 1.3.16. *Let P be an orthomodular poset and M a suborthoposet of P. If M, with the order and orthocomplementation inherited from P and the lattice operations $\vee, \wedge$ induced by the aforementioned order, is a Boolean algebra, then, all suprema and infima of pairs of elements of M as calculated in P exist and are in M(hence they coincide with the ones calculated in M).*

PROOF. Let $a, b \in M$. Since M is a Boolean algebra, $a \leftrightarrow_M b$. According to Proposition 1.2.3, there exist $c = a \wedge_M b, a_1, b_1$ mutually orthogonal elements of M such that $a = a_1 \vee_M c$ and $b = b_1 \vee_M c$. We recall that for orthogonal elements of M, the supremum exists in P and it belongs to M, hence it coincides with the supremum taken in M. Therefore, $a \leftrightarrow_P b$ and c, a_1, b_1 are mutually orthogonal elements of P such that $a = a_1 \vee c$ and $b = b_1 \vee c$. Using Proposition 1.2.3 again, we conclude that $c = a \wedge_P b$. It follows that every pair of elements in M have infimum in P which belongs to M. The dual statement for suprema follows, since M is closed under orthocomplementation. $\square$

The above result justifies the following definition:

DEFINITION 1.3.17. A suborthoposet of an orthomodular poset P which is a Boolean algebra with the induced from P order, orthocomplementation and lattice operations, is called a *Boolean subalgebra* of the orthomodular poset P.

Clearly, if P is an orthomodular lattice, this notion of Boolean subalgebra coincides with the one defined in 1.3.2.

Remark 1.3.18. Let us remark that the statement in Corollary 1.3.6 cannot be extended to orthomodular posets. More precisely, in an orthomodular poset it is possible to have a set of pairwise compatible elements which does not admit an enlargement to a Boolean subalgebra. Indeed, returning to Example 1.3.15, Δ is a paiwise compatible subset of the orthomodular poset P. Let us assume there exists a Boolean subalgebra B of P of such that $\Delta \subseteq B$. Then, according to Proposition 1.3.16, B contains all infima of pairs in Δ, as calculated in P. For instance we have $\{1,2,3,4\} \wedge \{1,2,5,6\} = \{1,2\} \in B$. Since B is a Boolean algebra, this implies $\{1,2\} \leftrightarrow \{1,3,5,7\}$, which is false. It follows that our assumption was wrong, and there is no such Boolean algebra.

In [**53**, Section 1.3] the conditions for a subset of pairwise compatible elements of an orthomodular poset P to admit an enlargement to a Boolean subalgebra of P are studied. We are only interested in presenting here the emerging conclusion. As it turns out, the following notion plays a key role in this matter.

Definition 1.3.19. (see [**53**, Definition 1.3.26]) An orthomodular poset P is *regular* if for every set $\{a, b, c\} \subset P$ of pairwise compatible elements we have $a \leftrightarrow b \vee c$.

Let us remark that the supremum $b \vee c$ has to exist, according to Proposition 1.2.3. We now present the previously announced result.

Proposition 1.3.20. (see [**53**, Proposition 1.3.29]) *An orthomodular poset P is regular if and only if every pairwise compatible subset of it admits an enlargement to a Boolean subalgebra of P.*

In view of Theorem 1.2.8, it is clear that an orthomodular lattice is a regular orthomodular poset. However, a regular orthomodular poset need not be an orthomodular lattice, as one can see by considering Ω_{even} for $\Omega = \{1,2,3,4,5,6\}$ (see Example 1.3.15 for the definition of Ω_{even}).

Definition 1.3.21. Let P be an orthomodular poset. An element $a \in P$ is *central* if it is compatible with every other element of P. The set of central elements of P is the *center* of P, denoted henceforth by $\widetilde{C}(P)$.

Let us remark that in an orthomodular poset P, $K(P) = \widetilde{C}(P)$.

Proposition 1.3.22. *The center of an orthomodular poset P is a Boolean subalgebra of P.*

PROOF. According to Lemma 1.2.2 and Theorem 1.2.8, $\widetilde{C}(P)$ is a subalgebra of P, hence an orthomodular poset in its own right. According to Corollary 1.2.12, the conclusion follows. $\square$

COROLLARY 1.3.23. *An orthomodular poset is a Boolean algebra if and only if it coincides with its center.*

REMARK 1.3.24. The center of an orthomodular lattice is the set-theoretical intersection of its blocks.

1.4. Atomicity. Covering and exchange properties. Lattices with dimension

DEFINITION 1.4.1. Let P be a poset with least element $\mathbf{0}$. The element $\alpha \in P$ is an *atom* of P if $\alpha \neq \mathbf{0}$ and $\mathbf{0} \leq a \leq \alpha$ implies $a = \mathbf{0}$ or $a = \alpha$. We shall denote the set of atoms of P by $\Omega(P)$.

REMARK 1.4.2. Let us remark that atoms are the minimal non-zero elements.

DEFINITION 1.4.3. A poset P with least element $\mathbf{0}$ is *atomic* if every its element dominates (at least) an atom of P. It is *atomistic* if every element is the supremum of the atoms it dominates. The set of atoms dominated by an element $a \in P$ will be denoted by Ω_a.

PROPOSITION 1.4.4. *Every atomic orthomodular lattice is atomistic.*

PROOF. Let L be an atomic orthomodular lattice and $a \in L$, $a \neq \mathbf{0}$. Since $\alpha \leq a$ for all $\alpha \in \Omega_a$, we only have to prove that every element $c \in L$ which is also an upper bound for Ω_a is greater than or equal to a.

Let $c \geq \alpha$ for all $\alpha \in \Omega_a$. Then, $\alpha \leq a \wedge c$ for all $\alpha \in \Omega_a$ and therefore, $\Omega_a = \Omega_{a \wedge c}$. Since $a \wedge c \leq a$, according to the orthomodular law, $a = (a \wedge c) \vee \big(a \wedge (a \wedge c)'\big)$. Let us assume that $a \wedge (a \wedge c)' \neq \mathbf{0}$. The lattice L is atomic, hence there exists an atom $\beta \in \Omega(L)$ such that $\beta \leq a \wedge (a \wedge c)'$. It follows $\beta \leq a$, hence $\beta \in \Omega_a = \Omega_{a \wedge c}$ and $\beta \leq a \wedge c$. On the other hand, $\beta \leq (a \wedge c)'$, which is a contradiction. It follows that $a \wedge (a \wedge c)' = \mathbf{0}$. We conclude that $a = a \wedge c$, which is equivalent to $a \leq c$. $\square$

REMARK 1.4.5.
(1) Two distinct atoms of an orthomodular lattice are compatible if and only if they are orthogonal (according to Lemma 1.2.2 (2)).
(2) Since in a Boolean algebra all atoms are compatible, they must be pairwise orthogonal as well.
(3) If L is an orthomodular lattice and $a \in L$, $\alpha \in \Omega(L)$, then $\alpha \leftrightarrow a$ if and only if $\alpha \leq a$ or $\alpha \leq a'$.

The following assertion can be deduced easily from Remark 1.4.5.

PROPOSITION 1.4.6. *If B is a Boolean subalgebra of the orthomodular lattice L, $\omega \in L$ and $\omega \le a \in \Omega(B)$, then $\omega \leftrightarrow B$.*

PROPOSITION 1.4.7. *Let L be an atomic orthomodular lattice. For every element $a \in L$, there exists a maximal family $\{\alpha_i\}_{i\in I}$ of mutually orthogonal atoms in Ω_a. Then, $a = \bigvee_{i\in I} \alpha_i$.*

PROOF. Let $a \in L$. The existence of a maximal family of orthogonal atoms from Ω_a follows from Zorn's lemma in a standard way. Let us prove $a = \bigvee_{i\in I} \alpha_i$, for such a maximal family $\{\alpha_i\}_{i\in I}$. Clearly, $\bigvee_{i\in I} \alpha_i \le a$. To prove the equality, using Proposition 1.2.5 (2), it suffices to show that $a \wedge (\bigvee_{i\in I} \alpha_i)' = \mathbf{0}$. Let us suppose, to the contrary, that $a \wedge (\bigvee_{i\in I} \alpha_i)' \neq \mathbf{0}$.The lattice L is atomic, hence there exists an atom $\beta \in \Omega(L)$ such that $\beta \le a \wedge (\bigvee_{i\in I} \alpha_i)'$. Then, $\beta \in \Omega_a$ and $\beta' \ge \bigvee_{i\in I} \alpha_i \ge \alpha_i$, for all $i \in I$. This contradicts to the maximality of the orthogonal family $\{\alpha_i\}_{i\in I}$ in Ω_a. □

DEFINITION 1.4.8. Let L be an atomic orthomodular lattice and $a \in L$. A maximal family $\{\alpha_i\}_{i\in I}$ of mutually orthogonal atoms in Ω_a is a *basis* of a. A basis of $\mathbf{1} \in L$ is also called a *basis of the lattice L*.

REMARK 1.4.9. For every element of an atomic orthomodular lattice, the basis is not unique, in general.

The direct implication from the following theorem can be found, e.g., in [**38**, Ch.1, Section 4, Lemma 2]

THEOREM 1.4.10. *Let L be an orthomodular lattice and B a Boolean subalgebra of L. If B is a block of L then the atoms of B are atoms of L. Conversely, if B is atomic and its atoms are atoms of L, it is a block of L.*

PROOF. For the direct implication, let us assume that $\beta \in \Omega(B)$ and there exists $a \in L$ such that $\mathbf{0} < a < \beta$ (i.e., $\beta \notin \Omega(L)$). Then $a \notin B$. However, according to Remark 1.4.5 (3), $\beta \le b$ or $\beta \le b'$ for every $b \in B$. Consequently, $a < b$ or $a < b'$ for every $b \in B$ and therefore, $a \leftrightarrow B$. This contradicts to the maximality of B as a Boolean subalgebra of L.

For the converse implication, let us assume that B is not maximal, as a Boolean subalgebra of L. There exists then a block B' of L such that $B \subsetneq B'$, and let $b \in B' \setminus B$. Then $b \leftrightarrow \beta$ for all $\beta \in \Omega(B)$, by Proposition 1.3.5. Since B is atomic, from Proposition 1.4.4 follows that $\bigvee \Omega(B) = \mathbf{1}$. According to Theorem 1.2.8, $b = b \wedge \mathbf{1} = b \wedge \big(\bigvee \Omega(B)\big) = \bigvee_{\beta\in\Omega(B)}(b \wedge \beta)$. Since the atoms of B are atoms of L, for every $\beta \in \Omega(B)$ we have $b \wedge \beta = \mathbf{0}$ or $b \wedge \beta = \beta$. It follows that b is the join of a subset of $\Omega(B)$, hence $b \in B$—a contradiction. □

We will now introduce, following mainly the exposition in [**46**], the covering and the exchange properties, as well as the notion of dimension in orthomodular lattices.

Definition 1.4.11. Let L be a lattice and $a, b \in L$. The element b *covers* a (denoted by $a \lessdot b$) if $a < b$ and $a \leq c \leq b$ implies $c = a$ or $c = b$ (i.e., b is a minimal upper bound for a).

Definition 1.4.12. Let L be an atomic lattice. If, for every $a \in L$ and $p \in \Omega(L)$, $a \wedge p = \mathbf{0}$ implies $a \lessdot a \vee p$, then L has the *covering property*.

Definition 1.4.13. Let L be an atomic lattice. If, for every $a \in L$ and $p, q \in \Omega(L)$, $p \leq a \vee q$ and $a \wedge p = \mathbf{0}$ imply $q \leq a \vee p$, then L has the *exchange property*.

Theorem 1.4.14. (see [**46**, Theorem 7.10]) *An atomic orthomodular lattice has the covering property if and only if it has the exchange property.*

Before introducing dimension, we need the following result.

Lemma 1.4.15. [**46**, Lemma 8.3] *Let p_i and q_i $(i = 1, 2, \dots, n)$ be atoms of the atomic orthomodular lattice L with the covering property. If $(p_1 \vee p_2 \vee \dots \vee p_{i-1}) \wedge p_i = \mathbf{0}$ for $i = 2, \dots, n$ and $p_i \leq \bigvee_{j=1}^n q_j$ for $i = 1, \dots, n$, then $\bigvee_{i=1}^n p_i = \bigvee_{j=1}^n q_j$.*

Definition 1.4.16. An element of an atomic orthomodular lattice with the covering property is *finite* if it is $\mathbf{0}$ or the join of a finite number of atoms of L.

It is easy to see that, for a finite non-zero element a of an atomic orthomodular lattice with the covering property L, there exists a finite set of atoms $\{p_i \in \Omega(L)\colon\, i = 1, \dots, n\}$, $(p_1 \vee p_2 \vee \dots \vee p_{i-1}) \wedge p_i = \mathbf{0}$ for $i = 2, \dots, n$ such that $a = \bigvee_{i=1}^n p_i$.

Theorem 1.4.17. [**46**, Theorem 8.4] *Let L be an atomic orthomodular lattice with the covering property and $a \in L$ be a finite non-zero element. The finite cardinal n of the set of atoms $\{p_i \in \Omega(L)\colon$ $i = 1, \dots, n\}$ such that $(p_1 \vee p_2 \vee \dots \vee p_{i-1}) \wedge p_i = \mathbf{0}$ for $i = 2, \dots, n$ and $a = \bigvee_{i=1}^n p_i$ is uniquely determined.*

Definition 1.4.18. Let L be an atomic orthomodular lattice with the covering property and $a \in L$ be a finite non-zero element. The uniquely determined finite cardinal n of at set of atoms $\{p_i \in \Omega(L)\colon$ $i = 1, \dots, n\}$ such that $(p_1 \vee p_2 \vee \dots \vee p_{i-1}) \wedge p_i = \mathbf{0}$ for $i = 2, \dots, n$ and $a = \bigvee_{i=1}^n p_i$ is called the *dimension* (or *height*) of a, denoted by $h(a)$. If $\mathbf{1} \in L$ is finite, then $h(1)$ is the *dimension of* L and L is said to have *finite dimension.*

Remark 1.4.19. Obviously enough, in an atomic orthomodular lattice with the covering property, the dimension of a finite element coincides with the number of elements of a basis of that element. In particular, if the lattice is finite dimensional, its dimension coincides with the number of elements of a basis of the lattice. For an example, in the lattice $\mathcal{P}(H)$, described in Example 1.1.14, which is atomic and has the covering property, the finite elements are the projections onto finite dimensional subspaces of the Hilbert space H. Moreover, their dimension as elements of the lattice coincides with the dimension of their range as a closed subspace of H.

1.5. Morphisms in orthomodular structures

Let us introduce now several types of morphisms for orthomodular structures. There are a few equivalent possibilities to define such morphisms, depending on which conditions are used in the definitions and which conditions are obtained as properties. Our choice here is not the most common in the literature. Instead, we tried to give a definition that can be used with minimal changes in the different structures that we are interested in.

DEFINITION 1.5.1. Let L_1, L_2 be orthomodular posets. A mapping $h : L_1 \to L_2$ is a *morphism of orthomodular posets* if the following conditions are satisfied:

(1) $h(\mathbf{1}) = \mathbf{1}$;
(2) $a \perp b$ implies $h(a) \perp h(b)$, for every $a, b \in L_1$;
(3) $h(a \vee b) = h(a) \vee h(b)$, for every pair of orthogonal elements $a, b \in L_1$.

DEFINITION 1.5.2. Let L_1, L_2 be orthomodular lattices (Boolean algebras). A mapping $h : L_1 \to L_2$ is a *morphism of orthomodular lattices* (*Boolean algebras*, respectively) if the following conditions are satisfied:

(1) $h(\mathbf{1}) = \mathbf{1}$;
(2) $a \perp b$ implies $h(a) \perp h(b)$, for every $a, b \in L_1$;
(3) $h(a \vee b) = h(a) \vee h(b)$, for every $a, b \in L_1$.

REMARK 1.5.3. It can be easily checked, using condition (3) in Definition 1.5.1 or Definition 1.5.2, respectively, and the orthomodular law, that a morphism preserves order (i.e., $a \leq b$ implies $h(a) \leq h(b)$, for all elements $a, b \in L_1$). Also, a morphism preserves the orthocomplement, in the sense that $h(a') = h(a)'$, for every $a \in L_1$. Indeed, for $a \in L_1$, we have $a \perp a'$, hence $h(a) \perp h(a')$. On the other hand, $\mathbf{1} = h(\mathbf{1}) = h(a \vee a') = h(a) \vee h(a')$. It follows that $h(a') = h(a)'$.

DEFINITION 1.5.4. A morphism $h : L_1 \to L_2$ of orthomodular posets (or orthomodular lattices, or Boolean algebras, respectively) is an *embedding* if, for every $a, b \in L_1$, $h(a) \perp h(b)$ implies $a \perp b$ (i.e., the converse of the condition (2) in Definition 1.5.1 or Definition 1.5.2, respectively, holds).

DEFINITION 1.5.5. A morphism $h : L_1 \to L_2$ of orthomodular posets (or orthomodular lattices, or Boolean algebras, respectively) is an *isomorphism* if it is bijective and its inverse $h^{-1} : L_2 \to L_1$ is also a morphism.

DEFINITION 1.5.6. Let L be an orthomodular poset (or an orthomodular lattice, or a Boolean algebra). An isomorphism $h : L \to L$ is an *automorphism* of L.

REMARK 1.5.7. At a first glance, the condition in Definition 1.5.4 (that for every $a, b \in L_1$, $h(a) \perp h(b)$ implies $a \perp b$) seems unexpected. At a closer look, one will discover that it implies that for every $a, b \in L_1$, $h(a) \leq h(b)$ implies $a \leq b$ (see the following Proposition 1.5.8). This, in turn, entails the injectivity of h. Moreover, it entails the fact that, if we restrict the function's codomain to its image $h(L_1)$, which is a suborthoposet (subalgebra, respectively) of L_2, it becomes a bijective mapping whose inverse is also a morphism. In view of Definition 1.5.5, this means that $h : L_1 \to h(L_1) \subseteq L_2$ is an isomorphism. Finally, it follows that a surjective embedding is an isomorphism.

Automorphisms can be characterized as follows:

PROPOSITION 1.5.8. *Let L be an orthomodular poset (or an orthomodular lattice, or a Boolean algebra). A mapping $h : L \to L$ is an automorphism if and only if it satisfies the following conditions:*

(1) $h(\mathbf{1}) = \mathbf{1}$*;*
(2) $a \perp b$ *implies* $h(a) \perp h(b)$*, for all* $a, b \in L$*;*
(3) $a \leq b$ *if and only if* $h(a) \leq h(b)$*, for all* $a, b \in L$*;*
(4) h *is surjective.*

PROOF. In view of Definitions 1.5.3, 1.5.4, 1.5.5, 1.5.6 and Remark 1.5.7, an automorphism is a surjective mapping $h : L \to L$ such that $h(\mathbf{1}) = \mathbf{1}$, $a \perp b$ if and only if $h(a) \perp h(b)$, for every $a, b \in L$, and $h(a \vee b) = h(a) \vee h(b)$, for every (orthogonal—if L is an orthomodular poset) $a, b \in L$.

For the direct implication, we only have to prove that an automorphism h fulfills condition (3). If $a \leq b$, according to the orthomodular law, there exists $c \in L$, $c \perp a$ such that $a \vee c = b$. Hence, $h(b) = h(a \vee c) = h(a) \vee h(c) \geq h(a)$. Notice now, that since $b \perp b'$, it follows that $h(b) \perp h(b')$ and $\mathbf{1} = h(\mathbf{1}) = h(b \vee b') = h(b) \vee h(b')$ and therefore, $h(b') = h(b)'$. If $h(a) \leq h(b)$, then $h(a) \perp h(b)'$. In view of the previous observation, this entails that $h(a) \perp h(b')$, hence $a \perp b'$, i.e., $a \leq b$.

The converse implication requires that we prove, for a mapping fulfilling conditions (1)-(4), that $h(a \vee b) = h(a) \vee h(b)$, for every (orthogonal—if L is an orthomodular poset) $a, b \in L$, and also that $h(a) \perp h(b)$ implies $a \perp b$, for all $a, b \in L$. Since h preserves order, clearly $h(a), h(b) \leq h(a \vee b)$. Let $c \geq h(a), h(b)$. Then, h being surjective, there exists $d \in L$ such that $c = h(d)$. It follows that $d \geq a, b$, hence $d \geq a \vee b$ and therefore, $c = h(d) \geq h(a \vee b)$. We conclude that $h(a \vee b) = h(a) \vee h(b)$. For the second assertion, let $h(a) \perp h(b)$. Then $h(a) \leq h(b)'$. By the same argument as before, $h(b') = h(b)'$, hence $h(a) \leq h(b')$ and it follows that $a \leq b'$, i.e., $a \perp b$. □

PROPOSITION 1.5.9. *Let L be an orthomodular lattice and $h : L \to L$ be an automorphism. Then, for every pair of elements $a, b \in L$, $a \leftrightarrow b$ if and only if $h(a) \leftrightarrow h(b)$.*

PROOF. According to Definition 1.5.2 and Remark 1.5.3, h preserves suprema, and the orthocomplement. Therefore, it must also preserve the infima, by de Morgan's law (Remark 1.1.8). Then, for a pair of elements $a, b \in L$ the following chain of equalities holds: $h\big((a \wedge b) \vee (a \wedge b')\big) = \big(h(a) \wedge h(b)\big) \vee \big(h(a) \wedge h(b')\big) = \big(h(a) \wedge h(b)\big) \vee \big(h(a) \wedge h(b)'\big)$. It follows, using Corollary 1.2.6, the chain of equalities just mentioned and Corollary 1.2.6 again, that $a \leftrightarrow b$ if and only if $a = (a \wedge b) \vee (a \wedge b')$ if and only if $h(a) = h\big((a \wedge b) \vee (a \wedge b')\big) = \big(h(a) \wedge h(b)\big) \vee \big(h(a) \wedge h(b)'\big)$ if and only if $h(a) \leftrightarrow h(b)$. □

COROLLARY 1.5.10. *Let L be an orthomodular lattice and $h : L \to L$ be an automorphism. For every block B of L, $h(B)$ is also a block of L.*

The following result shed some light on the structure of automorphisms of an atomic complete orthomodular lattice. It can be summarized by the statement that every automorphism of such a lattice is uniquely determined by its restriction to the set of atoms, which is a bijective map that preserves orthogonality both ways.

THEOREM 1.5.11. *Let L be an atomic complete orthomodular lattice. For every automorphism h of L, its restriction to $\Omega(L)$ is $\chi : \Omega(L) \to \Omega(L)$ satisfying the conditions:*

(1) *χ bijective;*

(2) *$\alpha \perp \beta$ if and only if $\chi(\alpha) \perp \chi(\beta)$, for all $\alpha, \beta \in \Omega(L)$.*

Conversely, for every mapping $\chi : \Omega(L) \to \Omega(L)$ satisfying the above conditions (1), (2) there exists a unique automorphism h of L such that its restriction to $\Omega(L)$ is χ.

PROOF. Let $h : L \to L$ be an automorphism, and χ be its restriction to $\Omega(L)$. If α is an atom, we will prove that $\chi(\alpha)$ is an atom as well. According to Remark 1.5.7, $a \leq b$ if and only if $h(a) \leq h(b)$, for all $a, b \in L$. Let us assume that $\chi(\alpha)$ is not an atom. There exists then an element $b \in L$ such that $\mathbf{0} < b < \chi(\alpha) = h(\alpha)$. Due to the surjectivity of h, there exists $a \in L$ such that $h(a) = b$. Therefore, $\mathbf{0} < h(a) < h(\alpha)$, hence $\mathbf{0} < a < \alpha$, in contradiction with α being an atom. We are now entitled to write $\chi : \Omega(L) \to \Omega(L)$. Clearly, condition (2) is satisfied by χ, since it is fulfilled by h for arbitrary elements (including atoms). Also the injectivity of χ follows from the injectivity of h. For the first part of the theorem, we only have to prove the surjectivity of $\chi : \Omega(L) \to \Omega(L)$. Let β be an arbitrary atom of L. Since $h : L \to L$ is surjective, there exists $a \in L$ such that $h(a) = \beta$. However, a must be an atom (otherwise there exists $\alpha \in L$ such that $\mathbf{0} < \alpha < a$, which implies $\mathbf{0} < h(\alpha) < h(a) = \beta$, contradicting the fact that β is an atom). It follows that $h(a) = \chi(a) = \beta$, for some $a \in \Omega(L)$, hence the surjectivity of χ.

Conversely, let $\chi : \Omega(L) \to \Omega(L)$ be a bijective mapping such that $\alpha \perp \beta$ if and only if $\chi(\alpha) \perp \chi(\beta)$, for all $\alpha, \beta \in \Omega(L)$. We define $h : L \to L$ by $h(a) = \bigvee_{\alpha \in \Omega_a} \chi(\alpha)$, for every $a \in L$. Clearly, if $\alpha \in \Omega(L)$, then $h(\alpha) = \chi(\alpha)$, therefore χ is the restriction of h to $\Omega(L)$. We have to prove h is an automorphism.

First, $h(\mathbf{1}) = \bigvee_{\alpha \in \Omega(L)} \chi(\alpha) = \bigvee \Omega(L) = \mathbf{1}$. Let us prove now that $a \perp b$, if and only if $h(a) \perp h(b)$, for every $a, b \in L$. Indeed, the following assertions are equivalent: $a \perp b$, $\alpha \perp \beta$ for every $\alpha \in \Omega_a$ and every $\beta \in \Omega_b$, $\chi(\alpha) \perp \chi(\beta)$ for every $\alpha \in \Omega_a$ and every $\beta \in \Omega_b$, $\bigvee_{\alpha \in \Omega_a} \chi(\alpha) \perp \bigvee_{\beta \in \Omega_b} \chi(\beta)$, $h(a) \perp h(b)$. It is easy to see that h also preserves order. Indeed, $a \leq b$ implies $\Omega_a \subseteq \Omega_b$, hence $h(a) = \bigvee_{\alpha \in \Omega_a} \chi(\alpha) \leq \bigvee_{\beta \in \Omega_b} \chi(\beta) = h(b)$.

To prove the bijectivity of h, we will construct its inverse. Let $\xi = \chi^{-1}$. Then, $\xi : \Omega(L) \to \Omega(L)$ has exactly the same properties as χ, and we can define $g : L \to L$ by $g(a) = \bigvee_{\alpha \in \Omega_a} \xi(\alpha)$, for every $a \in L$. Consequently, g has all the properties of h. We assert that $g = h^{-1}$. To prove it, a few more steps are necessary. Let $a \in L$ and $\{\alpha_i\}_{i \in I}$ a basis of a. Then, according to Proposition 1.4.7, $a = \bigvee_{i \in I} \alpha_i$. Since χ preserves orthogonality, $\{\chi(\alpha_i)\}_{i \in I}$ is an orthogonal family of atoms. Also, $\chi(\alpha_i) \leq h(a)$, for all $i \in I$. We will show that $\{\chi(\alpha_i)\}_{i \in I}$ is a basis of $h(a)$, and for that, the maximality of the family $\{\chi(\alpha_i)\}_{i \in I}$ is required. Let us suppose, to the contrary, that $\beta \in \Omega_{h(a)}$, $\beta \perp \chi(\alpha_i)$ for all $i \in I$. By the surjectivity of χ, there exists an atom α such that $\chi(\alpha) = \beta$. Hence, $\chi(\alpha) \perp \chi(\alpha_i)$ for all $i \in I$, and therefore, $\alpha \perp \alpha_i$ for all $i \in I$. Since $a = \bigvee_{i \in I} \alpha_i$, it follows that $\alpha \perp a$, hence $\beta \perp h(a)$, in contradiction to $\beta \in \Omega_{h(a)}$. We have proved that $\{\chi(\alpha_i)\}_{i \in I}$ is a basis of $h(a)$, hence $h(a) = \bigvee_{i \in I} \chi(\alpha_i)$. Since the analogue result holds for g and ξ, we find that $g(h(a)) = g(\bigvee_{i \in I} \chi(\alpha_i)) = \bigvee_{i \in I} \xi(\chi(\alpha_i)) = \bigvee_{i \in I} \alpha_i = a$. Similarly, $h(g(a)) = h(\bigvee_{i \in I} \xi(\alpha_i)) = \bigvee_{i \in I} \chi(\xi(\alpha_i)) = \bigvee_{i \in I} \alpha_i = a$, and it follows that $g = h^{-1}$. It is now clear that $h(a) \leq h(b)$ implies $g(h(a) \leq g(h(b)$ (due to the fact that g preserves order), hence $a \leq b$, for all $a, b \in L$ and according to Proposition 1.5.8, h is an automorphism.

Let us now prove that h is the only automorphism that extends χ to L. Let $h_1 : L \to L$ be an automorphism such that its restriction to $\Omega(L)$ is χ, and let $a \in L$. For every $\alpha \in \Omega_a$, $\chi(\alpha) = h_1(\alpha) \leq h_1(a)$, therefore,

$h(a) = \bigvee_{\alpha\in\Omega_a} \chi(\alpha) \leq h_1(a)$. According to Proposition 1.2.5 (2), to prove the equality, it suffices to show that $\left(\bigvee_{\alpha\in\Omega_a} \chi(\alpha)\right)' \wedge h_1(a) = \mathbf{0}$. Let us assume the contrary. Then, there exists $\beta \in \Omega(L)$ such that $\beta \leq \left(\bigvee_{\alpha\in\Omega_a} \chi(\alpha)\right)' \wedge h_1(a)$ and there exists $\delta \in \Omega(L)$ such that $\chi(\delta) = \beta$. It follows that $\chi(\delta) \perp \chi(\alpha)$, for all $\alpha \in \Omega_a$, and therefore, $\delta \perp \alpha$, for all $\alpha \in \Omega_a$, hence $\delta \perp \bigvee_{\alpha\in\Omega_a} \alpha = a$. Then, $\delta \leq a'$ which implies $\beta = \chi(\delta) = h_1(\delta) \leq h_1(a') = h_1(a)'$. On the other hand, $\beta \leq h_1(a)$, which is a contradiction, since β is an atom, and therefore $\beta \neq \mathbf{0}$. $\square$

Remark 1.5.12. Theorem 1.5.11 offers a method to construct an automorphism of an atomic complete orthomodular lattice L by defining, on the set of atoms, a bijection χ which preserves orthogonality both ways and extending it to a lattice automorphism by putting, as above, $h(a) = \bigvee_{\alpha\in\Omega_a} \chi(\alpha)$, for every $a \in L$.

Example 1.5.13. We return to the prototypical example of the lattice of projectors/subspaces of a Hilbert space (see Example 1.1.14). Let H be a Hilbert space and $\mathcal{P}(H)$ the lattice of projection operators on H. It is well known that the unitary operators are the automorphisms of a Hilbert space. It is interesting to note that they induce corresponding automorphisms of the lattice structure of $\mathcal{P}(H)$. More precisely, we assert that every unitary operator $U : H \to H$ induces an automorphism $h : \mathcal{P}(H) \to \mathcal{P}(H)$ defined by $h(P) = UPU^*$, for every $P \in \mathcal{P}(H)$ (where U^* denotes the adjoint of U). Let us briefly justify this assertion. First, it is easy to see that $h(P)$ is a projector if P is one, since $h(P)^* = (UPU^*)^* = UP^*U^* = UPU^* = h(P)$ and $h(P)^2 = (UPU^*)(UPU^*) = UP^2U^* = UPU^* = h(P)$. To verify that h is an automorphism it is convenient to check the conditions (1)–(4) from Proposition 1.5.8. This is straightforward, if we consider the following facts: (a) $P \leq Q$ if and only if $PQ = P$, for every $P, Q \in \mathcal{P}(H)$; (b)$P \perp Q$ if and only if $PQ = \mathbf{0}$, for every $P, Q \in \mathcal{P}(H)$; (c) $h^{-1} : \mathcal{P}(H) \to \mathcal{P}(H)$ defined by $h^{-1}(P) = U^*PU$, for every $P \in \mathcal{P}(H)$ is the inverse of h.

$\mathcal{P}(H)$ is an atomic complete orthomodular lattice. According to Theorem 1.5.11, every automorphism of such a lattice is completely determined by its action on the atoms of the lattice. Let us see how the automorphism h acts on the atoms of $\mathcal{P}(H)$. An atom of $\mathcal{P}(H)$ is a projector $P_e : H \to H$, $P_e x = \langle x, e\rangle e$, where $e \in H$, $\|e\| = 1$ and $\langle\cdot,\cdot\rangle$ denotes the inner product of H (i.e., it is a projection on a 1-dimensional subspace generated by a vector $e \in H$). It follows that $h(P_e)x = UP_eU^*x = U\langle U^*x, e\rangle e = \langle x, Ue\rangle Ue = P_{Ue}x$ for all $x \in H$. This means that the atom P_e is transformed by h into the atom P_{Ue}, thereby justifying our claim that h is induced by the unitary U. Considering the properties of unitary operators, it follows that the restriction of h to atoms is bijective and preserves orthogonality, as expected, according to Theorem 1.5.11. Conversely, by the same theorem, the automorphism defined by $h(P) = UPU^*$ for all $P \in \mathcal{P}(H)$ is the only one that transforms the atom P_e of $\mathcal{P}(H)$ into P_{Ue}, for all $e \in H$, $\|e\| = 1$.

CHAPTER 2

Understanding the Logic of Quantum Mechanics in Classical Terms

The problem of embedding quantum logics into classical ones is very old. Its origin can be traced back to a well known article of Einstein, Podolsky and Rosen (EPR) [**15**]. In this historic paper, the authors conjectured that a "completion" of quantum mechanical formalism, leading to its "embedding" into a larger, classical and deterministic theory (from the algebraic and logic point of view) is possible. Such a theory would have to reproduce the results of quantum mechanics.

Their conjecture was based on the assumption that "elements of physical reality" exist whether or not they are actually observed. A formalization of EPR's notion of "elements of physical reality" can be given in terms of two-valued states (valuations) which take only values of 0 and 1, corresponding to the classical logical "false" and "true" notions, respectively.

The first to obtain a result that contradicts the EPR conjecture were Kochen and Specker [**39**]. Their result suggests the impossibility to "complete" quantum physics by introducing noncontextual hidden parameter models. Indeed, any truth value assignment to quantum propositions of a standard Hilbert space quantum logic of dimension higher than two (represented by an orthomodular lattice) prior to the actual measurement leads to a contradiction. On the other hand, for classical propositional systems (identifiable with Boolean algebras) it is always possible to prove the existence of separable valuations or truth assignments. Hence, no embedding from a quantum logic into some Boolean algebra can exist. In fact, the question of existence of such an embedding can be translated into the following one [**6**]: "How far might a classical understanding of quantum mechanics be, in principle, possible?". Although Kochen and Specker result mentioned before suggests an answer to this question in the negative, one may obtain a different answer by abandoning some of the restrictions on (i.e., by weakening) the embedding notion.

It is our aim in this chapter to explore the various possibilities to obtain such embeddings of quantum logics into classical ones. We intend to discuss in detail the different approaches and results obtained concerning this matter by e.g., Kochen and Specker [**39**], Zierler and Schlessinger [**64**], Calude, Hertling and Svozil [**6**], Harding and Ptak [**35**], thus offering an overview of what can be achieved in terms of classical understanding of quantum mechanics.

Let us briefly sketch the main ideas we intend to develop in our exposition. It is a well known result, due to M.H.Stone, that every Boolean algebra can be represented as a subalgebra of a powerset Boolean algebra (see, e.g., [**57**]). However, this is not the case with all orthomodular posets. A necessary and sufficient condition for an orthomodular poset to have such a representation is to possess a full set of two-valued states. This proves to be a quite restrictive condition, as it can be shown that for a Hilbert space of dimension higher than three, the orthomodular lattice of its projection operators cannot admit even one two-valued state, let alone a full set. It follows that in general we cannot embed orthomodular lattices or posets into Boolean algebras if we ask that such an embedding should preserve the join (i.e., suprema) of orthogonal elements (which is the natural first step when relaxing the standard requirement that the embedding should preserve all joins).

It follows that we have to ask even less of our embedding. A possibility to do this was explored by Zierler and Schlessinger [**64**]. They only asked for an embedding to preserve the join of central elements of an orthomodular lattice. In such a case, the attempt to embed a quantum logic into a classical one proves successful. However, this result is often not very useful, since the center of an orthomodular lattice can be very "poor", or even trivial – in which case the embedding notion used here becomes too weak to be interesting.

A result of Harding and Ptak [**35**] brings great improvement to the properties we can ask from such an embedding. Namely, for every Boolean subalgebra B of an orthomodular lattice L, we can find an embedding of L in a Boolean algebra of subsets of a certain set such that the join is preserved on B and also for every pair of elements such that at least one is central in L. Finally, the authors also show by an example that this result can hardly be improved.

2.1. The impossibility of embedding a quantum logic into a classical one

Let us show why it is impossible to have a "proper" embedding of a non-Boolean orthomodular lattice (a quantum logic) into a Boolean algebra (a classical logic).

Let L be an orthomodular lattice, B be a Boolean algebra and $h : L \to B$ an embedding. Let us assume there exist $a, b \in L$ that aren't compatible. Then $a \neq (a \wedge b) \vee (a \wedge b')$ and using the injectivity of h, it follows that $h(a) \neq h\big((a \wedge b) \vee (a \wedge b')\big)$. On the other hand, using to the properties of embeddings, we find that $h\big((a \wedge b) \vee (a \wedge b')\big) = \big(h(a) \wedge h(b)\big) \vee \big(h(a) \wedge h(b')\big) = \big(h(a) \wedge h(b)\big) \vee \big(h(a) \wedge h(b)'\big)$, hence

$h(a) \neq \big(h(a) \wedge h(b)\big) \vee \big(h(a) \wedge h(b)'\big)$, which means that $h(a)$ and $h(b)$ aren't compatible. However, B is Boolean and $h(a), h(b) \in B$, hence $h(a) \leftrightarrow h(b)$. We have reached a contradiction, therefore, our assumption that there exist $a, b \in L$ incompatible doesn't hold. Since L is an orthomodular lattice with the property that every pair of elements is compatible, it must be Boolean.

In what follows, we will try to weaken the notion of embedding, in order to make it possible for an orthomodular lattice to be embedded (in this weaker sense) into a Boolean algebra. A natural idea to overcome the above mentioned contradiction is to only ask that an embedding preserves the join for orthogonal elements. In such a case, we can as well generalize our discussion to orthomodular posets instead of orthomodular lattices.

2.2. A characterization of orthomodular posets that can be embedded into Boolean algebras

Before stating the main result of this section, let us introduce a few important notions.

DEFINITION 2.2.1. An orthomodular poset $(P, \subseteq, ^c, \emptyset, X)$, where X is a nonempty set, $P \subseteq 2^X$, order is defined by set-theoretical inclusion, the orthocomplement of an element $A \in P$ is the set-theoretical complement of A relative to X (denoted by A^c) and $\emptyset$ and X are the least and greatest elements of P, respectively, is a *set orthomodular poset* .

DEFINITION 2.2.2. A set orthomodular poset with the property that, for every $A, B \in P$, $A \cup B \in P$ whenever $A \cap B = \emptyset$ is a *concrete orthomodular poset.*

REMARK 2.2.3. It can be shown that every orthomodular poset can be represented as a set orthomodular poset. However, not every orthomodular poset has a representation as a concrete orthomodular poset, as we shall soon prove.

DEFINITION 2.2.4. Let P be an orthomodular poset. A mapping $s : P \to [0, 1]$ such that:

(1) $s(\mathbf{1}) = 1$;
(2) $s(a \vee b) = s(a) + s(b)$, whenever $a, b \in P$, $a \perp b$

is a *state* on P. If, moreover, the range of s is $\{0, 1\}$, then s is a *two-valued state* on P.

It is not difficult to see that a state preserves order on an orthomodular poset and that for every element $a \in P$, $s(a') = 1 - s(a)$. In particular, it follows that $s(\mathbf{0}) = 0$.

DEFINITION 2.2.5. A set $\mathcal{S}$ of states on an orthomodular poset P is *full* (or *order determining*) if, for every $a, b \in P$ with $a \nleq b$, there exists a state $s \in \mathcal{S}$ such that $s(a) \not\leq s(b)$ (i.e., $s(a) \leq s(b)$ for all $s \in \mathcal{S}$ implies $a \leq b$).

THEOREM 2.2.6 (see [**53**, Theorem 2.2.1] or [**27**]). *An orthomodular poset has a representation as a concrete orthomodular poset if and only if it has a full set of two-valued states.*

PROOF. "$\Rightarrow$" Let P be an orthomodular poset that has a representation as a concrete orthomodular poset, i.e., it is isomorphic to a concrete orthomodular poset $(M, \subseteq, ^c, \emptyset, \mathcal{S})$, where $\mathcal{S} \neq \emptyset$ and $M \subseteq 2^{\mathcal{S}}$ and A^c denotes set-theoretical complement of $A \in M$ in $\mathcal{S}$. Let us denote this isomorphism by $h : P \to M$.

We will define on $2^{\mathcal{S}}$ a family of states "carried by a point" - for every point in $\mathcal{S}$. More precisely, we will consider the family of states $(\varphi_x)_{x \in \mathcal{S}}$ defined by:

$$\varphi_x : 2^{\mathcal{S}} \to \{0,1\},\ \varphi_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

It is a straightforward verification that $\varphi_x \circ h$ is a two-valued state on P, for every $x \in \mathcal{S}$. Moreover, it is easy to see that $(\varphi_x \circ h)_{x \in \mathcal{S}}$ is a full set of two-valued states on P.

"$\Leftarrow$" Conversely, let $\mathcal{S}$ be a full set of two-valued states on an orthomodular poset P. We define $h : P \to 2^{\mathcal{S}}$ by $h(a) = \{s \in \mathcal{S} : s(a) = 1\}$, and we shall prove that if we restrict the codomain of h to its image $h(P)$, we obtain an isomorphism from P onto the concrete orthomodular poset $(h(P), \subseteq, ^c, \emptyset, \mathcal{S})$.

First let us notice that $h(\mathbf{0}) = \emptyset$ and $h(\mathbf{1}) = \mathcal{S}$. Secondly, if $a, b \in P$, $a \leq b$, then $s(a) \leq s(b)$ for every $s \in \mathcal{S}$. Therefore, if $s \in h(a)$, then $s \in h(b)$ for all $s \in \mathcal{S}$, i.e., $h(a) \subseteq h(b)$. Conversely, if $h(a) \subseteq h(b)$, then $s(a) = 1$ implies $s(b) = 1$ for every $s \in \mathcal{S}$. It follows that $s(a) \leq s(b)$ or every $s \in \mathcal{S}$, hence $a \leq b$, because $\mathcal{S}$ is a full set of states. We have just proved that h preserves order, both ways. Further, we will prove that h preserves orthocomplementation. Indeed, for every $a \in P$, $h(a') = \{s \in \mathcal{S} \colon s(a') = 1\} = \{s \in \mathcal{S} \colon s(a) = 0\} = \mathcal{S} \setminus h(a) = h(a)^c$. Moreover, h preserves orthogonality, both ways, since the following assertions are equivalent for every $a, b \in P$: $a \perp b$, $a \leq b'$, $h(a) \leq h(b')$, $h(a) \leq h(b)^c$, $h(a) \cap h(b) = \emptyset$, $h(a) \perp h(b)$.

Finally, let $a, b \in P$, $a \perp b$. Since $a, b \leq a \vee b$, it follows that $h(a), h(b) \subseteq h(a \vee b)$, hence $h(a) \cup h(b) \subseteq h(a \vee b)$. To prove the converse inclusion, let us consider $s \in \mathcal{S}$ such that $s \notin h(a) \cup h(b)$. Then, $s(a) = s(b) = 0$ and therefore, $s(a \vee b) = s(a) + s(b) = 0$. It follows that $s \notin h(a \vee b)$. We have proved that $h(a \vee b) = h(a) \cup h(b) \in h(P)$ for all $a, b \in P$, $a \perp b$, which concludes our proof. $\square$

Let us return to the problem of embedding an orthomodular poset into a Boolean algebra (embedding which should preserve the join for orthogonal elements only). We notice that an orthomodular poset has a concrete representation if and only if it can be embedded into a Boolean algebra and therefore, Theorem 2.2.6 gives in fact a necessary and sufficient condition for the existence of such an embedding—the existence of a full set of two-valued states defined on the orthomodular poset.

As it turns out, this condition is quite restrictive. For instance, it can be shown that the standard Hilbert space orthomodular lattice of projectors, for a Hilbert space H of dimension higher than 2, has *no* two-valued states, not to mention a full set. This is a consequence of Gleason's well known theorem. In particular, for $H = \mathbb{R}^3$, a geometric proof of this fact can be found in [**6**].

We must conclude that in general, we cannot embed orthomodular posets or lattices into Boolean algebras, if we expect that such an embedding to preserve the join of orthogonal elements. In the next section, we discuss what happens if we further weaken the embedding notion, asking the join to be preserved only for central elements.

2.3. The Zierler-Schlessinger theory

Definition 2.3.1. Let L be an orthomodular lattice and B a Boolean algebra. A mapping $h : L \to B$ is a *Z-embedding* of L into B if the following conditions are satisfied:

(1) $h(\mathbf{1}) = \mathbf{1}$;
(2) $h(a') = h(a)'$ for every $a \in L$;
(3) $a \perp b$ if and only if $h(a) \perp h(b)$, for every $a, b \in L$;
(4) $h(a \vee b) = h(a) \vee h(b)$, for every pair of central elements $a, b \in L$.

Remark 2.3.2. It can be easily checked that if h preserves orthocomplements, then the preservation of orthogonality is equivalent to the preservation of order, i.e., in the presence of condition (2) of the above definition, condition (3) is equivalent to the following condition:

(3') $a \leq b$ if and only if $h(a) \leq h(b)$, for every $a, b \in L$.

Following a well known result of N. Zierler and M. Schlessinger [**64**], we shall prove that, for every orthomodular lattice, it is possible to construct a Z-embedding into a power set Boolean algebra. The proof goes along the same lines as in the Boolean case (i.e., the case of Stone's theorem -see, e.g., [**57**]). However, some notions need to be adapted to the non-Boolean framework offered by orthomodular lattices. Therefore, we shall need the notions of Z-ideal and Z-state.

For the remainder of this section, L will denote an orthomodular lattice, and $\widetilde{C}(L)$ its center.

Definition 2.3.3. A mapping $s : L \to [0, 1]$ such that:

(1) $s(\mathbf{1}) = 1$;
(2) $s(a') = 1 - s(a)$, for every $a \in L$;

(3) $a \leq b$ implies $s(a) \leq s(b)$, for every $a, b \in L$;
(4) $s(a \vee b) = s(a) + s(b)$, whenever $a, b \in \widetilde{C}(L)$ and $a \perp b$

is a *Z-state* on L. If $s : L \to \{0, 1\}$, then we call s a *two-valued Z-state.*

Let us remark that, since states fulfill the conditions (2) and (3) in Definition 2.3.3, a state is always a Z-state, but the converse doesn't hold, in general. We shall define the notion of full set of Z-states on an orthomodular lattice L in the same way it was defined for states.

Definition 2.3.4. A set $\mathcal{S}$ of Z-states on an orthomodular lattice L is *full* (or *order determining*) if, for every $a, b \in L$ with $a \not\leq b$, there exists a Z-state $s \in \mathcal{S}$ such that $s(a) \not\leq s(b)$ (i.e., $s(a) \leq s(b)$ for all $s \in \mathcal{S}$ implies $a \leq b$).

Definition 2.3.5. Let B be a Boolean algebra. A subset $F \subset B$ is a *filter* of B if:

(1) for every $a \in B$ and $b \in F$, $a \geq b$ implies $a \in F$;
(2) $a, b \in F$ implies $a \wedge b \in F$;
(3) $a \in F$ implies $a' \notin F$.

Definition 2.3.6. Let B be a Boolean algebra. A subset $I \subset B$ is an *ideal* of B if:

(1) for every $a \in B$ and $b \in I$, $a \leq b$ implies $a \in I$;
(2) $a, b \in I$ implies $a \vee b \in I$;
(3) $a \in I$ implies $a' \notin I$.

Let us notice that condition (3) in Definitions 2.3.5 and 2.3.6, respectively, is equivalent to the fact that $F \subsetneq B$ and $I \subsetneq B$, respectively. Therefore, the notions of filter and ideal, as given in Definitions 2.3.5 and 2.3.6, correspond to the *proper* filters and ideals, as usually defined in the literature.

Let us remark that the notion of filter is dual to that of ideal, hence all the statements dual to statements that are true for ideals are true for filters. For more details about ideals and filters in Boolean algebras, we refer to, e.g., [**57**].

Definition 2.3.7. A subset $I \subset L$ is a *Z-ideal* of the orthomodular lattice L if:

(1) for every $a \in L$ and $b \in I$, $a \leq b$ implies $a \in I$;
(2) $a, b \in I \cap \widetilde{C}(L)$ implies $a \vee b \in I\big(\cap \widetilde{C}(L)\big)$;
(3) $a \in I$ implies $a' \notin I$.

Lemma 2.3.8. *If I is a Z-ideal of L and $a, a' \notin I$, then $J = \{x \in L\colon x \leq m \vee n, m \in I \cap \widetilde{C}(L), n \in \langle a \rangle \cap \widetilde{C}(L)\} \cup I \cup \langle a \rangle$ is a Z-ideal of L that includes I and a (where $\langle a \rangle = \{x \in L\colon x \leq a\}$).*

Proof. Let us check that J satisfies conditions (1)–(3) from Definition 2.3.7. Obviously, condition (1) is fulfilled. To verify condition

(2), let $u, v \in J \cap \widetilde{C}(L)$. Let us denote, for simplicity, $W = \{x \in L\colon x \leq m \vee n, m \in I \cap \widetilde{C}(L), n \in \langle a \rangle \cap \widetilde{C}(L)\}$. There are four possible cases:

- u, v are in the same subset of J: W, I or $\langle a \rangle$;
- $u \in \langle a \rangle$, $v \in I$;
- $u \in \langle a \rangle$, $v \in W$;
- $u \in I$, $v \in W$.

In all cases, the desired result is easily verified.

To check condition (3), two steps are necessary. First, we shall prove that $a' \notin J$ (hence J is a strict subset of L). We know $a' \notin I$ and clearly $a' \notin \langle a \rangle$. Assuming that $a' \in W$, there exist $m \in I \cap \widetilde{C}(L)$ and $n \in \langle a \rangle \cap \widetilde{C}(L)$ such that $a' \leq m \vee n$. It follows that $a' \leq m \vee a$ and therefore, $\mathbf{1} = a \vee a' \leq m \vee a$, hence $m' \wedge a' = \mathbf{0}$. However, $m \in \widetilde{C}(L)$ implies $a' \leftrightarrow m$, hence $a' = (a' \wedge m) \vee (a' \wedge m') = (a' \wedge m)$. It follows that $a' \leq m$ and since $m \in I$, $a' \in I$ as well—in contradiction to the hypothesis. We conclude that our assumption doesn't hold and therefore $a' \notin J$.

In the second step, we check that $u \in J$ implies $u' \notin J$. Let us assume to the contrary that $u, u' \in J$. Again, we can distinguish four cases:

- u, u' are in the same subset of J: W, I or $\langle a \rangle$;

  Since I, $\langle a \rangle$ are Z-ideals, they cannot contain u and u' simultaneously. Then $u, u' \in W$ and there exist $m, m_1 \in I \cap \widetilde{C}(L)$ and $n, n_1 \in \langle a \rangle \cap \widetilde{C}(L)$ such that $u \leq m \vee n$ and $u' \leq m_1 \vee n_1$. Hence $\mathbf{1} = u \vee u' \leq (m \vee m_1) \vee (n \vee n_1)$, and since $m \vee m_1 \in I \cap \widetilde{C}(L)$ and $n \vee n_1 \in \langle a \rangle \cap \widetilde{C}(L)$, it follows that $\mathbf{1} \in W \subset J$—in contradiction to what we proved in the first step ($J \neq L$).
- $u \in \langle a \rangle$, $u' \in I$;

  We have $u \leq a$, hence $a' \leq u' \in I$ and therefore $a' \in I$—a contradiction.
- $u \in \langle a \rangle$, $u' \in W$;

  Similarly, $u \leq a$ implies $a' \leq u' \in W$, hence $a' \in W \subset J$—in contradiction to what we proved in the first step.
- $u \in I$, $u' \in W$.

  Since $u' \in W$, there exist $m \in I \cap \widetilde{C}(L)$ and $n \in \langle a \rangle \cap \widetilde{C}(L)$ such that $u' \leq m \vee n$, hence $m' \wedge n' \leq u \in I$, and $m' \wedge n' \in I \cap \widetilde{C}(L) \subset J \cap \widetilde{C}(L)$. On the other hand, $n \in J \cap \widetilde{C}(L)$ and it follows that $(m' \wedge n') \vee n \in J \cap \widetilde{C}(L)$, since we already proved that J fulfills condition (2) from Definition 2.3.7. Since m, n are central, $(m' \wedge n') \vee n = m' \vee n \in J$, hence $m' \in J \cap \widetilde{C}(L)$. However, $m \in J \cap \widetilde{C}(L)$ and applying condition (2) again,

$\mathbf{1} = m \vee m' \in J$—in contradiction to what we proved in the first step ($J \neq L$).

Since in all the cases we have obtained contradictions, our assumption is false and condition (3) from the definition of Z-ideals is fulfilled by J. □

The following two corollaries can be easily derived from Lemma 2.3.8.

COROLLARY 2.3.9. *If I is a maximal Z-ideal in L and $a \in L$, then either $a \in I$ or $a' \in I$.*

COROLLARY 2.3.10. *If $a, b \in L$ such that $a \not\leq b$, then there exists a maximal Z-ideal I in L such that $b \in I, a \notin I$.*

The following couple of lemmas show the one-to-one correspondence between two-valued Z-states and maximal Z-ideals.

LEMMA 2.3.11. *Let $s : L \to \{0, 1\}$ be a two-valued Z-state and let $I = \mathrm{Ker}(s)$. Then, I is a maximal Z-ideal in L.*

PROOF. First, we shall check conditions (1)–(3) from Definition 2.3.7. For (1), let $b \in I$ and $a \leq b$. Then, $s(a) \leq s(b) = 0$, hence $a \in I$. To verify condition (2), let $a, b \in I \cap \widetilde{C}(L)$. Since $a \leftrightarrow b$, there exist mutually orthogonal elements $a_1, b_1, c \in \widetilde{C}(L)$ such that $a = a_1 \vee c$ and $b = b_1 \vee c$. It follows that $a \vee b = a_1 \vee b_1 \vee c = a_1 \vee b$, and a_1, b are central orthogonal elements. Therefore, $s(a \vee b) = s(a_1) + s(b) \leq s(a) + s(b) = 0$, hence $a \vee b \in I$. Condition (3), as well as the maximality of the Z-ideal I follow from the condition that, for every $a \in L$, $s(a) = 1 - s(a')$ and the converse of the statement in Corollary 2.3.9, which clearly holds. □

LEMMA 2.3.12. *Let I be a maximal Z-ideal in L and let $s : L \to \{0, 1\}$ be defined by $s(a) = 0$ whenever $a \in I$ and $s(a) = 1$ otherwise. Then, s is a two-valued Z-state on L.*

PROOF. Let us check conditions (1)–(4) from Definition 2.3.3. First, $\mathbf{1} \notin I$, hence $s(\mathbf{1}) = 1$. For the second condition, there are two cases to consider. If $a \in I$, then $a' \notin I$, hence $s(a) + s(a') = 1$. If $a \notin I$, then, according to Corollary 2.3.9, $a' \in I$ and again $s(a) + s(a') = 1$. Let us verify condition (3). Let $a \leq b$. If $b \in I$, then $a \in I$ and $s(a) = s(b) = 0$. If $b \notin I$, then $s(b) = 1 \geq s(a)$. To check condition (4), let us consider $a, b \in \widetilde{C}(L)$, $a \perp b$. We have to prove that $s(a \vee b) = s(a) + s(b)$. We shall distinguish three cases:

- $a, b \in I$. Then $a \vee b \in I$, hence $s(a) + s(b) = 0 = s(a \vee b)$.
- $a \in I$, $b \notin I$. It follows $a \vee b \notin I$ and therefore $s(a) + s(b) = 1 = s(a \vee b)$.
- $a, b \notin I$. We shall derive a contradiction, showing that this case isn't in fact possible. Indeed, $a \perp b$, hence $a \leq b'$. However,

in view of Corollary 2.3.9, $b \notin I$ implies $b' \in I$. It follows $a \in I$—a contradiction.

$\square$

Finally, we are ready to prove the result of N. Zierler and M. Schlessinger [**64**] that we have announced in the beginning of this section.

THEOREM 2.3.13. *For every orthomodular lattice L, there exist a Z-embedding into a power set Boolean algebra.*

PROOF. We shall first built the power set Boolean algebra and the Z-embedding. Let $\mathcal{S}$ be the set of all two-valued Z-states on L. We define $h : L \to 2^{\mathcal{S}}$ by $h(a) = \{s \in \mathcal{S}\colon\ s(a) = 1\}$ and we shall prove that h is a Z-embedding of L into the power set Boolean algebra $(2^{\mathcal{S}}, \subseteq, {}^c, \emptyset, \mathcal{S})$.

Clearly, $h(\mathbf{0}) = \emptyset$ and $h(\mathbf{1}) = \mathcal{S}$. Moreover, it is easy to see that for every $a \in L$, $h(a') = \{s \in \mathcal{S}\colon\ s(a') = 1\} = \{s \in \mathcal{S}\colon s(a) = 0\} = \mathcal{S} \setminus h(a) = h(a)^c$. In view of Remark 2.3.2, instead of proving that h preserves orthogonality, we can equivalently prove it preserves order (i.e., it fulfills condition (3') in the aforementioned remark). Obviously, $a \le b$ implies $s(a) \le s(b)$ for every $s \in \mathcal{S}$, hence $h(a) \subseteq h(b)$. Conversely, $h(a) \subseteq h(b)$ means that $s \in h(a)$ implies $s \in h(b)$ for all $s \in \mathcal{S}$, i.e., $s(a) = 1$ implies $s(b) = 1$ for all $s \in \mathcal{S}$. It follows that $s(a) \le s(b)$ for every $s \in \mathcal{S}$. Now we only need to prove that $\mathcal{S}$ is a full set of Z-states to obtain that $a \le b$, hence h satisfies (3'). Indeed, according to Corollary 2.3.10, if $a \not\le b$, then there exists a maximal Z-ideal I in L such that $b \in I, a \notin I$. According to Lemma 2.3.12, there exists a two-valued Z-state on L corresponding to the maximal Z-ideal I, defined by $s(a) = 0$ whenever $a \in I$ and $s(a) = 1$ otherwise. Therefore, $s(b) = 0$ and $s(a) = 1$, hence $s(a) \not\le s(b)$. It follows that $\mathcal{S}$ is a full set of Z-states.

Finally, we need to prove that h fulfills condition (4) in Definition 2.3.1, i.e., preserves the join of central elements. Let $a, b \in \widetilde{C}(L)$. Clearly, $a, b \le a \vee b$, hence $h(a), h(b) \subseteq h(a \vee b)$ and therefore $h(a) \cup h(b) \subseteq h(a \vee b)$. Let us prove the inverse inclusion. Since $a \leftrightarrow b$ and $\widetilde{C}(L)$ is a Boolean subalgebra of L, there exist mutually orthogonal elements $a_1, b_1, c \in \widetilde{C}(L)$ such that $a = a_1 \vee c$ and $b = b_1 \vee c$. Let $s \in h(a \vee b)$. Then, $s(a \vee b) = s(a_1 \vee c \vee b_1) = s(a_1) + s(c) + s(b_1) = 1$, and it follows that $s \in h(a_1) \cup h(c) \cup h(b_1) \subseteq h(a) \cup h(b)$. This concludes the proof of condition (4) and of the theorem. $\square$

A few remarks on this result are in order. First, let us notice that if L is a Boolean algebra, then $\widetilde{C}(L) = L$ and the Z-embedding we have built is just the embedding from Stone's theorem.

In the opposite situation, when $\widetilde{C}(L) = \{\mathbf{0}, \mathbf{1}\}$, the notion of Z-embedding becomes very "weak", as it doesn't preserve the join (except for trivial cases). In this case, the theorem justifies (after an easy generalization to orthomodular posets) our previous assertion according to which every orthomodular poset has a representation as a set orthomodular poset (Remark 2.2.3).

In many cases, the center of an orthomodular lattice is rather "poor" or even trivial. In these cases, the above theorem gives us a type of embedding that preserves the join of a very limited number of elements. The next section is devoted to a result of J. Harding and P. Pták [**35**] that substantially improves the properties that can be obtained for an embedding of an orthomodular lattice into a Boolean algebra.

2.4. The result of Harding and Pták

We intend to prove the following theorem (see [**35**]):

THEOREM 2.4.1. *Let L be an orthomodular lattice and let B be a Boolean subalgebra of L. There exist a set $\mathcal{S}$ and a mapping $h : L \to 2^{\mathcal{S}}$ such that:*

(1) $h(\mathbf{1}) = \mathcal{S}$;
(2) $h(a') = h(a)^c$;
(3) $a \perp b$ *if and only if* $h(a) \cap h(b) = \emptyset$;
(4) $h(a \vee b) = h(a) \cup h(b)$, *for every* $a, b \in B$;
(5) $h(a \vee b) = h(a) \cup h(b)$ *whenever* $a \in \widetilde{C}(L)$.

Let us notice that, in view of Remark 2.3.2, condition (3) in the above theorem can be replaced by the following condition:

(3') $a \leq b$ if and only if $h(a) \leq h(b)$, for every $a, b \in L$.

Before we can prove Theorem 2.4.1, some preparatives are necessary. Our steps will be along the same lines as in the previous section, but some changes are required. First, we need to define appropriate versions for the notions of ideal and state.

DEFINITION 2.4.2. Let L be an orthomodular lattice. A mapping $s : L \to [0, 1]$ such that:

(1) $s(\mathbf{1}) = 1$;
(2) $s(a') = 1 - s(a)$, for every $a \in L$;
(3) $a \leq b$ implies $s(a) \leq s(b)$, for every $a, b \in L$;
(4) $s(a \vee b) = s(a) + s(b)$, whenever $a \in \widetilde{C}(L)$ and $a \perp b$

is a *centrally additive state* on L. If $s : L \to \{0, 1\}$, then we call s a *two-valued centrally additive state.*

DEFINITION 2.4.3. Let L be an orthomodular lattice and B a Boolean subalgebra of L. A mapping $s : L \to [0, 1]$ such that:

(1) $s(\mathbf{1}) = 1$;
(2) $s(a') = 1 - s(a)$, for every $a \in L$;
(3) $a \leq b$ implies $s(a) \leq s(b)$, for every $a, b \in L$;
(4) $s(a \vee b) = s(a) + s(b)$, whenever $a, b \in B$ and $a \perp b$

is a B-*state* on L. If $s : L \to \{0,1\}$, then we call s a *two-valued* B-*state*.

Let us remark that a state is always a centrally additive state and a B-state, but the converse statements do not hold, in general.

DEFINITION 2.4.4. Let L be an orthomodular lattice and B a Boolean subalgebra of L. A set $\mathcal{S}$ of B-states (centrally additive states, respectively) on L is *full* (or *order determining*) if, for every $a, b \in L$ with $a \not\leq b$, there exists a B-state (centrally additive state, respectively) $s \in \mathcal{S}$ such that $s(a) \not\leq s(b)$ (i.e., $s(a) \leq s(b)$ for all $s \in \mathcal{S}$ implies $a \leq b$).

DEFINITION 2.4.5. A subset $I \subset L$ is a *central ideal* of the orthomodular lattice L if:

(1) for every $a \in L$ and $b \in I$, $a \leq b$ implies $a \in I$;
(2) $a, b \in I$, $b \in \widetilde{C}(L)$ implies $a \vee b \in I$;
(3) $a \in I$ implies $a' \notin I$;
(4) I includes a maximal ideal of $\widetilde{C}(L)$.

LEMMA 2.4.6 ([**35**, Lemma 3]). *If I is a central ideal of an orthomodular lattice L and $a' \notin I$, then $J = I \cup \{x \in L\colon\ x \leq m \vee a, m \in I \cap \widetilde{C}(L)\}$ is a central ideal of L that includes I and a.*

PROOF. Let us check that J satisfies conditions (1)–(4) from Definition 2.4.5. Let us denote, for simplicity, $W = \{x \in L\colon\ x \leq m \vee a, m \in I \cap \widetilde{C}(L)\}$. Condition (1) is easily verified, since I is a central ideal.

To check condition (3), two steps are necessary. First, we shall prove that $a' \notin J$ (hence J is a strict subset of L). We know $a' \notin I$. Assuming that $a' \in W$, there exist $m \in I \cap \widetilde{C}(L)$ such that $a' \leq m \vee a$. It follows that $a' \leq m \vee a$ and therefore, $\mathbf{1} = a \vee a' \leq m \vee a$, hence $m' \wedge a' = \mathbf{0}$. However, $m \in \widetilde{C}(L)$ implies $a' \leftrightarrow m$, hence $a' = (a' \wedge m) \vee (a' \wedge m') = (a' \wedge m)$. It follows that $a' \leq m$ and since $m \in I$, $a' \in I$ as well—in contradiction to the hypothesis. We conclude that our assumption doesn't hold and therefore $a' \notin J$.

In the second step, we check that $u \in J$ implies $u' \notin J$. Let us assume to the contrary that $u, u' \in J$. We can distinguish the following cases:

- $u, u' \in I$—not possible since I is a central ideal.
- $u, u' \in W$. Then there exist $m, n \in I \cap \widetilde{C}(L)$ such that $u \leq m \vee a$ and $u' \leq n \vee a$. It follows that $\mathbf{1} = u \vee u' \leq m \vee n \vee a$. Since $m \vee n \in I \cap \widetilde{C}(L)$, it results that $\mathbf{1} \in W \subseteq J$—a contradiction.
- $u \in I$, $u' \in W$. There exists $m \in I \cap \widetilde{C}(L)$ such that $u' \leq m \vee a$. Therefore, $m' \wedge a' \leq u \in I$, hence $m' \wedge a' \in I$. Since $m \in I \cap \widetilde{C}(L)$ and I is a central ideal, $m \vee (m' \wedge a') \in I$ and therefore, $m \vee a' \in I$, hence $a' \in I$—a contradiction.

Since in all the cases we have obtained contradictions, our assumption is false and condition (3) from the definition of central ideals is fulfilled by J.

To verify condition (4), we shall first prove that $J \cap \widetilde{C}(L)$ is an ideal in $\widetilde{C}(L)$ (which is a Boolean algebra). Indeed, if $x, y \in \widetilde{C}(L)$, $x \leq y$ and $y \in J$, then $x \in J$ (according to condition (1), that we already verified for J). Secondly, if $x, y \in J \cap \widetilde{C}(L)$, we have to show that $x \vee y \in J \cap \widetilde{C}(L)$. The following cases are possible:

- $x, y \in I$. Then $x \vee y \in I \subseteq J$, since $x, y \in \widetilde{C}(L)$ and I is a central ideal.
- $x \in I$, $y \in W$. Then there exists $m \in I \cap \widetilde{C}(L)$ such that $y \leq m \vee a$ and therefore, $x \vee y \leq x \vee m \vee a$. However, $x \vee m \in I \cap \widetilde{C}(L)$, hence $x \vee y \in W \subseteq J$.
- $x, y \in W$. Then there exist $m, n \in I \cap \widetilde{C}(L)$ such that $x \leq m \vee a$, $y \leq n \vee a$ and therefore, $x \vee y \leq m \vee n \vee a$. However, $m \vee n \in I \cap \widetilde{C}(L)$, hence $x \vee y \in W \subseteq J$.

The third and last condition for $J \cap \widetilde{C}(L)$ to be an ideal in $\widetilde{C}(L)$, namely that for an element $x \in \widetilde{C}(L)$, $x \in J$ implies $x' \notin J$ is easily verified.

Let us now remark that $I \cap \widetilde{C}(L)$ is an ideal in $\widetilde{C}(L)$. Since I is a central ideal, it follows that I includes a maximal ideal of $\widetilde{C}(L)$, hence $I \cap \widetilde{C}(L)$ is a maximal ideal in $\widetilde{C}(L)$. On the other hand, we just proved that $J \cap \widetilde{C}(L)$ is an ideal in $\widetilde{C}(L)$ and since $I \cap \widetilde{C}(L) \subseteq J \cap \widetilde{C}(L)$, it follows that $I \cap \widetilde{C}(L) = J \cap \widetilde{C}(L)$ and therefore $J \cap \widetilde{C}(L)$ is a maximal ideal of $\widetilde{C}(L)$.

We have yet to prove that J fulfills condition (2). Let $u, v \in J$, $v \in \widetilde{C}(L)$. We will show that $u \vee v \in J$. The following cases may occur:

- $u, v \in I$. Since I is a central ideal, it follows that $u \vee v \in I \subseteq J$.
- $u, v \in W$. Then there exist $m, n \in I \cap \widetilde{C}(L)$ such that $u \leq m \vee a$, $v \leq n \vee a$ and therefore, $u \vee v \leq m \vee n \vee a$. However, $m \vee n \in I \cap \widetilde{C}(L)$, hence $u \vee v \in W \subseteq J$.
- $u \in W$, $v \in I$. Then there exists $m \in I \cap \widetilde{C}(L)$ such that $u \leq m \vee a$ and therefore, $u \vee v \leq v \vee m \vee a$. However, $v \vee m \in I \cap \widetilde{C}(L)$, hence $u \vee v \in W \subseteq J$.
- $u \in I$, $v \in W$. In this case, we need to make use of the previously proven fact that $I \cap \widetilde{C}(L) = J \cap \widetilde{C}(L)$. Indeed, $v \in W \cap \widetilde{C}(L) \subseteq J \cap \widetilde{C}(L) = I \cap \widetilde{C}(L)$ and it follows that $u \vee v \in I \subseteq J$.

This concludes the proof of the lemma. $\square$

COROLLARY 2.4.7. *A central ideal I of an orthomodular lattice L is maximal if and only if $a \notin I$ implies $a' \in I$, for every $a \in L$.*

Like in the previous section, we will now establish a one-to-one correspondence between a certain type of states and a certain type of ideals (in the following two lemmas).

LEMMA 2.4.8. *Let L be an orthomodular lattice and B a Boolean subalgebra of L. If $s : L \to \{0, 1\}$ is a two-valued centrally additive B-state and $I = \mathrm{Ker}(s)$, then I is a maximal central ideal of L that includes a maximal ideal of B.*

PROOF. First, let us check conditions (1)–(4) from Definition 2.4.5. For (1), let $b \in I$ and $a \leq b$. Then, $s(a) \leq s(b) = 0$, hence $a \in I$. To verify condition (2), let $a, b \in I$ and $b \in \widetilde{C}(L)$. Since $a \leftrightarrow b$, there exist mutually orthogonal elements $a_1, b_1, c \in \widetilde{C}(L)$ such that $a = a_1 \vee c$ and $b = b_1 \vee c$. It follows that $a \vee b = a_1 \vee b_1 \vee c = a_1 \vee b$, with a_1, b orthogonal elements and $b \in \widetilde{C}(L)$. Therefore, $s(a \vee b) = s(a_1) + s(b) \leq s(a) + s(b) = 0$, hence $a \vee b \in I$. Condition (3) is obviously fulfilled, since $s(a') = 1 - s(a) = 1$ whenever $a \in I$. To verify condition (4), let us notice first that $I \cap \widetilde{C}(L)$ is an ideal in $\widetilde{C}(L)$. It is also a maximal one, since $a \in \widetilde{C}(L) \setminus I$ implies $s(a) = 1$, hence $s(a') = 0$ and therefore $a' \in I \cap \widetilde{C}(L)$. It follows that I is a central ideal of L. Moreover, since $a \notin I$ implies $s(a) = 1$, hence $s(a') = 0$ and thereby $a' \in I$, we conclude, according to Corollary 2.4.7, that I is a maximal central ideal.

We have yet to prove that I includes a maximal ideal of B (namely $I \cap B$). This verification is straightforward and uses the same technique as above, hence we omit it. $\square$

LEMMA 2.4.9. *Let L be an orthomodular lattice and B a Boolean subalgebra of L. If I is a maximal central ideal in L which includes a maximal ideal of B and $s : L \to \{0, 1\}$ is defined by $s(a) = 0$ whenever $a \in I$ and $s(a) = 1$ otherwise, then s is a two-valued centrally additive B-state on L.*

PROOF. Firstly, $\mathbf{1} \notin I$, hence $s(\mathbf{1}) = 1$. Further, if $a \in I$, then $a' \notin I$, hence $s(a)+s(a') = 1$ while if $a \notin I$, according to Corollary 2.4.7, $a' \in I$ and again $s(a) + s(a') = 1$. Let us verify that s preserves order. Let $a \leq b$. If $b \in I$, then $a \in I$ and $s(a) = s(b) = 0$. If $b \notin I$, then $s(b) = 1 \geq s(a)$. To check that $s(a \vee b) = s(a) + s(b)$, whenever $a \in \widetilde{C}(L)$ and $a \perp b$, let us consider $a \in \widetilde{C}(L)$, $b \in L$ such that $a \perp b$. We shall distinguish three cases:

- $a, b \in I$. Then $a \vee b \in I$, hence $s(a) + s(b) = 0 = s(a \vee b)$.
- $a \in I$, $b \notin I$. It follows $a \vee b \notin I$ and therefore $s(a) + s(b) = 1 = s(a \vee b)$.
- $a, b \notin I$. We shall derive a contradiction, showing that this case isn't in fact possible. Indeed, $a \perp b$, hence $a \leq b'$. However, in view of Corollary 2.4.7, $b \notin I$ implies $b' \in I$. It follows $a \in I$—a contradiction.

The proof that $s(a \vee b) = s(a) + s(b)$, whenever $a, b \in B$ and $a \perp b$ is similar, and therefore we omit it. $\square$

We have established a one-to-one correspondence between maximal central ideals which include a maximal ideal of B and the two-valued centrally additive B-states. Since the proof of Theorem 2.4.1 will follow a similar pattern as the proof of Theorem 2.3.13, it will be necessary to prove that the two-valued centrally additive B-states form a full set. Hence, the following lemma.

LEMMA 2.4.10 ([**35**, Lemma 5]). *Let L be an orthomodular lattice and B a Boolean subalgebra of L, such that $\widetilde{C}(L) \subseteq B$ and let $a, b \in L$ such that $a \nleq b$. There exists then a central ideal I such that $I \cap B$ is a maximal ideal in B and $a', b \in I$.*

PROOF. Let $X = \{x \in B\colon\ a \leq x\}$, $Y = \{y \in B\colon\ b' \leq y\}$, $Z = \{z \in \widetilde{C}(L)\colon\ a \leq z \vee b\}$ and let $W = X \cup Y \cup Z$. The set W generates a proper filter F in B. Assuming the contrary would imply that there exist $x \in X$, $y \in Y$ and $z \in Z$ such that $x \wedge y \wedge z = \mathbf{0}$. Since z is central, $z \leftrightarrow x \wedge y$, hence $x \wedge y = (x \wedge y \wedge z) \vee (x \wedge y \wedge z') = x \wedge y \wedge z'$ and therefore $x \wedge y \leq z'$ or equivalently, $z \leq x' \vee y'$. On the other hand, $a \leq x$ and therefore $a \wedge z \leq x \wedge (x' \vee y') = x \wedge y' \leq y' \leq b$. Since $a \wedge z \leq z$, it follows that $a \wedge z \leq b \wedge z$. However, $a \leq z \vee b$ implies $a \wedge z' \leq (z \vee b) \wedge z' = b \wedge z'$, where the last equality is due to the centrality of z. Using again the fact that z is central, we find that $a = (a \wedge z) \vee (a \wedge z') \leq (b \wedge z) \vee (b \wedge z') = b$—in contradiction to the hypothesis. It follows that F is a proper filter in B.

Let Q be the maximal ideal of B which is the dual of the maximal filter of B that includes F. It follows that $Q \cap W = \emptyset$. We define $I_0 = \{x \in L\colon\ x \leq p \text{ for some } p \in Q\}$, with the intention to prove that I_0 is a central ideal and $I_0 \cap B$ is a maximal ideal in B. Clearly, $y \leq x \in I_0$ implies $y \in I_0$. Let $x, y \in I_0$ and $y \in \widetilde{C}(L)$. Then there exist $p_1, p_2 \in Q$ such that $x \leq p_1$ and $y \leq p_2$. It follows that $x \vee y \leq p_1 \vee p_2 \in Q$, hence $x \vee y \in I_0$. The third condition is easily verified, since $x, x' \in I_0$ would imply $\mathbf{1} \in Q$—a contradiction, since Q is proper. For the fourth condition, let us notice that $Q \cap \widetilde{C}(L)$ is a maximal ideal of $\widetilde{C}(L)$ and $Q \cap \widetilde{C}(L) \subseteq I_0 \cap \widetilde{C}(L)$, hence $I_0 \cap \widetilde{C}(L) = Q \cap \widetilde{C}(L)$ is a maximal ideal of $\widetilde{C}(L)$. Moreover, $I_0 \cap B = Q$ is a maximal ideal of B.

We now show that $a, b' \notin I_0$. Indeed, assuming $a \in I_0$ implies $a \leq x$ for some $x \in Q$, hence $x \in Q \cap W = \emptyset$—a contradiction. Similarly, assuming $b' \in I_0$ implies $b' \leq y$ for some $y \in Q$, hence $y \in Q \cap W = \emptyset$—a contradiction. According to Lemma 2.4.6, the set $I_1 = I_0 \cup \{x \in L\colon x \leq m \vee b, m \in I_0 \cap \widetilde{C}(L)\}$ is a central ideal containing b. Moreover, $a \notin I_1$, since assuming the opposite leads to $a \leq z \vee b$, with $z \in I_0 \cap \widetilde{C}(L)$ and further to $z \in Q \cap W = \emptyset$—a contradiction. Applying Lemma 2.4.6

again, we find that the set $I_2 = I_1 \cup \{x \in L\colon\ x \leq n \vee a', n \in I_1 \cap \widetilde{C}(L)\}$ is a central ideal containing a' and b, which concludes the proof. $\square$

COROLLARY 2.4.11 (see [**35**, Theorem 6]). *Let L be an orthomodular lattice and B a Boolean subalgebra of L and let $a, b \in L$ such that $a \nleq b$. There exists then $s : L \to \{0, 1\}$ a two-valued centrally additive B-state such that $s(a) = 1$ and $s(b) = 0$ (i.e., the set of two-valued centrally additive B-states is a full set).*

PROOF. The result follows easily, using Lemma 2.4.10 and the one-to-one correspondence between maximal central ideals which include a maximal ideal of B and the two-valued centrally additive B-states. Since we do not assume here that $\widetilde{C}(L) \subseteq B$, we can, if necessary, take the subalgebra generated by $\widetilde{C}(L) \cup B$ instead of B. $\square$

We are now able to prove the result we announced at the beginning of this section.

PROOF. (Theorem 2.4.1) Let $\mathcal{S}$ be the set of two-valued centrally additive B-states on L. We define the mapping $h : L \to 2^{\mathcal{S}}$, $h(a) = \{s \in \mathcal{S}\colon\ s(a) = 1\}$. One can now easily verify, following the same steps as in the proof of Theorem 2.3.13, that the mapping h fulfills conditions (1)–(5) required. $\square$

CHAPTER 3

Spectral Automorphisms in Orthomodular Lattices

In this chapter we present original research results that were published in articles [**36**] and [**11**]. We develop the theory of spectral automorphisms in orthomodular lattices and obtain in this framework some results that are analogues of the ones in the spectral theory in Hilbert spaces.

3.1. Introduction

In quantum mechanics, the Hilbert space formalism might be physically justified in terms of some axioms based on the orthomodular lattice mathematical structure [**51**]. Since the framework of orthomodular lattices/quantum logics is, in a sense, more general than that of Hilbert space, which is supported by specific mathematical structures, we intend to investigate what amount of "quantum physics" is already contained in the mathematical structure of orthomodular lattices. In other words, the problem is to have some look on the dependence of several fundamental physical facts on Hilbert space specific tools.

Towards this end, we introduced the notion of spectral automorphism and studied their fundamental mathematical properties and connections with some quantum problems. We have also used spectral automorphisms in our attempt to clarify the physical meaning of some currently used Hilbert-space mathematical objects.

A very well known result in the theory of Hilbert space [**1, 55**] states that there exists a one-to-one correspondence between three sets:

- the set of selfadjoint operators
- the set of spectral measures
- the set of all one-parameter strongly continuous groups of unitary operators.

This observation is consistently used in the following considerations.

Before the definition enounced, let us explain the motivation and the origin of the notion of spectral automorphism. One of the most fundamental objects in the Hilbert-space quantum theory is that of the spectrum of an observable/selfadjoint operator. Its definition depends strictly on the fact that the observable in the Hilbert-space framework is a linear operator in a vector space endowed with a topology generated by a scalar product. On the other hand - and this is crucial for our approach - observables in orthomodular lattices are quite similar as mathematical objects to spectral measures, which define the Hilbert-space observables/selfadjoint operators. However, a notable difference consists in the fact that the spectral measures have as values orthogonal projectors in a Hilbert space but any observable of an orthomodular lattice has as values elements of that orthomodular lattice. This observation leads to the idea that considering the properties of lattice-automorphisms might be a "substitute" of Hilbert-space technique in studying the spectral properties of observables in the more general framework of orthomodular lattices.

3.2. Definition and basic facts about spectral automorphisms

Let H be a Hilbert space and $\mathcal{P}(H)$ the orthomodular lattice of projection operators on H. According to a version of Wigner's theorem due to Wright [**63**], automorphisms of $\mathcal{P}(H)$ are of the form $\varphi_U : \mathcal{P}(H) \to \mathcal{P}(H)$, $\varphi_U(P) = UPU^{-1}$, with U being a unitary or an antiunitary operator on H. Let us assume that U is unitary and B_U is the Boolean subalgebra of $\mathcal{P}(H)$ that is the range of the spectral measure associated to U. Then $P \in \mathcal{P}(H)$ is φ_U-invariant if and only if $UP = PU$ if and only if P commutes with B_U (i.e., commutes with every projection operator in B_U) if and only if $P \leftrightarrow B_U$ (for the last two equivalences, see [**34**]). This inspired in a natural way the definition of spectral automorphisms in orthomodular lattices.

Definition 3.2.1. Let L be an orthomodular lattice and φ be an automorphism of L. The automorphism φ is *spectral* if there is a Boolean subalgebra B of L such that

$$\varphi(a) = a \text{ if and only if } a \leftrightarrow B\,. \tag{P1}$$

A Boolean subalgebra of L satisfying condition (P1) is a *spectral algebra* of φ. The set of φ-invariant elements of L is denoted by L_φ.

Proposition 3.2.2. *Let L be an orthomodular lattice and $\varphi : L \to L$ be an automorphism. Then L_φ is a subalgebra of L.*

Proof. The statement is a straightforward consequence of the properties of φ. □

The next proposition is important, since it leads to the definition of the spectrum of a spectral automorphism.

Proposition 3.2.3. *For any spectral automorphism, there exists the greatest Boolean subalgebra having the property* (P1)

Proof. Let φ be a spectral automorphism of the orthomodular lattice L and $\{B_i; i \in I\}$ the set of all Boolean subalgebras of L having the property (P1) with respect to φ. Then, for any $i, j \in I$, $i \neq j$ we have $B_i \leftrightarrow B_j$. Indeed, if $a \in B_i$ then $a \leftrightarrow B_i$, hence $\varphi(a) = a$

and therefore, $a \leftrightarrow B_j$. According to Proposition 1.3.8, $\left[\bigcup_{i\in I} B_i\right]$ is a Boolean algebra. Let us check that it satisfies (P1). Clearly, if $a \in L$ such that $a \leftrightarrow \left[\bigcup_{i\in I} B_i\right]$, then $a \leftrightarrow B_i$ for all $i \in I$, hence $\varphi(a) = a$. Conversely, if $\varphi(a) = a$, then $a \leftrightarrow B_i$ for all $i \in I$. Since for any $i, j \in I$, $i \neq j$ we have $B_i \leftrightarrow B_j$, it follows that $\{a\} \cup \bigcup_{i\in I} B_i$ is a set of pairwise compatible elements, therefore $\left[\{a\} \cup \bigcup_{i\in I} B_i\right]$ is a Boolean algebra, according to Corollary 1.3.7. As a belongs to this Boolean algebra, $a \leftrightarrow \left[\{a\} \cup \bigcup_{i\in I} B_i\right]$ and since $\left[\bigcup_{i\in I} B_i\right] \subseteq \left[\{a\} \cup \bigcup_{i\in I} B_i\right]$, it follows that $a \leftrightarrow \left[\bigcup_{i\in I} B_i\right]$. □

Definition 3.2.4. If $\varphi : L \to L$ is a spectral automorphism, the greatest Boolean subalgebra having the property (P1) is called *the spectrum of* φ and will be denoted by σ_φ.

A trivial example shows that in general there are several Boolean subalgebras having the property (P1) with respect to a given automorphism. Indeed, let L be a Boolean algebra and $\mathrm{id} : L \to L$ its identity. Obviously, id is an automorphism, and any Boolean subalgebra of L has the property (P1). It follows that id is spectral and its spectrum is L. We can prove even more:

Proposition 3.2.5. *If L is a Boolean algebra, then its identity* $\mathrm{id} : L \to L$ *is the only spectral automorphism of L.*

Proof. Let φ be a spectral automorphism of the Boolean algebra L and σ_φ its spectrum. According to Remark 1.2.7, every element of L is compatible with σ_φ, hence every element is φ-invariant. Therefore, $\varphi = \mathrm{id}$. □

A simple, but very important consequence of this fact is:

Corollary 3.2.6. *If an orthomodular lattice has a nontrivial spectral automorphism, then it cannot be Boolean.*

Lemma 3.2.7. *Let L be an orthomodular lattice and $\varphi : L \to L$ a spectral automorphism. Then $\sigma_\varphi \subseteq L_\varphi$.*

Proof. Let $a \in \sigma_\varphi$. Then, according to Corollary 1.3.23, $a \leftrightarrow \sigma_\varphi$, hence $a \in L_\varphi$, according to (P1). □

Lemma 3.2.8. *Let L be an orthomodular lattice and $\varphi : L \to L$ a spectral automorphism. Then $\sigma_\varphi \subseteq \widetilde{C}(L_\varphi)$.*

Proof. Let $a \in \sigma_\varphi$. Then, according to Definition 3.2.4, $a \leftrightarrow L_\varphi$ and according to Lemma 3.2.7, $a \in L_\varphi$. Therefore, $a \in \widetilde{C}(L_\varphi)$. □

Lemma 3.2.9. *Let L be an orthomodular lattice and $\varphi : L \to L$ a spectral automorphism. Then $a \in L_\varphi$ if and only if $a \leftrightarrow \widetilde{C}(L_\varphi)$.*

Proof. Let $a \in L_\varphi$. If $b \in \widetilde{C}(L_\varphi)$, then $a \leftrightarrow b$ by Definition 1.3.21. Conversely, if $a \leftrightarrow \widetilde{C}(L_\varphi)$, then, by Lemma 3.2.8, it follows that $a \leftrightarrow \sigma_\varphi$ and therefore, by (P1), $a \in L_\varphi$. □

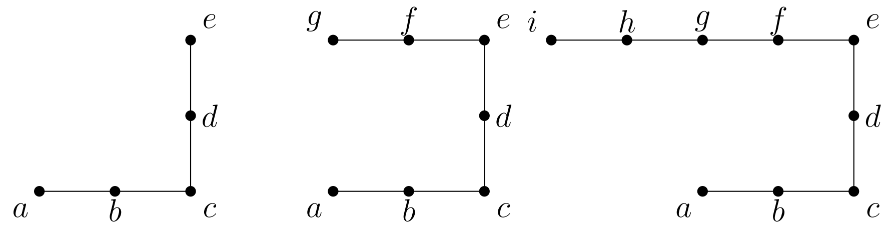

FIGURE 1. Greechie diagrams of orthomodular lattices used in Examples 3.2.12, 3.2.13 and 3.8.9.

The following proposition is a characterization of spectral automorphisms and their spectra.

PROPOSITION 3.2.10. *The automorphism $\varphi : L \to L$ is spectral if and only if there is a Boolean subalgebra B of L such that $L_\varphi = K(B)$. In this case, $\sigma_\varphi = \widetilde{C}(L_\varphi)$.*

PROOF. The first statement is an obvious reformulation of the property (P1). If φ is spectral, $\sigma_\varphi \subseteq \widetilde{C}(L_\varphi)$ according to Lemma 3.2.8. On the other hand, $\widetilde{C}(L_\varphi)$ has the property (P1), by Lemma 3.2.9. Since $\widetilde{C}(L_\varphi)$ is a Boolean algebra, by Proposition 1.3.22,and since the spectrum of φ is the greatest Boolean algebra with this property, the second assertion is proved. □

Let us remark the obvious fact that $L_\varphi \subseteq K(\widetilde{C}(L_\varphi))$, hence the following corollary holds:

COROLLARY 3.2.11. *The automorphism $\varphi : L \to L$ is spectral if and only if $K(\widetilde{C}(L_\varphi)) \subseteq L_\varphi$*

This means that L_φ must be "sufficiently large" for including the commutant of its center. Similarly, we might say that the center $\widetilde{C}(L_\varphi)$ must have "sufficiently many" elements for its commutant to be included in the lattice of φ-invariant elements.

Let us discuss some examples of spectral automorphisms. We shall use the technique of Greechie diagrams—see, e.g., [**38**]. A Greechie diagram of an orthomodular lattice L consists of a set of points and a set of lines such that points are in one-to-one correspondence with atoms of L and lines are in one-to-one correspondence with blocks of L. An automorphism of a finite orthomodular lattice is completely determined by its values on atoms.

EXAMPLE 3.2.12. Let L be the orthomodular lattice described by the first Greechie diagram in Fig. 1. It is the union of two blocks, one determined by atoms $\{a, b, c\}$ and the other determined by atoms $\{c, d, e\}$. Let φ be an automorphism of L such that a, b, c are φ-invariant and d, e are permuted by φ. Then the set L_φ of φ-invariant elements of L is the block of L determined by atoms $\{a, b, c\}$. Since L_φ is already Boolean, it coincides with its center $\widetilde{C}(L_\varphi)$ and since it is a block, $x \leftrightarrow L_\varphi$ if and only if $x \in L_\varphi$ for every $x \in L$. Therefore φ is spectral and $\sigma_\varphi = L_\varphi = \widetilde{C}(L_\varphi)$.

EXAMPLE 3.2.13. Let L be the orthomodular lattice described by the second Greechie diagram in Fig. 1. It is the union of three blocks, the first determined by atoms $\{a, b, c\}$, the second determined by atoms $\{c, d, e\}$ and the last determined by atoms $\{e, f, g\}$. Let φ be an automorphism of L such that a, b, c, d, e are φ-invariant and f, g are permuted by φ. Then the set L_φ of φ-invariant elements of L is not a block but it is precisely the lattice from Example 3.2.12 which is the union of blocks determined by atoms $\{a, b, c\}$ and $\{c, d, e\}$. Its center is $\widetilde{C}(L_\varphi) = \{\mathbf{0}, \mathbf{1}, c, c'\}$ and for an element $x \in L$, $x \leftrightarrow \widetilde{C}(L_\varphi)$ if and only if $x \leftrightarrow c$ if and only if $x \in L_\varphi$. Therefore φ is spectral and $\sigma_\varphi = \widetilde{C}(L_\varphi) = \{\mathbf{0}, \mathbf{1}, c, c'\}$.

EXAMPLE 3.2.14. Let H be an n-dimensional complex Hilbert space and $\mathcal{P}(H)$ be the set of its projection operators. Let Q be a 1-dimensional projection on H and Q' be its orthogonal complement. We define $U : H \to H$ as the symmetry of H with respect to the hyperplane corresponding to Q'. It is easy to see that U is a unitary operator, therefore $\varphi : \mathcal{P}(H) \to \mathcal{P}(H)$ defined by $\varphi(P) = UPU^{-1}$ is an automorphism of $\mathcal{P}(H)$. $\mathcal{B} = \{\mathbf{0}, Q, Q', \mathbf{1}\}$ is a Boolean subalgebra of $\mathcal{P}(H)$ fulfilling condition (P1) in Definition 3.2.1. Indeed, the set of φ-invariant elements, as well as the set of elements that are compatible with $\mathcal{B}$, is $\mathcal{P}_0 \cup \mathcal{P}_0'$, where $\mathcal{P}_0 = \{A \in \mathcal{P}(H) \colon A \leq Q'\}$ and $\mathcal{P}_0'$ denotes the set of orthocomplements of the elements of $\mathcal{P}_0$.

3.3. Spectral automorphisms in products and horizontal sums

We shall discuss the construction of spectral automorphisms in products and horizontal sums of orthomodular lattices. This will show new ways to obtain spectral automorphisms, proving the richness of this class of automorphisms. A factorization of the spectra of spectral automorphisms is also discussed.

Let us begin by recalling the basic facts about products and horizontal sums of orthomodular lattices (see, e.g., [**53**]). In what follows, we refer to the smallest possible orthomodular lattice $\{\mathbf{0}, \mathbf{1}\}$ as the *trivial* orthomodular lattice. The *product* $\prod_{i \in I} L_i$ of a nonempty collection $(L_i)_{i \in I}$ of orthomodular lattices is obtained by endowing their Cartesian product with the "component-wise" partial order and orthocomplementation (i.e., for $a, b \in \prod_{i \in I} L_i$, we have $a \leq b$ if $a_i \leq b_i$ for all $i \in I$ and $a = b'$ if $a_i = b_i'$ for all $i \in I$). The *horizontal sum* of a nonempty collection $(L_i)_{i \in I}$ of orthomodular lattices is constructed as the disjoint union of all L_i's with identifying their least (greatest) elements to obtain the least (greatest) element of the horizontal sum, the order and orthocomplementation in the horizontal sum are inherited from L_i. Two elements $a_i \in L_i$, $a_j \in L_j$ with $i \neq j$ and such that $a_i, a_j \notin \{\mathbf{0}, \mathbf{1}\}$ are incomparable. It follows that, if b_i, b_j are not elements of the same summand (hence, $b_i, b_j \notin \{\mathbf{0}, \mathbf{1}\}$), then, in the horizontal sum, $b_i \wedge b_j = \mathbf{0}$ and $b_i \vee b_j = \mathbf{1}$. A summand that is not a horizontal sum of at least two nontrivial orthomodular lattices is *minimal*. (Every orthomodular lattice is the horizontal sum of itself and of an arbitrary collection of trivial orthomodular lattices). Every summand is a subalgebra of the horizontal sum. Both the product and the horizontal sum of a collection of orthomodular lattices are orthomodular lattices.

LEMMA 3.3.1. *Let $(L_i)_{i \in I}$ be a collection of orthomodular lattices and let L be their product. The elements $a = (a_i)_{i \in I}, b = (b_i)_{i \in I}$ of L are compatible if and only if $a_i \leftrightarrow b_i$ for all $i \in I$.*

PROOF. We have $a \leftrightarrow b$ if and only if there are pairwise orthogonal elements $c, d, e \in L$, $c = (c_i)_{i \in I}$, $d = (d_i)_{i \in I}$, $e = (e_i)_{i \in I}$, such that $a = c \vee d$ and $b = c \vee e$. This happens if and only if for every $i \in I$, there are pairwise orthogonal elements $c_i, d_i, e_i \in L_i$ such that $a_i = c_i \vee d_i$ and $b_i = c_i \vee e_i$, which is equivalent to $a_i \leftrightarrow b_i$, for all $i \in I$. $\square$

THEOREM 3.3.2. *Let L be the product of a collection $(L_i)_{i \in I}$ of orthomodular lattices and, for every $i \in I$, φ_i be an automorphism of L_i. Let us define the mapping $\varphi \colon L \to L$ by $\varphi\big((a_i)_{i \in I}\big) = \big(\varphi_i(a_i)\big)_{i \in I}$. Then:*

(1) φ is an automorphism of L;
(2) φ is spectral if and only if φ_i is spectral for every $i \in I$; in this case, $\sigma_\varphi = \prod_{i \in I} \sigma_{\varphi_i}$.

PROOF. (1) It is a routine verification.

(2) An element $(a_i)_{i \in I} \in L$ is φ-invariant if and only if $\varphi_i(a_i) = a_i$ for every $i \in I$, i.e., $L_\varphi = \prod_{i \in I} L_{\varphi_i}$. According to Lemma 3.3.1, $\widetilde{C}(L_\varphi) = \prod_{i \in I} \widetilde{C}(L_{\varphi_i})$ and $K\big(\widetilde{C}(L_\varphi)\big) = \prod_{i \in I} K\big(\widetilde{C}(L_{\varphi_i})\big)$. According to Corollary 3.2.11, φ is spectral if and only if $K\big(\widetilde{C}(L_\varphi)\big) = L_\varphi$, i.e., $\prod_{i \in I} K\big(\widetilde{C}(L_{\varphi_i})\big) = \prod_{i \in I} L_{\varphi_i}$, i.e., $K\big(\widetilde{C}(L_{\varphi_i})\big) = L_{\varphi_i}$ for every $i \in I$, i.e., φ_i is spectral for every $i \in I$. Moreover, in such a case, $\sigma_\varphi = \widetilde{C}(L_\varphi) = \prod_{i \in I} \widetilde{C}(L_{\varphi_i}) = \prod_{i \in I} \sigma_{\varphi_i}$. $\square$

We shall turn now to spectral automorphisms in horizontal sums of orthomodular lattices. Let us begin with some preparatory remarks.

REMARK 3.3.3. Let L be an orthomodular lattice and φ be a spectral automorphism of L. If $L_\varphi = \{\mathbf{0}, \mathbf{1}\}$ then $L = \{\mathbf{0}, \mathbf{1}\}$. This follows from the fact that $\sigma_\varphi = \widetilde{C}(L_\varphi) = \{\mathbf{0}, \mathbf{1}\}$ and $\{\mathbf{0}, \mathbf{1}\} \leftrightarrow L$, hence $L \subseteq L_\varphi = \{\mathbf{0}, \mathbf{1}\}$.

REMARK 3.3.4. Let L be the horizontal sum of a collection $(L_i)_{i\in I}$ of orthomodular lattices. If $a_i \in L_i \setminus \{\mathbf{0},\mathbf{1}\}$ and $a_j \in L_j \setminus \{\mathbf{0},\mathbf{1}\}$ for some $i,j \in I$, $i \neq j$, then $a_i \not\leftrightarrow a_j$. Indeed, $a_i \wedge a_j = \mathbf{0}$ and $a_i \wedge a_j' = \mathbf{0}$, hence $(a_i \wedge a_j) \vee (a_i \wedge a_j') = \mathbf{0} \neq a_i$, and thus $a_i \not\leftrightarrow a_j$, according to Corollary 1.2.6. Hence, every nontrivial block (i.e., every block if L is nontrivial) of L is a subset of exactly one summand.

LEMMA 3.3.5. *Let L be the horizontal sum of a collection $(L_i)_{i\in I}$ of orthomodular lattices such that every summand is minimal, φ be an automorphism of L such that $\varphi(L_i)\cap L_i \neq \{\mathbf{0},\mathbf{1}\}$ for some $i \in I$. Then the restriction φ_i of φ to L_i is an automorphism of L_i.*

PROOF. The horizontal sum L is nontrivial, hence, according to Remark 3.3.4, every block of L is a subset of exactly one summand. Let us denote $L_{i,j} = \bigcup\{B \subseteq L_i\colon \text{B is a block in } L,\ \varphi(B) \subseteq L_j\}$. We will prove that L_i is the horizontal sum of $(L_{i,j})_{j\in I}$.

First, let us show that $L_i = \bigcup_{j\in I} L_{i,j}$. Let us remark that the union on the right-hand side of the equality includes all the blocks of L_i, since, according to Corollary 1.5.10, for every block B in L_i (which is also a block in L), $\varphi(B)$ is a block in L, and according to Remark 3.3.4, it is a subset of exactly one summand L_j. Since L_i is the union of its blocks, the equality $L_i = \bigcup_{j\in I} L_{i,j}$ follows.

Second, we will prove that for $j \neq k$, $L_{i,j} \cap L_{i,k} = \{\mathbf{0},\mathbf{1}\}$. Let us assume to the contrary that there exists $a \in L_{i,j} \cap L_{i,k} \setminus \{\mathbf{0},\mathbf{1}\}$. Then, there exist the blocks $B_1, B_2 \subseteq L_i$ such that $a \in B_1, B_2$ and $\varphi(B_1) \subseteq L_j$, $\varphi(B_2) \subseteq L_k$. It follows that $\varphi(a) \in L_j \cap L_k = \{\mathbf{0},\mathbf{1}\}$, and since φ is an automorphism, that $a \in \{\mathbf{0},\mathbf{1}\}$—a contradiction.

Finally, we show that if $a_j \in L_{i,j} \setminus \{\mathbf{0},\mathbf{1}\}$ and $a_k \in L_{i,k} \setminus \{\mathbf{0},\mathbf{1}\}$, $j \neq k$, then a_j and a_k are incomparable. Let us assume $a_j \leq a_k$ and seek a contradiction. According to Lemma 1.2.2 (1), $a_j \leftrightarrow a_k$. Since φ is an automorphism, $\varphi(a_j), \varphi(a_k) \notin \{\mathbf{0},\mathbf{1}\}$ and by Proposition 1.5.9, $\varphi(a_j) \leftrightarrow \varphi(a_k)$. Since $a_j \in L_{i,j} \setminus \{\mathbf{0},\mathbf{1}\}$, there exists a block B_j in L_i such that $a_j \in B_j$ and $\varphi(B_j) \subseteq L_j$. Therefore, $\varphi(a_j) \in L_j$. Similarly, we find that $\varphi(a_k) \in L_k$. We have thus found $\varphi(a_j) \in L_j$ and $\varphi(a_k) \in L_k$ such that $\varphi(a_j) \leftrightarrow \varphi(a_k)$ and $\varphi(a_j), \varphi(a_k) \notin \{\mathbf{0},\mathbf{1}\}$, in contradiction to Remark 3.3.4.

We can conclude now that L_i is the horizontal sum of $(L_{i,j})_{j\in I}$. Since L_i is a minimal summand and, due to the condition $\varphi(L_i)\cap L_i \neq \{\mathbf{0},\mathbf{1}\}$, $L_{i,i}$ is nontrivial, we obtain $L_{i,i} = L_i$ and therefore $\varphi(L_i) \subseteq L_i$. By applying the same reasoning to φ^{-1}, which also satisfies the condition $\varphi^{-1}(L_i) \cap L_i \neq \{\mathbf{0},\mathbf{1}\}$, we find that $\varphi^{-1}(L_i) \subseteq L_i$. Hence φ_i is a bijection on L_i and, since φ is an automorphism of L, φ_i is an automorphism of L_i. □

THEOREM 3.3.6. *Let L be the horizontal sum of a collection $(L_i)_{i\in I}$ of orthomodular lattices such that every summand is minimal and φ be an automorphism of L.*

(1) *If φ is spectral then there is an $i \in I$ such that $L_\varphi \subset L_i$, $\sigma_\varphi \subset L_i$ and the restriction φ_i of φ to L_i is a spectral automorphism of L_i with $L_{\varphi_i} = L_\varphi$ and $\sigma_{\varphi_i} = \sigma_\varphi$.*

(2) *If $L_\varphi \neq \{\mathbf{0}, \mathbf{1}\}$ and there is an $i \in I$ such that $L_\varphi \subset L_i$ and the restriction φ_i of φ to L_i is spectral then φ is a spectral automorphism of L and $L_\varphi = L_{\varphi_i}$, $\sigma_\varphi = \sigma_{\varphi_i}$.*

Proof. (1) Let us assume that L is nontrivial (otherwise there is nothing to prove). The spectrum σ_φ of φ is a Boolean subalgebra of L. According to Remark 3.3.4, there is an $i \in I$ such that $\sigma_\varphi \subset L_i$. For every $a \in L_\varphi$, we have $a \leftrightarrow \sigma_\varphi$ and, according to Remark 3.3.4, $a \in L_i$. Hence, $L_\varphi \subset L_i$. According to Remark 3.3.3, there is an $a \in L_\varphi \setminus \{\mathbf{0}, \mathbf{1}\}$ and, obviously, $\varphi(a) = a \in L_i$. Hence, $\varphi(L_i) \cap L_i \neq \{\mathbf{0}, \mathbf{1}\}$ and, according to Lemma 3.3.5, the restriction φ_i of φ to L_i is an automorphism of L_i. Since $L_\varphi \subset L_i$, $L_{\varphi_i} = L_\varphi$ and $\widetilde{C}(L_{\varphi_i}) = \widetilde{C}(L_\varphi) = \sigma_\varphi$. Therefore, since φ is spectral, φ_i is spectral, too, and $\sigma_{\varphi_i} = \widetilde{C}(L_{\varphi_i}) = \sigma_\varphi$.

(2) Let us prove first that $\varphi(L_i) \cap L_i \neq \{\mathbf{0}, \mathbf{1}\}$. According to the hypothesis, $L_\varphi \neq \{\mathbf{0}, \mathbf{1}\}$, hence there exists $a \in L_\varphi \setminus \{\mathbf{0}, \mathbf{1}\} \subseteq L_i$. It follows that $a \in L_i$ and $\varphi(a) \in \varphi(L_i)$. Since $a = \varphi(a) \notin \{\mathbf{0}, \mathbf{1}\}$, it follows that $a \in \varphi(L_i) \cap L_i \setminus \{\mathbf{0}, \mathbf{1}\}$ and thus $\varphi(L_i) \cap L_i \neq \{\mathbf{0}, \mathbf{1}\}$.

According to Lemma 3.3.5, the restriction φ_i of φ to L_i is an automorphism of L_i which we assume by hypothesis to be spectral. Since $a \in L_\varphi = L_{\varphi_i}$ if and only if $a \leftrightarrow \sigma_{\varphi_i}$, φ is spectral and $\sigma_\varphi = \widetilde{C}(L_\varphi) = \widetilde{C}(L_{\varphi_i}) = \sigma_{\varphi_i}$. □

Theorem 3.3.7. *Let L be an orthomodular lattice, φ be a spectral automorphism of L and $a \in L \setminus \{\mathbf{0}, \mathbf{1}\}$ be φ-invariant. Let us denote by φ_a the restriction of φ to $[\mathbf{0}, a]$ (see Example 1.1.15) and $B_x = x \wedge \sigma_\varphi = \{x \wedge b \colon b \in \sigma_\varphi\}$ for every $x \in L$. Then:*

(1) φ_a is a spectral automorphism of $[\mathbf{0}, a]$ and B_a is its spectral algebra;
(2) if $a \in \sigma_\varphi$, then $\sigma_{\varphi_a} = B_a$;
(3) σ_φ is isomorphic to the product $B_a \times B_{a'}$.

Proof. (1) Let us denote by ${}^*\colon\ b \mapsto b' \wedge a$ the orthocomplementation in $[\mathbf{0}, a]$ and let us verify that φ_a is an automorphism of $[\mathbf{0}, a]$. For every $b \in [\mathbf{0}, a]$, $\varphi_a(b) = \varphi(b) \leq \varphi(a) = a$. Hence, φ_a is a mapping into $[\mathbf{0}, a]$. Since φ is an automorphism of L, for every $b \in [\mathbf{0}, a]$ there is a $c \in L$ such that $\varphi(c) = b \leq a = \varphi(a)$ and therefore $c \leq a$. Hence, φ_a is a mapping onto $[\mathbf{0}, a]$. Since φ is an automorphism of L, $\varphi_a, \varphi_a^{-1}$ preserve the ordering and, for every $b \in [\mathbf{0}, a]$, $\varphi_a(b^*) = \varphi(b' \wedge a) = \varphi(b)' \wedge \varphi(a) = \varphi_a(b)^*$.

Clearly, $B_a \supseteq \{\mathbf{0}, a\}$ and is closed under the operation $\wedge$. Moreover, for every $b \in \sigma_\varphi$, $a \leftrightarrow b$ and, using Proposition 1.2.11, $(a \wedge b)^* = (a \wedge b)' \wedge a = \big((a \wedge b) \vee a'\big)' = \big((a \vee a') \wedge (b \vee a')\big)' = (b \vee a')' = b' \wedge a \in B_a$. Hence B_a

is closed under the operation * and therefore B_a is a subalgebra of $[\mathbf{0}, a]$. For every $b_1, b_2 \in \sigma_\varphi$, elements a, b_1, b_2 pairwise commute. It follows that, since $b_1 \leftrightarrow \{a, b_2\}$, we have $b_1 \leftrightarrow (a \wedge b_2)$, by Corollary 1.2.9 (2). Similarly, since $a \leftrightarrow \{a, b_2\}$, $a \leftrightarrow (a \wedge b_2)$. Therefore, $(a \wedge b_2) \leftrightarrow \{a, b_1\}$ and using the same Corollary 1.2.9 (2) once more, we conclude that $a \wedge b_1 \leftrightarrow a \wedge b_2$ and therefore B_a is a Boolean subalgebra of $[\mathbf{0}, a]$.

Let us prove now that B_a is a spectral algebra of φ_a. Every φ_a-invariant $c \in [\mathbf{0}, a]$ is φ-invariant, hence $c \leftrightarrow \sigma_\varphi$. Since $c \leq a$, we have $c \leftrightarrow a$ and therefore, by Corollary 1.2.9 (2), $c \leftrightarrow a \wedge b$ for every $b \in \sigma_\varphi$, i.e., $c \leftrightarrow B_a$. Conversely, let $c \in [\mathbf{0}, a]$ with $c \leftrightarrow B_a$. Hence $c \leftrightarrow a \wedge b$ for every $b \in \sigma_\varphi$. Since $c \leq a \leq a \vee b' = (a' \wedge b)'$, $c \leftrightarrow a' \wedge b$, by Lemma 1.2.2. Using Theorem 1.2.8, the fact that $a \leftrightarrow b$ and Corollary 1.2.6, we find $c \leftrightarrow (a \wedge b) \vee (a' \wedge b) = b$ for every $b \in \sigma_\varphi$. Hence c is φ-invariant and therefore φ_a-invariant.

(2) Since B_a is a spectral algebra of φ_a, $B_a \subseteq \sigma_{\varphi_a}$. Let $b \in \sigma_{\varphi_a} = \widetilde{C}(L_{\varphi_a})$. Since $a \in \sigma_\varphi = \widetilde{C}(L_\varphi)$, for every $c \in L_\varphi$ we have $c \leftrightarrow a$, i.e., $c = (c \wedge a) \vee (c \wedge a')$. Since $c \wedge a \in L_{\varphi_a}$ and $b \in \widetilde{C}(L_{\varphi_a})$, we obtain $b \leftrightarrow (c \wedge a)$ and since $b \leq a$ and $(c \wedge a') \leq a'$, it follows $b \perp (c \wedge a')$ and therefore, by Lemma 1.2.2, $b \leftrightarrow (c \wedge a')$. Since $b \leftrightarrow (c \wedge a), (c \wedge a')$, according to Theorem 1.2.8, $b \leftrightarrow (c \wedge a) \vee (c \wedge a') = c$ for every $c \in L_\varphi$. Hence, $b \in \widetilde{C}(L_\varphi) = \sigma_\varphi$ and, since $b \leq a$, $b = a \wedge b$, i.e., $b \in B_a$.

(3) We shall prove that $f\colon \sigma_\varphi \to B_a \times B_{a'}$ defined by $f(b) = (a \wedge b, a' \wedge b)$ is an isomorphism and $g\colon B_a \times B_{a'} \to \sigma_\varphi$ with $g(c_1, c_2) = c_1 \vee c_2$ is its inverse. Since $a \leftrightarrow \sigma_\varphi$, we get $g\big(f(b)\big) = (a \wedge b) \vee (a' \wedge b) = b$ for every $b \in \sigma_\varphi$. For every $c_1 \in B_a$ and $c_2 \in B_{a'}$ there are $b_1, b_2 \in \sigma_\varphi$ such that $c_1 = b_1 \wedge a$ and $c_2 = b_2 \wedge a'$. Then, $f\big(g(c_1, c_2)\big) = f\big((b_1 \wedge a) \vee (b_2 \wedge a')\big) = \big(a \wedge ((b_1 \wedge a) \vee (b_2 \wedge a')), a' \wedge ((b_1 \wedge a) \vee (b_2 \wedge a'))\big)$. Since $b_1 \wedge a \leq a$, $b_2 \wedge a' \leq a'$, $a \perp (b_2 \wedge a')$ and $a' \perp (b_1 \wedge a)$, it follows from Lemma 1.2.2 (1) and (2) that $\{a, a'\} \leftrightarrow \{b_1 \wedge a, b_2 \wedge a'\}$, and we obtain, according to Proposition 1.2.11, $\big(a \wedge ((b_1 \wedge a) \vee (b_2 \wedge a')), a' \wedge ((b_1 \wedge a) \vee (b_2 \wedge a'))\big) = (a \wedge b_1, a' \wedge b_2) = (c_1, c_2)$. Hence, f and g are bijective mappings and $g = f^{-1}$. Clearly, f preserve the operation $\wedge$, g preserves the operation $\vee$ and therefore both preserve the ordering. It remains to prove that f preserves the orthocomplementation. Let $^\sharp$ denote the orthocomplementation in $B_a \times B_{a'}$ and * the orthocomplementations in $B_a, B_{a'}$. For every $b \in \sigma_\varphi$, we obtain $f(b)^\sharp = (a \wedge b, a' \wedge b)^\sharp = \big((a \wedge b)^*, (a' \wedge b)^*\big) = (a \wedge b', a' \wedge b') = f(b')$. □

Question 3.3.8. Is it possible to omit the condition $a \in \sigma_\varphi$ in Theorem 3.3.7 (2)?

3.4. Characterizations of spectral automorphisms

Theorem 3.4.1. *Let L be an orthomodular lattice. An automorphism φ of L is spectral if and only if $a \wedge b \in L_\varphi$ for every $a \in \widetilde{C}(L_\varphi)$ and $b \in K\big(\widetilde{C}(L_\varphi)\big)$.*

PROOF. "⇒" If φ is spectral then, according to Corollary 3.2.11, $K\big(\widetilde{C}(L_\varphi)\big) \subseteq L_\varphi$. If $a \in \widetilde{C}(L_\varphi)$ and $b \in K\big(\widetilde{C}(L_\varphi)\big)$ then $a, b \in L_\varphi$ and, since L_φ is a subalgebra of L, we obtain $a \wedge b \in L_\varphi$.

"⇐" Let $b \in K\big(\widetilde{C}(L_\varphi)\big)$. For every $a \in \widetilde{C}(L_\varphi)$, we have, by Proposition 1.3.22, that $a' \in \widetilde{C}(L_\varphi)$ and, due to the hypothesis, $a \wedge b, a' \wedge b \in L_\varphi$ and, since $b \leftrightarrow a$ and L_φ is a subalgebra of L, $b = (a \wedge b) \vee (a' \wedge b) \in L_\varphi$. Hence, $K\big(\widetilde{C}(L_\varphi)\big) \subseteq L_\varphi$ and, according to Corollary 3.2.11, φ is spectral. $\square$

DEFINITION 3.4.2. Let L be an orthomodular lattice and φ an automorphism of L. An element $a \in L$ is *totally φ-invariant* if $\varphi(b) = b$ for every $b \in L$ with $b \leq a$.

LEMMA 3.4.3. *Let L be a complete orthomodular lattice and φ be an automorphism of L such that $\widetilde{C}(L_\varphi)$ is atomic with the set $\Omega(\widetilde{C}(L_\varphi))$ of atoms. Then $\bigvee \Omega(\widetilde{C}(L_\varphi)) = \mathbf{1}$.*

PROOF. Let us denote, for simplicity, the set $\Omega(\widetilde{C}(L_\varphi))$ by A. Since L is complete, $\bigvee A$ exists in L. Since φ is an automorphism of L and $A \subseteq L_\varphi$, $\varphi(\bigvee A) = \bigvee_{a \in A} \varphi(a) = \bigvee_{a \in A} a = \bigvee A$ and therefore $\bigvee A \in L_\varphi$. Since $L_\varphi \leftrightarrow A$, according to Theorem 1.2.8, $L_\varphi \leftrightarrow \bigvee A$ and therefore $\bigvee A \in \widetilde{C}(L_\varphi)$. Since $\widetilde{C}(L_\varphi)$ is atomic, $\bigvee A = \mathbf{1}$. $\square$

THEOREM 3.4.4. *Let L be a complete orthomodular lattice and φ be an automorphism of L such that $\widetilde{C}(L_\varphi)$ is atomic. Then φ is spectral if and only if all atoms of $\widetilde{C}(L_\varphi)$ are totally φ-invariant.*

PROOF. "⇒" Suppose φ is spectral. Let a be an atom of $\widetilde{C}(L_\varphi)$ and $b \in L$ with $b \leq a$. According to Propositions 1.4.6 and 3.2.10, $b \leftrightarrow \widetilde{C}(L_\varphi) = \sigma_\varphi$ and therefore b is φ-invariant. Hence, a is totally φ-invariant.

"⇐" Let $b \in K\big(\widetilde{C}(L_\varphi)\big)$ and A be the set of atoms of $\widetilde{C}(L_\varphi)$. Then, according to Lemma 3.4.3, $\bigvee A = \mathbf{1}$ and, according to Theorem 1.2.8, since $b \leftrightarrow A$, $b = b \wedge \bigvee A = \bigvee_{a \in A}(b \wedge a)$. Since φ is an automorphism and $b \wedge a$ is φ-invariant for every $a \in A$, we obtain $\varphi(b) = \bigvee_{a \in A} \varphi(b \wedge a) = \bigvee_{a \in A}(b \wedge a) = b$ and therefore $b \in L_\varphi$. Hence, $K\big(\widetilde{C}(L_\varphi)\big) \subseteq L_\varphi$ and therefore, according to Proposition 3.2.10, φ is spectral. $\square$

COROLLARY 3.4.5. *Let L be a complete orthomodular lattice and φ be an automorphism of L such that $\widetilde{C}(L_\varphi)$ is atomic. If all atoms of $\widetilde{C}(L_\varphi)$ are atoms of L then φ is spectral.*

PROOF. It follows easily from Theorem 3.4.4 because atoms of $\widetilde{C}(L_\varphi)$ are φ-invariant and, being atoms of L, they are totally φ-invariant. $\square$

THEOREM 3.4.6. *Let L be a complete orthomodular lattice and φ be an automorphism of L such that L_φ and $\widetilde{C}(L_\varphi)$ are atomic. If φ is spectral then all atoms of L_φ are atoms of L.*

PROOF. First, let us prove that every atom of L_φ is dominated by an atom of $\widetilde{C}(L_\varphi)$. Let us suppose that b is an atom of L_φ that is not dominated by any atom of $\widetilde{C}(L_\varphi)$ and seek a contradiction. For every atom a of $\widetilde{C}(L_\varphi)$, since $b \in L_\varphi$, we have $b \leftrightarrow a$ and therefore $b = (b \wedge a) \vee (b \wedge a')$. Since b is an atom in L_φ, according to Remark 1.4.5 (3), either $b \leq a$ or $b \leq a'$. Since we supposed that the first inequality is not satisfied, we obtain $b \leq a'$, i.e., $a \leq b'$, hence $\bigvee\{a \in L\colon\ a \text{ is an atom in } \widetilde{C}(L_\varphi)\} \leq b'$. According to Lemma 3.4.3, $\bigvee\{a \in L\colon\ a \text{ is an atom in } \widetilde{C}(L_\varphi)\} = \mathbf{1}$ and therefore $b = \mathbf{0}$, which is a contradiction.

According to Theorem 3.4.4, all atoms of $\widetilde{C}(L_\varphi)$ are totally φ-invariant. Since all atoms of L_φ are dominated by atoms of $\widetilde{C}(L_\varphi)$, all atoms of L_φ are totally φ-invariant and therefore they are atoms of L. $\square$

3.5. C-maximal Boolean subalgebras of an OML

We intend to look into the following matter: what could be the condition for a Boolean subalgebra of an orthomodular lattice L to be the spectrum of an automorphism of L? It is clear, considering Proposition 3.2.10 and Corollary 3.2.11, that for this to happen, the Boolean algebra B must satisfy $B = \widetilde{C}(K(B))$. Since we always have $B \subseteq \widetilde{C}(K(B))$, the condition reduces to the inverse inclusion. Therefore, we give the following:

DEFINITION 3.5.1. Let L be an orthomodular lattice. A Boolean subalgebra $B \subseteq L$ satisfying $\widetilde{C}(K(B)) \subseteq B$ is said to be *C-maximal* (i.e. maximal with respect to its commutant).

C-maximality is, thus, a necessary condition for a Boolean subalgebra of an orthomodular lattice to be the spectrum of an automorphism of that orthomodular lattice.

EXAMPLE 3.5.2. Any maximal Boolean subalgebra (block) of an orthomodular lattice is C-maximal, since it coincides with its commutant. On the other hand it is simple to find Boolean subalgebras which are not maximal. Indeed, let us consider L an orthomodular lattice having only two blocks, B_1, B_2 and having also the property $B_1 \cap B_2 \neq \{\mathbf{0}, \mathbf{1}\}$. Obviously, $\widetilde{C}(L) = B_1 \cap B_2$ and $K(\{\mathbf{0}, \mathbf{1}\}) = L$. Therefore, $\{\mathbf{0}, \mathbf{1}\}$ is not C-maximal.

THEOREM 3.5.3. *A Boolean subalgebra of an orthomodular lattice is C-maximal if and only if it coincides with its bicommutant (i.e., the commutant of its commutant).*

PROOF. Let B be a Boolean subalgebra of an orthomodular lattice L. Suppose $B = K(K(B))$. This means that B is the set of *all* elements that are compatible with $K(B)$. Since $B \subseteq K(B)$, it is the center of $K(B)$, which means it is C-maximal.

Conversely, if B is C-maximal, we need to prove $K(K(B)) \subseteq B$, since the inverse inclusion is trivial. Let $b \in K(K(B))$. Then we can write:

$$b \leftrightarrow K(B) \tag{3.1}$$

Since $B \subseteq K(B)$, according to (3.1), $b \leftrightarrow B$, and therefore

$$b \in K(B) \tag{3.2}$$

Now, if we put together (3.1) and (3.2), we deduce that $b \in \widetilde{C}(K(B)) = B$, which concludes the proof. □

THEOREM 3.5.4. *If the automorphism φ is spectral, then the following assertions are true:*

(1) $\widetilde{C}(L_\varphi) = K(L_\varphi)$;
(2) $\widetilde{C}(L_\varphi)$ *is C-maximal.*

PROOF. It is obvious that $\widetilde{C}(L_\varphi) \subseteq K(L_\varphi)$. We shall prove the inverse inclusion. Let $a \in K(L_\varphi)$. Since $a \leftrightarrow L_\varphi$, it follows that $a \leftrightarrow \sigma_\varphi$ (due to the fact that $\sigma_\varphi \subseteq L_\varphi$, according to Lemma 3.2.7). But according to the definition property of σ_φ, this means $a \in L_\varphi$, hence $a \in \widetilde{C}(L_\varphi)$. This concludes the proof of the first statement.

The second assertion results easily because of Corollary 3.2.11. We have $L_\varphi = K(\widetilde{C}(L_\varphi))$, hence $\widetilde{C}(L_\varphi) = \widetilde{C}(K(\widetilde{C}(L_\varphi)))$. It follows that $\widetilde{C}(L_\varphi)$ is C-maximal. □

3.6. Spectral automorphisms and physical theories

Since orthomodular lattices have been considered in this work – and not only – the roots of any physical theory, let us examine the importance of spectral automorphisms for some fundamental problems concerning their classification and also their structure.

Let us begin by reminding that in our discourse a physical theory is an orthomodular lattice, which is occasionally assumed possessing some supplementary properties, like having atoms, satisfying the covering law or being complete.

Let us see now what is the essential difference between classical and non-classical physical theories from the point of view of spectral automorphisms. In Proposition 3.2.5, it has been shown that a classical theory/Boolean algebra has no nontrivial spectral automorphisms, that is, the only spectral automorphism is the identity of the theory in question. So if a theory has at least one nontrivial spectral automorphism, it has to be non-classic.

We do not intend to prove here very general results concerning non-classical theories, but we will show that in such theories there are states, which are invariant under spectral automorphisms. For doing this there will be considered some particular cases of spectral automorphisms and non-classical theories.

Let us consider a theory L, which is atomic, complete and has the covering property (Definition 1.4.12). Assume also that this theory has a spectral automorphism φ whose spectrum is atomic. Below we will show that examples of such theories exist, such as a finite dimensional quantum logic. Then, given $a \in \Omega(\sigma_\varphi)$ and $\omega \in \Omega(L)$, $\omega \leq a$, by Proposition 1.4.6, $\omega \in K(\sigma_\varphi)$. Moreover, φ being spectral, it follows that ω is invariant under φ. Since L is atomic, complete and has the covering property, it follows that each $a \in \Omega(\sigma_\varphi)$ has a basis of atoms of L, let us denote it by $\mathcal{B}_a$ (see Proposition 1.4.7). The conclusion is that $\bigcup_{a\in\Omega(\sigma_\varphi)}\mathcal{B}_a$ is a basis of L, whose elements are invariant under φ. It is almost obvious that this result is an analogue of the spectral theorem for unitary operators having purely point spectrum and also for the corresponding selfadjoint operator. If L has a finite dimension, it is easy to probe that all spectral automorpisms have atomic spectra, which means that the just proved result may be applied with no modifications.

3.7. Spectral automorphisms and Piron's theorem

Piron's representation theorem (see [**50**]) is still the only fundamental result allowing us to consider that non-classical theories are based on the Hilbert space formalism. However, a very delicate problem remains to discuss: is the Hilbert space real, complex or quaternionic? This problem is, in a sense, still open. Our intention is to prove that choosing one of the first two possibilities depends strongly on a property of the symmetries of the theory in question.

Let us remind a very important result concerning physical theories: any symmetry of a physical theory is represented by an automorphism of that theory [**51**]. Now if we apply the last result of the previous section to spectral symmetries (i.e. symmetries that are represented by automorphisms that are spectral) we reach the following interesting conclusion: if admitted that a finite dimensional physical theory, represented by the lattice of projectors of a finite dimensional Hilbert space, must have spectral symmetries other than simple reflexions relative to a hyperplane, then the Hilbert space in question cannot be *real.* Indeed, a well-known theorem affirms that no symmetry in such a space, with the exception of the reflexions relative to hyperplanes, has a complete system of invariant one-dimensional subspaces (see, for instance, [**52**]). Therefore, if there are physical motivations for admitting that spectral symmetries (other than reflexions relative to a hyperplane) must exist in a theory, then the real Hilbert spaces have to be excluded from those able to support quantum theories.

3.8. Spectral families of automorphisms and a Stone-type theorem

DEFINITION 3.8.1. Let L be an orthomodular lattice and Φ be a family of automorphisms of L. The family Φ is *spectral* if there is a Boolean subalgebra B of L such that:

(P2) $\quad (\varphi(a) = a \text{ for every } \varphi \in \Phi)$ if and only if $a \leftrightarrow B$.

A Boolean algebra B satisfying condition (P2) is a *spectral algebra of* Φ. The set of Φ-invariant elements of L (which is a subalgebra of L) is denoted by L_Φ.

Let us remark that a family of (more than one) spectral automorphisms need not be a spectral family of automorphisms—see Example 3.8.5.

PROPOSITION 3.8.2. *For every spectral family Φ of automorphisms of an orthomodular lattice L there exists the greatest spectral algebra of the family Φ.*

PROOF. Let $\{B_i \colon\ i \in I\}$ be the set of spectral algebras of the family Φ. Since B_i are Boolean algebras, if $a \in B_i$ then $a \leftrightarrow B_i$, hence $a \in L_\Phi$ and $a \leftrightarrow B_j$, for every $i, j \in I$. Hence $B_i \leftrightarrow B_j$ for every $i, j \in I$ and the subalgebra of L generated by $\{B_i \colon\ i \in I\}$ is a Boolean algebra, according to 1.3.8. Let us check that it satisfies (P2). If $a \in L$ such that $a \leftrightarrow \left[\bigcup_{i\in I} B_i\right]$, then $a \leftrightarrow B_i$ for all $i \in I$, hence $\varphi(a) = a$ for every $\varphi \in \Phi$. Conversely, if $\varphi(a) = a$ for every $\varphi \in \Phi$, then $a \leftrightarrow B_i$ for all $i \in I$. Since for any $i, j \in I$, $i \neq j$ we have $B_i \leftrightarrow B_j$, it follows that $\{a\} \cup \bigcup_{i\in I} B_i$ is a set of pairwise compatible elements, therefore $\left[\{a\} \cup \bigcup_{i\in I} B_i\right]$ is a Boolean algebra, according to Corollary 1.3.7. As a belongs to this Boolean algebra, $a \leftrightarrow \left[\{a\} \cup \bigcup_{i\in I} B_i\right]$ and since $\left[\bigcup_{i\in I} B_i\right] \subseteq \left[\{a\} \cup \bigcup_{i\in I} B_i\right]$, it follows that $a \leftrightarrow \left[\bigcup_{i\in I} B_i\right]$.

We conclude that $\left[\bigcup_{i\in I} B_i\right]$ satisfies condition (P2). Obviously, it is the greatest spectral algebra of the family Φ. ☐

DEFINITION 3.8.3. Let Φ be a spectral family of automorphisms of an orthomodular lattice. The *spectrum σ_Φ of the family* Φ is the greatest spectral algebra of the family Φ.

PROPOSITION 3.8.4. *Let Φ be a spectral family of automorphisms of an orthomodular lattice L. Then:*

(1) $\sigma_\Phi = \widetilde{C}(L_\Phi)$;
(2) $\sigma_\Phi = \widetilde{C}\big(K(\sigma_\Phi)\big)$ *(i.e., σ_Φ is C-maximal);*
(3) $\sigma_\Phi = K\big(K(\sigma_\Phi)\big)$.

PROOF. (1) According to Definition 3.8.1, B is a spectral algebra of Φ if and only if $L_\Phi = K(B)$; in such a case $B \leftrightarrow B$ and therefore

$B \subseteq L_\Phi$ and $B \subseteq \widetilde{C}(L_\Phi)$. In particular, $L_\Phi = K(\sigma_\Phi)$ and $\sigma_\Phi \subseteq \widetilde{C}(L_\Phi)$. Obviously, $L_\Phi \subseteq K\big(\widetilde{C}(L_\Phi)\big)$ and, since $\sigma_\Phi \subseteq \widetilde{C}(L_\Phi)$, $K\big(\widetilde{C}(L_\Phi)\big) \subseteq K(\sigma_\Phi) = L_\Phi$, hence $\widetilde{C}(L_\Phi)$ is a spectral algebra of Φ. Since $\sigma_\Phi \subseteq \widetilde{C}(L_\Phi)$ and σ_Φ is the greatest spectral algebra of Φ, we obtain $\sigma_\Phi = \widetilde{C}(L_\Phi)$.

(2) According to part (1) and Definition 3.8.1, $\sigma_\Phi = \widetilde{C}(L_\Phi) = \widetilde{C}\big(K(\sigma_\Phi)\big)$.

(3) According to 3.5.3, $\sigma_\Phi = \widetilde{C}\big(K(\sigma_\Phi)\big)$ if and only if $\sigma_\Phi = K\big(K(\sigma_\Phi)\big)$. □

Example 3.8.5. Let L be the lattice from Example 3.2.12, φ, ψ be automorphisms of L such that a, b, c are φ-invariant, d, e are permuted by φ, c, d, e are ψ-invariant and a, b are permuted by ψ. Then $\Phi = \{\varphi, \psi\}$ is a nonspectral family of spectral automorphisms. Indeed, we have shown in Example 3.2.12 that φ is spectral, and, similarly, ψ is spectral, too. On the other hand, $L_\Phi = L_\varphi \cap L_\psi = \{\mathbf{0}, c, c', \mathbf{1}\} = \widetilde{C}(L_\Phi)$, hence $K(\widetilde{C}(L_\Phi)) = L \neq L_\Phi$ and therefore, according to Proposition 3.8.4, Φ is not a spectral family.

Theorem 3.8.6. *Let L be an orthomodular lattice and Φ be a family of spectral automorphisms of L. Then Φ is a spectral family if and only if $\sigma_\varphi \leftrightarrow \sigma_\psi$ for every $\varphi, \psi \in \Phi$. In this case, the spectrum σ_Φ of the family contains all spectra σ_φ, $\varphi \in \Phi$.*

Proof. "$\Rightarrow$" Let $\varphi \in \Phi$. For every $a \in K(\sigma_\Phi) = L_\Phi$ we have $\varphi(a) = a$ and therefore $a \leftrightarrow \sigma_\varphi$. Hence, $\sigma_\varphi \leftrightarrow K(\sigma_\Phi)$, and thus, using Proposition 3.8.4, $\sigma_\varphi \subseteq K\big(K(\sigma_\Phi)\big) = \sigma_\Phi$. Since σ_Φ is a Boolean algebra, we get $\sigma_\varphi \leftrightarrow \sigma_\psi$ for every $\varphi, \psi \in \Phi$.

"$\Leftarrow$" Let us denote by B the subalgebra of L generated by $\bigcup_{\varphi \in \Phi} \sigma_\varphi$. A subalgebra of an orthomodular lattice generated by a family of pairwise commuting Boolean algebras is a Boolean algebra, according to Proposition 1.3.8. The following statements are equivalent: $\varphi(a) = a$ for every $\varphi \in \Phi$, $a \leftrightarrow \sigma_\varphi$ for every $\varphi \in \Phi$, $a \leftrightarrow \bigcup_{\varphi \in \Phi} \sigma_\varphi$, $a \leftrightarrow B$. Hence B is a spectral algebra and Φ is a spectral family. □

From the very beginning, the purpose of introducing and studying spectral automorphisms has been to construct something similar to the Hilbert space spectral theory without using the specific instruments available in a Hilbert space setting, but using only the abstract orthomodular lattice structure. The next result is intended as an analogue of the Stone theorem concerning strongly continuous uniparametric groups of unitary operators. Before stating it, we should notice the following easily verifiable facts:

Remark 3.8.7. (1) The identity $\mathrm{id} : L \to L$ is a spectral automorphism and $\sigma_{\mathrm{id}} = \widetilde{C}(L)$.

(2) The inverse φ^{-1} of a spectral automorphism φ of L is also spectral and $\sigma_{\varphi^{-1}} = \sigma_\varphi$.

THEOREM 3.8.8. *Let L be an orthomodular lattice and Φ be a family of spectral automorphisms of L. If Φ is an Abelian group and $\varphi(L_\psi) = L_{\varphi\psi}$ for every $\varphi, \psi \in \Phi$ with $\psi \notin \{\mathrm{id}, \varphi^{-1}\}$ then:*

(1) $L_\varphi = L_\psi$ for every $\varphi, \psi \in \Phi \setminus \{\mathrm{id}\}$;
(2) $\sigma_\varphi = \sigma_\psi$ for every $\varphi, \psi \in \Phi \setminus \{\mathrm{id}\}$;
(3) Φ is a spectral family.

PROOF. (1) Let $\varphi, \psi \in \Phi$. The following statements are equivalent for every $a \in L$: $a \in L_\varphi$, $\varphi(a) = a$, $\psi\big(\varphi(a)\big) = \psi(a)$, $\varphi\big(\psi(a)\big) = \psi(a)$, $\psi(a) \in L_\varphi$. Hence $\psi(L_\varphi) = L_\varphi$.

Let $\varphi, \psi \in L \setminus \{\mathrm{id}\}$ be different. If $\psi = \varphi^{-1}$ then, obviously, $L_\varphi = L_\psi$. Let us suppose that $\psi \neq \varphi^{-1}$ and let us denote $\chi = (\varphi\psi)^{-1} = \psi^{-1}\varphi^{-1}$. Since $\chi \in \Phi \setminus \{\mathrm{id}, \varphi^{-1}, \psi^{-1}\}$, we obtain $L_\varphi = L_{\chi^{-1}\varphi\chi} = \chi^{-1}\big(\varphi(L_\chi)\big) = \chi^{-1}\big(\psi(L_\chi))\big) = L_{\chi^{-1}\psi\chi} = L_\psi$.

(2) According to part (1) and Proposition 3.8.4 (1), $\sigma_\varphi = \widetilde{C}(L_\varphi) = \widetilde{C}(L_\psi) = \sigma_\psi$ for every $\varphi, \psi \in \Phi \setminus \{\mathrm{id}\}$.

(3) According to Proposition 3.2.10, $\sigma_{\mathrm{id}} = \widetilde{C}(L)$ and therefore $\sigma_{\mathrm{id}} \leftrightarrow L$. Since $\sigma_\varphi \subseteq L$ for every $\varphi \in \Phi$, it follows that $\sigma_{\mathrm{id}} \leftrightarrow \sigma_\varphi$ for every $\varphi \in \Phi$. Since, according to part (2), the set $\{\sigma_\varphi \colon \varphi \in \Phi\}$ is at most 2-element, $\sigma_\varphi \leftrightarrow \sigma_\psi$ for every $\varphi, \psi \in \Phi$. According to Theorem 3.8.6, Φ is a spectral family. $\square$

Let us show a non-trivial example of an abelian group of spectral automorphisms satisfying the conditions of the Theorem 3.8.8.

EXAMPLE 3.8.9. Let L be the orthomodular lattice described by the third Greechie diagram in Fig. 1. L is the union of three blocks, the first determined by atoms $\{a, b, c\}$, the second determined by atoms $\{c, d, e\}$ and the last determined by atoms $\{e, f, g, h, i\}$. Let φ be an automorphism of L such that a, b, c, d, e are φ-invariant and φ performs a cyclic permutation on the atoms f, g, h, i (i.e., $\varphi(f) = g$, $\varphi(g) = h$, $\varphi(h) = i$ and $\varphi(i) = f$). Clearly, $\varphi^4 = \mathrm{id}$ and $\Phi = \{\mathrm{id}, \varphi, \varphi^2, \varphi^3\}$ is an abelian group of automorphisms of L. $L_\varphi = L_{\varphi^2} = L_{\varphi^3} = L_\Phi$ is the set-theoretical union of the blocks determined by $\{a, b, c\}$ and $\{c, d, e\}$. $C(L_\varphi) = \widetilde{C}(L_{\varphi^2}) = \widetilde{C}(L_{\varphi^3}) = \widetilde{C}(L_\Phi) = \{\mathbf{0}, c, c', \mathbf{1}\}$ is the spectrum of automorphisms $\varphi, \varphi^2, \varphi^3$, and therefore of the family Φ, because for $x \in L$, $x \leftrightarrow \{\mathbf{0}, c, c', \mathbf{1}\}$ if and only if $x \leftrightarrow c$ if and only if $x \in L_\Phi$.

Let us remark that Theorem 3.8.8 gives purely algebraic conditions for a family of automorphisms to have a spectrum. The last hypothesis (namely, that $\varphi(L_\psi) = L_{\varphi\psi}$) can be seen as a replacement for the continuity condition in the original Stone theorem.

Part 2

Unsharp Quantum Logics

Preliminaries

In this second part of the thesis, we shall move our investigations from the framework of orthomodular posets or lattices—which may be considered as representing "sharp" quantum logics—to the more general framework of effect algebras—regarded as "unsharp" quantum logics. Before immersing into the mathematical universe of effect algebras, let us explain briefly why they have become important in the contemporary theory of quantum measurement. The reader interested in a more detailed physical interpretation of the mathematical notions presented here should refer to e.g. [**4, 5, 41, 44, 43**]. An interesting description of how a quantum system with all its ingredients like observables, states and symmetries can be represented using effect algebras (with compression bases) can also be found in [**21**].

According to the operational approach to quantum mechanics (see, e.g., [**44, 43, 41**]), states of a quantum system, which intuitively correspond to a complete knowledge of the system, are described in terms of preparation procedures (or classes of them). Moreover, effects, which can be thought of as yes-no measurements that may be unsharp, are defined as equivalence classes of so-called effect apparatuses—i.e., instruments that perform yes-no measurements. As we shall see, we can regard the measurement of any observable as a combination of yes-no measurements—which is why effects play such an important rôle in the modern theory of quantum measurements.

According to the Hilbert space formalism of quantum mechanics, the various quantities and relations pertaining to the quantum system are representable in terms of operators defined on a complex separable Hilbert space H—the so-called state space of the system. More precisely:

- states are represented by density operators ρ on H, i.e., trace class operators of trace 1.
- effects are represented by effect operators A on H, i.e., elements of the set $\mathcal{E}(H)$ of self-adjoint operators on H lying between the null operator and the identity operator. Let us remark that, in conventional quantum mechanics, yes-no measurements were considered to be represented by projection operators on H, the so-called "decision effects" [**44**], which form a proper subset $\mathcal{P}(H)$ of $\mathcal{E}(H)$ (see also Example 4.2.8(a)). While projections are interpreted as "sharp" events, the effects

can be "unsharp" or "fuzzy", and for instance may fail to satisfy the principle of excluded middle, since the greatest lower bound of effects A and A' ("non-A") may be different from the null effect. Motivation for replacing projection operators with the more general effect operators in representing quantum effects is by now well estabilished. We refer to, e.g., [**4, 41**]. An argument concerning this matter will also be presented here in Chapter 5, in connection to the notion of sequential product.

- the probability for the occurrence of an effect A (denoted with the same letter as the corresponding operator) when the system is prepared in a state ρ (again, denoted with the same letter as the corresponding density operator) is $p_\rho(A) = \mathrm{tr}(\rho A)$ (where tr() denotes, of course, the trace).
- observable are represented as normalised positive operator valued measures (POV-measures). A POV-measure is a mapping $E : \mathcal{F} \to \mathcal{E}(H)$, where $(X, \mathcal{F})$ is a measurable space (with X interpreted as the set of possible outcomes of the measurements performed on the observable), mapping which satisfies $E(X) = \mathbf{1}$ and $E(\cup M_i) = \sum E(M_i)$ for all disjoint sequences (M_i) in $\mathcal{F}$ (the series converges in weak operator topology). For comparation, let us remark that, according to conventional quantum mechanics, observables were defined as projection valued measures (PV-measures), which can be regarded as a particular case of POV-measures taking values in the set $\mathcal{P}(H)$. It should be mentioned that usually, the measurable space $(X, \mathcal{F})$ is just the real Borel space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, in which case a PV-measure is uniquely associated, by the spectral theorem, to a self-adjoint operator. This allows, for "conventional" observables defined as PV-measures to be represented by (or identified with) self-adjoint operators on the state space H. One of the interesting consequences of the transition from conventional observables, defined as self-adjoint operators or the corresponding PV-measures to the observables defined as POV-measures is the possibility of measuring together observables that do not commute (see [**42**]).

The interpretation of the above representation of observables as POV-measures is as follows (see [**41, 21, 4**]). X represents the set of possible outcomes of the measurements performed on the observable. For a set $M \in \mathcal{F}$, the question if a measurement of the observable yields a value contained in M or not is a yes-no measurement, thereby described by an effect operator $E(M)$. If the result of the measurement performed on the observable belongs to M, we say that the effect $E(M)$ is *observed*, whereas in the opposite case it is *non-observed*. In both cases, $E(M)$ is *tested* by the observable, since it is contained in its range. An effect in $\mathcal{E}(H)$ that is not in the range of the POV-measure associated to an observable is *not tested* by that observable. The probability of the measured value of the observable to be in M when the system is in the state ρ is $p_\rho(E(M)) = \mathrm{tr}(\rho E(M))$. Therefore, the map $E : M \mapsto E(M)$ completely describe the "statistics" of the observable in any state ρ of the system.

A set of effects is *simultaneously testable* or *coexistent* if there exist an observable whose range includes it [**44, 42**]. Later on, we shall discuss in details the properties of the important notion of coexistence in abstract effect algebras, which is the generalization of compatibility/commutativity as defined in orthomodular posets and orthomodular lattices.

The set $\mathcal{E}(H)$ of effects can be endowed with a family of morphisms $(J_P)_{P\in\mathcal{P}(H)}$ indexed by the projection operators $P \in \mathcal{P}(H)$ and defined by $J_P(A) = PAP$, called *compressions.* They interact in useful ways with states, observables and symmetries. The family $(J_P)_{P\in\mathcal{P}(H)}$ is said to form a *compression base* of $\mathcal{E}(H)$.

Just like the structure of orthomodular lattice was introduced as an abstraction of the important features of the lattice $\mathcal{P}(H)$ of projection operators of a Hilbert space H, the algebraic structure of effect algebra is an abstraction of the essential features of the set $\mathcal{E}(H)$ of effect operators and compression base effect algebras (CB-effect algebras) are an abstraction of the essential features of the set $\mathcal{E}(H)$ endowed with the family of morphisms $(J_P)_{P\in\mathcal{P}(H)}$. Other structures that generalize $\mathcal{E}(H)$ are D-posets, introduced by F. Kôpka and F. Chovanec [**40**] and weak orthoalgebras, introduced by R. Giuntini and H. Greuling [**24**]. They all turned out to be equivalent structures. In order to avoid confusion and make our exposition easier to follow, we will present all the results in terms of the effect algebra structure, even those that were originally formulated by their authors in terms of another equivalent structure.

We conclude these introductory remarks with the words of D. Foulis [**21**], which, we believe, could best motivate our study of CB-effect algebras:

" A physical system $\mathcal{S}$, understood as the subject of experimental investigation, is appropriately represented by a CB-effect algebra E, which hosts the observables, states, symmetries and other ingredients of a physical theory for $\mathcal{S}$. Moreover, physical systems are classified by the CB-effect algebras that represent them. "

CHAPTER 4

Basics on Effect Algebras

The first chapter of this second part of the thesis is devoted to an introduction to unsharp quantum logics—represented here by effect algebras or, equivalently, by D-posets—and their basic properties. Special (isotropic, sharp, principal and central) elements are introduced and the conditions for an effect algebra to be an orthoalgebra, an orthomodular poset or even a Boolean algebra are discussed. The coexistence relation, which generalizes the compatibility defined for orthomodular posets to effect algebras (and bears the corresponding physical meaning), is discussed, as well as its connection to central elements. We then introduce orthogonal systems, orthocompleteness and weak orthocompleteness and also atomicity and related properties in the framework of effect algebras. Finally, we introduce various types of maps on effect algebras. The facts presented in this chapter can be found, e.g., in the book of Dvurečenskij and Pulmannová [**14**] which gathers many of the recent results in the field of quantum structures. A selection of research articles which cover many of the facts presented here would include [**22, 23, 24, 26, 40**].

4.1. Effect algebras. Basic definitions and properties

Definition 4.1.1. An *effect algebra* is a system $(E, \oplus, \mathbf{0}, \mathbf{1})$ such that E is a set, $\mathbf{0}$ and $\mathbf{1}$ are distinct elements of E and $\oplus$ is a partial binary operation on E, and the following conditions hold for every $a, b, c \in E$ (the equalities should be understood in the sense that if one side exists, the other side exists as well):

(EA1) $a \oplus b = b \oplus a$ (commutativity)
(EA2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity)
(EA3) for every $a \in E$, there exists a unique $a' \in E$ such that $a \oplus a' = \mathbf{1}$ (orthosupplement)
(EA4) if $a \oplus \mathbf{1}$ is defined, then $a = \mathbf{0}$ (zero-unit law).

As it is usual, we will often refer to the effect algebra E instead of $(E, \oplus, \mathbf{0}, \mathbf{1})$, for brevity. An orthogonality and a partial order relation are defined in an effect algebra as follows:

Definition 4.1.2. Let E be an effect algebra. Elements $a, b \in E$ are called *orthogonal* (denoted by $a \perp b$) if the sum $a \oplus b$ is defined. We write $a \leq b$ if there is an element $c \in E$ such that $a \oplus c = b$.

The next couple of propositions gives a list of basic properties that hold in effect algebras. We shall omit their simple verification, which can be found, however, in [**22**].

PROPOSITION 4.1.3. *Let E be an effect algebra. For every $a, b \in E$ the following properties hold:*

(1) $a'' = a$
(2) $a \leq b$ *implies* $b' \leq a'$
(3) $\mathbf{1}' = \mathbf{0}$ *and* $\mathbf{0}' = \mathbf{1}$.

PROPOSITION 4.1.4. *Let E be an effect algebra. For every $a, b, c \in E$ the following properties hold:*

(1) $\mathbf{0} \leq a \leq \mathbf{1}$.
(2) $a \oplus \mathbf{0} = a$.
(3) $a \perp b$ *if and only if* $a \leq b'$.
(4) *If $a \leq b$ and $c \in E$ is such that $a \oplus c = b$, then c is uniquely determined by the elements a and b, namely $c = (a \oplus b')'$. We will then denote $c = b \ominus a$.*
(5) *"$\leq$" is a partial order on E.*
(6) $a \oplus b = a \oplus c$ *implies* $b = c$ *(cancellation law).*
(7) $a \oplus b \leq a \oplus c$ *implies* $b \leq c$ *(cancellation law).*

The following result gives some of the basic properties, which can be found in [**22**], of the partial binary operation "$\ominus$" introduced above.

PROPOSITION 4.1.5. *Let E be an effect algebra and $a, b, c \in E$. Then:*

(D1) *if* $a \leq b$ *then* $b \ominus a \leq b$;
(D2) *if* $a \leq b$ *then* $b \ominus (b \ominus a) = a$;
(D3) *if* $a \leq b \leq c$ *then* $c \ominus b \leq c \ominus a$;
(D4) *if* $a \leq b \leq c$ *then* $(c \ominus a) \ominus (c \ominus b) = b \ominus a$;
(D5) $(a \ominus b) \ominus c = (a \ominus c) \ominus b$ *(if one side exists, the other exists as well).*

PROOF. (D1) and (D2) follow from $a \oplus (b \ominus a) = b$. (D3) and (D4) follow from $c \ominus a = (b \ominus a) \oplus (c \ominus b)$, which in turn follows from $a \oplus (b \ominus a) \oplus (c \ominus b) = b \oplus (c \ominus b) = c$. To prove (D5), assume that $(a \ominus b) \ominus c$ exists and let $d = (a \ominus b) \ominus c$. Then $a = d \oplus c \oplus b$, hence $a \ominus c = d \oplus b$ and therefore $(a \ominus c) \ominus b = d = (a \ominus b) \ominus c$. □

REMARK 4.1.6. A partially ordered set (poset) with a partial binary operation "$\ominus$" such that $b \ominus a$ is defined whenever $a \leq b$, and "$\ominus$" satisfies conditions (D1)–(D4) is called a *D*-poset. As mentioned previously, the D-poset structure is equivalent to the effect algebra structure (and Proposition 4.1.5 proves one of the implications).

DEFINITION 4.1.7. Let E be an effect algebra and $F \subset E$. If $\mathbf{0}, \mathbf{1} \in F$, and F is closed to $\oplus$ and to orthosupplementation, then $(F, \oplus|_{F \times F}, \mathbf{0}, \mathbf{1})$ is a *sub-effect algebra* of E.

PROPOSITION 4.1.8. *Let E be an effect algebra and F be a sub-effect algebra of E. If $a, b \in F$, $a \leq b$, then $b \ominus a \in F$.*

PROOF. According to Proposition 4.1.4 (4), $b \ominus a = (a \oplus b')'$. Since $a, b \in F$ and, by Definition 4.1.7, F is closed under $\oplus$ and to orthosupplementation, it follows that $(a \oplus b')' \in F$, hence $b \ominus a \in F$. □

4.2. Special elements. Coexistence

DEFINITION 4.2.1. An element a of an effect algebra E is called:

- *isotropic* if $a \perp a$;
- *sharp* ($a \in E_S$) if $a \wedge a' = \mathbf{0}$;
- *principal* if for every orthogonal pair $b, c \in E$, $b, c \leq a$ we have $b \oplus c \leq a$;
- *central* if a, a' are principal and for every $b \in E$, there are $b_1, b_2 \in E$ such that $b_1 \leq a$, $b_2 \leq a'$ and $b = b_1 \oplus b_2$.

PROPOSITION 4.2.2. *In an effect algebra, the following assertions hold:*

(1) *every central element is principal;*
(2) *every principal element is sharp;*
(3) *every nonzero sharp element is nonisotropic.*

PROOF. Let E be an effect algebra. Since by definition every central element is principal, let us first prove that every principal element is sharp. Assume $a \in E$ is principal and $b \leq a, a'$. Then $a, b \leq a$ and $a \perp b$, hence $a \oplus b \leq a$ and by the cancellation law, $b = \mathbf{0}$. It follows that $a \wedge a' = \mathbf{0}$, hence a is sharp.

Let us prove now that every sharp nonzero element is nonisotropic. Assume $a \in E$, $a \neq \mathbf{0}$ and $a \perp a$. Then $a \leq a'$, hence $a \wedge a' = a \neq \mathbf{0}$ and it follows that a is not sharp. □

It is not difficult to find examples which show that in general, the converse statements do not hold (see also Example 4.2.8).

DEFINITION 4.2.3. An *orthoalgebra* is an effect algebra whose only isotropic element is $\mathbf{0}$.

DEFINITION 4.2.4. Let E be an effect algebra and let us denote by na the sum of n copies of an element $a \in E$, if it exists. We call E *Archimedean* if $\sup\{n \in \mathbb{N} \colon na \text{ is defined}\} < \infty$ for every nonzero element $a \in E$.

Let us remark that every orthoalgebra is Archimedean since no nonzero element is orthogonal to itself.

THEOREM 4.2.5. *Let E be an effect algebra. The following conditions are equivalent:*

(1) *E is an orthoalgebra;*
(2) *every element of E is sharp;*

(3) *$a \oplus b$ is a minimal upper bound of a, b for any orthogonal pair $a, b \in E$.*

Proof. "(1) $\Rightarrow$ (2)" Let $a \in E$ and assume $b \leq a, a'$. Then $b \leq a \leq b'$, hence $b \perp b$ and since E is an orthoalgebra, it follows $b = \mathbf{0}$. We conclude that $a \wedge a' = \mathbf{0}$, hence a is sharp.
"(2) $\Rightarrow$ (1)" This follows trivially from the last statement of Proposition 4.2.2.
"(2) $\Rightarrow$ (3)" Let $a, b \in E$ be an orthogonal pair. Clearly $a \oplus b \geq a, b$. Let us assume that there is another upper bound $d \in E$ of a, b such that $a \oplus b \geq d \geq a, b$. Let $e = (a \oplus b) \ominus d$. Then, using (D3) from Proposition 4.1.5, $e = (a \oplus b) \ominus d \leq (a \oplus b) \ominus b = a$ and $e = (a \oplus b) \ominus d \leq (a \oplus b) \ominus a = b \leq a'$. Since $e \leq a, a'$ and a is sharp, it follows that $e = \mathbf{0}$, hence $d = a \oplus b$.
"(3) $\Rightarrow$ (2)" According to the hypothesis, for every element $a \in E$, $\mathbf{1} = a \oplus a'$ is a minimal upper bound of a, a'. Therefore $\mathbf{1} \geq d \geq a, a'$ for some $d \in E$ entails that $d = \mathbf{1}$. Since the only upper bound of a, a' is $\mathbf{1}$, we conclude that $a \vee a' = \mathbf{1}$, hence, by de Morgan's law, $a \wedge a' = \mathbf{0}$. □

Remark 4.2.6. Let $(E, \leq, ', \mathbf{0}, \mathbf{1})$ be an orthomodular poset and define $a \oplus b = a \vee b$ for every orthogonal (i.e. $a \leq b'$) pair of elements $a, b \in E$. It is a routine verification that $(E, \oplus, \mathbf{0}, \mathbf{1})$ is an effect algebra (even an orthoalgebra) and, moreover, the order and supplement in the effect algebra coincide with the order and complement in the orthomodular poset.

Theorem 4.2.7. *Let E be an effect algebra. The following conditions are equivalent:*

(1) *E is an orthomodular poset (with the effect algebra order and supplementation);*
(2) *every element of E is principal;*
(3) *$a \oplus b$ is the lowest upper bound of a, b (i.e., $a \oplus b = a \vee b$) for every orthogonal pair $a, b \in E$.*
(4) *if $a, b, c \in E$ are mutually orthogonal elements, then $a \oplus b \oplus c$ exists in E (coherence).*

Proof. "(1) $\Rightarrow$ (3)" If E is an orthomodular poset, $a \vee a' = \mathbf{1}$ or equivalently $a \wedge a' = \mathbf{0}$ for every $a \in E$, which means that every element of E is sharp. By Theorem 4.2.5, this implies that $a \oplus b$ is a minimal upper bound of a, b for any orthogonal pair $a, b \in E$. But for every such pair, $a \vee b$ exists in E and since $a \oplus b \geq a \vee b \geq a, b$ it follows that $a \oplus b = a \vee b$.
"(3) $\Rightarrow$ (1)" We have to check the conditions of Definition 1.1.10. We will verify only the non-trivial ones, namely that $a \vee a' = \mathbf{1}$ for every $a \in E$ and the orthomodular law. Since for any orthogonal elements $a, b \in E$, $a \oplus b$ is their lowest upper bound, $a \oplus b$ is also a minimal

upper bound, hence according to Theorem 4.2.5, every element of E is sharp, i.e. $a \wedge a' = \mathbf{0}$ or $a \vee a' = \mathbf{1}$. For the orthomodular law, it suffices to recall that if $a \leq b$, there exists $c \in E$ such that $a \perp c$ and $a \oplus c = b$ and by our assumption, $a \oplus c = a \vee c$.
"(2) $\Rightarrow$ (3)" Let $a, b \in E$, $a \perp b$. Then $a \oplus b \geq a, b$. For every $c \in E$, $c \geq a, b$ we have $c \geq a \oplus b$, since c is principal. It follows that $a \oplus b = a \vee b$.
"(3) $\Rightarrow$ (2)" For every $a, b, c \in E$ such that $a \perp b$ and $a, b \leq c$ we have $a \vee b \leq c$, hence $a \oplus b \leq c$ and therefore c is principal.
"(2) $\Rightarrow$ (4)" Let $a, b, c \in E$ be mutually orthogonal. Then $a, b \leq c'$, $a \perp b$ and c' is principal by our assumption, hence $a \oplus b \leq c'$ or equivalently $a \oplus b \perp c$ which means that $a \oplus b \oplus c$ exists.
"(4) $\Rightarrow$ (2)" Let $a, b, c \in E$ such that $a \perp b$ and $a, b \leq c$. It follows that $a \perp c'$ and $b \perp c'$. According to our hypothesis, $a \oplus b \oplus c'$ exists, hence $a \oplus b \perp c'$ or equivalently $a \oplus b \leq c$. □

Let us mention that the above characterizations of orthoalgebras and orthomodular posets (Theorems 4.2.5 and 4.2.7) are based on results in [**22, 23**].

Example 4.2.8. Let us now present a few classical examples of effect algebras.
(a) We begin of course with the prototypical example of $\mathcal{E}(H)$. Let H be a separable, complex Hilbert space and let $\mathbf{0}, \mathbf{1}$ denote the zero and identity operators on H. An ordering is defined on the set of bounded self-adjoint operators on H by:

$$A \leq B \iff \langle Ax, x\rangle \leq \langle Bx, x\rangle \text{ for all } x \in H \tag{4.1}$$

where $\langle\cdot,\cdot\rangle$ denotes of course the inner product on H. Then $\mathcal{E}(H)$ is defined as the set of self-adjoint operators on H lying between $\mathbf{0}$ and $\mathbf{1}$, in the sense of (4.1)– the so-called *effect operators*. A partial binary operation "$\oplus$" is defined on $\mathcal{E}(H)$ by $A \oplus B = A + B$, whenever $A + B \in \mathcal{E}(H)$. Then $(\mathcal{E}(H), \oplus, \mathbf{0}, \mathbf{1})$ is the so-called *standard Hilbert space effect algebra*. The supplement of an effect operator $A \in \mathcal{E}(H)$ is $A' = \mathbf{1} - A$. Let us remark that the effect algebra order relation defined on $\mathcal{E}(H)$ according to Definition 4.1.2 coincides with the order relation defined in (4.1). It is worth noting that there are nonzero isotropic effects , e.g. $\frac{1}{2}\mathbf{1} \in \mathcal{E}(H)$ (which is even its own supplement). Therefore, in view of Theorem 4.2.5, $\mathcal{E}(H)$ is not an orthoalgebra. Its sharp elements are in fact the projection operators (self-adjoint idempotents) on H, which form an orthomodular lattice which is also a sub-effect algebra of $\mathcal{E}(H)$, denoted, as previously mentioned, by $\mathcal{P}(H)$.
(b) The $[0, 1]$ interval of the real numbers can be organized as an effect algebra, with "$\oplus$" defined by $a \oplus b = a + b$ for every pair $a, b \in [0, 1]$ such that $a + b \in [0, 1]$ and of course 0 and 1 as zero and unit effects.

In fact, for a 1-dimensional Hilbert space H, the effect algebra $\mathcal{E}(H)$ can be identified with $[0,1]$ organized as described. Moreover, this is the only case when $\mathcal{E}(H)$ is lattice ordered (even totally ordered), with the effect algebra order coinciding with the usual real numbers order.
(c) The following example, due to R. Wright, is the simplest orthoalgebra which is not an orthomodular poset (all its elements are sharp, but not all are principal). Let $E = \{\mathbf{0}, \mathbf{1}, a, b, c, d, e, f, a', b', c', d', e', f'\}$ and let "$\oplus$" be a partial binary operation on E such that $(E, \oplus, \mathbf{0}, \mathbf{1})$ is an effect algebra and, in addition to the obvious relations, the following ones hold:

$$a \oplus b \oplus c = \mathbf{1}, \qquad c \oplus d \oplus e = \mathbf{1}, \qquad e \oplus f \oplus a = \mathbf{1}.$$

We can easily see that e.g. a' is not principal, since $c, e \leq a'$ and $c \oplus e = d'$ exists in E, but $d' \leq a'$ fails. On the other hand, let us show that all the elements of E are sharp. Assume $u \leq a, a'$ for some $u \in E$. Since a is a minimal nonzero element, it is only possible that $u = a$ or $u = \mathbf{0}$. The first case leads us to $a \leq a'$, but since $a \oplus a$ is not defined, this is not true. It then follows that $u = \mathbf{0}$, hence $a \wedge a' = \mathbf{0}$ and therefore a, a' are sharp elements. Applying the same reasoning to b, c, d, e, f and their supplements, we find that all elements of E are sharp.

Let us now turn to the important property of coexistence. Recall that two effects are coexistent (or simultaneously testable) if they are in the range of the same observable. Mathematically, this translates (see [**4**]) into the following property:
Two effect operators $A, B \in \mathcal{E}(H)$ on the Hilbert space H coexist if there exists $A_1, B_1, C \in \mathcal{E}(H)$ such that $A = A_1 \oplus C$, $B = B_1 \oplus C$ and $A_1 \oplus B_1 \oplus C$ exists in $\mathcal{E}(H)$.

Taking this property to abstract effect algebras leads us to the following definition:

Definition 4.2.9. Let E be an effect algebra and $a, b \in E$. Elements a and b *coexist in* E if there are $a_1, b_1, c \in E$ such that $a = a_1 \oplus c$, $b = b_1 \oplus c$ and $a_1 \oplus b_1 \oplus c$ exists in E. In this case, we write $a \leftrightarrow_E b$ or, if there is no risk of confusion about E, just $a \leftrightarrow b$. Elements $a_1, b_1, c \in E$ fulfilling this conditions are called a *Mackey decomposition* of a, b. For a subset M of E, we write $a \leftrightarrow M$ if $a \leftrightarrow b$ for all $b \in M$. The *commutant* of M in E is the set $K_E(M) = \{a \in E \colon a \leftrightarrow M\}$. If there is no possibility of confusion concerning E, we shall simply denote it by $K(M)$.

Remark 4.2.10. Recall that in an orthomodular poset P, elements a and b are compatible (or commute) if there exist $a_1, b_1, c \in P$ mutually orthogonal elements such that $a = a_1 \vee c$ and $b = b_1 \vee c$. In view of Theorem 4.2.7, it is then clear that coexistence generalizes compatibility to effect algebras and that the two notions coincide in orthomodular posets. This justifies the use of the same notation for coexistence and compatibility and also for the commutant with respect to coexistence or compatibility. In what follows, a pair of coexistent elements of an effect algebra will be sometimes called compatible or commuting, if the pair belongs to an orthomodular poset.

Furthermore, as the following Proposition 4.2.13 states, the notion of center of an effect algebra, defined as the set of its central elements, generalizes the notion of center in orthomodular posets, defined as the set of its elements which are compatible with all the others.

We shall first need the following well known characterization of coexistence:

LEMMA 4.2.11. *Let E be an effect algebra and $a, b \in E$. Then, $a \leftrightarrow b$ if and only if there exist $b_1, b_2 \in E$ such that $b_1 \leq a$, $b_2 \leq a'$ and $b = b_1 \oplus b_2$.*

PROOF. Let a, b be elements of E. There exist $b_1, b_2 \in E$ such that $b = b_1 \oplus b_2$ and $b_1 \leq a$, $b_2 \leq a'$ if and only if $b = b_1 \oplus b_2$, $a = b_1 \oplus a_1$ (where $a_1 = a \ominus b_1$) and $a \perp b_2$, i.e. $a_1 \oplus b_1 \oplus b_2 = a \oplus b_2$ exists. This however is the same as $a \leftrightarrow b$. □

A obvious corollary follows:

COROLLARY 4.2.12. *Let E be an effect algebra and $a, b \in E$. Then, $a \leftrightarrow b$ if and only if $a' \leftrightarrow b$.*

PROPOSITION 4.2.13. *Let E be an effect algebra. Then:*

(1) *an element of $a \in E$ is central if and only if it coexists with all elements of E and a, a' are principal;*
(2) *if E is an orthomodular poset, the set of central elements of E coincides with the center of E as an orthomodular poset.*

PROOF. (1) The result follows directly from Lemma 4.2.11.

(2) This is almost trivial, considering the fact that every element of an orthomodular poset is principal, according to Theorem 4.2.7, and the above remark that coexistence is the same as commutativity/compatibility in orthomodular posets. □

In view of the above result, we shall henceforth denote the center (i.e., the set of central elements) of an effect algebra E by $\widetilde{C}(E)$.

Recall that the center of an orthomodular poset is a Boolean algebra. In view of the above Proposition 4.2.13, it is legitimate to ask ourselves if the same is not true in effect algebras in general. The answer to this question is in the affirmative, as the following result from [**26**] asserts. We omit here the proof which can be found in the aforementioned paper.

THEOREM 4.2.14. [**26**, Theorem 5.4] *The center $\widetilde{C}(E)$ of an effect algebra E is a sub-effect algebra of E and as an effect algebra in its own right, $\widetilde{C}(E)$ forms a Boolean algebra. Furthermore, if $a, b \in \widetilde{C}(E)$, then $a \wedge b$ and $a \vee b$ as calculated in $\widetilde{C}(E)$ are also the infimum and supremum of a and b as calculated in E.*

REMARK 4.2.15. As a consequence of Theorems 4.2.5, 4.2.7 and 4.2.14, we can conclude that an effect algebra is:

- an orthoalgebra if and only if its every element is sharp
- an orthomodular poset if and only if its every element is principal
- a Boolean algebra if and only if its every element is central

4.3. Substructures in effect algebras

In the first part of this thesis, we have defined Boolean subalgebras in the context of ortholattices (Definition 1.3.2) and orthoposets (Definition 1.3.17). We shall now define them in the more general framework of effect algebras.

DEFINITION 4.3.1. A *Boolean subalgebra* of an effect algebra E is a sub-effect algebra of E which is a Boolean algebra with $'$ and with the operations $\vee$, $\wedge$ induced by the order in E.

We have seen that orthomodular posets can be considered special cases of effect algebras (Remark 4.2.15). It is then clear that the above definition should generalize Definition 1.3.17—and it is easy to see that indeed it does.

However, in the case of orthomodular posets, Proposition 1.3.16 guarantees that all suprema and infima of pairs of elements of a Boolean subalgebra exist and are the same if calculated in the orthomodular poset or the subalgebra. It is a natural question if the analogue assertion holds for a Boolean subalgebra of an effect algebra. As we shall see, the answer is in the affirmative if we ask the effect algebra to be an orthoalgebra. More precisely, the following result holds:

PROPOSITION 4.3.2. *Let E be an orthoalgebra and let $F \subseteq E$ be a Boolean subalgebra of E. Then:*

(1) *If $a, b \in F$ and $a \wedge_E b$ exists, then $a \wedge_E b = a \wedge_F b$;*
(2) *If $a, b \in F$ and $a \vee_E b$ exists, then $a \vee_E b = a \vee_F b$;*

PROOF. Let $a, b \in F$. Since F is a Boolean algebra, we have $a \leftrightarrow_F b$. Therefore, there exist the mutually orthogonal elements $a_1, b_1, c \in F$ such that $a = a_1 \oplus c$, $b = b_1 \oplus c$ and $a_1 \oplus b_1 \oplus c$ exist in F. Since F is a Boolean algebra, it is clear that $a \wedge_F b = c$ and $a \vee_F b = a_1 \vee b_1 \vee c = a_1 \oplus b_1 \oplus c$.

(1) Let us assume that $a \wedge_E b$ exists. Then $c = a \wedge_F b \leq a \wedge_E b$, because $F \subseteq E$. There exists then an element $d \in E$ such that $c \oplus d =$

$a \wedge_E b$. But $a \wedge_E b \leq a, b$, hence $c \oplus d \leq a_1 \oplus c, b_1 \oplus c$. By cancellation law, we find that $d \leq a_1, b_1$. However, $a_1 \perp b_1$, hence $b_1 \leq a_1'$ and it follows that $d \leq a_1, a_1'$. Since E is an orthoalgebra, $d = \mathbf{0}$ and therefore $c = a \wedge_F b = a \wedge_E b$.

(2) If $a \vee_E b$ exists, then $a \vee_E b \leq a \vee_F b = a_1 \oplus b_1 \oplus c$. There exists then an element $d \in E$ such that $(a \vee_E b) \oplus d = a_1 \oplus b_1 \oplus c$. But $a, b \leq a \vee_E b$, hence $a_1 \oplus c \oplus d, b_1 \oplus c \oplus d \leq a_1 \oplus b_1 \oplus c$. Again, by cancellation, we find that $d \leq b_1, a_1$ and it follows that $d = \mathbf{0}$ and therefore $a \vee_E b = a \vee_F b = a_1 \oplus b_1 \oplus c$. $\square$

The following Remark is rather obvious, but very useful:

Remark 4.3.3. If F is a sub-effect algebra of an effect algebra E, the following statements hold:

(1) if $a \in F$ and a is a principal element of E, then it is also principal in F;
(2) if $a, b \in F$ coexist in F, then they coexist in E.

Of course, the converse statements need not be true.

4.4. Important classes of effect algebras

Let us present now a few important properties that an effect algebra may fulfill and their correlations.

Definition 4.4.1. An effect algebra that is a lattice with respect to its usual order relation, is called a *lattice effect algebra*.

Definition 4.4.2. An *MV-effect algebra* is a lattice effect algebra E such that $a \wedge b = \mathbf{0}$ implies $a \perp b$ for every $a, b \in E$.

Remark 4.4.3. Coexistence in lattice effect algebras has interesting properties which can be found, e.g., in [**14**]. A review on this subject is also included in [**9**]. Let us just mention here that the blocks in lattice effect algebras (maximal subsets of mutually coexisting elements) are MV-effect algebras, which in turn were proven to be equivalent to the well known MV-algebras.

Definition 4.4.4. Let E be an effect algebra. A system $(a_i)_{i \in I}$ of elements of E is *orthogonal* if $\bigoplus_{i \in F} a_i$ is defined for every finite set $F \subset I$. A *majorant* of an orthogonal system $(a_i)_{i \in I}$ is an upper bound of $\{\bigoplus_{i \in F} a_i \colon F \subset I \text{ is finite}\}$. The *sum* of an orthogonal system is its least majorant (if it exists).

Definition 4.4.5. An effect algebra E is *orthocomplete* if every orthogonal system of its elements has a sum. An effect algebra E is *weakly orthocomplete* if every orthogonal system in E has a sum or no minimal majorant.

Definition 4.4.6. An effect algebra E has the *maximality* property if the set $\{a, b\}$ has a maximal lower bound for every set $\{a, b\} \subseteq E$.

REMARK 4.4.7. The maximality property was introduced by Tkadlec [**58**]. In [**61**, Theorem 2.2], he proved that effect algebras with the maximality property or the ones that are weakly orthocomplete are common generalizations of lattice effect algebras and orthocomplete effect algebras. Since finite or chain-finite effect algebras are orthocomplete (see [**60**, Theorem 4.1]), they also must satisfy the maximality property.

DEFINITION 4.4.8. Let E be an effect algebra. A minimal non-zero element of E is called an *atom*. E is *atomic* if every non-zero element dominates an atom. E is *atomistic* if every non-zero element is the supremum of the atoms it dominates. E is *determined by atoms* if, for different $a, b \in E$, the sets of atoms dominated by a and b are different.

The relations between these notions are outlined in the following result.

LEMMA 4.4.9 ([**59**, Lemma 2.2]). *Every atomistic effect algebra is determined by atoms. Every effect algebra determined by atoms is atomic.*

Examples showing that the converse implications do not hold can be found in [**25, 59**].

PROPOSITION 4.4.10 ([**59**, Corollary 2.6]). *Every atomic effect algebra in which every its atom is sharp is an orthoalgebra.*

4.5. Morphisms of effect algebras

DEFINITION 4.5.1. Let E and E' be effect algebras and let $\varphi : E \to E'$ be a map. We call φ an *additive* map if $a \perp b$ implies $\varphi(a) \perp \varphi(b)$ and $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$, for every $a, b \in E$.

DEFINITION 4.5.2. Let E and E' be effect algebras and let $\varphi : E \to E'$ be a map. Then φ is a *morphism* of effect algebras if it is additive and $\varphi(\mathbf{1}_E) = \mathbf{1}_{E'}$.

DEFINITION 4.5.3. A morphism φ of effect algebras which preserves the infimum (i.e., $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$, whenever $a \wedge b$ exists) is a $\wedge$-*morphism*.

DEFINITION 4.5.4. A bijective morphism φ such that φ^{-1} is also a morphism is an *isomorphism*.

PROPOSITION 4.5.5. *Let E and E' be effect algebras and let $\varphi : E \to E'$ be a map. Then φ is an isomorphism of effect algebras if and only if it is bijective and, for every $a, b \in E$, $a \perp b$ if and only if $\varphi(a) \perp \varphi(b)$, in which case $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$. Moreover, if φ is an isomorphism, it is also a $\wedge$-morphism.*

PROOF. The main point in proving both statements is to show that a bijective map φ with the property that for every $a, b \in E$, $a \perp b$ if and only if $\varphi(a) \perp \varphi(b)$ in which case $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$ preserves order both ways, i.e., it satisfies $a \leq b$ if and only if $\varphi(a) \leq \varphi(b)$, for every $a, b \in E$. Let us prove this first.

Indeed, let $a, b \in E$, $a \leq b$. Then, there exists $c \in E$ such that $a \oplus c = b$ and it follows that $\varphi(a) \oplus \varphi(c) = \varphi(b)$. Hence $\varphi(a) \leq \varphi(b)$. Conversely, if $\varphi(a) \leq \varphi(b)$, there exists an element $d \in E'$ such that $\varphi(a) \oplus d = \varphi(b)$. However, since φ is surjective, there is an element $e \in E$ such that $\varphi(e) = d$. It follows that $\varphi(a) \oplus \varphi(e) = \varphi(a \oplus e) = \varphi(b)$, and by the injectivity of φ, $a \oplus e = b$, and therefore $a \leq b$.

Now let us prove the first statement of the proposition. The direct implication is straightforward. Conversely, we need to prove that a bijective map φ such that for every $a, b \in E$, $a \perp b$ if and only if $\varphi(a) \perp \varphi(b)$ in which case $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$ is a morphism. This is evident except for the fact that $\varphi(\mathbf{1}_E) = \mathbf{1}_{E'}$. Let us assume that $\varphi(\mathbf{1}_E) = d \in E'$ and by surjectivity of φ, there exists $c \in E$ such that $\varphi(c) = \mathbf{1}_{E'}$. Then, $\varphi(\mathbf{1}_E) \leq \varphi(c)$ and since φ preserves order, it follows that $\mathbf{1}_E \leq c$, hence $c = \mathbf{1}_E$ and $\varphi(\mathbf{1}_E) = \mathbf{1}_{E'}$. The proof that φ^{-1} is also a morphism is now straightforward.

The second statement is a direct consequence of the fact proven in the begining that an isomorphism preserves order both ways (and of its bijectivity). $\square$

Definition 4.5.6. Let E be an effect algebra. An isomorphism $\varphi : E \to E$ is an *automorphism*.

Remark 4.5.7. Clearly, the results in Proposition 4.5.5 hold also if we replace E' by E and the word “isomorphism” by “automorphism”.

CHAPTER 5

Sequential, compressible and compression base effect algebras

The notion of a sequential product defined in general effect algebras was introduced by Gudder and Greechie [**31**]. This sequential product satisfies a set of physically motivated axioms as it formalizes the case of sequentially performed measurements. The existence of a sequential product in an effect algebra proves to be a restrictive condition, far from being met by all effect algebras. An effect algebra on which a sequential product is defined is called a sequential effect algebra.

Gudder [**28**] introduced compressions on effect algebras and also compressible effect algebras. He was inspired by the work of David Foulis on compressions and compressible groups [**16, 17, 18, 19**]. Although the important examples of effect algebras proved to be compressible, it seems that the concept of compressible algebras is too restrictive, leaving out a rather large class of effect algebras.

As it turns out, the two notions (sequential effect algebra and compressible effect algebra) are somehow related, since the sequential product with a sharp element (of a sequential effect algebra) defines a compression. Although the restrictions imposed by the existence of a sequential product seem stronger than those determined by compressibility, neither of the two notions is a generalization of the other, as an example of a noncompressible sequential effect algebra confirms [**28**].

Again generalizing the work of Foulis in unital groups [**20**], Gudder introduced a common generalization of both sequential and compressible effect algebras, namely effect algebras having a compression base or CB-effect algebras [**29**]. Indeed, he proves that the compressions of a compressible effect algebra form a compresion base and also that a sequential effect algebra is naturally endowed with a maximal compression base. Then, he generalize many of the results that hold for sequential and compressible effect algebras to effect algebras having a compression base. The investigation on compression base effect algebras is continued by Pulmannová in [**54**], with many important properties. Among them, the fact that the set of projections of a compression base effect algebra is a regular orthomodular poset, or the projection cover property and its implications for the set of projections of a compression base effect algebra will be of special interest for our work.

In this chapter, we present the important (and useful, for our further work) established facts concerning sequential, compressible and compression base effect algebras, laying the foundation for the new results that will be the presented in the following chapters. For complete details and proofs that are omitted here, we refer to [**28, 29, 31, 54**].

5.1. Sequential effect algebras

As previously mentioned, effect algebras appeared as an abstraction of the structure of the standard Hilbert space effect algebra $\mathcal{E}(H)$ of effect operators (see Example 4.2.8 (a)), which is important in the theory of quantum measurements (see, e.g., [**4, 5, 41, 43, 44**]). The notion of sequential product in arbitrary effect algebras (and consequently, the notion of sequential effect algebra) appeared in a similar manner, as an abstraction of the sequential product that was defined in $\mathcal{E}(H)$ for the purpose of studying the frequently occurring situation of quantum measurements that are performed sequentially [**33, 32, 30**].

Let us briefly sketch here, for readers convenience, the reasoning that leads to the definition of sequential product on the set of effect operators $\mathcal{E}(H)$. We will follow the exposition from [**33**], to which we also refer for further details.

Let A, B denote yes-no measurements (with only two possible outcomes) that may be unsharp, called *effects* ([**4, 5**]). As previously stated, in quantum mechanics, effects are represented by effect operators, i.e., elements of $\mathcal{E}(H)$. A *density operator* is an operator $\rho \in \mathcal{E}(H)$ of trace class such that $\operatorname{tr}(\rho) = 1$. Density operators represent quantum states, and the probability that the effect A occurs when the system is in the state ρ is $p_\rho(A) = \operatorname{tr}(A\rho)$. Let $A, B \in \mathcal{E}(H)$, ρ be a density operator, and denote by $A \circ B$ the sequential measurement in which A is executed first and B second. It is then reasonable to consider that

$$p_\rho(A \circ B) = p_\rho(A) p_\rho(B|A), \tag{5.1}$$

where $p_\rho(B|A)$ denotes the *conditional probability of B given A* in the state ρ. If $A^{1/2}$ denotes the unique positive square root of A, then

$$p_\rho(B|A) = \frac{\operatorname{tr}(BA^{1/2}\rho A^{1/2})}{\operatorname{tr}(A\rho)} = \frac{\operatorname{tr}(A^{1/2}BA^{1/2}\rho)}{\operatorname{tr}(A\rho)} \tag{5.2}$$

when $\operatorname{tr}(A\rho) \neq 0$ (see [**5, 30, 32**]). Let us notice that equation (5.2) generalizes the von Neumann–Lüders formula in [**48**], which holds when $A, B \in \mathcal{P}(H)$:

$$p_\rho(B|A) = \frac{\operatorname{tr}(BA\rho A)}{\operatorname{tr}(A\rho)}$$

On the other hand, $A^{1/2}BA^{1/2} \in \mathcal{E}(H)$, because $0 \le \langle A^{1/2}BA^{1/2}x, x\rangle = \langle BA^{1/2}x, A^{1/2}x\rangle \le \langle A^{1/2}x, A^{1/2}x\rangle = \langle Ax, x\rangle \le \langle x, x\rangle$ for every $x \in H$.

It follows from equations (5.1) and (5.2) that

$$p_\rho(A \circ B) = \mathrm{tr}(A^{1/2}BA^{1/2}\rho) = p_\rho(A^{1/2}BA^{1/2}),$$

for every quantum state ρ. It is then natural to define

$$A \circ B = A^{1/2}BA^{1/2}$$

and call it the *sequential product* of effects A and B. Effects A and B are called *sequentially independent* if $A \circ B = B \circ A$. In this case, we write $A \mid B$.

Let us remark, in view of these facts, the importance of considering unsharp effects. Indeed, even if $A, B \in \mathcal{P}(H)$, $A \circ B = ABA$ is not a projection operator unless $AB = BA$. However, $0 \leq ABA \leq I$ holds, therefore $ABA \in \mathcal{E}(H)$.

The following interesting result is proved in [**33**, Corollary 2.2, Theorem 2.3, Corollary 2.4].

Theorem 5.1.1. (1) *For $A, B \in \mathcal{E}(H)$, $A \circ B = B \circ A$ if and only if $AB = BA$.*
(2) *For $A, B \in \mathcal{E}(H)$, if $A \circ B \in \mathcal{P}(H)$, then $AB = BA$.*
(3) *For $A, B \in \mathcal{P}(H)$, $A \circ B \in \mathcal{P}(H)$ if and only if $AB = BA$.*

The next result we present shows that if $dim(H) \geq 3$, then the effect algebra structure of $\mathcal{E}(H)$ is determined by the sequential product. More precisely, we have:

Theorem 5.1.2 ([**31**, Theorem 2.7]). *Suppose that $dim(H) \geq 3$. If $\varphi : \mathcal{E}(H) \to \mathcal{E}(H)$ is a bijection satisfying $\varphi(A \circ B) = \varphi(A) \circ \varphi(B)$ for all $A, B \in \mathcal{E}(H)$, then φ is an effect algebra isomorphism.*

Various properties of the sequential product defined on the Hilbert space effect algebra are proved in [**33, 32**]. The essential ones—that are also physically motivated, according to [**31**]—are used as axioms for a sequential product in abstract effect algebras (and therefore, for sequential effect algebras).

Definition 5.1.3. A *sequential product* on an effect algebra $(E, \oplus, \mathbf{0}, \mathbf{1})$ is a binary operation $\circ$ on E such that for every $a, b, c \in E$, the following conditions hold:

(S1) $a \circ (b \oplus c) = (a \circ b) \oplus (a \circ c)$ if $b \oplus c$ exists;
(S2) $\mathbf{1} \circ a = a$;
(S3) if $a \circ b = \mathbf{0}$ then $a \mid b$ (where $a \mid b$ denotes $a \circ b = b \circ a$);
(S4) if $a \mid b$ then $a \mid b'$ and $a \circ (b \circ c) = (a \circ b) \circ c$;
(S5) if $c \mid a, b$ then $c \mid a \circ b$ and $c \mid (a \oplus b)$ (if $a \oplus b$ exists).

An effect algebra E endowed with a sequential product is called a *sequential effect algebra*.

Definition 5.1.4. Let E be a sequential effect algebra. Elements $a, b \in E$ are *sequentially independent* if $a \circ b = b \circ a$. In this case, we will write $a \mid b$. If $a \mid b$ for all $a, b \in E$, then E is a *commutative* sequential effect algebra.

Some examples of sequential effect algebras are given in [**31**]:

EXAMPLE 5.1.5.
- Every Boolean algebra, endowed with the sequential product $a \circ b = a \wedge b$ is a sequential effect algebra.
- Let X be a nonempty set and $\mathcal{F} \subseteq [0,1]^X$ such that:
  (1) $f_0, f_1 \in \mathcal{F}$, where $f_0(x) = 0$ and $f_1(x) = 1$ for all $x \in X$;
  (2) $f \in \mathcal{F}$ implies $f_1 - f \in \mathcal{F}$;
  (3) $f, g \in \mathcal{F}$ and $f + g \leq f_1$ implies $f + g \in \mathcal{F}$;
  (4) $f, g \in \mathcal{F}$ implies $fg \in \mathcal{F}$.
  Then, $\mathcal{F}$ is a sequential effect algebra if we define $f \oplus g = f + g$ for $f, g \in \mathcal{F}$ such that $f + g \leq f_1$ and $f \circ g = fg$.
- Obviously, the Hilbert space effect algebra $\mathcal{E}(H)$ is a sequential effect algebra if we define the sequential product by $A \circ B = A^{1/2}BA^{1/2}$.

The following results give some of the important properties of the sequential product. They are proved in [**31**].

PROPOSITION 5.1.6 (see [**31**, Lemma 3.1, Lemma 3.3, Theorem 3.4]). *Let E be a sequential effect algebra. For every $a, b, c \in E$, the following properties hold:*

(1) $a \circ \mathbf{0} = \mathbf{0} \circ a = \mathbf{0}$;
(2) $a \circ \mathbf{1} = \mathbf{1} \circ a = a$;
(3) $a \circ b \leq a$;
(4) $a \leq b$ *implies* $c \circ a \leq c \circ b$;
(5) $a \circ b = \mathbf{0}$ *implies* $a \perp b$.

If a is a sharp element of E, then:

(6) $a \leq b$ *if and only if* $a \circ b = b \circ a = a$;
(7) $b \leq a$ *if and only if* $a \circ b = b \circ a = b$;
(8) $a \circ b = \mathbf{0}$ *if and only if* $a \perp b$;
(9) $a \mid b$ *implies* $a \circ b = a \wedge b$;
(10) $a \perp b$ *implies* $a \oplus b = a \vee b = (a' \circ b')'$.

For the remainder of this chapter, let E_S denote the set of sharp elements of the effect algebra E. The following characterization of the sharp elements of a sequential effect algebra holds:

LEMMA 5.1.7 ([**31**, Lemma 3.2]). *Let E be a sequential effect algebra. The following statements are equivalent:*

(1) $a \in E_S$;
(2) $a \circ a' = \mathbf{0}$;
(3) $a \circ a = a$.

DEFINITION 5.1.8. The *sequential center* of the sequential effect algebra E is the set $C(E) = \{a \in E \colon a \mid b \text{ for all } b \in E\}$.

THEOREM 5.1.9 (see [**31**, Theorem 3.6, Lemma 4.2, Corollary 3.5, Theorem 4.4]). *Let E be a sequential effect algebra and let $a, b \in E$. Then:*

(1) *$a \mid b$ implies that a and b coexist; if $b \in E_S$ then the converse implication holds as well;*
(2) *a is principal if and only if $a \in E_S$;*
(3) *E_S is a sub-effect algebra of E that is an orthomodular poset;*
(4) *$\widetilde{C}(E) = C(E) \cap E_S$.*

LEMMA 5.1.10 ([**31**, Lemma 5.1]). *An effect algebra E that, with the binary operations $\vee$, $\wedge$ induced by the partial order on E, and with the orthosupplementation $'$ is a Boolean algebra admits a unique sequential product $a \circ b = a \wedge b$.*

LEMMA 5.1.11 ([**31**, Lemma 5.2]). *Let E be a sequential effect algebra.*

(1) *If $a \in E$ is an atom, then $a \leq b$ or $a \leq b'$ for every $b \in E$.*
(2) *If $a, b \in E$ are distinct atoms, then $a \perp b$.*

THEOREM 5.1.12 ([**31**, Theorem 5.3]). *An atomistic orthoalgebra admits a sequential product if and only if it is Boolean.*

5.2. Compressible effect algebras

Let us present now the basic facts about retractions, compressions and compressible effect algebras. We omit the proofs, which can be found, along with many other results, in [**28**].

DEFINITION 5.2.1. Let E be an effect algebra and $J : E \to E$ be an additive map. If $a \leq J(\mathbf{1})$ implies $J(a) = a$, then J is a *retraction*. In this case, $J(\mathbf{1})$ is the *focus* of J. An element $p \in E$ is a *projection* if it is the focus of a retraction.

Let us remark, anticipating some terminology that will be defined shortly, that usually, the term "projection" is used for retraction foci in compressible or compression base effect algebras. Since we shall work with retraction foci also in arbitrary effect algebras, we extend its use to this more general framework.

Let us notice that, being additive, J preserves order, hence $J(a) = a$ implies $a \leq J(\mathbf{1})$.

PROPOSITION 5.2.2. *Let E be an effect algebra and $J : E \to E$ be a retraction. If $a \leq b$, then $J(a) \leq J(b)$ and $J(b \ominus a) = J(b) \ominus J(a)$.*

PROOF. If $a \leq b$, then $a \oplus (b \ominus a) = b$. Since J is additive, it follows that $J(a) \perp J(b \ominus a)$ and $J(b) = J(a) \oplus J(b \ominus a)$. We conclude that $J(a) \leq J(b)$ and $J(b \ominus a) = J(b) \ominus J(a)$. □

For a map $J : E \to E$, we shall denote in the following $\mathrm{Ker}(J) = \{a \in E \colon J(a) = \mathbf{0}\}$.

Proposition 5.2.3 (see [**28**, Lemma 3.1, Lemma 3.2]). *Let E be an effect algebra and $J : E \to E$ be a retraction with focus p. Then:*

(1) *J is idempotent;*
(2) $[\mathbf{0}, p'] \subseteq \mathrm{Ker}(J)$*;*
(3) *p is principal and therefore sharp;*
(4) *$p \le a$ implies $J(a) = p$;*
(5) $J(E) = [\mathbf{0}, p]$.

Definition 5.2.4. Let E be an effect algebra and $J : E \to E$ be a retraction. If $J(a) = \mathbf{0}$ implies $a \le J(\mathbf{1})'$, then J is a *compression*.

Proposition 5.2.5 ([**28**, Lemma 3.3]). *Let E be an effect algebra and $J : E \to E$ be a retraction with focus p. The following statements are equivalent:*

(1) *J is a compression;*
(2) *$J(a) = p$ implies $p \le a$;*
(3) $\mathrm{Ker}(J) = [\mathbf{0}, p']$.

Definition 5.2.6. Let E be an effect algebra and $I, J : E \to E$ be retractions. Then J is *direct* if $J(a) \le a$ for every $a \in E$. Retractions I and J are *supplementary* if $\mathrm{Ker}(I) = J(E)$ and $\mathrm{Ker}(J) = I(E)$.

Theorem 5.2.7 (see [**28**, Theorem 3.1]). *Let E be an effect algebra and $J : E \to E$ be a retraction with focus p.*

(1) *If J has a supplement I, then I and J are compressions and p' is the focus of I.*
(2) *If J is direct, then J has a supplement and therefore it is a compression.*

Definition 5.2.8. An effect algebra E is *compressible* if every retraction on E is uniquely determined by its focus and has a supplement.

Let E be a compressible effect algebra. According to Theorem 5.2.7, every retraction on E is a compression. The set of projections in E will be denoted by $P(E)$ or just P, when no confusion is possible. For a projection $p \in P$, J_p will denote the unique compression on E with focus p. By the same Theorem 5.2.7, p' is also a projection and $J_{p'}$ is the supplement of J_p.

Remark 5.2.9. If E is a sequential effect algebra, the sequential product with a sharp (and therefore principal) element $p \in E_S$ defines a compression with focus p by $J_p(a) = p \circ a$ [**29**]. If, moreover, E is compressible, then $J_p : E \to E$, $J_p(a) = p \circ a$ is the unique compression on E with focus p. The close relation between sequential and compressible effect algebras becomes now evident.

In view of the above Remark 5.2.9, in a compressible effect algebra E we will use the following *notation*: instead of $J_p(a)$ we will write $p \circ a$. However, it should be remembered that $p \circ a$ only makes sense (unlike in a sequential effect algebra) if p is a projection (i.e., $p \in P(E)$). Moreover, for $p, q \in P(E)$, we will write $p \mid q$ for $p \circ q = q \circ p$ which means of course that $J_p(q) = J_q(p)$. Also, we will denote $C(p) = \{a \in E\colon\ a = J_p(a) \oplus J_{p'}(a)\}$, for every $p \in P(E)$.

The following lemma summarizes the basic properties of compressions that were given in the beginning of this section, using the new notation " $\circ$ ". If E is a compressible effect algebra and P the set of its compressions, we can also regard " $\circ$ " as a partial binary operation $\circ : P \times E \to E$ whose properties are given below.

Lemma 5.2.10 (see [**28**, Lemmas 3.1–3.3 and 4.1]). *Let E be an effect algebra, J a compression on E with the focus p and let us denote $p \circ a = J(a)$ for every $a \in E$. Then, for every $a, b \in E$:*

(1) *p, p' are principal and hence sharp;*
(2) *$p \circ (a \oplus b) = (p \circ a) \oplus (p \circ b)$;*
(3) *$p \circ a \leq p \circ b$ whenever $a \leq b$;*
(4) *$p \circ \mathbf{0} = \mathbf{0}, p \circ \mathbf{1} = p$;*
(5) *$p \circ a = a$ if $a \leq p$;*
(6) *$p \circ a \leq p$; $p \circ a = p$ if and only if $p \leq a$;*
(7) *$p \circ a = \mathbf{0}$ if and only if $p \perp a$ ($a \leq p'$).*

Remark 5.2.11. In a compressible sequential effect algebra, the partial binary operation $\circ : P \times E \to E$ defined above is a restriction of the sequential product $\circ : E \times E \to E$. However, not every compressible effect algebra can become a sequential effect algebra by extending the product $\circ : P \times E \to E$ to a sequential product on E (see [**28**, Section 5]).

Proposition 5.2.12 (see [**28**, Corollary 4.4, Corollary 4.5]). *Let E be a compressible effect algebra with P the set of projections and $p, q \in P$.*

(1) *If $p \perp q$, then $p \oplus q = p \vee q = (p' \circ q')'$.*
(2) *P is a sub-effect algebra of E that is an orthomodular poset.*

Theorem 5.2.13 (see [**28**, Theorem 4.2, Corollary 4.3]). *Let E be a compressible effect algebra and $p, q \in P$. The following statements are equivalent:*

(1) *$p \circ q = q \circ p$;*
(2) *$p \in C(q)$;*
(3) *$p \circ q \in P$;*
(4) *p and q coexist.*

If any of these conditions hold, then $p \wedge q = p \circ q = q \circ p$ is the greatest lower bound of p and q in both E and P.

Theorem 5.2.14 (see [**28**, Lemma 4.2, Lemma 4.3]). *Let E be a compressible effect algebra and p a projection. Then:*

(1) *$C(p) = J_p(E) \oplus \mathrm{Ker}(J_p) = [\mathbf{0}, p] \oplus [\mathbf{0}, p']$;*

(2) *J_p is a direct retraction if and only if $C(p) = E$;*
(3) *$C(p) = E$ if and only if $p \in \widetilde{C}(E)$.*

Proof. Since the first two assertions are proved in the cited article, let us only prove the last one.

If $C(p) = E$, then $a = J_p(a) \oplus J_{p'}(a)$, for every $a \in E$. According to Lemma 5.2.10, p, p' are principal, $J_p(a) \leq p$ and $J_{p'}(a) \leq p'$, hence p is a central element of E.

Conversely, let us suppose that p is a central element of E. For every $a \in E$ there are $a_1 \leq p$, $a_2 \leq p'$ such that $a = a_1 \oplus a_2$. Hence $J_p(a) = J_p(a_1 \oplus a_2) = J_p(a_1) \oplus J_p(a_2) = a_1 \oplus \mathbf{0} = a_1$ and similarly $J_{p'}(a) = a_2$. Thus $a = a_1 \oplus a_2 = J_p(a) \oplus J_{p'}(a)$ and it follows that $a \in C(p)$. □

5.3. Compression bases in effect algebras

Effect algebras with compression bases are a common generalization of compressible and sequential effect algebras. They prove to be very useful, because they allow us to work with a well structured set of compressions on almost any effect algebra, not only on the ones in the rather restrictive categories that we mentioned (compressible or sequential). Our presentation of compression base effect algebras is based on [**29, 54**].

Definition 5.3.1. Let E be an effect algebra. A sub-effect algebra F of E is *normal* if, for every $a, b, c \in E$ such that $a \oplus b \oplus c$ exists in E and $a \oplus b, b \oplus c \in F$, it follows that $b \in F$.

Remark 5.3.2. Let us remark that the definition property of a normal sub-effect algebra F of an effect algebra E implies that if two elements of F coexist in E, they coexist in F as well ([**29**, Lemma 3.1]).

Definition 5.3.3. Let E be an effect algebra. A system $(J_p)_{p \in P}$ of compressions on E indexed by a normal sub-effect algebra P of E is called a *compression base* for E if the following conditions hold:

(1) Each compression J_p has the focus p.
(2) If $p, q, r \in P$ and $p \oplus q \oplus r$ is defined in E, then $J_{p \oplus r} \circ J_{r \oplus q} = J_r$.

Let us remark here the obvious fact that every effect algebra has a trivial compression base $\{J_{\mathbf{0}}, J_{\mathbf{1}}\}$ where $J_{\mathbf{0}}(a) = \mathbf{0}$, $J_{\mathbf{1}}(a) = a$ for every $a \in E$.

It is easy to see that if $\mathcal{J}_1$ and $\mathcal{J}_2$ are compression bases for E, then $\mathcal{J}_1 \cap \mathcal{J}_2$ is also a compression base for E. If $\{\mathcal{J}_\alpha\}$ is a chain of compression bases for E then $\bigcup_\alpha \mathcal{J}_\alpha$ is also a compression base for E. As a consequence, according to Zorn lemma, every effect algebra has a maximal compression base. If J_p and $J_{p'}$ are compressions, they are contained in a maximal compression base.

EXAMPLE 5.3.4. (1) Let us present the *canonical* compression base for the Hilbert space effect algebra $\mathcal{E}(H)$. The set of sharp elements of this effect algebra is $\mathcal{P}(H)$, the set of projection operators on H. For every $P \in \mathcal{P}(H)$, let us define $J_P : \mathcal{E}(H) \to \mathcal{E}(H)$ by $J_P(A) = PAP$ for every $A \in \mathcal{E}(H)$. Then $(J_P)_{P\in\mathcal{P}(H)}$ is a compression base for $\mathcal{E}(H)$. Clearly, the focus of each compression J_P is P, therefore the set of projections (in the sense of foci of compressions) of $\mathcal{E}(H)$ is just $\mathcal{P}(H)$. One can notice that the compressions in this compression base are just the ones derived from the sequential product on $\mathcal{E}(H)$, by $J_P(A) = P \circ A = PAP$.

(2) In every effect algebra E the center $\widetilde{C}(E)$ is a normal sub-effect algebra and $(J_p)_{p\in\widetilde{C}(E)}$ with $J_p(a) = p \wedge a$ is a compression base (see [**54**, Example 3.4]).

THEOREM 5.3.5 ([**29**, Theorems 3.3 and 3.4]). (1) *If E is a compressible effect algebra, then the set $P(E)$ of its projections is a normal sub-effect algebra of E and $(J_p)_{p\in P(E)}$ is a compression base for E.*

(2) *If E is a sequential effect algebra, then the set E_S of its sharp elements is a normal sub-effect algebra of E. If, for every $p \in E_S$, J_p is the compression on E defined by $J_p(a) = p \circ a$, for every $a \in E$, then $(J_p)_{p\in E_S}$ is a maximal compression base for E.*

For an effect algebra E with a compression base $(J_p)_{p\in P}$ we will maintain, from now on, the notations introduced in the previous section, namely:

- $p \circ a = J_p(a)$ for every $p \in P$ and $a \in E$;
- $p \mid q$ if $p, q \in P$ and $p \circ q = q \circ p$ (i.e., $J_p(q) = J_q(p)$);
- $C(p) = \{a \in E \colon a = J_p(a) \oplus J_{p'}(a)\}$ for every $p \in P$.

The properties of compressible effect algebras, as stated in Proposition 5.2.12 and Theorems 5.2.13, 5.2.14 are generalizable to compression base effect algebras almost without modifications. More precisely, the following results hold:

PROPOSITION 5.3.6 (see [**29**, Lemma 3.5, Theorem 3.7]). *Let $(J_p)_{p\in P}$ be a compression base for the effect algebra E and let $p, q \in P$.*

(1) *If $p \perp q$, then $p \oplus q = p \vee q = (p' \circ q')'$.*
(2) *P is an orthomodular poset.*
(3) *$J_{p'}$ is a supplement of J_p for every $p \in P$.*

THEOREM 5.3.7 (see [**29**, see Theorem 4.2, Corollary 4.3]). *Let E be an effect algebra with a compression base $(J_p)_{p\in P}$ and $p, q \in P$. The following statements are equivalent:*

(1) *$p \circ q = q \circ p$;*

(2) *$p \in C(q)$;*
(3) *$p \circ q \in P$;*
(4) *p and q coexist.*

If any of these conditions hold, then $p \wedge q = p \circ q = q \circ p$ is the greatest lower bound of p and q in both E and P.

THEOREM 5.3.8 (see [**29**, Lemma 4.1] and proof of Theorem 5.2.14). *Let E be an effect algebra with a compression base $(J_p)_{p\in P}$ and let $p \in P$. Then:*

(1) *$C(p) = J_p(E) \oplus \mathrm{Ker}(J_p) = [\mathbf{0}, p] \oplus [\mathbf{0}, p']$;*
(2) *J_p is a direct retraction if and only if $C(p) = E$;*
(3) *$C(p) = E$ if and only if $p \in \widetilde{C}(E)$.*

We shall need the following result.

THEOREM 5.3.9 ([**29**, Theorem 3.6]). *Let E be an effect algebra with a compression base $(J_p)_{p\in P}$. For every $p, q \in P$, the following statements are equivalent:*

(1) *$p \leq q$;*
(2) *$J_q \circ J_p = J_p$;*
(3) *$q \circ p = p$;*
(4) *$J_p \circ J_q = J_p$;*
(5) *$p \circ q = p$.*

In [**54**], Pulmannová studied many properties of compression base effect algebras. We shall reproduce here two results that are particularly important for our work. As we already know from Proposition 5.3.6, if $(J_p)_{p\in P}$ is a compression base for the effect algebra E, then P is an orthomodular poset. However, this result admits the following improvement:

PROPOSITION 5.3.10 (see [**54**, Theorem 4.5, Corollary 4.2]). *Let E be an effect algebra and $(J_p)_{p\in P}$ be a compression base for E.*

(1) *If $p, q \in P$ and $p \leftrightarrow q$, then $C(p) \cap C(q) \subseteq C(p \wedge q) \cap C(p \vee q)$.*
(2) *If $p, q, r \in P$ are pairwise compatible, then $p \leftrightarrow r \wedge q$ and $p \leftrightarrow r \vee q$, hence P is a regular orthomodular poset.*

REMARK 5.3.11. Let E be an effect algebra and $(J_p)_{p\in P}$ be a compression base for E. In view of Lemma 4.2.11 and of Theorem 5.3.8 (1), if $a \in E$, $p \in P$, then $a \in C(p)$ if and only if $a \leftrightarrow p$. Therefore the result of Proposition 5.3.10 (1) can be restated as follows: if $p, q \in P$, $a \in E$ and $p \leftrightarrow q$, then $a \leftrightarrow \{p, q\}$ implies $a \leftrightarrow \{p \wedge q, p \vee q\}$.

The second result from [**54**] that we are interested in involves the projection cover property for compression base effect algebras.

DEFINITION 5.3.12. A compression base $(J_p)_{p\in P}$ on the effect algebra E has the *projection cover property* if for every element $a \in E$

there exists the least element $b \in P$ (the *projection cover* of a) with $b \geq a$.

THEOREM 5.3.13 (see [**54**, Theorem 5.1]). *Let E be an effect algebra with a compression base $(J_p)_{p\in P}$ that has the projection cover property. Then P is an orthomodular lattice.*

CHAPTER 6

Spectral automorphisms in CB-effect algebras

In the third chapter we have introduced spectral automorphisms (see also [**36**]). They resulted from our attempt to construct, in the abstract framework of orthomodular lattices, an analogue of the spectral theory in Hilbert spaces.

Since compression base effect algebras are currently considered as the appropriate mathematical structures for representing physical systems, including observables, states and symmetries [**21**], it is only natural that we pursue the goal of generalizing spectral automorphisms, along with most of their properties, to the framework of compression base effect algebras. We obtain characterizations of spectral automorphisms as well as necessary conditions for an automorphism to be spectral. In order to evaluate how well our theory performs in practice, we apply it to an example of a spectral automorphism on the standard effect algebra of a finite-dimensional Hilbert space and we show the consequences of spectrality of an automorphism for the unitary Hilbert space operator that generates it.

The last section is devoted to spectral families of automorphisms and their properties. An attempt to clarify the connection between spectral automorphisms and the classical discussion of unitary time evolution of a system as one parameter family of automorphisms on the associated logic of the system was made in the third chapter (see also [**11**]). We take this attempt to effect algebras with compression bases. An effect algebra version of the Stone-type theorem in [**11**] is obtained.

The results presented here are accepted for publication in [**8**].

6.1. Spectral automorphisms: the idea and definitions

Before defining spectral automorphisms in the context of compression base effect algebras, let us see what are the facts, in the framework of standard Hilbert space effect algebra, which suggest this notion. Let H be a Hilbert space and $\mathcal{E}(H)$ the corresponding standard effect algebra. Automorphisms of $\mathcal{E}(H)$ are of the form $\varphi_U : \mathcal{E}(H) \to \mathcal{E}(H)$, $\varphi_U(A) = UAU^{-1}$, where U is a unitary or antiunitary Hilbert space operator [**21**]. An element $A \in \mathcal{E}(H)$ is φ_U-invariant if and only if $\varphi_U(A) = UAU^{-1} = A$, i.e., operators U and A commute. Let B_U be the Boolean algebra of projection operators that is the image of the projection-valued spectral measure associated to U. Then, operators

A and U commutes if and only if A commutes with B_U (i.e., with every projection operator in B_U) [**34**]. We are therefore led to the following definition of spectral automorphisms in compression base effect algebras:

Definition 6.1.1. Let E be an effect algebra and $(J_p)_{p\in P}$ be a compression base for E. An automorphism $\varphi : E \to E$ is *spectral* if there exists a Boolean subalgebra B of P with the property:

(P1) $$\varphi(a) = a \text{ if and only if } a \leftrightarrow B$$

Before we can formulate, as it would be expected, the definition of the spectrum of a spectral automorphism as the greatest Boolean subalgebra of P fulfilling (P1), some more work is required. Indeed, it is not clear at all if such a Boolean algebra exists. In order to prove that it does, we will make use of a number of well known properties of compatibility in orthomodular posets. For complete details, we refer to [**53**, Section 1.3.].

Recall that in orthomodular lattices, every subset of pairwise compatible elements is a subset of a Boolean subalgebra of the lattice. However, this is not the case in orthomodular posets, unless they satisfy the regularity property, according to Proposition 1.3.20. It is therefore fortunate that we can take advantage of the result in Proposition 5.3.10, which asserts that the set of projections (i.e., compression foci) of a compression base effect algebra is a regular orthomodular poset.

Although it might be considered as a known fact, the content of the next lemma is tailored to suit our needs, as it will be used several times throughout this chapter. The construction that is used in its proof is inspired from [**53**, Proposition 1.3.23].

Lemma 6.1.2. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and $M \subseteq P$ be a set of pairwise compatible elements. Then there exists the smallest Boolean subalgebra B of P such that $M \subseteq B$. Moreover, for this Boolean subalgebra B, $K(M) = K(B)$ holds.*

Proof. According to Propositions 1.3.20 and 5.3.10, there exists a Boolean subalgebra of P which includes M. The smallest Boolean subalgebra of P which includes M is then just the set-theoretical intersection of all such Boolean algebras.

Let B be the smallest Boolean subalgebra of P which includes M and $a \in E$. For the rest of the proof, let us keep in mind that, according to Remark 4.2.10, coexistence, for elements of P, is the same thing as compatibility, as defined in orthomodular posets. Clearly, if $a \leftrightarrow B$, then $a \leftrightarrow M$, since $M \subseteq B$. Let us prove the converse. Let $Q \subseteq M$ be a finite set, and let us denote $Q' = Q \cup \{q' \in E\colon\ q \in Q\}$. Since $Q \subseteq P$ and P is a sub-effect algebra of E, it follows that $Q' \subseteq P$. By Corollary 4.2.12, elements of Q' are pairwise compatible. Let F_Q denote the set of infima of all subsets of Q' and let $B(Q)$ be the set of suprema of all subsets of F_Q. Furthermore, let $\widetilde{B} = \bigcup\{B(Q) \colon Q \subseteq M,\ Q \text{ finite}\}$. We omit here the routine verification of the fact that $\widetilde{B}$ is a Boolean algebra, which can be found in [**53**, Proposition 1.3.23]. Since $\widetilde{B}$ is a Boolean algebra that includes M, and every Boolean algebra that includes M must include $\widetilde{B}$ (obviously), it follows that $B = \widetilde{B}$. Since, for every $Q \subseteq M$, Q finite, we have $B(Q) \subseteq B$ and B is a Boolean algebra, it follows that the elements of $B(Q)$ are pairwise compatible (and so are the elements of F_Q and Q') for every such Q. If $a \leftrightarrow M$, then $a \leftrightarrow Q$ and by Corollary 4.2.12, $a \leftrightarrow Q'$. Since elements of Q' are pairwise compatible, it follows from Proposition 5.3.10 that $a \leftrightarrow F_Q$. Since elements of F_Q are pairwise compatible, it follows, again by Proposition 5.3.10, that $a \leftrightarrow B(Q)$, for every $Q \subseteq M$, Q finite. It follows that $a \leftrightarrow B$, which concludes our proof. □

Let us now state the result that will allow us to define the spectrum of a spectral automorphism.

PROPOSITION 6.1.3. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and $\varphi : E \to E$ be a spectral automorphism. There exists the greatest Boolean subalgebra $B \subseteq P$ satisfying* (P1).

PROOF. Let $\{B_i \colon i \in I\}$ be the set of all Boolean subalgebras of P satisfying (P1). We will prove that $\bigcup_{i\in I} B_i$ is a set of pairwise compatible elements of P. Indeed, for every $i, j \in I$, from $a \in B_i$ it follows that $a \leftrightarrow B_i$ and, due to (P1), $\varphi(a) = a$. Applying (P1) again for B_j, we conclude that $a \leftrightarrow B_j$. Hence $B_i \leftrightarrow B_j$ for every $i, j \in I$ and every pair of elements of $\bigcup_{i\in I} B_i$ is compatible. According to Lemma 6.1.2, there exists the smallest Boolean subalgebra B of P containing $\bigcup_{i\in I} B_i$.

We will now prove that B satisfies (P1), hence being the greatest Boolean subalgebra of P with this property. Clearly, if $a \leftrightarrow B$, then $a \leftrightarrow B_i$ for every $i \in I$, hence $\varphi(a) = a$. Conversely, let us assume $\varphi(a) = a$. It follows that $a \leftrightarrow B_i$ for every $i \in I$, hence $a \leftrightarrow \bigcup_{i\in I} B_i$. In view of the second assertion of Lemma 6.1.2, we conclude that $a \leftrightarrow B$. □

DEFINITION 6.1.4. Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and $\varphi : E \to E$ be a spectral automorphism. The greatest Boolean subalgebra of P fulfilling (P1) is the *spectrum* of the automorphism φ, denoted by σ_φ^P.

As we already mentioned, spectral automorphisms in effect algebras with compression base, as defined in this section, are intended as generalizations of spectral automorphisms in orthomodular lattices, as defined in the first part of this thesis. However, this is not a generalization in the strict sense of the term. Let us explain more precisely what we mean.

Let us consider E an effect algebra endowed with a compression base $(J_p)_{p\in P}$. Assume that E is an orthomodular lattice (with the effect algebra order and orthosupplementation defined on E). It is then a natural question if in this case, the notions of spectral automorphism and its spectrum, as given in Definitions 3.2.1, 3.2.4, coincide with the corresponding notions from Definitions 6.1.1,6.1.4. The answer is positive in what concerns spectrality, as can be easily verified. Indeed, if $\varphi : E \to E$ a spectral automorphism, in the sense of Definition 6.1.1, then it is also a spectral automorphism of the orthomodular lattice E, in the sense of Definition 3.2.1, since σ_φ^P is a spectral algebra for φ. However, its spectrum σ_φ, as defined in 3.2.4 is not necessarily the same as its spectrum σ_φ^P in the sense of the above Definition 6.1.4, depending on the set P of projections. Indeed, as we shall prove in the following section (Theorem 6.2.7), $\sigma_\varphi^P = \widetilde{C}(E_\varphi)\cap P$, while according to Proposition 3.2.10, $\sigma_\varphi = \widetilde{C}(E_\varphi)$. Therefore, $\sigma_\varphi^P \subseteq \sigma_\varphi$ and the inclusion is strict, unless $\widetilde{C}(E_\varphi) \subseteq P$.

An interesting special case to consider is when the effect algebra E is, in addition to being an orthomodular lattice, a sequential effect algebra. Then, according to Theorem 5.3.5 (2), there exists a compression base $(J_p)_{p\in P}$ of E such that $P = E$. In this case, we have $\sigma_\varphi^P = \sigma_\varphi$.

Proposition 6.1.5. *Let E be an effect algebra and $(J_p)_{p\in P}$ be a compression base for E. If $P \subseteq \widetilde{C}(E)$, then the identity is the only spectral automorphism of E.*

Proof. Let $\varphi : E \to E$ be a spectral automorphism. Then $\sigma_\varphi^P \subseteq P \subseteq \widetilde{C}(E)$. It follows that $a \leftrightarrow \sigma_\varphi^P$ for every $a \in E$ and, due to (P1), every element of E is φ-invariant. $\square$

Remark 6.1.6. As a particular case, if E is a Boolean algebra, then its identity is its only spectral automorphism. Therefore, the presence of nontrivial spectral automorphisms allows us to distinguish between classical (Boolean) and nonclassical theories.

Before ending this section, let us discuss an example of a spectral automorphism.

Example 6.1.7. Consider H a 3-dimensional complex Hilbert space, $\mathcal{E}(H)$ the corresponding standard effect algebra and $\mathcal{P}(H)$ the set of its projection operators or, equivalently, the set of its subspaces. Let $\mathcal{E}(H)$ be endowed with its canonical compression base $(J_P)_{P\in\mathcal{P}(H)}$. Let $P \in \mathcal{P}(H)$ be a 1-dimensional projector and denote S_P the corresponding subspace (i.e., its range). Then P' is its orthogonal complement and define $U : H \to H$ as the symmetry of H with respect to the plane $S_{P'}$ corresponding to P'. Clearly U is a unitary operator, hence $\varphi : \mathcal{E}(H) \to \mathcal{E}(H)$ defined by $\varphi(A) = UAU^{-1}$ is an automorphism of $\mathcal{E}(H)$. We assert that $\mathcal{B} = \{\mathbf{0}, P, P', \mathbf{1}\}$ is a Boolean subalgebra of $\mathcal{P}(H)$ satisfying (P1). To prove this assertion, let us notice that $A \in \mathcal{E}(H)$ is φ-invariant if and only if A commutes with U (as operators). On the other hand, $A \leftrightarrow \mathcal{B}$ if and only if $A \leftrightarrow P$ and it is well known, since P is a projection operator, that this is equivalent to the fact that A and P commute as operators (see, e.g., [**41**]). However, by a classic Hilbert space theory result [**34**, Section 27, Theorem 2], A and P commutes if and only if the range of P reduces A (i.e., S_P and $S_{P'}$ are invariant under A). To complete our proof, we only need to show that A and U commute if and only if S_P and $S_{P'}$ are invariant under A. In order to do that, we need to remark that, considering the definition of U, $x \in S_P$ if and only if $Ux = -x$ and $x' \in S_{P'}$ if and only if $Ux' = x'$. Now, let us assume that $AU = UA$. If $x \in S_P$ then $UAx = AUx = -Ax$ and therefore $Ax \in S_P$. Also, if $x' \in S_{P'}$ then $UAx' = AUx' = Ax'$ and therefore $Ax' \in S_{P'}$. Conversely, let us assume that $A(S_P) \subseteq S_P$ and $A(S_{P'}) \subseteq S_{P'}$. For an arbitrary $y \in H$, there exist $x \in S_P$ and $x' \in S_{P'}$ such that $y = x + x'$. Then $AUy = A(Ux+Ux') = A(-x+x') = -Ax+Ax' = UAx+UAx' = UAy$, which proves the commutativity. We conclude that φ is a spectral automorphism.

6.2. Characterizations and properties of spectral automorphisms

For an automorphism φ of an effect algebra E, we will denote by E_φ the set of φ-invariant elements of E. Due to the definition properties of automorphisms, it is clear that E_φ is a sub-effect algebra of E.

The fact that the spectrum σ_φ^P of a spectral automorphism $\varphi : E \to E$ fulfills (P1) can be written equivalently in the useful form of the equality $E_\varphi = K(\sigma_\varphi^P)$.

Let E be an effect algebra and $M, N \subseteq E$. The following properties of commutants can be easily verified: (1) $M \subseteq K(K(M))$ and (2) if $M \subseteq N$ then $K(N) \subseteq K(M)$.

PROPOSITION 6.2.1. *Let E be an effect algebra and $(J_p)_{p\in P}$ be a compression base for E. If $M \subseteq P$ then the commutant $K(M)$ of M is a sub-effect algebra of E.*

PROOF. Clearly, $\mathbf{0}, \mathbf{1} \in K(M)$. According to Corollary 4.2.12, $a \leftrightarrow b$ if and only if $a' \leftrightarrow b$ for every $a, b \in E$. Hence $a \in K(M)$ if and only if $a' \in K(M)$. It remains to prove that, for every orthogonal pair of elements $a, b \in E$, if $a, b \leftrightarrow M$, then $a \oplus b \leftrightarrow M$. Towards this end, we will use the characterization of coexistence given in Lemma 4.2.11. Let $c \in M$. Since $a, b \leftrightarrow c$, there exist $a_1, b_1 \leq c$ and $a_2, b_2 \leq c'$ such that $a = a_1 \oplus a_2$ and $b = b_1 \oplus b_2$. On the other hand, since $a \perp b$, it follows, using Proposition 4.1.4 (3), that $a_1, a_2 \leq a \leq b' \leq b_1', b_2'$ and therefore $a_1 \perp b_1$, $a_2 \perp b_2$. It follows that $a_1 \oplus b_1 \leq c$ and $a_2 \oplus b_2 \leq c'$, since c, c'

are in P and therefore, according to Lemma 5.2.10, they are principal elements of E. Taking into account that $a \oplus b = (a_1 \oplus b_1) \oplus (a_2 \oplus b_2)$, the desired conclusion that $a \oplus b \leftrightarrow c$ follows. $\square$

LEMMA 6.2.2. *Let E be an effect algebra, $(J_p)_{p \in P}$ be a compression base for E and F be a sub-effect algebra of E. Then $\widetilde{C}(F) \cap P$ is a Boolean subalgebra of P.*

PROOF. First, let us remark that since F is a sub-effect algebra of E, it makes sense to speak about its center $\widetilde{C}(F)$, which is even a Boolean algebra, according to Theorem 4.2.14. It is then clear that $\widetilde{C}(F) \cap P$ is a sub-effect algebra of E (and of P as well). All its elements are principal in E, according to Lemma 5.2.10, and therefore in all sub-effect algebras of E, in view of Remark 4.3.3 (1). It follows from Remark 4.2.15 that $\widetilde{C}(F) \cap P$ is an orthomodular poset. To prove it is a Boolean algebra, we need to show that its elements are pairwise coexistent. Let $a, b \in \widetilde{C}(F) \cap P$. Since $\widetilde{C}(F)$ is a Boolean algebra, a and b coexist in $\widetilde{C}(F)$. Let $a = a_1 \oplus c$, $b = b_1 \oplus c$ be a Mackey decomposition of a, b in $\widetilde{C}(F)$. Then $a_1 \oplus c \oplus b_1$ exists in $\widetilde{C}(F)$, hence in E, $a = a_1 \oplus c \in P$ and $b = b_1 \oplus c \in P$. Since P is a normal sub-effect algebra of E, it follows that $c \in P$. Since $a, b, c \in P$ and $a_1 = a \ominus c$, $b_1 = b \ominus c$, it follows, by Proposition 4.1.8, that $a_1, b_1 \in P$. This proves that a_1, b_1, c is a Mackey decomposition of a, b in $\widetilde{C}(F) \cap P$, hence a, b coexist in $\widetilde{C}(F) \cap P$, which concludes our proof. $\square$

COROLLARY 6.2.3. *Let E be an effect algebra, $(J_p)_{p \in P}$ be a compression base for E and $\varphi : E \to E$ be an automorphism. Then $\widetilde{C}(E_\varphi) \cap P$ is a Boolean subalgebra of P.*

PROOF. Follows from Lemma 6.2.2 and the fact that E_φ is a sub-effect algebra of E. $\square$

The following lemma and corollary, that will be useful in the sequel, are related to [**26**, Theorem 4.2 and Lemma 5.2]. However, the statements we prove are slightly more general and could be interesting in their own right.

LEMMA 6.2.4. *Let E be an effect algebra, $\{e_1, e_2, \ldots, e_n\}$ be an orthogonal set of its elements (i.e., the sum $\bigoplus_{i=1}^n e_i$ exists) and consider $p \in E$ such that $p = \bigoplus_{i=1}^n p_i$ with $p_i \leq e_i$. If e_j is principal for some $j \in \{1, 2, \ldots, n\}$, then $p \wedge e_j$ exists in E and $p_j = p \wedge e_j$.*

PROOF. Let us assume e_j is principal. Clearly $p_j \leq e_j, p$, and for an arbitrary $x \in E$, $x \leq e_j, p$, we have to prove that $x \leq p_j$. Let us denote by $q_i = e_i \ominus p_i$ for all $i \in \{1, 2, \ldots, n\}$. Then $\bigoplus_{i=1}^n e_i = \bigoplus_{i=1}^n (p_i \oplus q_i) = \left(\bigoplus_{i=1}^n p_i\right) \oplus \left(\bigoplus_{i=1}^n q_i\right) = p \oplus q$, where $q = \bigoplus_{i=1}^n q_i$. It follows that $q_j \leq q \leq p' \leq x'$, hence $x \perp q_j$. Since $x, q_j \leq e_j$ and e_j

is principal, it results that $x \oplus q_j \leq e_j = p_j \oplus q_j$ and therefore, by the cancellation law, $x \leq p_j$. $\square$

COROLLARY 6.2.5. *If a, a' are principal elements of the effect algebra E, $b \in E$ and $a \leftrightarrow b$, then $a \wedge b$ and $a' \wedge b$ exist in E and $b = (a \wedge b) \oplus (a' \wedge b)$.*

PROOF. According to Lemma 4.2.11, $a \leftrightarrow b$ if and only if there exist $b_1, b_2 \in E$ such that $b_1 \leq a$, $b_2 \leq a'$ and $b = b_1 \oplus b_2$. Since $\{a, a'\}$ form an orthogonal set of elements of E and a, a' are principal, it follows from Lemma 6.2.4 that $a \wedge b$ and $a' \wedge b$ exist in E and $b_1 = a \wedge b$, $b_2 = a' \wedge b$. Thus, $b = b_1 \oplus b_2 = (a \wedge b) \oplus (a' \wedge b)$. $\square$

LEMMA 6.2.6. *Let E be an effect algebra, $(J_p)_{p \in P}$ be a compression base for E and $\varphi : E \to E$ be an automorphism. If a, a' are principal in E, $a, b \in E_\varphi$ and a, b coexist in E, then they coexist in E_φ as well.*

PROOF. According to Corollary 6.2.5, $a \wedge b$ and $a' \wedge b$ exist in E and $b = (a \wedge b) \oplus (a' \wedge b)$. Since $a, a', b \in E_\varphi$, according to Proposition 4.5.5, $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b) = a \wedge b$ and $\varphi(a' \wedge b) = \varphi(a') \wedge \varphi(b) = a' \wedge b$, and therefore $a \wedge b, a' \wedge b \in E_\varphi$. Considering that $a \wedge b \leq a$ and $a' \wedge b \leq a'$, it follows that a, b coexist in E_φ, according to Lemma 4.2.11. $\square$

The following theorem allows us to find the spectrum of a spectral automorphism.

THEOREM 6.2.7. *Let E be an effect algebra and $(J_p)_{p \in P}$ be a compression base for E. If $\varphi : E \to E$ is a spectral automorphism, then $\sigma_\varphi^P = \widetilde{C}(E_\varphi) \cap P$.*

PROOF. Let $s \in \sigma_\varphi^P$. Since σ_φ^P is a Boolean algebra, s commutes with it and, according to (P1), $s \in E_\varphi$. Since $s \in \sigma_\varphi^P \subseteq P$, s and s' are principal elements of E, according to Lemma 5.2.10, and of E_φ as well, according to Remark 4.3.3 (1). Clearly, $s \leftrightarrow K(\sigma_\varphi^P)$ and using (P1) again, $K(\sigma_\varphi^P) = E_\varphi$, hence $s \leftrightarrow E_\varphi$, i.e., s coexists with every element of E_φ in E. To show that $s \in \widetilde{C}(E_\varphi)$, we need this coexistence to take place in E_φ as well. This happens, according to Lemma 6.2.6. It follows that $s \in \widetilde{C}(E_\varphi) \cap P$, which proves that $\sigma_\varphi^P \subseteq \widetilde{C}(E_\varphi) \cap P$.

To prove the converse inclusion, it suffices to show that $\widetilde{C}(E_\varphi) \cap P$ satisfies (P1), since it is a Boolean subalgebra of P, according to Corollary 6.2.3, and σ_φ^P is the greatest Boolean subalgebra of P with this property. Let $a \in E$ be φ-invariant, i.e., $a \in E_\varphi$. Then $a \leftrightarrow \widetilde{C}(E_\varphi)$, hence $a \leftrightarrow \widetilde{C}(E_\varphi) \cap P$. Conversely, since $\widetilde{C}(E_\varphi) \cap P \supseteq \sigma_\varphi^P$, $a \leftrightarrow \widetilde{C}(E_\varphi) \cap P$ implies $a \leftrightarrow \sigma_\varphi^P$ and, due to the spectrality of φ, this entails that $\varphi(a) = a$. $\square$

COROLLARY 6.2.8. *Let E be an effect algebra, $(J_p)_{p \in P}$ be a compression base for E and $\varphi : E \to E$ be an automorphism. Then φ is* *spectral if and only if* $K(\widetilde{C}(E_\varphi) \cap P) \subseteq E_\varphi$ *(the converse inclusion is always true).*

Proof. According to Theorem 6.2.7, if φ is spectral, then $\sigma_\varphi^P = \widetilde{C}(E_\varphi) \cap P$, and due to (P1), $K(\sigma_\varphi^P) = E_\varphi$. Conversely, if $K(\widetilde{C}(E_\varphi) \cap P) \subseteq E_\varphi$, then $K(\widetilde{C}(E_\varphi) \cap P) = E_\varphi$ and therefore $\widetilde{C}(E_\varphi) \cap P$ is a Boolean subalgebra of P (according to Corollary 6.2.3) which satisfies (P1), and it follows that φ is spectral. $\square$

The following result characterizes spectral automorphisms in compression base effect algebras.

Theorem 6.2.9. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and $\varphi : E \to E$ be an automorphism. Then φ is spectral if and only if $a \wedge b \in E_\varphi$ for every $a \in \widetilde{C}(E_\varphi) \cap P$, $b \in K(\widetilde{C}(E_\varphi) \cap P)$.*

Proof. "$\Leftarrow$" Let us denote $\widetilde{C}(E_\varphi) \cap P = B$. According to Corollary 6.2.3, B is a Boolean subalgebra of P. Let b be an element of E such that $b \leftrightarrow B$. For every $a \in B$ we have $a' \in B$, a, a' are principal in E, according to Lemma 5.2.10, and $a \leftrightarrow b$, therefore, according to Corollary 6.2.5, $b = (a \wedge b) \oplus (a' \wedge b)$. It then follows from our hypothesis that $a \wedge b, a' \wedge b \in E_\varphi$, hence $b \in E_\varphi$. Conversely, if $b \in E_\varphi$, clearly $b \leftrightarrow \widetilde{C}(E_\varphi) \cap P = B$. It follows that B is a Boolean subalgebra of P satisfying (P1), hence φ is a spectral automorphism.

"$\Rightarrow$" Let us assume φ is spectral and $b \in K(\widetilde{C}(E_\varphi) \cap P)$, $a \in \widetilde{C}(E_\varphi) \cap P$. Then a, a' are principal elements of E, according to Lemma 5.2.10, $a \leftrightarrow b$ and, according to Corollary 6.2.5, the infimum $a \wedge b$ exists in E. Then, according to Proposition 4.5.5, $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b) = a \wedge b$, since $a \in E_\varphi$ and, according to Corollary 6.2.8, $b \in E_\varphi$ as well. It follows that $a \wedge b \in E_\varphi$. $\square$

The search for the conditions that a Boolean algebra must fulfill in order to be the spectrum of a spectral automorphism leads to the following notion.

Definition 6.2.10. Let E be an effect algebra and $(J_p)_{p\in P}$ be a compression base for E. A Boolean subalgebra $B \subseteq P$ is *C-maximal* if $\widetilde{C}(K(B)) \cap P \subseteq B$.

Let us remark that, according to Proposition 6.2.1, $K(B)$ is an effect algebra, therefore its center exists. It is easy to see that, for example, every block (i.e., maximal Boolean subalgebra) of P is C-maximal.

The following results from [**54**] will be used for the proof of our next theorem.

Lemma 6.2.11. (see [**54**, Lemma 4.1, Theorem 4.5, Corollary 4.1]) *Let E be an effect algebra and $(J_p)_{p\in P}$ be a compression base for E.*

(1) *If* $p, q \in P$ *and* $p \leftrightarrow q$*, then, for every* $a \in E$*,* $a \leftrightarrow \{p, q\}$ *implies* $a \leftrightarrow p \wedge q$*.*
(2) *If* $p_1, p_2 \in P$ *are orthogonal and* $a \in E$*,* $a \leftrightarrow \{p_1, p_2\}$*, then* $(p_1 \oplus p_2) \wedge a = (p_1 \wedge a) \oplus (p_2 \wedge a)$*.*

THEOREM 6.2.12. *Let* E *be an effect algebra and* $(J_p)_{p \in P}$ *be a compression base for* E*. A Boolean subalgebra* $B \subseteq P$ *is C-maximal if and only if* $B = K(K(B)) \cap P$*.*

PROOF. "$\Leftarrow$" Let B be a Boolean subalgebra of P such that $B = K(K(B)) \cap P$, and let $a \in \widetilde{C}(K(B)) \cap P$. Then $a \in P$ and $a \leftrightarrow K(B)$, hence $a \in K(K(B)) \cap P = B$. It follows that $\widetilde{C}(K(B)) \cap P \subseteq B$, hence B is C-maximal.

"$\Rightarrow$" Let B be a Boolean subalgebra of P such that $\widetilde{C}(K(B)) \cap P \subseteq B$. The inclusion $B \subseteq K(K(B)) \cap P$ is trivial. Let $a \in K(K(B)) \cap P$. It suffices to prove that $a \in \widetilde{C}(K(B))$. Since $a \leftrightarrow K(B)$ and $B \subseteq K(B)$, also $a \leftrightarrow B$, hence $a \in K(B)$. Since $a, a' \in P$ are principal elements of E, and, according to Remark 4.3.3 (1), also of $K(B)$, we only need to prove that a coexists with every element of $K(B)$ in $K(B)$ too (not just in E). Let $b \in K(B)$. Since $a \in K(K(B))$, $a \leftrightarrow b$ in E and, according to Corollary 6.2.5, $a \wedge b$ and $a' \wedge b$ exist in E and $b = (a \wedge b) \oplus (a' \wedge b)$. To prove $a \leftrightarrow b$ in $K(B)$, it suffices to show that $a \wedge b, a' \wedge b \in K(B)$. We will only prove that $a \wedge b \in K(B)$, the proof for $a' \wedge b$ being analogous. Let $d \in B$. We have to show that $d \leftrightarrow a \wedge b$ in E. Let us remark that, although a, b, d are pairwise coexistent, b need not be in P and therefore we cannot use the regularity of P. We shall, instead, use Lemma 6.2.11 (1). Indeed, $a, d \in P$, $a \leftrightarrow d$ (in E and in P, by Remark 5.3.2, since P is a normal sub-effect algebra of E) and $b \leftrightarrow \{a, d\}$, hence $b \leftrightarrow a \wedge d$. Similarly, $b \leftrightarrow a \wedge d'$. On the other hand, applying Corollary 6.2.5 in P, we find that $a \wedge d$ and $a \wedge d'$ exist in P and $a = (a \wedge d) \oplus (a \wedge d')$. Applying Lemma 6.2.11 (2) with $a \wedge d, a \wedge d'$ as p_1, p_2, we find that $\big((a \wedge d) \oplus (a \wedge d')\big) \wedge b = \big((a \wedge d) \wedge b\big) \oplus \big((a \wedge d') \wedge b\big)$. Therefore, $a \wedge b = \big((a \wedge d) \oplus (a \wedge d')\big) \wedge b = \big((a \wedge b) \wedge d\big) \oplus \big((a \wedge b) \wedge d'\big)$, hence $a \wedge b \leftrightarrow d$, according to Lemma 4.2.11. $\square$

Let us apply the just proved result to the spectrum of a spectral automorphism.

COROLLARY 6.2.13. *Let* E *be an effect algebra,* $(J_p)_{p \in P}$ *be a compression base for* E *and* $\varphi : E \to E$ *be a spectral automorphism. Then:*
(1) $\sigma_\varphi^P = \widetilde{C}(E_\varphi) \cap P$ *is C-maximal;*
(2) $\sigma_\varphi^P = K(K(\sigma_\varphi^P)) \cap P$*;*
(3) $\sigma_\varphi^P = K(E_\varphi) \cap P$*.*

PROOF. (1) According to Theorem 6.2.7, $\sigma_\varphi^P = \widetilde{C}(E_\varphi) \cap P$ and, according to (P1), $E_\varphi = K(\sigma_\varphi^P)$. It results that $\sigma_\varphi^P = \widetilde{C}(K(\sigma_\varphi^P)) \cap P$. Since σ_φ^P is a Boolean subalgebra of P, its C-maximality follows.

(2) It follows directly from (1) and Theorem 6.2.12.

(3) It is a direct result of (2) and the fact that $K(\sigma_\varphi^P) = E_\varphi$, since φ is spectral. $\square$

6.3. An application of spectral automorphisms to $\mathcal{E}(H)$

The notion of spectral automorphism was introduced with the declared intention to obtain an analogue of the Hilbert space spectral theory in the abstract setting of compression base effect algebras. It is time to see if this attempt was succesful, by applying the abstract theory to the particular case of the standard Hilbert space effect algebra. Therefore, we devote this section to the proof of a "spectral theorem" in $\mathcal{E}(H)$, for a finite-dimensional Hilbert space H.

Before we can prove the main result of this section, some preparations are needed. It is, of course, a well known fact that the set $\mathcal{P}(H)$ of projection operators (or equivalently, the set of closed subspaces) of a Hilbert space H forms an atomic complete orthomodular lattice (see, e.g., [**38**, Section 5]). Its atoms are the 1-dimensional subspaces, or the corresponding projectors. Let us denote in the sequel by $\hat{e}$ the 1-dimensional subspace generated by $e \in H$, $\|e\| = 1$ and $P_{\hat{e}}$ the corresponding projection operator, i.e., $P_{\hat{e}} : H \to H$, $P_{\hat{e}}x = \langle x, e\rangle e$ (where $\langle \cdot, \cdot \rangle$ denotes the inner product of H).

LEMMA 6.3.1. *Let H be a Hilbert space. For every automorphism $\varphi : \mathcal{E}(H) \to \mathcal{E}(H)$ defined by $\varphi(A) = UAU^{-1}$, where U is a unitary operator on H, and every atom $P_{\hat{e}} \in \mathcal{P}(H)$, we have $\varphi(P_{\hat{e}}) = P_{\widehat{Ue}}$.*

PROOF. As previously mentioned, for $e \in H$, $\|e\| = 1$, $P_{\hat{e}}$ is defined by $P_{\hat{e}} : H \to H$, $P_{\hat{e}}x = \langle x, e\rangle e$. Since U is unitary, we have $\|Ue\| = 1$ and U^{-1} is also the adjoint of U. Then $\varphi(P_{\hat{e}})x = UP_{\hat{e}}U^{-1}x = U\langle U^{-1}x, e\rangle e = \langle x, Ue\rangle Ue = P_{\widehat{Ue}}x$ for every $x \in H$. $\square$

THEOREM 6.3.2. *Let H be an n-dimensional Hilbert space, $\mathcal{E}(H)$ be its standard effect algebra and $(J_P)_{P \in \mathcal{P}(H)}$ be the canonical compression base for $\mathcal{E}(H)$. Let $U : H \to H$ be a unitary operator and $\varphi : \mathcal{E}(H) \to \mathcal{E}(H)$ be the automorphism defined by $\varphi(A) = UAU^{-1}$. If φ is spectral, then:*

1. *There is an orthogonal basis $\{e_1, e_2, \dots, e_n\}$ of H such that for every $i \in \{1, 2, \dots, n\}$, $Ue_i = \lambda_i e_i$ where λ_i is a scalar, $|\lambda_i| = 1$.*
2. *There exists a partition Π of the set $\{1, 2, \dots, n\}$ such that any φ-invariant atom of $\mathcal{P}(H)$ is a 1-dimensional subspace in exactly one of the subspaces $\bigvee_{j \in J} \hat{e}_j$, $J \in \Pi$.*
3. *If the subalgebra $\mathcal{E}(H)_\varphi$ of φ-invariant elements of $\mathcal{E}(H)$ is Boolean, then the spectrum $\sigma_\varphi^{\mathcal{P}(H)} = \mathcal{E}(H)_\varphi \cap \mathcal{P}(H)$ is a block in $\mathcal{P}(H)$. In this case all eigenvalues of U are distinct and $\Pi = \big\{\{1\}, \{2\}, \dots, \{n\}\big\}$.*

(4) *The spectrum* $\sigma_\varphi^{\mathcal{P}(H)}$ *is the Boolean algebra generated by* $\{\bigvee_{j\in J}\hat{e}_j : J \in \Pi\}$.

(5) *If the effect* $A \in \mathcal{E}(H)$ *is* φ*-invariant and* $P \in \mathcal{P}(H)$ *is the smallest projection that dominates* A *(namely the projection on the range of* A*), then* P *is* φ*-invariant too.*

(6) *If* A *is a* φ*-invariant nonzero effect dominated by an atom of* $\mathcal{P}(H)$*, then the range of* A *is included in one of the subspaces* $\bigvee_{j\in J}\hat{e}_j$, $J \in \Pi$.

PROOF. (1) φ is a spectral automorphism, hence $\mathcal{E}(H)_\varphi = K(\sigma_\varphi^{\mathcal{P}(H)})$. Since $\sigma_\varphi^{\mathcal{P}(H)}$ is a Boolean subalgebra of $\mathcal{P}(H)$, according to Corollary 1.3.6 there exists a block B_0 of $\mathcal{P}(H)$ such that $\sigma_\varphi^{\mathcal{P}(H)} \subseteq B_0$. Obviously $B_0 \subseteq K(\sigma_\varphi^{\mathcal{P}(H)})$ and it follows that $B_0 \subseteq \mathcal{E}(H)_\varphi \cap \mathcal{P}(H)$. According to Theorem 1.4.10, the atoms of B_0 are atoms of $\mathcal{P}(H)$, i.e., 1-dimensional subspaces/projectors of H. Let $\mathcal{B}$ be the set of all atoms of B_0. Since B_0 is Boolean, it follows that its atoms are mutually orthogonal and therefore the corresponding 1-dimensional subspaces and the vectors that generate these subspaces are orthogonal. Since H is n-dimensional, it follows that there are at most n atoms in $\mathcal{B}$. However, since $\bigvee \mathcal{B} = \mathbf{1}$, it follows that there must be exactly n atoms in $\mathcal{B}$. Let $\mathcal{B} = \{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\}$. In view of the previous arguments, it is clear that $\{e_1, e_2, \ldots, e_n\}$ is an orthogonal basis of H. Recall now that $\mathcal{B} = \{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\} \subseteq B_0 \subseteq \mathcal{E}(H)_\varphi$, hence $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n$ are φ-invariant. Then, according to Lemma 6.3.1, for every $i \in \{1, 2, \ldots, n\}$, $\varphi(P_{\hat{e}_i}) = P_{\widehat{Ue_i}} = P_{\hat{e}_i}$. It follows that $\widehat{Ue_i} = \hat{e}_i$, hence $Ue_i = \lambda_i e_i$ for some scalar λ_i (which must be of modulus 1 since U is unitary), for every $i \in \{1, 2, \ldots, n\}$.

(2) Let $\hat{e} \in \mathcal{P}(H)$ be a φ-invariant atom such that $\hat{e} \notin \mathcal{B}$. Since $\{e_1, e_2, \ldots, e_n\}$ is a basis of H, there exists $J \subseteq \{1, 2, \ldots, n\}$ such that $e = \sum_{j\in J} a_j e_j$, with $a_j \neq 0$ for all $j \in J$. Due to the φ-invariance of $\hat{e}$, it follows that there exists a scalar λ such that $Ue = \lambda e$ and since $Ue_j = \lambda_j e_j$ for every $j \in J$, we find that $\sum_{j\in J} a_j(\lambda - \lambda_j)e_j = 0$. We conclude that $\lambda_j = \lambda$ for all $j \in J$, and $\bigvee_{j\in J}\hat{e}_j$ is the corresponding eigenspace. It is now clear that each element J of the partition Π that we are looking for corresponds to a distinct eigenspace of U.

(3) If $\mathcal{E}(H)_\varphi$ is Boolean, according to Theorem 6.2.7, $\sigma_\varphi^{\mathcal{P}(H)} = \mathcal{E}(H)_\varphi \cap \mathcal{P}(H)$. Let $A \in \mathcal{E}(H)$ such that $A \leftrightarrow \mathcal{E}(H)_\varphi \cap \mathcal{P}(H)$. Then $A \leftrightarrow \sigma_\varphi^{\mathcal{P}(H)}$, hence $A \in \mathcal{E}(H)_\varphi$. It follows that $\sigma_\varphi^{\mathcal{P}(H)} = \mathcal{E}(H)_\varphi \cap \mathcal{P}(H)$ is a block of $\mathcal{P}(H)$, hence $\mathcal{B} = \{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\}$ are the only φ-invariant atoms in $\mathcal{P}(H)$. This implies that eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ corresponding to vectors $\{e_1, e_2, \ldots, e_n\}$ are distinct. Indeed, if more than one of these vectors correspond to the same eigenvalue, than any subspace of their corresponding eigenspace is φ-invariant, in contradiction to our previous assertion.

(4) Since $\mathcal{B} = \{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\}$ is the set of all the atoms of B_0 and $\sigma_\varphi^{\mathcal{P}(H)} \subseteq B_0$, it follows that every element, and in particular every atom of $\sigma_\varphi^{\mathcal{P}(H)}$ is a supremum of a subset of $\mathcal{B}$. Considering the fact that $\sigma_\varphi^{\mathcal{P}(H)}$ is a Boolean algebra and therefore its atoms are mutually orthogonal and their supremum is $\mathbf{1}$, we conclude that there exists a partition Π_1 of $\{1, 2, \ldots, n\}$ such that the atoms of $\sigma_\varphi^{\mathcal{P}(H)}$ are $\left\{\bigvee_{i\in I} \hat{e}_i \colon I \in \Pi_1\right\}$. We have to prove that $\Pi_1 = \Pi$. Let $\omega \in H$, $\widehat{\omega} \leq \bigvee_{i\in I} \hat{e}_i$ for some $I \in \Pi_1$. According to Proposition 1.4.6, $\widehat{\omega} \leftrightarrow \sigma_\varphi^{\mathcal{P}(H)}$ and since φ is spectral, $\widehat{\omega}$ is φ-invariant. Therefore, all 1-dimensional subspaces dominated by $\bigvee_{i\in I} \hat{e}_i$ are φ-invariant and it follows that $\bigvee_{i\in I} \hat{e}_i$ is included in some eigenspace of U. Then there exists $J \in \Pi$ such that $I \subseteq J$. Since $\sum_{I\in\Pi_1} \mathrm{card}(I) = \sum_{J\in\Pi} \mathrm{card}(J) = n$, we only need to prove that there are no distinct $I_1, I_2 \in \Pi_1$ such that $I_1, I_2 \subseteq J$ for some $J \in \Pi$. Indeed, if that would be the case, we could choose $\omega_1, \omega_2 \in H$ such that $\widehat{\omega_1} \leq \bigvee_{i\in I_1} \hat{e}_i$ and $\widehat{\omega_2} \leq \bigvee_{i\in I_2} \hat{e}_i$. Then let $\omega = \omega_1 + \omega_2 \in H$ and we have $\widehat{\omega} \nleq \bigvee_{i\in I_1} \hat{e}_i$, $\widehat{\omega} \nleq \bigvee_{i\in I_2} \hat{e}_i$ but $\widehat{\omega} \leq \bigvee_{j\in J} \hat{e}_j$, which in turn implies $\widehat{\omega}$ is φ-invariant, hence $\widehat{\omega} \leftrightarrow \sigma_\varphi^{\mathcal{P}(H)}$ and $\widehat{\omega} \leftrightarrow \bigvee_{i\in I} \hat{e}_i$ for every $I \in \Pi_1$. Since $\widehat{\omega}$ is an atom of $\mathcal{P}(H)$ that is neither included nor orthogonal to $\bigvee_{i\in I_1} \hat{e}_i, \bigvee_{i\in I_2} \hat{e}_i$, this is a contradiction.

(5) Let $A \in \mathcal{E}(H)$ be φ-invariant and $P \in \mathcal{P}(H)$ be the projection on the range of A, which is the smallest projection that dominates A. We have to prove that P is also φ-invariant. Since the automorphism φ is order-preserving, $\varphi(P)$ must be the smallest projection that dominates $\varphi(A)$, namely the projection on the range of $\varphi(A)$. Since $\varphi(A) = A$, it follows that $\varphi(P) = P$.

(6) Follows from (5) and (2). □

Remark 6.3.3. The properties (1)–(6) from Theorem 6.3.2 were derived only from the fact that φ is spectral, without any other information except for the properties of unitary operators.

6.4. Spectral families of automorphisms

Let E denote, for the rest of this section, an effect algebra endowed with a compression base $(J_p)_{p\in P}$ and let Φ be a family of automorphisms of E.

Definition 6.4.1. The family Φ of automorphisms of E is called a *spectral family of automorphisms* if there exists a Boolean subalgebra B_Φ of P satisfying:

(P2) $\qquad \varphi(a) = a$, for all $\varphi \in \Phi$ if and only if $a \leftrightarrow B_\Phi$

In the sequel, for a family Φ of automorphisms of E, we denote $E_\Phi = \{a \in E \colon \varphi(a) = a, \text{ for all } \varphi \in \Phi\}$. Let us remark that $E_\Phi = \bigcap_{\varphi\in\Phi} E_\varphi$ and therefore it is a sub-effect algebra of E.

PROPOSITION 6.4.2. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a spectral family of automorphisms of E. There exists the greatest Boolean subalgebra B_Φ of P satisfying* (P2).

PROOF. The proof relies heavily on Lemma 6.1.2 and it is completely similar to the proof of Proposition 6.1.3 (except instead of one automorphism we have a family of automorphisms), therefore we omit it. □

DEFINITION 6.4.3. Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a spectral family of automorphisms of E. The *spectrum* (denoted by σ_Φ^P) of the spectral family Φ of automorphisms is the greatest Boolean subalgebra B of P fulfilling (P2).

LEMMA 6.4.4. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a spectral family of automorphisms of E. Then $\widetilde{C}(E_\Phi)\cap P$ is a Boolean subalgebra of P.*

PROOF. Follows from Lemma 6.2.2 and the fact that E_Φ is a sub-effect algebra of E. □

LEMMA 6.4.5. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a family of automorphisms of E. If a, a' are principal in E, $a, b \in E_\Phi$ and a, b coexist in E, then they coexist in E_Φ as well.*

PROOF. The proof is analogous to the proof of Lemma 6.2.6. □

THEOREM 6.4.6. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a spectral family of automorphisms of E. Then $\sigma_\Phi^P = \widetilde{C}(E_\Phi)\cap P$.*

PROOF. Property (P2) fulfilled by σ_Φ is equivalent to $E_\Phi = K(\sigma_\Phi)$. Let $a \in \sigma_\Phi$. Then $a \leftrightarrow \sigma_\Phi^P$ and therefore $a \in K(\sigma_\Phi) = E_\Phi$. Moreover, since $a \in \sigma_\Phi$, we obtain $a \leftrightarrow K(\sigma_\Phi) = E_\Phi$, i.e., a coexists with every element of E_Φ in E. According to Lemma 6.4.5 and since $a, a' \in P$ are principal elements of E, it follows that a coexists with every element of E_Φ in E_Φ as well, hence $a \in \widetilde{C}(E_\Phi)\cap P$, and we conclude that $\sigma_\Phi \subseteq \widetilde{C}(E_\Phi)\cap P$.

For the converse inclusion, since, according to Lemma 6.4.4, $\widetilde{C}(E_\Phi)\cap P$ is a Boolean subalgebra of P, it suffices to prove it fulfills (P2). If $\varphi(a) = a$, for all $\varphi \in \Phi$, it follows $a \in E_\Phi$, hence $a \leftrightarrow \widetilde{C}(E_\Phi)\cap P$. For the converse implication, since $\widetilde{C}(E_\Phi)\cap P \supseteq \sigma_\Phi$, $a \leftrightarrow \widetilde{C}(E_\Phi)\cap P$ implies $a \leftrightarrow \sigma_\Phi$ and therefore, since σ_Φ satisfies (P2), $\varphi(a) = a$ for all $\varphi \in \Phi$. □

COROLLARY 6.4.7. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a family of automorphisms of E. Then Φ is a spectral family if and only if $K(\widetilde{C}(E_\Phi)\cap P) \subseteq E_\Phi$ (the converse inclusion is trivially satisfied).*

PROPOSITION 6.4.8. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a spectral family of automorphisms of E. Then:*

1. $\sigma_\Phi^P = \widetilde{C}(E_\Phi) \cap P$ *is C-maximal;*
2. $\sigma_\Phi^P = K(K(\sigma_\Phi^P)) \cap P$*;*
3. $\sigma_\Phi^P = K(E_\Phi) \cap P$*.*

PROOF. (1) The result follows from Theorem 6.4.6 and the fact that $E_\Phi = K(\sigma_\Phi^P)$, since Φ is a spectral family of automorphisms.

(2) It is a direct consequence of (1) and Theorem 6.2.12.

(3) The result follows from (2) and the fact that $K(\sigma_\Phi^P) = E_\Phi$. $\square$

THEOREM 6.4.9. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a family of spectral automorphisms of E. Then Φ is a spectral family of automorphisms if and only if the spectra of the automorphisms in the family are pairwise compatible, i.e., $\sigma_\varphi^P \leftrightarrow \sigma_\psi^P$ for every $\varphi, \psi \in \Phi$. In this case, σ_Φ^P includes all spectra of automorphisms in the family.*

PROOF. "$\Rightarrow$" Let us assume Φ is a spectral family of automorphisms of E, with the spectrum σ_Φ^P. We will prove that $\sigma_\varphi^P \subseteq \sigma_\Phi^P$, for all $\varphi \in \Phi$. Since σ_Φ^P is a Boolean algebra, it will follow that $\sigma_\varphi^P \leftrightarrow \sigma_\psi^P$ for every $\varphi, \psi \in \Phi$. Let $\varphi \in \Phi$ and $b \in \sigma_\varphi^P$. For every $a \in K(\sigma_\Phi^P)$, (P2) implies $\varphi(a) = a$ for all $\varphi \in \Phi$, hence $a \leftrightarrow \sigma_\varphi^P$ and in particular $a \leftrightarrow b$. It follows that $b \in K(K(\sigma_\Phi^P))$, and since $b \in P$, we find that $b \in K(K(\sigma_\Phi^P)) \cap P$. However, according to Proposition 6.4.8, $K(K(\sigma_\Phi^P)) \cap P = \sigma_\Phi^P$, and therefore $\sigma_\varphi^P \subseteq \sigma_\Phi^P$.

"$\Leftarrow$" Conversely, let us assume $\sigma_\varphi^P \leftrightarrow \sigma_\psi^P$ for every $\varphi, \psi \in \Phi$. Then $\bigcup_{\varphi\in\Phi} \sigma_\varphi^P$ is a set of pairwise compatible elements of P. According to Lemma 6.1.2, there exists the smallest Boolean subalgebra $B \subseteq P$ which includes $\bigcup_{\varphi\in\Phi} \sigma_\varphi^P$. Moreover, by the same Lemma 6.1.2, $a \leftrightarrow \bigcup_{\varphi\in\Phi} \sigma_\varphi^P$ if and only if $a \leftrightarrow B$. It follows that $\varphi(a) = a$ for all $\varphi \in \Phi$ if and only if $a \leftrightarrow \sigma_\varphi^P$ for all $\varphi \in \Phi$ if and only if $a \leftrightarrow \bigcup_{\varphi\in\Phi} \sigma_\varphi^P$ if and only if $a \leftrightarrow B$, which means that B is a Boolean subalgebra of P fulfilling (P2). We conclude that Φ is a spectral family of automorphisms. $\square$

REMARK 6.4.10. The following useful facts hardly require verification:

1. If $\varphi : E \to E$ is a spectral automorphism, then φ^{-1} is also a spectral automorphism, and $\sigma_\varphi^P = \sigma_{\varphi^{-1}}^P$.
2. The identity $id_E : E \to E$, $id_E(a) = a$ for all $a \in E$ is a spectral automorphism and its spectrum is $\sigma_{id_E}^P = \widetilde{C}(E) \cap P$.

THEOREM 6.4.11. *(A "replica" of Stone's Theorem on strongly continuous uniparametric groups of unitary operators.) Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a family of spectral automorphisms of E. If the following conditions are fulfilled:*

(i) Φ is an abelian group; (ii) $\varphi(E_\psi) = E_{\varphi\psi}$ for every $\varphi, \psi \in \Phi$ such that $\psi \notin \{id_E, \varphi^{-1}\}$, then:

(1) $E_\varphi = E_\psi$ *for all* $\varphi, \psi \in \Phi \setminus \{id_E\}$*;*
(2) $\sigma_\varphi^P = \sigma_\psi^P$ *for all* $\varphi, \psi \in \Phi \setminus \{id_E\}$*;*
(3) Φ *is a spectral family.*

Proof. The proofs for (1) and (2) are completely similar to the ones for Theorem 3.8.8 (1) and (2), respectively, therefore we omit them.

(3) According to Theorem 6.4.9, Φ is a spectral family of automorphisms if and only if their spectra are pairwise compatible. Since all spectra except the spectrum of identity coincide, we only have to prove that $\sigma_{id_E}^P \leftrightarrow \sigma_\varphi^P$ for some $\varphi \in \Phi \setminus \{id_E\}$. We will even prove that $\sigma_{id_E}^P \subseteq \sigma_\varphi^P$. Let $\varphi \in \Phi \setminus \{id_E\}$. Obviously $K(E) \subseteq K(E_\varphi)$, hence, according to Corollary 6.2.13, $\sigma_{id_E}^P = K(E) \cap P \subseteq K(E_\varphi) \cap P = \sigma_\varphi^P$. $\square$

Let us remark that Theorem 6.4.11 generalizes Theorem 3.8.8 to spectral automorphisms in CB-effect algebras. Its proof is similar to the one for the case of orthomodular lattice spectral automorphisms.

Remark 6.4.12. An abelian group $\{\varphi_t\}_t$ of automorphisms of the standard effect algebra $\mathcal{E}(H)$ of a Hilbert space H is generated, e.g., by a one-parameter abelian group $\{U_t\}_t$ of unitaries on H by taking $\varphi_t(A) = U_t A U_t^{-1}$.

CHAPTER 7

Atomic effect algebras with compression bases

In this chapter, we focus on atomic compression base effect algebras and the consequences of atoms being foci (so-called projections) of the compressions in the compression base. Tkadlec [**59**] proved various conditions for an atomic sequential effect algebra or its set of sharp elements to be a Boolean algebra. We generalize some of these conditions to the case of effect algebras having a compression base, and also present some new ones for this more general framework. The role of the set of sharp elements of the SEA will be played by the orthomodular poset of foci (or projections) of the effect algebra's compression base.

In the first section we establish some properties of atoms in effect algebras endowed with a compression base, mainly regarding coexistence and centrality. Then, in the second section, we introduce the notion of projection-atomicity which aims to be an analogue, in the framework of effect algebras with a compression base, for the property of an effect algebra of having sharp atoms—used in sequential effect algebras. Consequences of projection-atomicity are studied, some of which generalize results obtained in [**59**]. A few conditions for an atomic compression base effect algebra to be a Boolean algebra are established. Finally, we apply these results to the particular case of sequential effect algebras and find a sufficient condition for them to be Boolean algebras that strengthens previous results by Gudder and Greechie [**31**] and Tkadlec [**59**]. The results presented here have been published in [**10**].

7.1. Atoms and centrality

Let us reiterate that for an effect algebra E with a compression base $(J_p)_{p\in P}$ we maintain, from now on, the notations introduced in Chapter 5, namely:

- $p \circ a = J_p(a)$ for every $p \in P$ and $a \in E$;
- $p \mid q$ if $p, q \in P$ and $p \circ q = q \circ p$ (i.e., $J_p(q) = J_q(p)$);
- $C(p) = \{a \in E\colon\ a = J_p(a) \oplus J_{p'}(a)\}$ for every $p \in P$.

Proposition 7.1.1. *Let E be an effect algebra. If p is an atom in E that is the focus of a compression and $a \in E$ then $p \leq a$ or $p \leq a'$.*

Proof. Since $p \circ a \leq p$ and p is an atom, either $p \circ a = \mathbf{0}$ or $p \circ a = p$. Therefore, $p \leq a'$ or $p \leq a$, according to Lemma 5.2.10. $\square$

COROLLARY 7.1.2. *Distinct atoms that are foci of compressions in an effect algebra are orthogonal.*

COROLLARY 7.1.3. *Let E be an effect algebra with a compression base $(J_p)_{p\in P}$. If $p, q \in P$ and p is an atom in E then $p \mid q$.*

PROOF. According to Proposition 7.1.1, $p \leq q$ or $p \leq q'$. In the first case, according to Theorem 5.3.9, $p \circ q = p = q \circ p$, hence $p \mid q$. If $p \leq q'$, then $p \perp q$ and, according to Lemma 5.2.10, $p \circ q = \mathbf{0} = q \circ p$, hence $p \mid q$. $\square$

PROPOSITION 7.1.4. *Let E be an effect algebra with a compression base $(J_p)_{p\in P}$ and $p, q, r \in P$ such that $p \leq q \circ r$ and $p \mid r$. Then $p \leq r \circ q$.*

PROOF. According to Lemma 5.2.10, $p \leq q$. According to Lemma 5.2.10, Theorem 5.3.9, the assumption and Lemma 5.2.10 again, $p = p \circ (q \circ r) = J_p\big(J_q(r)\big) = J_p(r) = p \circ r = r \circ p \leq r \circ q$. $\square$

PROPOSITION 7.1.5. *Let E be an effect algebra with a compression base $(J_p)_{p\in P}$ and $p \in P$ be an atom in E. For every $q, r \in P$, $p \leq q \circ r$ if and only if $p \leq r \circ q$.*

PROOF. This is a straightforward consequence of Corollary 7.1.3 and Proposition 7.1.4. $\square$

THEOREM 7.1.6. *Let E be an effect algebra with a compression base $(J_p)_{p\in P}$. If E is determined by atoms and every atom is in P then P is a Boolean algebra.*

PROOF. Let $q, r \in P$. According to Proposition 7.1.5, $q \circ r$ and $r \circ q$ dominate the same set of atoms (since all atoms are in P). Since E is determined by atoms, this means, in view of Definition 4.4.8, that $q \circ r = r \circ q$ and hence, according to Theorem 5.3.7, q, r coexist. According to Lemma 5.3.6, P is an orthomodular poset. Hence, P is an orthomodular poset with every pair of its elements coexistent and therefore a Boolean algebra, according to Corollary 1.2.12. $\square$

Let us remark that the conclusion of the above theorem cannot be improved to the statement that E is a Boolean algebra. The effect algebra in Example 7.2.13 satisfies the hypotheses (it is even atomistic), however it is not a Boolean algebra.

LEMMA 7.1.7. *Let E be an effect algebra with a compression base $(J_p)_{p\in P}$. If $p \in P$ is an atom in E then $C(p) = E$.*

PROOF. Let $a \in E$. First, let us remark that $J_p(a) \oplus J_{p'}(a)$ exists, since, according to Proposition 5.2.3, $J_p(a) \leq p$ and $J_{p'}(a) \leq p'$ and therefore, $J_{p'}(a) \leq p' \leq J_p(a)'$, i.e., $J_p(a) \perp J_{p'}(a)$.

According to Proposition 7.1.1, we have $p \leq a$ or $p \leq a'$.

If $p \leq a$ (and therefore $a' \leq p'$), by Lemma 5.2.10, $J_p(a) = p$ and $J_{p'}(a') = a'$ and it follows that $J_p(a) \oplus J_{p'}(a) = p \oplus J_{p'}(\mathbf{1} \ominus a') =$

$= p \oplus \big(J_{p'}(\mathbf{1}) \ominus J_{p'}(a')\big) = p \oplus (p' \ominus a') = p \oplus \big((\mathbf{1} \ominus p) \ominus (\mathbf{1} \ominus a)\big) = p \oplus (a \ominus p) = a$, where we have used Proposition 5.2.2 for the second equality and the fact that $p \le a \le \mathbf{1}$ and Proposition 4.1.5 (D3), (D4) for the fifth equality.

If $p \le a'$ (and thus $a \le p'$), then $J_{p'}(a) = a$, $J_p(a) = \mathbf{0}$ and thus $J_p(a) \oplus J_{p'}(a) = a$. $\square$

Remark 7.1.8. Let E be an effect algebra with a compression base $(J_p)_{p\in P}$. The previous result implies that every atom $p \in P$ in E coexists with every element of E . Indeed, for every $a \in E = C(p)$, $a = J_p(a) \oplus J_{p'}(a)$. Since $J_p(a) \le p$, there exist $p_1 \in E$ such that $p = J_p(a) \oplus p_1$. Taking into account that $J_{p'}(a) \le p'$, it follows that the sum $J_{p'}(a) \oplus p = J_{p'}(a) \oplus J_p(a) \oplus p_1$ exists and therefore a and p coexist.

Remark 7.1.9. Recall that we have shown in Theorem 5.3.8 (3) that a compression focus $p \in P$ is central in a compression base effect algebra E if and only if $C(p) = E$. In particular, this is also true if E is a sequential effect algebra endowed with the compression base $(J_p)_{p\in E_S}$, $J_p(a) = p \circ a$.

Theorem 7.1.10. *Let E be an effect algebra with a compression base $(J_p)_{p\in P}$. Every $p \in P$ that is an atom in E is central in E.*

Proof. Let $p \in P$ be an atom in E. According to Lemma 7.1.7, $C(p) = E$ and, according to Theorem 5.3.8 (3), p is central in E. $\square$

7.2. Projection-atomic effect algebras

The following property is intended as a substitute, in the framework of atomic effect algebras having a compression base, for the property of an effect algebra of having all the atoms sharp:

Definition 7.2.1. An effect algebra E is *projection-atomic* if it is atomic and there is a compression base $(J_p)_{p\in P}$ of E such that P contains all atoms in E.

In view of the above definition, the result of Theorem 7.1.10 implies that atoms of a projection-atomic effect algebra are central. The converse also holds, as it will be shown in the next remark.

Remark 7.2.2. Pulmannová [**54**, Example 3.4] proved that for every effect algebra E the center $\widetilde{C}(E)$ is a normal sub-EA and $(J_p)_{p\in\widetilde{C}(E)}$ with $J_p(a) = p \wedge a$ is a compression base. Hence, every atomic effect algebra with all atoms central is projection-atomic.

Proposition 7.2.3. *Every projection-atomic effect algebra is an orthoalgebra.*

PROOF. Let E be a projection-atomic effect algebra. Then E is atomic and, according to Theorem 7.1.10, all its atoms are central, hence sharp. According to Proposition 4.4.10, E is an orthoalgebra. $\square$

We will need the following properties in the sequel:

DEFINITION 7.2.4. A subset M of a poset P is *downward directed* if for every $a, b \in M$ there is an element $c \in M$ such that $c \leq a, b$.

DEFINITION 7.2.5. An effect algebra E is *weakly distributive* if $a \wedge b = a \wedge b' = \mathbf{0}$ implies $a = \mathbf{0}$ for every $a, b \in E$.

Let us also recall that, according to Definition 4.4.6, an effect algebra E has the maximality property if $[\mathbf{0}, a] \cap [\mathbf{0}, b]$ has a maximal element for every $a, b \in E$.

REMARK 7.2.6. The maximality property generalizes several important properties of effect algebras. E.g., every chain-finite, orthocomplete or lattice effect algebra has the maximality property. For details and more properties generalized by the maximality property see [**60**, Theorem 4.1] and [**61**, Theorem 3.1].

THEOREM 7.2.7 ([**58**, Theorem 4.2]). *Every weakly distributive orthomodular poset with the maximality property is a Boolean algebra.*

LEMMA 7.2.8. *Every projection-atomic effect algebra is weakly distributive.*

PROOF. Let E be a projection-atomic effect algebra and $(J_p)_{p \in P}$ a compression base of E such that P contains all atoms in E. Suppose that E is not weakly distributive. Then there are $a, b \in E$ such that $a \neq \mathbf{0}$, $a \wedge b = \mathbf{0}$ and $a \wedge b' = \mathbf{0}$. Since E is projection-atomic, there is an atom $p \in P$ in E such that $p \leq a$. Then $p \not\leq b$ and $p \not\leq b'$, which contradicts Proposition 7.1.1. $\square$

LEMMA 7.2.9. *The set of upper bounds of a set of atoms in a projection-atomic effect algebra with the maximality property is downward directed.*

PROOF. Let E be a projection-atomic effect algebra with a compression base $(J_p)_{p \in P}$ such that P contains the set of atoms of E, $A \subset P$ be a set of atoms, a, b be upper bounds of A. According to the maximality property, there is a maximal $c \leq a, b$. Let us suppose that c is not an upper bound of A and seek a contradiction. Then there is an atom $d \in A$ such that $d \not\leq c$, hence, according to Proposition 7.1.1, $d \leq c'$ and therefore $d' \geq c$. Let us remark that, since d is an atom and $d \in P$, it follows that d is central, according to Theorem 7.1.10. Since, by Theorem 4.2.14, $\widetilde{C}(E)$ is a sub-effect algebra of E, from $d \in \widetilde{C}(E)$ follows $d' \in \widetilde{C}(E)$. According to Proposition 4.2.2, since d' is central, it is also principal. Since $d' \geq c$, $d \leq a, b$ and therefore $d' \geq a', b'$,

$c \perp a', b'$ and d' is principal, we obtain $d' \geq c \oplus a'$ and $d' \geq c \oplus b'$. Hence $d \leq (c \oplus a')' = a \ominus c$ and $d \leq (c \oplus b')' = b \ominus c$. Then, there exist elements $e, f \in E$ such that $d \oplus e = a \ominus c$ and $d \oplus f = b \ominus c$ and it follows that $c \oplus (d \oplus e) = c \oplus (a \ominus c)$ and $c \oplus (d \oplus f) = c \oplus (b \ominus c)$, since the sums in the right-hand side of the equalities exist. We conclude that $(c \oplus d) \oplus e = a$ and $(c \oplus d) \oplus f = b$, hence $c \oplus d \leq a, b$—which contradicts the maximality of c. □

LEMMA 7.2.10. *Every element in a projection-atomic effect algebra is a minimal upper bound of the set of atoms it dominates. Every projection-atomic effect algebra with the maximality property is atomistic.*

PROOF. Let E be a projection-atomic effect algebra, $a \in E$ and A_a be the set of atoms dominated by a. First, let us show that a is a minimal upper bound of A_a. Let us suppose that there is an upper bound $b < a$ of A_a and seek a contradiction. Then $a \ominus b \neq \mathbf{0}$ and since E is atomic, there is an atom $p \in A_a$ such that $p \leq a \ominus b = (b \oplus a')'$ and therefore $p' \geq b \oplus a' \geq b$. It follows that $p \leq b'$. On the other hand, b is an upper bound of A_a and $p \in A_a$, hence $p \leq b$, and since E is an orthoalgebra (Proposition 7.2.3), we obtain $p \leq b \wedge b' = \mathbf{0}$—a contradiction.

If E has the maximality property then, according to Lemma 7.2.9, the set of upper bounds of A_a is downward directed, hence $a = \bigvee A_a$. □

LEMMA 7.2.11. *Every projection-atomic effect algebra with the maximality property is an orthomodular poset.*

PROOF. Let E be a projection-atomic effect algebra with the maximality property, $a, b \in E$ with $a \perp b$ and A_a, A_b be the sets of atoms dominated by a and b respectively. According to Lemma 7.2.10, E is atomistic and therefore the set of upper bounds of $\{a, b\}$ is the set of upper bounds of $A_a \cup A_b$. According to Proposition 7.2.3, E is an orthoalgebra and therefore, by Theorem 4.2.5, $a \oplus b$ is a minimal upper bound of $\{a, b\}$. According to Lemma 7.2.9, the set of upper bounds of $A_a \cup A_b$ is downward directed, hence $a \oplus b$ is the least upper bound of $\{a, b\}$. Hence $a \oplus b = a \vee b$ for orthogonal $a, b \in E$ and therefore, according to Theorem 4.2.7, E is an orthomodular poset. □

THEOREM 7.2.12. *Every projection-atomic effect algebra with the maximality property is a Boolean algebra.*

PROOF. It follows from Lemma 7.2.8, Lemma 7.2.11 and Theorem 7.2.7. □

In view of Remark 4.4.7, it is a natural question if the maximality condition can be replaced in Theorem 7.2.12 by weak orthocompleteness. The answer is in the negative, as the following example based on Tkadlec [**59, 61**] shows.

EXAMPLE 7.2.13. Let X_1, X_2, X_3, X_4 be infinite and mutually disjoint sets, $X = \bigcup_{i=1}^{4} X_i$,

$$E' = \{\emptyset, X_1 \cup X_2, X_2 \cup X_3, X_3 \cup X_4, X_4 \cup X_1, X\},$$
$$E = \{(A \setminus F) \cup (F \setminus A) \colon A \in E' \text{ and } F \subset X \text{ is finite}\}.$$

For disjoint $A, B \in E$ we define $A \oplus B = A \cup B$. Then $(E, \oplus, \emptyset, X)$ is an orthomodular poset, the orthosupplement is the set theoretic complement in X and the partial ordering is the inclusion. E is atomic and the set of its atoms is $\{\{x\} \colon x \in X\}$. Let us put

$$P = \{F \subseteq X \colon F \text{ is finite or } X \setminus F \text{ is finite}\}$$

and for every $F \in P$ let us define $J_F \colon E \to E$ by $J_F(A) = F \cap A$ for every $A \in E$.

It is a straightforward verification that $\big(J_F\big)_{F \in P}$ is a compression base for E and that P contains all atoms, hence E is projection-atomic. E is weakly orthocomplete, because if an orthogonal system $(A_i)_{i \in I}$ has a minimal majorant $B \in E$ then $B = \bigcup_{i \in I} A_i$ is the sum of $(A_i)_{i \in I}$. Since all elements of $[\emptyset, X_2]$ are finite, $(X_1 \cup X_2) \wedge (X_2 \cup X_3)$ does not exist and therefore E is not a lattice (and hence not a Boolean algebra).

Let us remark that $P \neq E$—e.g., $X_1 \cup X_2 \in E \setminus P$.

In Chapter 5, we have presented the projection cover property for a compression base of an effect algebra (see Definition 5.3.12). It is now time to make use of it.

THEOREM 7.2.14. *Let E be a projection-atomic effect algebra. If a compression base on E for which all atoms are projections has the projection cover property, then E is a Boolean algebra.*

PROOF. Let $(J_p)_{p \in P}$ be a compression base on E that has the projection cover property and such that all atoms are in P. According to Theorem 5.3.13, P is an orthomodular lattice. Since P is atomic, it is atomistic, according to Proposition 1.4.4. Since all atoms are mutually orthogonal (see Corollary 7.1.2), every two elements of P are compatible, and hence P is a Boolean algebra.

It remains to prove that $E = P$. Let $a \in E$ and let us denote A_a the set of atoms in E dominated by a and $P_a = \{p \in P \colon p \leq a\}$. The set of projection upper bounds of a' is $P'_a = \{p' \in P \colon p \in P_a\}$ and, due to the projection cover property, there is a projection cover $\bigwedge P'_a \in P$ of a', hence $a \geq \bigvee P_a \in P$. Since a is a minimal upper bound of A_a (Lemma 7.2.10) and $\bigvee P_a$ is also an upper bound of A_a, it follows that $a = \bigvee P_a \in P$. □

COROLLARY 7.2.15. *Every atomic sequential orthoalgebra is a Boolean algebra.*

PROOF. According to Theorem 5.3.5 (2), every sequential effect algebra E has a maximal compression base $(J_p)_{p \in E_S}$, where E_S denotes the set of sharp elements of E. It follows that every sharp element of E is a projection (i.e., compression focus). Since E is an orthoalgebra, according to Theorem 4.2.5, all its elements are sharp, and therefore, all its elements are projections. Since E is atomic and all its atoms are projections, it follows that E, endowed with the compression base $(J_p)_{p\in E_S}$, is projection-atomic. On the other hand, since all the elements of E are projections, every element is its own projection cover (i.e., it is the smallest projection that dominates itself), therefore the compression base $(J_p)_{p\in E_S}$ has the projection cover property. According to Theorem 7.2.14, E is a Boolean algebra. □

Let us remark that the above corollary generalizes similar results obtained by Gudder and Greechie [**31**, Theorem 5.3] and Tkadlec [**59**, Theorems 5.4 and 5.6]. The first mentioned result assumes that the effect algebra is atomistic, the second assumes it has the maximality property and the third assumes it is determined by atoms.

Bibliography

1. Beals, R. : *Topics in Operator Theory*, The University of Chicago Press, Chicago, London (1971)
2. Beltrametti, E. G., Casinelli, G. (1981), *The Logic of Quantum Mechanics*, Addison-Wesley Publishing Company, Reading, Massachusetts
3. Birkhoff, G., Von Neumann, J.: The Logic of Quantum Mechanics *Annals of Mathematics***37**, No. 4, 823–843 (1936)
4. Busch, P., Grabowski, M., Lahti, P. J.: *Operational Quantum Physics.* Springer Verlag, Berlin (1995)
5. Busch, P., Lahti, P. J., Mittelstaedt: *The Quantum Theory of Measurement.* Lecture Notes in Physics, Springer Verlag, Berlin, Heidelberg, New York, London (1991)
6. Calude, C. S., Hertling, P. H., Svozil, K.: Embedding Quantum Universes into Classical Ones. *Foundations of Physics* **29**, No. 3, 249–379 (1999)
7. Caragheorgheopol, D.: An introduction to quantum logics. *Revista de logica* No.1 (2010)
8. Caragheorgheopol, D.: Spectral Automorphisms in CB-Effect Algebras. *Math. Slovaca*, to appear
9. Caragheorgheopol, D.: Unsharp Quantum Logics. *Revista de logica* No.4 (2010)
10. Caragheorgheopol, D., Tkadlec, J.: Atomic effect algebras with compression bases. *J. Math. Phys.* **52**, 013512 (2011), doi:10.1063/1.3533918
11. Caragheorgheopol, D., Tkadlec, J.: Characterizations of Spectral Automorphisms and a Stone-Type Theorem in Orthomodular Lattices. *Internat. J. Theoret. Phys.***50**, No.12, 3750–3760 (2011), doi:10.1007/s10773-011-0738-6 .
12. Chang, C. C.: Algebraic Analysis of many-valued logics. *Trans. Amer. Math. Soc.* **88**, 467–490 (1958)
13. Chovanec, F., Kôpka, F.: Boolean D-posets. *Tatra Mt. Math. Publ.* **10** 183–197 (1997)
14. Dvurečenskij, A., Pulmannová, S.: *New Trends in Quantum Structures.* Kluwer Academic, Bratislava (2000)
15. Einstein, A., Podolsky, B., Rosen, N.: Can quantum mechanical description of physical reality be considered complete? *Physical review* **47** 777–780 (1935)
16. Foulis, D. J.: Compressible groups. *Math. Slovaca* **53**, 433 (2003)
17. Foulis, D. J.: Compressions on partially ordered abelian groups. *Proc. Amer. Math. Soc.* **132**, 3581 (2004)
18. Foulis, D. J.: Compressible groups with general comparability. *Math. Slovaca* **55**, 409 (2005)
19. Foulis, D. J.: Spectral resolution in a Rickart comgroup *Rep. Math. Phys.* **54**, 319 (2004)
20. Foulis, D. J.: Compression bases in unital groups *Internat. J. Theoret. Phys.* **44**, 2191 (2005)

21. Foulis, D. J.: Observables, states and symmetries in the context of CB-effect algebras. *Rep. Math. Phys.* **60**, 329–346 (2007)
22. Foulis, D. J., Bennett, M. K.: Effect algebras and unsharp quantum logics. *Found. Phys.* **24**, 1331–1352 (1994)
23. Foulis, D., Greechie, R., Rüttimann, G.: Filters and supports in orthoalgebras. *Int. J. Theor. Phys.* **31**, 789–807 (1992)
24. Giuntini, R., Greuling, H.: Toward a formal language for unsharp properties. *Found. Phys.* **19** 931–945 (1989)
25. Greechie, R. J.: A particular non-atomistic orthomodular poset. *Commun. Math. Phys.* **14**, 326–328 (1969). doi:10.1007/BF01645388
26. Greechie, R. J., Foulis, D. J., Pulmannová, S.: The center of an effect algebra. *Order* **12**, 91–106 (1995)
27. Gudder, S.: *Stochastic Methods in Quantum Mechanics* Elsevier/North-Holland, Amsterdam (1979)
28. Gudder, S.: Compressible effect algebras *Rep. Math. Phys.* **54**, 93–114 (2004). doi:10.1016/S0034-4877(04)80008-9
29. Gudder, S.: Compression bases in effect algebras. *Demonstr. Math.* **39**, 43–54 (2006).
30. Gudder, S.: A histories approach to quantum mechanics. *J. Math. Phys.* **39**, 5772–5788 (1998)
31. Gudder, S., Greechie, R. J.: Sequential products on effect algebras. *Rep. Math. Phys.* **49**, 87–111 (2002). doi:10.1016/S0034-4877(02)80007-6
32. Gudder, S., Nagy, G.: Sequentially independent effects. *Proc. Amer. Math. Soc.* **130**, 1125–1130 (2001)
33. Gudder, S., Nagy, G.: Sequential quantum measurements. *J. Math. Phys.* **42**, 5212–5222 (2001)
34. Halmos, P. R. (1951), *Introduction to Hilbert Space and Theory of Spectral Multiplicity*, Chelsea Publishing Company, New York.
35. Harding, J., Ptak, P.: On the set representation of an orthomodular poset. *Colloquium Mathematicum* **89**, No.1-2, 233–240 (2001)
36. Ivanov, A. and Caragheorgheopol, D.: Spectral automorphisms in quantum logics. *Internat. J. Theoret. Phys.* **49**, No. 12, 3146–3152 (2010), doi:10.1007/s10773-010-0369-3.
37. Jenča, G., Riečanová, Z.: On sharp elements in lattice ordered effect algebras. *BUSEFAL* **80**, 24–29 (1999)
38. Kalmbach, G. (1983), *Orthomodular lattices*, Academic Press, London.
39. Kochen, S., Specker, E. P: The problem of hidden variables in quantum mechanics. *Journal of Mathematics and Mechanics* **17**, No.1, 59–87 (1967)
40. Kôpka, F., Chovanec, F.: D-posets. *Math. Slovaca* **44**, 21–34 (1994)
41. Kraus, K.: *States, Effects and Operations*, Lecture Notes in Physics, Vol.**190**, Springer Verlag, Berlin, Heidelberg, New York (1983).
42. Lahti, P., Pulmannová, S.: Coexistent observables and effects in quantum mechanics. *Rep. Math. Phys.* **39**, 339–351 (1997)
43. Ludwig, G.: *An Axiomatic Basis for Quantum Mechanics* Springer, New York (1986/87)
44. Ludwig, G.: *Foundations of Quantum Mechanics.* Springer Verlag, New York (1983)
45. Mackey, G. W.: *Mathematical Foundations of Quantum Mechanics.* W. A. Benjamin, Inc., New York (1963)
46. Maeda, F., Maeda, S. : *Theory of Symmetric Lattices.* Springer-Verlag Berlin, Heidelberg, New York (1970)

47. Mundici, D.: Interpretation of AFC*-algebras in Lukasiewicz sentential calculus. *J. Functional Anal.* **65**, 15–65 (1986)
48. v. Neumann, J.: *Mathematical Foundations of Quantum Mechanics.* Princeton University Press (1955)
49. Padmanabhan, R., Rudeanu, S.: *Axioms For Lattices And Boolean Algebras* World Scientific, Singapore (2008)
50. Piron, C. : 'Axiomatique Quantique'. *Helv. Phys. Acta* **37**, 439-468 (1964)
51. Piron, C. (1976), *Foundations of Quantum Physics.* W. A. Benjamin, Inc., Reading, Massachusetts
52. Postnikov, M. : *Lectures in Geometry, sem.II, Linear Algebra and Differential Geometry.* Mir Publishers Moscow (1982)
53. Pták, P. and Pulmannová, S. (1991) *Orthomodular Structures as Quantum Logics*, VEDA and Kluwer Acad. Publ., Bratislava and Dordrecht.
54. Pulmannová, S.: Effect algebras with compressions. Rep. Math. Phys. **58**, 301–324 (2006). doi:10.1016/S0034-4877(06)80054-6
55. Reed, M. and Simon, B.: *Methods of Modern Mathematical Physics, vol. I*, Acad. Press, New York (1975).
56. Riečanová, Z.: Generalization of blocks for D-lattice and lattice ordered effect algebras. *Internat. J. Theor. Phys.* **39**, 231–237 (2000)
57. Sikorski, R.: *Boolean Algebras (second edition)* Springer-Verlag, Berlin-Gottingen-Heidelberg-New York (1964).
58. Tkadlec, J.: Conditions that force an orthomodular poset to be a Boolean algebra. Tatra M. Math. Publ. **10**, 55–62 (1997).
59. Tkadlec, J.: Atomic sequential effect algebras. Internat. J. Theoret. Phys. **47**, 185–192 (2008). doi:10.1007/s10773-007-9492-1
60. Tkadlec, J.: Effect algebras with the maximality property. Algebra Universalis **61**, 187–194 (2009). doi:10.1007/s00012-009-0013-3
61. Tkadlec, J.: Common generalizations of orthocomplete and lattice effect algebras. Internat. J. Theoret. Phys., to appear. doi:10.1007/s10773-009-0108-9
62. Varadarajan, V. S. (1968), *Geometry of Quantum Theory, vol. I*, van Nostrand, Princeton.
63. Wright, R. (1977), The structure of projection-valued states: A generalization of Wigner's theorem, *Internat. J. Theoret. Phys.* **16**, No. 8, 567–573.
64. Zierler, N., Schlessinger, M.: Boolean embeddings of orthomodular sets and quantum logic. *Duke Mathematical Journal* **32**, 251–262, (1965).