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ABSTRACT

OF THE

DOCTORAL THESIS

BOOLEAN SUBALGEBRAS
AND
SPECTRAL AUTOMORPHISMS IN QUANTUM LOGICS

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To the memory of Alecu Ivanov,
my mentor and dear friend

Contents

Introduction

Quantum logics are logic-algebraic structures that arise in the study of the foundations of quantum mechanics.

According to the conventional Hilbert space formulation of quantum mechanics, states and observables are represented by operators in a Hilbert space associated to the quantum system under investigation, the so-called *state space*. Propositions, which represent yes-no experiments concerning the system, form an orthomodular lattice, isomorphic to the partially ordered set of projection operators $\mathcal{P}(H)$ on the state space H. It is called *the logic* associated to the quantum system. Thus, orthomodular lattices, which are sometimes assumed to be complete, atomic and to fulfill the covering property (just like $\mathcal{P}(H)$) bear the name of *quantum logics*. We propose and investigate the following question:

What amount of quantum mechanics is coded into the structure of the propositional system?

In other words, we intend to investigate to what extent some of the fundamental physical facts concerning quantum systems can be described in the more general framework of orthomodular lattices, without the support of Hilbert space–specific tools.

With this question in mind, we attempt to build, in abstract orthomodular lattices, something similar to the spectral theory in Hilbert space. For this purpose, we introduce and study *spectral automorphisms*.

According to the contemporary theory of quantum measurement, yes-no measurements that may be unsharp, called effects, are represented by so-called effect operators, self-adjoint positive operators on the state space H, smaller than identity. As an abstraction of the structure of the set of effect operators, the effect algebra structure is defined.

In the second part of the thesis, we move our investigation to the framework of unsharp quantum logics, represented by effect algebras. We generalize spectral automorphisms to effect algebras and obtain in this framework results that are analogous to the ones obtained in orthomodular lattices.

Finally, as a rather separate undertaking, we study atomic effect algebras endowed with a family of morphisms called *compression base*, analyzing the consequences of atoms being foci of compressions in the compression base. We then apply some of the obtained results to the particular case of effect algebras endowed with a sequential product.

The thesis is divided into two parts. The first part, composed of the first three chapters, is devoted to the study of "sharp" quantum logics, as represented by orthomodular lattices, arising from the conventional Hilbert space formulation of the quantum mechanics. In the second part, consisting of the chapters four to seven, we adopt the framework of "unsharp" quantum logics, represented by effect algebras, emerging from the contemporary theory of quantum measurement. A brief description of the chapters contents follows.

Chapter 1. The first chapter of the thesis is devoted to a presentation of orthomodular structures such as orthomodular posets and lattices and of their basic properties. The physically meaningful relation of compatibility is discussed. Blocks, commutants and center of such structures are covered. The last section of the chapter is dedicated to atomicity, as well as covering and exchange properties. The facts presented in this chapter are covered in various monographs, such as, e.g., [**38, 46, 51, 53, 62**].

Chapter 2. The second chapter contains a discussion of the problem concerning the possibility of embedding quantum logics into classical ones. The origin of this problem can be traced back to a famous paper of Einstein, Podolsky and Rosen, where authors conjectured that a "completion" of quantum mechanical formalism, leading to its "embedding" into a larger, classical and deterministic theory is possible.

We give an overview of classical and newer results concerning this matter by Kochen and Specker [**39**], Zierler and Schlessinger [**64**], Calude, Hertling and Svozil [**6**], Harding and Ptak [**35**] in a unitary treatment.

Chapter 3. The last chapter of the first part of the thesis consists of the original results obtained concerning spectral automorphisms in orthomodular lattices. First, we introduce spectral automorphisms. We define the spectrum of a spectral automorphism and study a few examples. Then, we analyze the possibility of constructing such automorphisms in products and horizontal sums of lattices. A factorization of the spectrum of a spectral automorphism is found. We give various characterizations, as well as necessary or sufficient conditions for an automorphism to be spectral or for a Boolean algebra to be its spectrum. Then, we prove that the presence of spectral automorphisms allows us to distinguish between classical and nonclassical theories. For finite dimensional quantum logics, we show that for every spectral automorphism there is a basis of invariant atoms. This is an analogue of the spectral theorem for unitary operators having purely point spectrum. An interesting consequence is that, if there are physical motivations for admitting that a finite dimensional theory must have spectral symmetries, it cannot be represented by the lattice of projections of a finite dimensional *real* Hilbert space. The last part of this chapter addresses the problem of the unitary time evolution of a system from the point of view of the spectral automorphisms theory. An analogue of the Stone theorem concerning strongly continuous one-parameter unitary groups is given. The results in this chapter have been published in the articles "Spectral automorphisms in quantum logics", by Ivanov and Caragheorgheopol (2010), and "Characterizations of spectral automorphisms and a Stone-type theorem in orthomodular lattices", by Caragheorgheopol and Tkadlec (2011), both appeared in *International Journal of Theoretical Physics*.

Chapter 4. In the fourth chapter, we present background information on unsharp quantum logics, as represented by effect algebras. We discuss special elements, coexistence relation, which generalizes compatibility from orthomodular posets, various substructures and important classes of effect algebras, as well as automorphisms in effect algebras. The facts presented in this chapter can be found, e.g., in the book of Dvurečenskij and Pulmanová [**14**] which gathers many of the recent results in the field of quantum structures.

Chapter 5. In the fifth chapter, we present sequential, compressible and compression base effect algebras, which will be needed in the sequel. They were introduced by Gudder [**28, 29**] and Gudder and Greechie [**31**].

Sequential product in effect algebras formalizes the case of sequentially performed measurements. The prototypical example of a sequential product is defined on the set $\mathcal{E}(H)$ of effects operators by $A \circ B = A^{1/2}BA^{1/2}$.

The set $\mathcal{E}(H)$ of effect operators can be endowed with a family of morphisms $(J_P)_{P\in\mathcal{P}(H)}$ defined by $J_P(A) = PAP$, called *compressions* and indexed by the projection operators $P \in \mathcal{P}(H)$ which are also called the *foci* of compressions. The family $(J_P)_{P\in\mathcal{P}(H)}$ is said to form a *compression base* of $\mathcal{E}(H)$. Inspired by the main features of the family $(J_P)_{P\in\mathcal{P}(H)}$, the notions of compression, compression base and compressible efect algebra were introduced in abstract effect algebras. As it turns out, compression base effect algebras generalize sequential, as well as compressible effect algebras.

Chapter 6. In this chapter, we generalize spectral automorphisms to compression base effect algebras, which are currently considered as the appropriate mathematical structures for representing physical systems [**21**]. We obtain characterizations of spectral automorphisms in compression base effect algebras and various properties of spectral automorphisms and of their spectra. In order to evaluate how well our theory performs in practice, we apply it to an example of a spectral automorphism on the standard effect algebra of a finite-dimensional Hilbert space and we show the consequences of spectrality of an automorphism for the unitary Hilbert space operator that generates it. In the last section, spectral families of automorphisms are discussed and an effect algebra version of the Stone-type theorem in Chapter 3 is obtained. The results of this chapter are included in the article "Spectral automorphisms in CB-effect algebras", by Caragheorgheopol, which was accepted for publication by *Mathematica Slovaca* and will appear in Volume 62, No. 6 (2012).

Chapter 7. The last chapter of the thesis contains original results concerning atomic compression base effect algebras and the consequences of atoms being foci of compressions. Part of our work generalizes results obtained by Tkadlec [**59**] in atomic sequential effect algebras. The notion of projection-atomicity is introduced and studied and conditions that force a compression base effect algebra or the set of compression foci to be Boolean are given. We apply some of these results to the important particular case of sequential effect algebra and strengthen previous results obtained by Gudder and Greechie [**31**] and Tkadlec [**59**]. The results of this chapter have been published in the article "Atomic effect algebras with compression bases", by Caragheorgheopol and Tkadlec, which appeared in *Journal of Mathematical Physics* (2011).

Part 1

Quantum Logics as Orthomodular Structures

CHAPTER 1

Basics on orthomodular structures

In this introductive chapter we present the main orthomodular structures which arise from quantum mechanics—most notably, orthomodular posets and lattices—and their properties. The different facts presented in this chapter are covered in various monographs, like, e.g., [**2, 38, 46, 49, 51, 53, 62**].

1.1. Definitions of orthomodular structures

DEFINITION 1.1.1. Let $(P, \leq)$ be a bounded poset. A unary operation ' on P such that, for every $a, b \in P$, the following conditions are fulfilled:

(1) $a \leq b$ implies $b' \leq a'$,
(2) $a'' = a$,
(3) $a \vee a' = \mathbf{1}$ and $a \wedge a' = \mathbf{0}$,

is an *orthocomplementation* on P.

DEFINITION 1.1.2. A bounded poset with an orthocomplementation is an *orthoposet*. An orthoposet which is a lattice is an *ortholattice*.

DEFINITION 1.1.3. A relation *orthogonal*, denoted by "$\perp$" is defined for elements a, b of an orthoposet by

$$a \perp b \iff a \leq b'$$

DEFINITION 1.1.4. An orthoposet (ortholattice) $(P, \leq, ')$ satisfies the *orthomodular law* if for every $a, b \in P$,

(OM1) $\quad a \leq b$ implies there exists $c \in P, c \perp a$ such that $b = a \vee c$

.

DEFINITION 1.1.5. An orthoposet with the property that every pair of orthogonal elements has supremum and satisfying the orthomodular law is an *orthomodular poset*. If, moreover, the supremum exists for every countable set of pairwise orthogonal elements, it is a *σ–complete orthomodular poset*.

DEFINITION 1.1.6. An ortholattice satisfying the orthomodular law is an *orthomodular lattice*.

1.2. Compatibility. Basic properties

Compatible pairs represent simultaneously verifiable events, hence their importance in the axiomatics of quantum theories.

DEFINITION 1.2.1. Let P be an orthomodular poset. Elements $a, b \in P$ are *compatible (in P)* if there exist mutually orthogonal elements $a_1, b_1, c \in P$ such that $a = a_1 \vee c$ and $b = b_1 \vee c$. In this case we will write $a \leftrightarrow_P b$, or just $a \leftrightarrow b$, when there's no risk of confusion. For M a subset of P, we shall write $a \leftrightarrow M$ when $a \leftrightarrow m$ for every $m \in M$.

LEMMA 1.2.2. *Let a and b be elements of an orthomodular poset P. Then:*

(1) *$a \leq b$ implies $a \leftrightarrow b$;*
(2) *$a \perp b$ if and only if $a \leftrightarrow b$ and $a \wedge b = \mathbf{0}$;*
(3) *the following are equivalent: $a \leftrightarrow b$, $a' \leftrightarrow b$, $a \leftrightarrow b'$, $a' \leftrightarrow b'$.*

THEOREM 1.2.3. [**38**, Ch. 1, Section 3, Proposition 4] *Let $(L, \leq, ')$ be an orthomodular lattice and M be a subset such that $\bigvee M$ exists. If $b \in L$ is such that $b \leftrightarrow M$, then:*

(1) $b \leftrightarrow \bigvee M$
(2) $b \wedge (\bigvee M) = \bigvee \{b \wedge m \colon m \in M\}$

PROPOSITION 1.2.4. [**53**, Proposition 1.3.11] *Let $(L, \leq, ')$ be an orthomodular lattice and $a, b, c \in L$. If $a \leftrightarrow b$ and $a \leftrightarrow c$, then $\{a, b, c\}$ is a distributive triple.*

COROLLARY 1.2.5. *An orthomodular poset is a Boolean algebra if and only if every pair of its elements is compatible.*

1.3. Orthomodular substructures

DEFINITION 1.3.1. A subset of an ortholattice is a *subalgebra* if it contains the least and greatest elements and it is closed under lattice operations $\vee, \wedge$ and orthocomplementation $'$.

DEFINITION 1.3.2. Let L be an ortholattice. A subalgebra of L which is a Boolean algebra with the induced operations from L is a *Boolean subalgebra* of L.

DEFINITION 1.3.3. The maximal Boolean subalgebras of an orthomodular lattice are called its *blocks*.

DEFINITION 1.3.4. A subset of an orthomodular poset is a *suborthoposet* if it contains the least and greatest elements and it is closed under orthocomplementation and under suprema of orthogonal pairs.

DEFINITION 1.3.5. A suborthoposet of an orthomodular poset P which is a Boolean algebra with the induced from P order, orthocomplementation and lattice operations, is called a *Boolean subalgebra* of the orthomodular poset P.

If P is an orthomodular lattice, this notion of Boolean subalgebra coincides with the one defined in 1.3.2.

DEFINITION 1.3.6. Let P be an orthomodular poset. An element $a \in P$ is *central* if it is compatible with every other element of P. The set of central elements of P is the *center* of P, denoted henceforth by $\widetilde{C}(P)$.

PROPOSITION 1.3.7. *The center of an orthomodular poset P is a Boolean subalgebra of P.*

DEFINITION 1.3.8. Let P be an orthomodular poset and $\Delta \subseteq P$. The *commutant* of Δ in P is the set $\{a \in P\colon\ a \leftrightarrow \Delta\}$. It will be denoted henceforth by $K_P(\Delta)$ or, if there is no possibility of confusion about P, simply by $K(\Delta)$.

1.4. Atomicity. Covering and exchange properties. Lattices with dimension

DEFINITION 1.4.1. Let L be a poset with least element $\mathbf{0}$. A minimal non-zero element of L is an *atom*. L is *atomic* if every its element dominates (at least) an atom of L. It is *atomistic* if every element is the supremum of the atoms it dominates. Let Ω_a denote the set of atoms dominated by an element $a \in L$, and $\Omega(L)$ denote the set of atoms of L.

PROPOSITION 1.4.2. *Every atomic orthomodular lattice is atomistic.*

PROPOSITION 1.4.3. *If B is a Boolean subalgebra of the orthomodular lattice L, a is an atom of B and $\omega \in L$ such that $\omega \leq a$, then $\omega \leftrightarrow B$.*

PROPOSITION 1.4.4. *Let L be an atomic orthomodular lattice. For every element $a \in L$, there exists a maximal family $\{\alpha_i\}_{i\in I}$ of mutually orthogonal atoms in Ω_a. Then, $a = \bigvee_{i\in I} \alpha_i$.*

DEFINITION 1.4.5. Let L be an atomic orthomodular lattice and $a \in L$. A maximal family $\{\alpha_i\}_{i\in I}$ of mutually orthogonal atoms in Ω_a is a *basis* of a. A basis of $\mathbf{1} \in L$ is also called a *basis of the lattice* L.

THEOREM 1.4.6 (see [**38**, Ch.1, Section 4, Lemma 2]). *Let L be an orthomodular lattice and B a Boolean subalgebra of L. If B is a block of L then the atoms of B are atoms of L. Conversely, if B is atomic and its atoms are atoms of L, it is a block of L.*

1.5. Morphisms in orthomodular structures

DEFINITION 1.5.1. Let L_1, L_2 be orthomodular posets. A mapping $h : L_1 \to L_2$ is a *morphism of orthomodular posets* if the following conditions are satisfied:

(1) $h(\mathbf{1}) = \mathbf{1}$;
(2) $a \perp b$ implies $h(a) \perp h(b)$, for every $a, b \in L_1$;
(3) $h(a\vee b) = h(a)\vee h(b)$, for every pair of orthogonal elements $a, b \in L_1$.

DEFINITION 1.5.2. Let L_1, L_2 be orthomodular lattices (Boolean algebras). A mapping $h : L_1 \to L_2$ is a *morphism of orthomodular lattices* (*Boolean algebras*, respectively) if it is a morphism of orthomodular posets and it preserves the join of arbitrary pairs of elements.

DEFINITION 1.5.3. A morphism $h : L_1 \to L_2$ of orthomodular posets (or orthomodular lattices, or Boolean algebras, respectively) is an *embedding* if, for every $a, b \in L_1$, $h(a) \perp h(b)$ implies $a \perp b$. It is an *isomorphism* if it is bijective and its inverse $h^{-1} : L_2 \to L_1$ is also a morphism. An isomorphism $h : L \to L$ is an *automorphism* of L.

PROPOSITION 1.5.4. *Let L be an orthomodular poset (or an orthomodular lattice, or a Boolean algebra). A mapping $h : L \to L$ is an automorphism if and only if it satisfies the following conditions:*

(1) $h(\mathbf{1}) = \mathbf{1}$;
(2) $a \perp b$ *implies* $h(a) \perp h(b)$, *for all* $a, b \in L$;
(3) $a \leq b$ *if and only if* $h(a) \leq h(b)$, *for all* $a, b \in L$;
(4) h *is surjective.*

The following result shows that every automorphism of an atomic complete orthomodular lattice is uniquely determined by its restriction to the set of atoms, which is a bijective map that preserves orthogonality both ways.

THEOREM 1.5.5 (see [**51**, Theorem 2.46]). *Let L be an atomic complete orthomodular lattice. For every automorphism h of L, its restriction to $\Omega(L)$ is $\chi : \Omega(L) \to \Omega(L)$ satisfying the conditions:*

(1) χ *bijective;*
(2) $\alpha \perp \beta$ *if and only if* $\chi(\alpha) \perp \chi(\beta)$, *for all* $\alpha, \beta \in \Omega(L)$.

Conversely, for every mapping $\chi : \Omega(L) \to \Omega(L)$ satisfying the above conditions (1), (2) there exists a unique automorphism h of L such that its restriction to $\Omega(L)$ is χ.

CHAPTER 2

Understanding the logic of quantum mechanics in classical terms

The problem of embedding quantum logics into classical ones is very old. Its origin can be traced back to a well known article of Einstein, Podolsky and Rosen (EPR) [**15**]. In this historic paper, the authors conjectured that a "completion" of quantum mechanical formalism, leading to its "embedding" into a larger, classical and deterministic theory (from the algebraic and logic point of view) is possible.

We explore various possibilities to embed quantum logics into classical ones. We discuss the different approaches and results obtained concerning this matter by e.g., Kochen and Specker [**39**], Zierler and Schlessinger [**64**], Calude, Hertling and Svozil [**6**], Harding and Ptak [**35**], thus offering an overview of what can be achieved in terms of classical understanding of quantum mechanics.

2.1. The impossibility of embedding a quantum logic into a classical one

Assuming that a "proper" embedding of a non-Boolean orthomodular lattice (a quantum logic) into a Boolean algebra (a classical logic) would exist, it would preserve compatibility. Since every pair of elements is compatible in a Boolean algebra, the same would have to be true in the orthomodular lattice, which contradicts to our assumption.

In what follows, we try to weaken the notion of embedding, in order to make it possible for an orthomodular lattice to be embedded (in this weaker sense) into a Boolean algebra. A natural idea to overcome the above mentioned contradiction is to only ask that an embedding preserves the join for orthogonal elements. In such a case, we can as well generalize our discussion to orthomodular posets instead of orthomodular lattices.

2.2. A characterization of orthomodular posets that can be embedded into Boolean algebras

Before stating the main result of this section, let us introduce a few important notions.

DEFINITION 2.2.1. An orthomodular poset $(P, \subseteq, ^c, \emptyset, X)$, where X is a nonempty set, $P \subseteq 2^X$, order is defined by set-theoretical inclusion, the orthocomplement of an element $A \in P$ is the set-theoretical complement of A relative to X (denoted by A^c) and $\emptyset$ and X are the least and greatest elements of P, respectively, is a *set orthomodular poset* .

DEFINITION 2.2.2. A set orthomodular poset with the property that, for every $A, B \in P$, $A \cup B \in P$ whenever $A \cap B = \emptyset$ is a *concrete orthomodular poset.*

DEFINITION 2.2.3. Let P be an orthomodular poset. A mapping $s : P \to [0, 1]$ such that:

(1) $s(\mathbf{1}) = 1$;
(2) $s(a \vee b) = s(a) + s(b)$, whenever $a, b \in P$, $a \perp b$

is a *state* on P. If, moreover, the range of s is $\{0, 1\}$, then s is a *two- valued state* on P.

DEFINITION 2.2.4. A set $\mathcal{S}$ of states on an orthomodular poset P is *full* (or *order determining*) if, for every $a, b \in P$ with $a \not\leq b$, there exists a state $s \in \mathcal{S}$ such that $s(a) \not\leq s(b)$ (i.e., $s(a) \leq s(b)$ for all $s \in \mathcal{S}$ implies $a \leq b$).

THEOREM 2.2.5 (see [**53**, Theorem 2.2.1] or [**27**]). *An orthomodular poset has a representation as a concrete orthomodular poset if and only if it has a full set of two-valued states.*

Let us remark that an orthomodular poset has a concrete representation if and only if it can be embedded into a Boolean algebra and therefore, Theorem 2.2.5 gives in fact a necessary and sufficient condition for the existence of such an embedding—the existence of a full set of two-valued states defined on the orthomodular poset. However, this condition is quite restrictive. For instance, the standard Hilbert space orthomodular lattice of projectors, for a Hilbert space H of dimension higher than 2, has *no* two-valued states, not to mention a full set.

We must conclude that in general, we cannot embed orthomodular posets or lattices into Boolean algebras, if we expect that such an embedding to preserve the join of orthogonal elements.

2.3. The Zierler-Schlessinger theory

Let us further weaken the embedding notion, asking the join to be preserved only for central elements.

DEFINITION 2.3.1. Let L be an orthomodular lattice and B a Boolean algebra. A mapping $h : L \to B$ is a *Z-embedding* of L into B if the following conditions are satisfied:

(1) $h(\mathbf{1}) = \mathbf{1}$;
(2) $h(a') = h(a)'$ for every $a \in L$;
(3) $a \perp b$ if and only if $h(a) \perp h(b)$, for every $a, b \in L$;
(4) $h(a \vee b) = h(a) \vee h(b)$, for every pair of central elements $a, b \in L$.

THEOREM 2.3.2 ([**64**]). *For every orthomodular lattice L, there exist a Z-embedding into a power set Boolean algebra.*

In many cases, the center of an orthomodular lattice is rather “poor” or even trivial. In these cases, the above theorem gives us a type of embedding that preserves the join of a very limited number of elements.

2.4. The result of Harding and Pták

The following result of J. Harding and P. Pták [**35**] substantially improves the properties that can be obtained for an embedding of an orthomodular lattice into a Boolean algebra.

THEOREM 2.4.1 ([**35**]). *Let L be an orthomodular lattice and let B be a Boolean subalgebra of L. There exist a set $\mathcal{S}$ and a mapping $h : L \to 2^{\mathcal{S}}$ such that:*

(1) $h(\mathbf{1}) = \mathcal{S}$;
(2) $h(a') = h(a)^c$;
(3) $a \perp b$ *if and only if* $h(a) \cap h(b) = \emptyset$;
(4) $h(a \vee b) = h(a) \cup h(b)$, *for every* $a, b \in B$;
(5) $h(a \vee b) = h(a) \cup h(b)$ *whenever* $a \in \widetilde{C}(L)$.

CHAPTER 3

Spectral automorphisms in orthomodular lattices

In this chapter we present original research results that were published in articles [**36**] and [**11**]. We develop the theory of spectral automorphisms in orthomodular lattices and obtain in this framework some results that are analogues of the ones in the spectral theory in Hilbert spaces.

3.1. Spectral automorphisms—the idea

Let H be a Hilbert space and $\mathcal{P}(H)$ the orthomodular lattice of projection operators on H. Automorphisms of $\mathcal{P}(H)$ are of the form $\varphi_U : \mathcal{P}(H) \to \mathcal{P}(H)$, $\varphi_U(P) = UPU^{-1}$, with U being a unitary or an antiunitary operator on H. Let U be unitary and let B_U be the Boolean subalgebra of $\mathcal{P}(H)$ that is the range of the spectral measure associated to U. Then $P \in \mathcal{P}(H)$ is φ_U-invariant if and only if $UP = PU$ if and only if P commutes with B_U (i.e., commutes with every projection operator in B_U) if and only if $P \leftrightarrow B_U$. This suggests the definition of spectral automorphisms in orthomodular lattices.

3.2. Definition and basic facts

Definition 3.2.1. Let L be an orthomodular lattice and φ be an automorphism of L. The automorphism φ is *spectral* if there is a Boolean subalgebra B of L such that

(P1) $$\varphi(a) = a \text{ if and only if } a \leftrightarrow B\,.$$

A Boolean subalgebra of L satisfying condition (P1) is a *spectral algebra* of φ. The set of φ-invariant elements of L is denoted by L_φ.

Proposition 3.2.2. *For any spectral automorphism, there exists the greatest Boolean subalgebra having the property* (P1)

Definition 3.2.3. If $\varphi : L \to L$ is a spectral automorphism, the greatest Boolean subalgebra having the property (P1) is called *the spectrum of* φ and will be denoted by σ_φ.

Corollary 3.2.4. *If an orthomodular lattice has a nontrivial spectral automorphism, then it cannot be Boolean.*

Proposition 3.2.5. *The automorphism $\varphi : L \to L$ is spectral if and only if there is a Boolean subalgebra B of L such that $L_\varphi = K(B)$. In this case, $\sigma_\varphi = \widetilde{C}(L_\varphi)$.*

Corollary 3.2.6. *The automorphism $\varphi : L \to L$ is spectral if and only if $K(\widetilde{C}(L_\varphi)) \subseteq L_\varphi$*

3.3. Spectral automorphisms in products and horizontal sums

We discuss the construction of spectral automorphisms in products and horizontal sums of orthomodular lattices. A factorization of the spectra of spectral automorphisms is also studied.

THEOREM 3.3.1. *Let L be the product of a collection $(L_i)_{i\in I}$ of orthomodular lattices and, for every $i \in I$, φ_i be an automorphism of L_i. Let us define the mapping $\varphi\colon L \to L$ by $\varphi\big((a_i)_{i\in I}\big) = \big(\varphi_i(a_i)\big)_{i\in I}$. Then:*

(1) φ is an automorphism of L;
(2) φ is spectral if and only if φ_i is spectral for every $i \in I$; in this case, $\sigma_\varphi = \prod_{i\in I} \sigma_{\varphi_i}$.

LEMMA 3.3.2. *Let L be the horizontal sum of a collection $(L_i)_{i\in I}$ of orthomodular lattices such that every summand is minimal, φ be an automorphism of L such that $\varphi(L_i) \cap L_i \neq \{\mathbf{0}, \mathbf{1}\}$ for some $i \in I$. Then the restriction φ_i of φ to L_i is an automorphism of L_i.*

THEOREM 3.3.3. *Let L be the horizontal sum of a collection $(L_i)_{i\in I}$ of orthomodular lattices such that every summand is minimal and φ be an automorphism of L.*

(1) *If φ is spectral then there is an $i \in I$ such that $L_\varphi \subset L_i$, $\sigma_\varphi \subset L_i$ and the restriction φ_i of φ to L_i is a spectral automorphism of L_i with $L_{\varphi_i} = L_\varphi$ and $\sigma_{\varphi_i} = \sigma_\varphi$.*

(2) *If $L_\varphi \neq \{\mathbf{0}, \mathbf{1}\}$ and there is an $i \in I$ such that $L_\varphi \subset L_i$ and the restriction φ_i of φ to L_i is spectral then φ is a spectral automorphism of L and $L_\varphi = L_{\varphi_i}$, $\sigma_\varphi = \sigma_{\varphi_i}$.*

THEOREM 3.3.4. *Let L be an orthomodular lattice, φ be a spectral automorphism of L and $a \in L \setminus \{\mathbf{0}, \mathbf{1}\}$ be φ-invariant. Let us denote by φ_a the restriction of φ to $[\mathbf{0}, a]$, and let $B_x = x \wedge \sigma_\varphi = \{x \wedge b\colon\ b \in \sigma_\varphi\}$ for every $x \in L$. Then:*

(1) φ_a is a spectral automorphism of $[\mathbf{0}, a]$ and B_a is its spectral algebra;
(2) if $a \in \sigma_\varphi$, then $\sigma_{\varphi_a} = B_a$;
(3) σ_φ is isomorphic to the product $B_a \times B_{a'}$.

QUESTION 3.3.5. Is it possible to omit the condition $a \in \sigma_\varphi$ in Theorem 3.3.4 (2)?

3.4. Characterizations of spectral automorphisms

THEOREM 3.4.1. *Let L be an orthomodular lattice. An automorphism φ of L is spectral if and only if $a \wedge b \in L_\varphi$ for every $a \in \widetilde{C}(L_\varphi)$ and $b \in K\big(\widetilde{C}(L_\varphi)\big)$.*

DEFINITION 3.4.2. Let L be an orthomodular lattice and φ an automorphism of L. An element $a \in L$ is *totally φ-invariant* if $\varphi(b) = b$ for every $b \in L$ with $b \leq a$.

THEOREM 3.4.3. *Let L be a complete orthomodular lattice and φ be an automorphism of L such that $\widetilde{C}(L_\varphi)$ is atomic. Then φ is spectral if and only if all atoms of $\widetilde{C}(L_\varphi)$ are totally φ-invariant.*

COROLLARY 3.4.4. *Let L be a complete orthomodular lattice and φ be an automorphism of L such that $\widetilde{C}(L_\varphi)$ is atomic. If all atoms of $\widetilde{C}(L_\varphi)$ are atoms of L then φ is spectral.*

THEOREM 3.4.5. *Let L be a complete orthomodular lattice and φ be an automorphism of L such that L_φ and $\widetilde{C}(L_\varphi)$ are atomic. If φ is spectral then all atoms of L_φ are atoms of L.*

3.5. C-maximal Boolean subalgebras of an OML

DEFINITION 3.5.1. Let L be an orthomodular lattice. A Boolean subalgebra $B \subseteq L$ satisfying $\widetilde{C}(K(B)) \subseteq B$ is said to be *C-maximal* (i.e. maximal with respect to its commutant).

THEOREM 3.5.2. *A Boolean subalgebra of an orthomodular lattice is C-maximal if and only if it coincides with its bicommutant.*

THEOREM 3.5.3. *If the automorphism φ is spectral, then the following assertions are true:*

(1) $\widetilde{C}(L_\varphi) = K(L_\varphi)$*;*
(2) $\widetilde{C}(L_\varphi)$ *is C-maximal.*

3.6. Spectral automorphisms and physical theories

Let us consider a theory represented by an orthomodular lattice L, which is atomic, complete and has the covering property. Assume also that this theory has a spectral automorphism φ whose spectrum is atomic. Examples of such theories exist, such as a finite dimensional quantum logic. Then, we can construct a basis of L, whose elements are invariant under φ. This result is an analogue of the spectral theorem for unitary operators having purely point spectrum.

3.7. Spectral automorphisms and Piron's theorem

Piron's representation theorem (see [**50**]) allows us to consider that non-classical theories are based on the Hilbert space formalism. However, a question remains: is the Hilbert space real, complex or quaternionic?

We show that, if there are physical motivations for admitting that spectral symmetries (other than simple reflexions relative to a hyperplane) must exist in a theory, then the real Hilbert spaces have to be excluded from those able to support quantum theories.

3.8. Spectral families of automorphisms and a Stone-type theorem

DEFINITION 3.8.1. Let L be an orthomodular lattice and Φ be a family of automorphisms of L. The family Φ is *spectral* if there is a Boolean subalgebra B of L such that:

$$\text{(P2)} \qquad (\varphi(a) = a \text{ for every } \varphi \in \Phi) \text{ if and only if } a \leftrightarrow B.$$

A Boolean algebra B satisfying condition (P2) is a *spectral algebra of* Φ. The set of Φ-invariant elements of L (which is a subalgebra of L) is denoted by L_Φ.

PROPOSITION 3.8.2. *For every spectral family Φ of automorphisms of an orthomodular lattice L there exists the greatest spectral algebra of the family Φ.*

DEFINITION 3.8.3. Let Φ be a spectral family of automorphisms of an orthomodular lattice. The *spectrum σ_Φ of the family* Φ is the greatest spectral algebra of the family Φ.

PROPOSITION 3.8.4. *Let Φ be a spectral family of automorphisms of an orthomodular lattice L. Then:*

(1) $\sigma_\Phi = \widetilde{C}(L_\Phi)$;

(2) $\sigma_\Phi = \widetilde{C}\big(K(\sigma_\Phi)\big)$ *(i.e., σ_Φ is C-maximal);*

(3) $\sigma_\Phi = K\big(K(\sigma_\Phi)\big)$.

THEOREM 3.8.5. *Let L be an orthomodular lattice and Φ be a family of spectral automorphisms of L. Then Φ is a spectral family if and only if $\sigma_\varphi \leftrightarrow \sigma_\psi$ for every $\varphi, \psi \in \Phi$. In this case, the spectrum σ_Φ of the family contains all spectra σ_φ, $\varphi \in \Phi$.*

The purpose of introducing and studying spectral automorphisms has been to construct something similar to the Hilbert space spectral theory without using the specific instruments available in a Hilbert space setting, but using only the abstract orthomodular lattice structure. The next result is intended as an analogue of the Stone theorem concerning strongly continuous uniparametric groups of unitary operators.

THEOREM 3.8.6. *Let L be an orthomodular lattice and Φ be a family of spectral automorphisms of L. If Φ is an Abelian group and $\varphi(L_\psi) = L_{\varphi\psi}$ for every $\varphi, \psi \in \Phi$ with $\psi \notin \{\mathrm{id}, \varphi^{-1}\}$, then:*

(1) $L_\varphi = L_\psi$ *for every* $\varphi, \psi \in \Phi \setminus \{\mathrm{id}\}$;

(2) $\sigma_\varphi = \sigma_\psi$ *for every* $\varphi, \psi \in \Phi \setminus \{\mathrm{id}\}$;

(3) Φ *is a spectral family.*

Part 2

Unsharp Quantum Logics

CHAPTER 4

Basics on effect algebras

In this second part of the thesis, we move our investigations from the framework of orthomodular posets or lattices—which may be considered as representing “sharp” quantum logics—to the more general framework of effect algebras—regarded as “unsharp” quantum logics.

The first chapter of this second part of the thesis is devoted to an introduction to effect algebras and their basic properties. The facts presented in this chapter can be found, e.g., in [**22, 23, 24, 26, 40, 14**].

4.1. Effect algebras. Basic definitions and properties

Definition 4.1.1. An *effect algebra* is an algebraic structure $(E, \oplus, \mathbf{0}, \mathbf{1})$ such that E is a set, $\mathbf{0}$ and $\mathbf{1}$ are distinct elements of E and $\oplus$ is a partial binary operation on E, and the following conditions hold for every $a, b, c \in E$ (the equalities should be understood in the sense that if one side exists, the other side exists as well):

(EA1) $a \oplus b = b \oplus a$ (commutativity)
(EA2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity)
(EA3) for every $a \in E$, there exists a unique $a' \in E$ such that $a \oplus a' = \mathbf{1}$ (orthosupplement)
(EA4) if $a \oplus \mathbf{1}$ is defined, then $a = \mathbf{0}$ (zero-unit law).

An orthogonality and a partial order relation are defined in an effect algebra as follows:

Definition 4.1.2. Let E be an effect algebra. Elements $a, b \in E$ are called *orthogonal* (denoted by $a \perp b$) if the sum $a \oplus b$ is defined. We write $a \leq b$ if there is an element $c \in E$ such that $a \oplus c = b$.

The next couple of propositions gives a list of basic properties that hold in effect algebras.

Proposition 4.1.3. *Let E be an effect algebra. For every $a, b \in E$ the following properties hold:*

(1) $a'' = a$
(2) $a \leq b$ *implies* $b' \leq a'$
(3) $\mathbf{1}' = \mathbf{0}$ *and* $\mathbf{0}' = \mathbf{1}$.

Proposition 4.1.4. *Let E be an effect algebra. For every $a, b, c \in E$ the following properties hold:*

(1) $\mathbf{0} \leq a \leq \mathbf{1}$.
(2) $a \oplus \mathbf{0} = a$.
(3) $a \perp b$ *if and only if* $a \leq b'$.

(4) *If* $a \leq b$ *and* $c \in E$ *is such that* $a \oplus c = b$*, then* c *is uniquely determined by the elements* a *and* b*, namely* $c = (a \oplus b')'$*. We will then denote* $c = b \ominus a$*.*
(5) *"*$\leq$*" is a partial order on* E*.*
(6) $a \oplus b = a \oplus c$ *implies* $b = c$ *(cancellation law).*
(7) $a \oplus b \leq a \oplus c$ *implies* $b \leq c$ *(cancellation law).*

Definition 4.1.5. Let E be an effect algebra and $F \subset E$. If $\mathbf{0}, \mathbf{1} \in F$, and F is closed to $\oplus$ and to orthosupplementation, then $(F, \oplus|_{F\times F}, \mathbf{0}, \mathbf{1})$ is a *sub-effect algebra* of E.

4.2. Special elements. Coexistence

Definition 4.2.1. An element a of an effect algebra E is called:

- *isotropic* if $a \perp a$;
- *sharp* ($a \in E_S$) if $a \wedge a' = \mathbf{0}$;
- *principal* if for every orthogonal pair $b, c \in E$, $b, c \leq a$ we have $b \oplus c \leq a$;
- *central* if a, a' are principal and for every $b \in E$, there are $b_1, b_2 \in E$ such that $b_1 \leq a$, $b_2 \leq a'$ and $b = b_1 \oplus b_2$.

Proposition 4.2.2. *In an effect algebra, the following assertions hold:*
(1) *every central element is principal;*
(2) *every principal element is sharp;*
(3) *every nonzero sharp element is nonisotropic.*

In general, the converse statements do not hold.

Definition 4.2.3. An *orthoalgebra* is an effect algebra whose only isotropic element is $\mathbf{0}$.

Definition 4.2.4. Let E be an effect algebra and let us denote by na the sum of n copies of an element $a \in E$, if it exists. We call E *Archimedean* if $\sup\{n \in \mathbb{N} \colon na \text{ is defined}\} < \infty$ for every nonzero element $a \in E$.

Remark 4.2.5. Let $(E, \leq, ', \mathbf{0}, \mathbf{1})$ be an orthomodular poset and define $a \oplus b = a \vee b$ for every orthogonal (i.e. $a \leq b'$) pair of elements $a, b \in E$. It is a routine verification that $(E, \oplus, \mathbf{0}, \mathbf{1})$ is an effect algebra (an orthoalgebra even) and, moreover, the order and supplement in the effect algebra coincide with the order and complement in the orthomodular poset.

Definition 4.2.6. Let E be an effect algebra and $a, b \in E$. Elements a and b *coexist in* E if there are $a_1, b_1, c \in E$ such that $a = a_1 \oplus c$, $b = b_1 \oplus c$ and $a_1 \oplus b_1 \oplus c$ exists in E. In this case, we write $a \leftrightarrow b$. For a subset M of E, we write $a \leftrightarrow M$ if $a \leftrightarrow b$ for all $b \in M$. The *commutant* of M in E is the set $K_E(M) = \{a \in E \colon a \leftrightarrow M\}$. If there is no possibility of confusion concerning E, we shall simply denote it by $K(M)$.

Proposition 4.2.7. *Let* E *be an effect algebra. Then:*
(1) *an element* $a \in E$ *is central if and only if it coexists with all elements of* E *and* a, a' *are principal;*
(2) *if* E *is an orthomodular poset, the set of central elements of* E *coincides with the center of* E *as an orthomodular poset.*

Remark 4.2.8. Coexistence generalizes compatibility to effect algebras (see Theorem 4.2.10). The two notions coincide in orthomodular posets. This justifies the use of the same notation for coexistence and compatibility and also for the commutant with respect to coexistence or compatibility.

The notion of center of an effect algebra, defined as the set of its central elements, generalizes the notion of center in orthomodular posets, defined as the set of its elements which are compatible with all the others. We shall denote the center of an effect algebra E by $\widetilde{C}(E)$.

Theorem 4.2.9. [**26**, Theorem 5.4] *The center $\widetilde{C}(E)$ of an effect algebra E is a sub-effect algebra of E and as an effect algebra in its own right, $\widetilde{C}(E)$ forms a Boolean algebra. Furthermore, if $a, b \in \widetilde{C}(E)$, then $a \wedge b$ and $a \vee b$ as calculated in $\widetilde{C}(E)$ are also the infimum and supremum of a and b as calculated in E.*

Theorem 4.2.10 (see [**22, 23**]). *An effect algebra is:*

1. *an orthoalgebra if and only if its every element is sharp if and only if $a \oplus b$ is a minimal upper bound of a, b for every orthogonal pair $a, b \in E$;*
2. *an orthomodular poset if and only if its every element is principal if and only if $a \oplus b = a \vee b$ for every orthogonal pair $a, b \in E$;*
3. *a Boolean algebra if and only if its every element is central.*

4.3. Substructures in effect algebras

Definition 4.3.1. A *Boolean subalgebra* of an effect algebra E is a sub-effect algebra of E which is a Boolean algebra with $'$ and with the operations $\vee$, $\wedge$ induced by the order in E.

Proposition 4.3.2. *Let E be an orthoalgebra and let $F \subseteq E$ be a Boolean subalgebra of E. Then:*

1. *If $a, b \in F$ and $a \wedge_E b$ exists, then $a \wedge_E b = a \wedge_F b$;*
2. *If $a, b \in F$ and $a \vee_E b$ exists, then $a \vee_E b = a \vee_F b$;*

4.4. Important classes of effect algebras

We present a few important properties that an effect algebra may fulfill and their correlations.

Definition 4.4.1. An effect algebra that is a lattice with respect to its usual order relation, is called a *lattice effect algebra.*

Definition 4.4.2. Let E be an effect algebra. A system $(a_i)_{i \in I}$ of elements of E is *orthogonal* if $\bigoplus_{i \in F} a_i$ is defined for every finite set $F \subset I$. A *majorant* of an orthogonal system $(a_i)_{i \in I}$ is an upper bound of $\{\bigoplus_{i \in F} a_i : F \subset I \text{ is finite}\}$. The *sum* of an orthogonal system is its least majorant (if it exists).

Definition 4.4.3. An effect algebra E is *orthocomplete* if every orthogonal system of its elements has a sum. An effect algebra E is *weakly orthocomplete* if every orthogonal system in E has a sum or no minimal majorant.

DEFINITION 4.4.4. An effect algebra E has the *maximality* property if the set $\{a, b\}$ has a maximal lower bound for every $a, b \in E$.

REMARK 4.4.5. The maximality property was introduced by Tkadlec [**58**]. In [**61**, Theorem 2.2], he proved that effect algebras with the maximality property or the ones that are weakly orthocomplete are common generalizations of lattice effect algebras and orthocomplete effect algebras. Since finite or chain-finite effect algebras are orthocomplete (see [**60**, Theorem 4.1]), they also must satisfy the maximality property.

DEFINITION 4.4.6. E is *determined by atoms* if, for different $a, b \in E$, the sets of atoms dominated by a and b are different.

LEMMA 4.4.7 ([**59**, Lemma 2.2]). *Every atomistic effect algebra is determined by atoms. Every effect algebra determined by atoms is atomic.*

Examples showing that the converse implications do not hold can be found in [**25, 59**].

4.5. Morphisms of effect algebras

DEFINITION 4.5.1. Let E and E' be effect algebras and let $\varphi : E \to E'$ be a map. We call φ an *additive* map if $a \perp b$ implies $\varphi(a) \perp \varphi(b)$ and $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$, for every $a, b \in E$. We call φ is a *morphism* of effect algebras if it is additive and $\varphi(\mathbf{1}_E) = \mathbf{1}_{E'}$. A morphism φ of effect algebras which preserves the infimum (i.e., $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$, whenever $a \wedge b$ exists) is a $\wedge$-*morphism*. A bijective morphism φ such that φ^{-1} is also a morphism is an *isomorphism*. An isomorphism $\varphi : E \to E$ is an *automorphism*.

PROPOSITION 4.5.2. *Let E and E' be effect algebras and let $\varphi : E \to E'$ be a map. Then φ is an isomorphism of effect algebras if and only if it is bijective and, for every $a, b \in E$, $a \perp b$ if and only if $\varphi(a) \perp \varphi(b)$, in which case $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$. Moreover, if φ is an isomorphism, it is also a $\wedge$-morphism.*

CHAPTER 5

Sequential, compressible and compression base effect algebras

In this chapter, we present the important established facts concerning sequential, compressible and compression base effect algebras, laying the foundation for the new results that will be the presented in the following chapters. Our presentation is based on [**28, 29, 31, 54**].

5.1. Sequential effect algebras

The notion of a sequential product defined in general effect algebras was introduced by Gudder and Greechie [**31**]. This sequential product satisfies a set of physically motivated axioms as it formalizes the case of sequentially performed measurements.

Definition 5.1.1. A *sequential product* on an effect algebra $(E, \oplus, \mathbf{0}, \mathbf{1})$ is a binary operation $\circ$ on E such that for every $a, b, c \in E$, the following conditions hold:

(S1) $a \circ (b \oplus c) = (a \circ b) \oplus (a \circ c)$ if $b \oplus c$ exists;
(S2) $\mathbf{1} \circ a = a$;
(S3) if $a \circ b = \mathbf{0}$ then $a \mid b$ (where $a \mid b$ denotes $a \circ b = b \circ a$);
(S4) if $a \mid b$ then $a \mid b'$ and $a \circ (b \circ c) = (a \circ b) \circ c$;
(S5) if $c \mid a, b$ then $c \mid a \circ b$ and $c \mid (a \oplus b)$ (if $a \oplus b$ exists).

An effect algebra E endowed with a sequential product is called a *sequential effect algebra*.

5.2. Compressible effect algebras

Definition 5.2.1. Let E be an effect algebra and $J : E \to E$ be an additive map. If $a \leq J(\mathbf{1})$ implies $J(a) = a$, then J is a *retraction*. In this case, $J(\mathbf{1})$ is the *focus* of J. If, moreover, $J(a) = \mathbf{0}$ implies $a \leq J(\mathbf{1})'$, then J is a *compression*. The focus of a retraction on E is a *projection*.

Proposition 5.2.2 (see [**28**, Lemma 3.1, Lemma 3.2, Lemma 3.3]). *Let E be an effect algebra and $J : E \to E$ be a compression with focus p. Then:*

(1) *J is idempotent;*
(2) *J preserves order;*
(2) $\mathrm{Ker}(J) = [\mathbf{0}, p']$*;*
(3) *$J(E) = [\mathbf{0}, p]$;*
(4) *p is principal and therefore sharp;*

Definition 5.2.3. An effect algebra E is *compressible* if every retraction on E is a compression and it is uniquely determined by its focus.

REMARK 5.2.4. If E is a sequential effect algebra, the sequential product with a sharp (and therefore principal) element $p \in E_S$ defines a compression with focus p by $J_p(a) = p \circ a$ [**29**]. If, moreover, E is compressible, then $J_p : E \to E$, $J_p(a) = p \circ a$ is the unique compression on E with focus p. The close relation between sequential and compressible effect algebras becomes now evident.

5.3. Compression bases in effect algebras

Effect algebras with compression bases are a common generalization of compressible and sequential effect algebras. Our presentation of compression base effect algebras is based on [**29, 54**].

DEFINITION 5.3.1. Let E be an effect algebra. A sub-effect algebra F of E is *normal* if, for every $a, b, c \in E$ such that $a \oplus b \oplus c$ exists in E and $a \oplus b, b \oplus c \in F$, it follows that $b \in F$.

DEFINITION 5.3.2. Let E be an effect algebra. A system $(J_p)_{p \in P}$ of compressions on E indexed by a normal sub-effect algebra P of E is called a *compression base* for E if the following conditions hold:

(1) Each compression J_p has the focus p.
(2) If $p, q, r \in P$ and $p \oplus q \oplus r$ is defined in E, then $J_{p \oplus r} \circ J_{r \oplus q} = J_r$.

THEOREM 5.3.3 ([**29**, Theorems 3.3 and 3.4]). (1) *If E is a compressible effect algebra, then the set $P(E)$ of its projections is a normal sub-effect algebra of E and $(J_p)_{p \in P(E)}$ is a compression base for E.*

(2) *If E is a sequential effect algebra, then the set E_S of its sharp elements is a normal sub-effect algebra of E. If, for every $p \in E_S$, J_p is the compression on E defined by $J_p(a) = p \circ a$, for every $a \in E$, then $(J_p)_{p \in E_S}$ is a maximal compression base for E.*

For an effect algebra E with a compression base $(J_p)_{p \in P}$ we will maintain, from now on, the following notations:

- $p \circ a = J_p(a)$ for every $p \in P$ and $a \in E$;
- $p \mid q$ if $p, q \in P$ and $p \circ q = q \circ p$ (i.e., $J_p(q) = J_q(p)$);
- $C(p) = \{a \in E\colon\ a = J_p(a) \oplus J_{p'}(a)\}$ for every $p \in P$.

DEFINITION 5.3.4. A compression base $(J_p)_{p \in P}$ on the effect algebra E has the *projection cover property* if for every element $a \in E$ there exists the least element $b \in P$ (the *projection cover* of a) with $b \geq a$.

THEOREM 5.3.5 (see [**54**, Theorem 5.1]). *Let E be an effect algebra with a compression base $(J_p)_{p \in P}$ that has the projection cover property. Then P is an orthomodular lattice.*

CHAPTER 6

Spectral automorphisms in CB-effect algebras

In the third chapter we have introduced spectral automorphisms (see also [**36**]). They resulted from our attempt to construct, in the abstract framework of orthomodular lattices, an analogue of the spectral theory in Hilbert spaces. We generalize spectral automorphisms to the framework of compression base effect algebras, currently considered as the appropriate mathematical structures for representing physical systems [**21**]. The results presented here are accepted for publication in [**8**].

6.1. Spectral automorphisms: the idea and definitions

Let H be a Hilbert space and $\mathcal{E}(H)$ the corresponding standard effect algebra. Automorphisms of $\mathcal{E}(H)$ are of the form $\varphi_U : \mathcal{E}(H) \to \mathcal{E}(H)$, $\varphi_U(A) = UAU^{-1}$, where U is a unitary or antiunitary Hilbert space operator [**21**]. An element $A \in \mathcal{E}(H)$ is φ_U-invariant if and only if $\varphi_U(A) = UAU^{-1} = A$, i.e., operators U and A commute. Let B_U be the Boolean algebra of projection operators that is the image of the projection-valued spectral measure associated to U. Then, operators A and U commutes if and only if A commutes with B_U (i.e., with every projection operator in B_U) [**34**]. We are therefore led to the following definition of spectral automorphisms in compression base effect algebras:

Definition 6.1.1. Let E be an effect algebra and $(J_p)_{p\in P}$ be a compression base for E. An automorphism $\varphi : E \to E$ is *spectral* if there exists a Boolean subalgebra B of P with the property:

(P1) $$\varphi(a) = a \text{ if and only if } a \leftrightarrow B$$

Proposition 6.1.2. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and $\varphi : E \to E$ be a spectral automorphism. There exists the greatest Boolean subalgebra $B \subseteq P$ satisfying* (P1).

Definition 6.1.3. Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and $\varphi : E \to E$ be a spectral automorphism. The greatest Boolean subalgebra of P fulfilling (P1) is the *spectrum* of the automorphism φ, denoted by σ_φ^P.

Proposition 6.1.4. *Let E be an effect algebra and $(J_p)_{p\in P}$ be a compression base for E. If $P \subseteq \widetilde{C}(E)$, then the identity is the only spectral automorphism of E.*

Remark 6.1.5. As a particular case, if E is a Boolean algebra, then its identity is its only spectral automorphism. Therefore, the presence of nontrivial spectral automorphisms allows us to distinguish between classical (Boolean) and nonclassical theories.

6.2. Characterizations and properties of spectral automorphisms

For an automorphism φ of an effect algebra E, we will denote by E_φ the set of φ-invariant elements of E. Due to the definition properties of automorphisms, it is clear that E_φ is a sub-effect algebra of E.

The following lemma and corollary, that will be useful in the sequel, are related to [**26**, Theorem 4.2 and Lemma 5.2]. However, the statements we prove are slightly more general and could be interesting in their own right.

LEMMA 6.2.1. *Let E be an effect algebra, $\{e_1, e_2, \ldots, e_n\}$ be an orthogonal set of its elements (i.e., the sum $\bigoplus_{i=1}^n e_i$ exists) and consider $p \in E$ such that $p = \bigoplus_{i=1}^n p_i$ with $p_i \leq e_i$. If e_j is principal for some $j \in \{1, 2, \ldots, n\}$, then $p \wedge e_j$ exists in E and $p_j = p \wedge e_j$.*

COROLLARY 6.2.2. *If a, a' are principal elements of the effect algebra E, $b \in E$ and $a \leftrightarrow b$, then $a \wedge b$ and $a' \wedge b$ exist in E and $b = (a \wedge b) \oplus (a' \wedge b)$.*

THEOREM 6.2.3. *Let E be an effect algebra and $(J_p)_{p \in P}$ be a compression base for E. If $\varphi : E \to E$ is a spectral automorphism, then $\sigma_\varphi^P = \widetilde{C}(E_\varphi) \cap P$.*

COROLLARY 6.2.4. *Let E be an effect algebra, $(J_p)_{p \in P}$ be a compression base for E and $\varphi : E \to E$ be an automorphism. Then φ is spectral if and only if $K(\widetilde{C}(E_\varphi) \cap P) \subseteq E_\varphi$ (the converse inclusion is always true).*

THEOREM 6.2.5. *Let E be an effect algebra, $(J_p)_{p \in P}$ be a compression base for E and $\varphi : E \to E$ be an automorphism. Then φ is spectral if and only if $a \wedge b \in E_\varphi$ for every $a \in \widetilde{C}(E_\varphi) \cap P$, $b \in K(\widetilde{C}(E_\varphi) \cap P)$.*

The search for the conditions that a Boolean algebra must fulfill in order to be the spectrum of a spectral automorphism leads to the following notion.

DEFINITION 6.2.6. Let E be an effect algebra and $(J_p)_{p \in P}$ be a compression base for E. A Boolean subalgebra $B \subseteq P$ is *C-maximal* if $\widetilde{C}(K(B)) \cap P \subseteq B$.

THEOREM 6.2.7. *Let E be an effect algebra and $(J_p)_{p \in P}$ be a compression base for E. A Boolean subalgebra $B \subseteq P$ is C-maximal if and only if $B = K(K(B)) \cap P$.*

COROLLARY 6.2.8. *Let E be an effect algebra, $(J_p)_{p \in P}$ be a compression base for E and $\varphi : E \to E$ be a spectral automorphism. Then:*

(1) *$\sigma_\varphi^P = \widetilde{C}(E_\varphi) \cap P$ is C-maximal;*
(2) *$\sigma_\varphi^P = K(K(\sigma_\varphi^P)) \cap P$;*
(3) *$\sigma_\varphi^P = K(E_\varphi) \cap P$.*

6.3. An application of spectral automorphisms to $\mathcal{E}(H)$

The notion of spectral automorphism was introduced with the declared intention to obtain an analogue of the Hilbert space spectral theory in the abstract setting of compression base effect algebras. It is time to see if this attempt was successful, by applying the abstract theory to the particular case of the standard Hilbert space effect algebra. Therefore, we devote this section to the proof of a "spectral theorem" in $\mathcal{E}(H)$, the set of self-adjoint operators between the null and the identity operators, for a finite-dimensional Hilbert space H.

Let us denote in the sequel by $\hat{e}$ the 1-dimensional subspace generated by $e \in H$, $\|e\| = 1$ and $P_{\hat{e}}$ the corresponding projection operator, i.e., $P_{\hat{e}} : H \to H$, $P_{\hat{e}}x = \langle x, e\rangle e$ (where $\langle\cdot,\cdot\rangle$ denotes the inner product of H).

THEOREM 6.3.1. *Let H be an n-dimensional Hilbert space, $\mathcal{E}(H)$ be its standard effect algebra and $(J_P)_{P\in\mathcal{P}(H)}$ be the canonical compression base for $\mathcal{E}(H)$. Let $U : H \to H$ be a unitary operator and $\varphi : \mathcal{E}(H) \to \mathcal{E}(H)$ be the automorphism defined by $\varphi(A) = UAU^{-1}$. If φ is spectral, then:*

(1) *There is an orthogonal basis $\{e_1, e_2, \ldots, e_n\}$ of H such that for every $i \in \{1, 2, \ldots, n\}$, $Ue_i = \lambda_i e_i$ where λ_i is a scalar, $|\lambda_i| = 1$.*
(2) *There exists a partition Π of the set $\{1, 2, \ldots, n\}$ such that any φ-invariant atom of $\mathcal{P}(H)$ is a 1-dimensional subspace in exactly one of the subspaces $\bigvee_{j\in J} \hat{e}_j$, $J \in \Pi$.*
(3) *If the subalgebra $\mathcal{E}(H)_\varphi$ of φ-invariant elements of $\mathcal{E}(H)$ is Boolean, then the spectrum $\sigma_\varphi^{\mathcal{P}(H)} = \mathcal{E}(H)_\varphi \cap \mathcal{P}(H)$ is a block in $\mathcal{P}(H)$. In this case all eigenvalues of U are distinct and $\Pi = \big\{\{1\}, \{2\}, \ldots, \{n\}\big\}$.*
(4) *The spectrum $\sigma_\varphi^{\mathcal{P}(H)}$ is the Boolean algebra generated by $\big\{\bigvee_{j\in J} \hat{e}_j : J \in \Pi\big\}$.*
(5) *If the effect $A \in \mathcal{E}(H)$ is φ-invariant and $P \in \mathcal{P}(H)$ is the smallest projection that dominates A (namely the projection on the range of A), then P is φ-invariant too.*
(6) *If A is a φ-invariant nonzero effect dominated by an atom of $\mathcal{P}(H)$, then the range of A is included in one of the subspaces $\bigvee_{j\in J} \hat{e}_j$, $J \in \Pi$.*

REMARK 6.3.2. The properties (1)–(6) from Theorem 6.3.1 were derived only from the fact that φ is spectral, without any other information except for the properties of unitary operators.

6.4. Spectral families of automorphisms

Let E denote, for the rest of this section, an effect algebra endowed with a compression base $(J_p)_{p\in P}$ and let Φ be a family of automorphisms of E.

DEFINITION 6.4.1. The family Φ of automorphisms of E is called a *spectral family of automorphisms* if there exists a Boolean subalgebra B_Φ of P satisfying:

(P2) $$\varphi(a) = a, \text{ for all } \varphi \in \Phi \text{ if and only if } a \leftrightarrow B_\Phi$$

In the sequel, we denote $E_\Phi = \{a \in E\colon \varphi(a) = a, \text{ for all } \varphi \in \Phi\}$. Let us remark that $E_\Phi = \bigcap_{\varphi\in\Phi} E_\varphi$ and therefore it's a sub-effect algebra of E.

PROPOSITION 6.4.2. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a spectral family of automorphisms of E. There exists the greatest Boolean subalgebra B_Φ of P satisfying* (P2).

DEFINITION 6.4.3. Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a spectral family of automorphisms of E. The *spectrum* (denoted by σ_Φ^P) of the spectral family Φ of automorphisms is the greatest Boolean subalgebra B of P fulfilling (P2).

THEOREM 6.4.4. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a spectral family of automorphisms of E. Then $\sigma_\Phi^P = \widetilde{C}(E_\Phi) \cap P$.*

COROLLARY 6.4.5. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a family of automorphisms of E. Then Φ is a spectral family if and only if $K(\widetilde{C}(E_\Phi) \cap P) \subseteq E_\Phi$ (the converse inclusion is trivially satisfied).*

PROPOSITION 6.4.6. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a spectral family of automorphisms of E. Then:*

(1) $\sigma_\Phi^P = \widetilde{C}(E_\Phi) \cap P$ *is C-maximal;*
(2) $\sigma_\Phi^P = K(K(\sigma_\Phi^P)) \cap P$*;*
(3) $\sigma_\Phi^P = K(E_\Phi) \cap P$*.*

THEOREM 6.4.7. *Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a family of spectral automorphisms of E. Then Φ is a spectral family of automorphisms if and only if the spectra of the automorphisms in the family are pairwise compatible, i.e., $\sigma_\varphi^P \leftrightarrow \sigma_\psi^P$ for every $\varphi, \psi \in \Phi$. In this case, σ_Φ^P includes all spectra of automorphisms in the family.*

THEOREM 6.4.8. *(A "replica" of Stone's Theorem on strongly continuous uniparametric groups of unitary operators.) Let E be an effect algebra, $(J_p)_{p\in P}$ be a compression base for E and Φ be a family of spectral automorphisms of E. If the following conditions are fulfilled:*
(i) Φ is an abelian group; (ii) $\varphi(E_\psi) = E_{\varphi\psi}$ for every $\varphi, \psi \in \Phi$ such that $\psi \notin \{id_E, \varphi^{-1}\}$, then:

(1) $E_\varphi = E_\psi$ *for all* $\varphi, \psi \in \Phi \setminus \{id_E\}$*;*
(2) $\sigma_\varphi^P = \sigma_\psi^P$ *for all* $\varphi, \psi \in \Phi \setminus \{id_E\}$*;*
(3) Φ *is a spectral family.*

Let us remark that Theorem 6.4.8 generalizes Theorem 3.8.6 to spectral automorphisms in CB-effect algebras.

CHAPTER 7

Atomic effect algebras with compression bases

In the first section we establish some properties of atoms in effect algebras endowed with a compression base, mainly regarding coexistence and centrality. Then, in the second section, we introduce the notion of projection-atomicity—which is the property of a compression base effect algebra of having the atoms compressions foci. Consequences of projection-atomicity are studied, some of which generalize results obtained in [**59**]. A few conditions for an atomic compression base effect algebra to be a Boolean algebra are established. Finally, we apply these results to the particular case of sequential effect algebras and find a sufficient condition for them to be Boolean algebras that strengthens previous results by Gudder and Greechie [**31**] and Tkadlec [**59**]. The results presented here have been published in [**10**].

7.1. Atoms and centrality

PROPOSITION 7.1.1. *Let E be an effect algebra. If p is an atom in E that is the focus of a compression and $a \in E$ then $p \leq a$ or $p \leq a'$.*

COROLLARY 7.1.2. *Distinct atoms that are foci of compressions in an effect algebra are orthogonal.*

THEOREM 7.1.3. *Let E be an effect algebra with a compression base $(J_p)_{p \in P}$. If E is determined by atoms and every atom is in P then P is a Boolean algebra.*

The conclusion of the above theorem cannot be improved to the statement that E is a Boolean algebra as we show by an example.

LEMMA 7.1.4. *Let E be an effect algebra with a compression base $(J_p)_{p \in P}$. If $p \in P$ is an atom in E then $C(p) = E$.*

THEOREM 7.1.5. *Let E be an effect algebra with a compression base $(J_p)_{p \in P}$. Every $p \in P$ that is an atom in E is central in E.*

7.2. Projection-atomic effect algebras

DEFINITION 7.2.1. An effect algebra E is *projection-atomic* if it is atomic and there is a compression base $(J_p)_{p \in P}$ of E such that P contains all atoms in E.

PROPOSITION 7.2.2. *Every projection-atomic effect algebra is an orthoalgebra.*

DEFINITION 7.2.3. A subset M of an effect algebra E is *downward directed* if for every $a, b \in M$ there is an element $c \in M$ such that $c \leq a, b$.

An effect algebra E is *weakly distributive* if $a \wedge b = a \wedge b' = \mathbf{0}$ implies $a = \mathbf{0}$ for every $a, b \in E$.

THEOREM 7.2.4 ([**58**, Theorem 4.2]). *Every weakly distributive orthomodular poset with the maximality property is a Boolean algebra.*

LEMMA 7.2.5. *Every projection-atomic effect algebra is weakly distributive.*

LEMMA 7.2.6. *The set of upper bounds of a set of atoms in a projection-atomic effect algebra with the maximality property is downward directed.*

LEMMA 7.2.7. *Every element in a projection-atomic effect algebra is a minimal upper bound of the set of atoms it dominates. Every projection-atomic effect algebra with the maximality property is atomistic.*

LEMMA 7.2.8. *Every projection-atomic effect algebra with the maximality property is an orthomodular poset.*

THEOREM 7.2.9. *Every projection-atomic effect algebra with the maximality property is a Boolean algebra.*

We can replace the maximality property in Theorem 7.2.9 by various stronger properties (see Remark 4.4.5), e.g., by the orthocompleteness. It cannot be replaced by the weak orthocompleteness, as we show by an example.

THEOREM 7.2.10. *Let E be a projection-atomic effect algebra. If a compression base on E for which all atoms are projections has the projection cover property, then E is a Boolean algebra.*

COROLLARY 7.2.11. *Every atomic sequential orthoalgebra is a Boolean algebra.*

The above corollary generalizes similar results obtained by Gudder and Greechie [**31**, Theorem 5.3] and Tkadlec [**59**, Theorems 5.4 and 5.6]. The first mentioned result assumes that the effect algebra is atomistic, the second assumes it has the maximality property and the third assumes it is determined by atoms.

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