On nodal domains, spectral minimal partitions and Aharonov-Bohm hamiltonians

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Given a bounded open set Ω in $\mathbb{R}^n$ (or in a Riemannian manifold) and a partition of Ω by k open sets ω_j, we can consider the quantity $\max_j \lambda(\omega_j)$ where $\lambda(\omega_j)$ is the ground state energy of the Dirichlet realization of the Laplacian in ω_j. If we denote by $\mathfrak{L}_k(\Omega)$ the infimum over all the k-partitions of $\max_j \lambda(\omega_j)$, a minimal k-partition is then a partition which realizes the infimum.
Although the analysis is rather standard when $k = 2$ (we find the nodal domains of a second eigenfunction), the analysis of higher k's becomes non trivial and quite interesting.

In this talk, we consider the two-dimensional case and discuss the properties of minimal spectral partitions, illustrate the difficulties by considering simple cases like the disc, the rectangle, the annulus, or the torus and explore the link with Aharonov-Bohm Hamiltonians. This work has started in collaboration with T. Hoffmann-Ostenhof and has been continued with coauthors : V. Bonnaillie-Noël, T. Hoffmann-Ostenhof, S. Terracini, G. Vial ....

We consider mainly two-dimensional Laplacians operators in bounded domains. We would like to analyze the relations between the nodal domains of the eigenfunctions of the Dirichlet Laplacians and the partitions by k open sets D_i which are minimal in the sense that the maximum over the D_i's of the ground state energy of the Dirichlet realization of the Laplacian in D_i is minimal.

Let Ω be a regular bounded domain. Let $H(\Omega)$ be the Laplacian $-\Delta$ on $\Omega \subset \mathbb{R}^2$ with Dirichlet boundary condition ($u_{\partial\Omega} = 0$). In the case of a Riemannian manifold we will consider the Laplace Beltrami operator.

We denote by $\lambda_j(\Omega)$ the increasing sequence of its eigenvalues and by u_j some associated orthonormal basis of eigenfunctions.

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We define for any $u \in C_0^0(\overline{\Omega})$

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The number of nodal domains of u is called $\mu(u)$. These $k = \mu(u)$ nodal domains define a partition of Ω.

The Courant nodal theorem says :

Theorem [Courant]

Let $k \geq 1$, λ_k be the k-th eigenvalue and $E(\lambda_k)$ the eigenspace of $H(\Omega)$ associated to λ_k. Then, $\forall u \in E(\lambda_k) \setminus \{0\}\ ,\ \mu(u) \leq k$.

Theorem [Pleijel]

There exists k_0 such that if $k \geq k_0$, then

$$\mu(u) < k\,,\ \forall u \in E(\lambda_k) \setminus \{0\}$$

The main points in the proof are the Faber-Krahn Inequality :

$$\lambda(\omega) \geq \frac{\pi j^2}{|\omega|} \ . \tag{2}$$

and the Weyl's law.

Partitions

We first introduce the notion of partition.

Definition 1

Let $1 \leq k \in \mathbb{N}$. We will call **partition** (or k-partition for indicating the cardinal of the partition) of Ω a family $\mathcal{D} = \{D_i\}_{i=1}^k$ of mutually disjoint sets such that

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We call it **open** if the D_i are open sets of Ω, **connected** if the D_i are connected.

We denote by $\mathfrak{O}_k$ the set of open connected partitions.

We now introduce the notion of spectral minimal partition sequence.

Definition 2

For any integer $k \geq 1$, and for $\mathcal{D}$ in $\mathfrak{O}_k$, we introduce

$$\Lambda(\mathcal{D}) = \max_i \lambda(D_i). \qquad (4)$$

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Then we define

$$\mathfrak{L}_k = \inf_{\mathcal{D} \in \mathfrak{O}_k} \Lambda(\mathcal{D}). \tag{5}$$

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Remark A

If $k = 2$, it is rather well known (see [HH1] or [CTV3]) that $\mathfrak{L}_2 = \lambda_2$ and that the associated minimal 2-partition is a nodal partition, i.e. a partition whose elements are the nodal domains of some eigenfunction corresponding to λ_2.

We discuss roughly the notion of regular and strong partition.

Definition 3

A partition $\mathcal{D} = \{D_i\}_{i=1}^{k}$ of Ω in $\mathfrak{O}_k$ is called strong if

$$\mathrm{Int}\,(\overline{\cup_i D_i}) \setminus \partial\Omega = \Omega\,. \tag{6}$$

Attached to a strong partition, we associate a closed set in $\overline{\Omega}$:

Definition 4 : "Boundary set"

$$N(\mathcal{D}) = \overline{\cup_i\,(\partial D_i \cap \Omega)}\,. \tag{7}$$

$N(\mathcal{D})$ plays the role of the nodal set (in the case of a nodal partition).

This leads us to introduce the set $\mathcal{R}(\Omega)$ of regular partitions (or nodal like) through the properties of its associated boundary set N, which should satisfy :

Definition 5

(i) Except finitely many distinct $x_i \in \Omega \cap N$ in the nbhd of which N is the union of $\nu_i(x_i)$ smooth curves ($\nu_i \geq 2$) with one end at x_i, N is locally diffeomorphic to a regular curve.
(ii) $\partial\Omega \cap N$ consists of a (possibly empty) finite set of points z_i. Moreover N is near z_i the union of ρ_i distinct smooth half-curves which hit z_i.
(iii) N has the equal angle meeting property

By equal angle meeting property, we mean that the half curves cross with equal angle at each critical point of N and also at the boundary together with the tangent to the boundary.

Partitions and associated graphs.

We say that D_i, D_j are neighbors or $D_i \sim D_j$, if $D_{i,j} := \mathrm{Int}\,(\overline{D_i \cup D_j}) \setminus \partial\Omega$ is connected.

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We recall that the graph associated to a collection of nodal domains of an eigenfunction is always bipartite.

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Here are two examples.

Figure 1

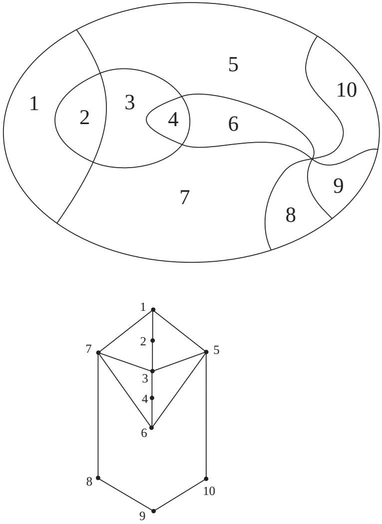

Figure: An example of regular strong bipartite partition with associated graph.

Figure 2

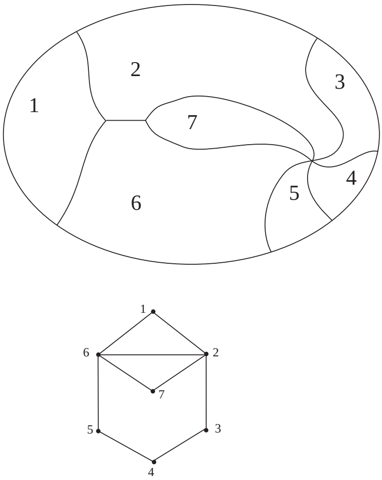

Figure: An example of regular strong nonbipartite partition with associated graph.

Main results

It has been proved by Conti-Terracini-Verzini [CTV1, CTV2, CTV3] and Helffer–Hoffmann-Ostenhof–Terracini [HHOT1] that

Theorem 1

For any k, there exists a minimal regular k-partition. Moreover any minimal k-partition has a regular representative.

Main results

It has been proved by Conti-Terracini-Verzini [CTV1, CTV2, CTV3] and Helffer–Hoffmann-Ostenhof–Terracini [HHOT1] that

Theorem 1

For any k, there exists a minimal regular k-partition. Moreover any minimal k-partition has a regular representative.

Other proofs of a somewhat weaker version of this statement have been given by Bucur-Buttazzo-Henrot [BBH], Caffarelli- F.H. Lin [CL].

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We have first the following converse theorem ([HH1], [HHOT1]) :

Theorem 2

If the graph of the minimal partition is bipartite this is a nodal partition.

A natural question is now to determine how general is the previous situation.

Surprisingly this only occurs in the so called Courant-sharp situation. We say that

Definition 6

u is Courant-sharp if

$$u \in E(\lambda_k) \setminus \{0\} \quad \text{and} \quad \mu(u) = k \ .$$

For any integer $k \geq 1$, we denote by L_k the smallest eigenvalue whose eigenspace contains an eigenfunction with k nodal domains. We set $L_k = \infty$, if there are no eigenfunctions with k nodal domains.
In general, one can show, that

$$\lambda_k \leq \mathfrak{L}_k \leq L_k \,. \tag{8}$$

The last result gives the full picture of the equality cases :

Theorem 3

Suppose $\Omega \subset \mathbb{R}^2$ is regular.
If $\mathfrak{L}_k = L_k$ or $\mathfrak{L}_k = \lambda_k$ then

$$\lambda_k = \mathfrak{L}_k = L_k \,.$$

In addition, one can find in $E(\lambda_k)$ a Courant-sharp eigenfunction.

This answers a question in [BHIM] (Section 7).

Examples of k-minimal partitions for special domains

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The case of a rectangle

Using Theorem 3, it is now easier to analyze the situation for the disk or for rectangles (at least in the irrational case), since we have just to check for which eigenvalues one can find associated Courant-sharp eigenfunctions.

In the case of the square, it is not to difficult to see that $\mathfrak{L}_3$ is strictly less than L_3. We observe indeed that λ_4 is Courant-sharp, so $\mathfrak{L}_4 = \lambda_4$, and there is no eigenfunction corresponding to $\lambda_2 = \lambda_3$ with three nodal domains (by Courant's Theorem).

Restricting to the half-rectangle and assuming that there is a minimal partition which is symmetric with one of the symmetry axes of the square perpendicular to two opposite sides, one is reduced to analyze a family of Dirichlet-Neumann problems.

Numerical computations performed by V. Bonnaillie-Noël and G. Vial lead to a natural candidate for a symmetric minimal partition.
See http://www.bretagne.ens-cachan.fr/math/Simulations/MinimalPartitions/

Figure 3

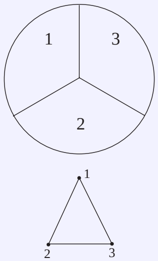
Figure: Trace on the half-square of the candidate for the 3-partition of the square. The complete structure is obtained from the half square by symmetry with respect to the horizontal axis.

This leads (with some success) to analyze the minimal partition with some topological type. If in addition, we introduce some symmetries, this leads to guess some candidates for minimal partitions.

The case of the disk

In the case of the disk, we have no proof that the minimal 3-partition is the "Mercedes star". But if we assume that the minimal 3-partition has a special topological type, then by going on the double covering of the punctured disk or by introducing a suitable Aharonov-Bohm Hamiltonian, one can show that it is indeed the Mercedes star.

The logo Mercedes and associated graph



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In the case of the square, we have no proof that the candidate described by Figure 3 is a minimal 3-partition.

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But if we assume that the minimal partition has one critical point and has the symmetry, then numerical computations lead to the Figure 3.
Numerics suggest more : the center of the square is the critical point of the partition.

Once this property is accepted, a double covering argument shows that this is the projection of a nodal partition on the covering. This point of view is explored numerically by Bonnaillie-Helffer [BH] and theoretically by Noris-Terracini [NT].

Looking for minimal 5-partitions.

Using the covering approach, we were able (BH+VB) to produce the following candidate for a minimal 5-partition of a specific topological type.

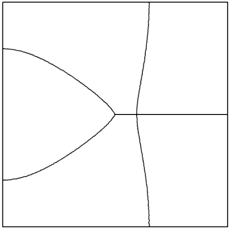

Figure: First candidate for the 5-partition of the square.

It is interesting to compare with other possible topological types of 5-minimal partitions. They can be classified by using Euler formula (see [HH2]). Inspired by numerical computations in [CyBaHo], one looks for a configuration which has the symmetries of the square and four critical points. We get two types of models that we can reduce to a Dirichlet-Neumann problem on a triangle corresponding to the eigth of the square. Moving the Neumann boundary on one side like in [BHV] leads to two candidates. One has a lower energy and coincides with the pictures in [CyBaHo].

$\lambda_{11}(\dot{\Omega}_X^{\mathcal{R}}) = 111.910$ $\quad$ $\lambda_2^{DN} = 104.294$ $\quad$ $\lambda_2^{DN} = 131.666$

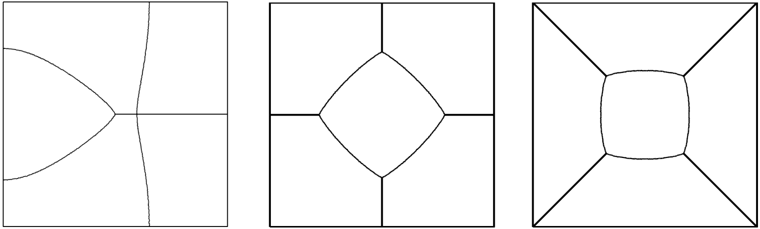

Figure: Three candidates for the 5-partition of the square.

Note that in the case of the disk a similar analysis leads to a different answer. The partition of the disk by five halfrays with equal angle has a lower energy that the minimal 5-partition with four singular points.

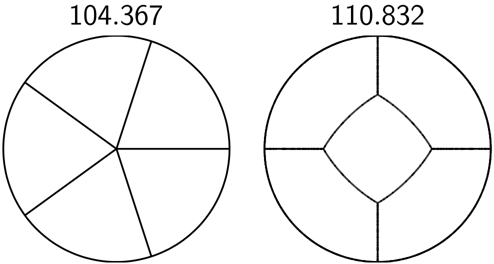

Figure: Two candidates for the 5-partition of the disk.

The Aharonov-Bohm Operator

Let us recall some definitions and results about the Aharonov-Bohm Hamiltonian (for short **AB**X-Hamiltonian) with a singularity at X introduced in [BHHO, HHOO] and motivated by the work of Berger-Rubinstein. We denote by $X = (x_0, y_0)$ the coordinates of the pole and consider the magnetic potential with renormalized flux at X

$$\frac{\Phi}{2\pi} = 1/2$$

$$\mathbf{A}^X(x,y) = (A_1^X(x,y), A_2^X(x,y)) = \frac{1}{2}\left(-\frac{y-y_0}{r^2}, \frac{x-x_0}{r^2}\right). \quad (9)$$

We know that the magnetic field vanishes identically in $\dot{\Omega}_X$. The **AB**X-Hamiltonian is defined by considering the Friedrichs extension starting from $C_0^\infty(\dot{\Omega}_X)$ and the associated differential operator is

$$-\Delta_{\mathbf{A}^X} := (D_x - A_1^X)^2 + (D_y - A_2^X)^2 \text{ with } D_x = -i\partial_x \text{ and } D_y = -i\partial_y. \tag{10}$$

Let K_X be the antilinear operator

$$K_X = e^{i\theta_X}\,\Gamma\,,$$

with $(x - x_0) + i(y - y_0) = \sqrt{|x - x_0|^2 + |y - y_0|^2}\, e^{i\theta_X}$, where Γ is the complex conjugation operator $\Gamma u = \bar{u}$ and

$$d\theta_X = 2\mathbf{A}_X\,.$$

The flux condition shows that $e^{i\theta_X}$ is univalued. Then we have

$$K_X\,\Delta_{\mathbf{A}^X} = \Delta_{\mathbf{A}^X}\,K_X\,.$$

We say that a function u is K_X-real, if it satisfies $K_X u = u$. Then the operator $-\Delta_{\mathbf{A}^X}$ is preserving the K_X-real functions and we can consider a basis of K_X-real eigenfunctions. Hence we only analyze the restriction of the $\mathbf{AB}X$-Hamiltonian to the K_X-real space $L^2_{K_X}$ where

$$L^2_{K_X}(\dot{\Omega}_X) = \{u \in L^2(\dot{\Omega}_X)\,,\ K_X\, u = u\}\,.$$

It was shown that the nodal set of such a K_X real eigenfunction has the same structure as the nodal set of an eigenfunction of the Laplacian except that an odd number of half-lines should met at X.

First we can extend our construction of an Aharonov-Bohm Hamiltonian in the case of a configuration with ℓ distinct points $X_1, \ldots, X_\ell$ (putting a (renormalized) flux $\frac{1}{2}$ at each of these points). We can just take as magnetic potential

$$\mathbf{A}^{\mathbf{X}} = \sum_{j=1}^{\ell} \mathbf{A}^{X_j} \,,$$

where $\mathbf{X} = (X_1, \ldots, X_\ell)$.

We can also construct (see [HHOO]) the antilinear operator $K_{\mathbf{X}}$, where θ_X is replaced by a multivalued-function $\phi_{\mathbf{X}}$ such that $d\phi_X = 2\mathbf{A}^{\mathbf{X}}$ and $e^{i\phi_{\mathbf{X}}}$ is univalued and C^∞. We can then consider the real subspace of the $K_{\mathbf{X}}$-real functions in $L^2_{K_{\mathbf{X}}}(\dot{\Omega}_{\mathbf{X}})$. It has been shown in [HHOO] (see in addition [AFT]) that the $K_{\mathbf{X}}$-real eigenfunctions have a regular nodal set (like the eigenfunctions of the Dirichlet Laplacian) with the exception that at each singular point X_j ($j = 1, \dots, \ell$) an odd number of half-lines should meet. In the case of one singular point, this fact was observed by Berger-Rubinstein [1] for the first eigenfunction. We denote by $L_k(\dot{\Omega}_{\mathbf{X}})$ the lowest eigenvalue (if any) such that there exists a $K_{\mathbf{X}}$-real eigenfunction with k nodal domains.

Toward a magnetic characterization of a minimal partition

We now discuss the following conjecture.

Conjecture

Let Ω be simply connected. Then

$$\mathfrak{L}_k(\Omega) = \inf_{\ell \in \mathbb{N}} \inf_{X_1, \dots, X_\ell} L_k(\dot{\Omega}_{\mathbf{X}}) \,.$$

Let us present a few examples illustrating the conjecture. When $k = 2$, there is no need to consider punctured Ω's. The infimum is obtained for $\ell = 0$. When $k = 3$, it is possible to show that it is enough, to minimize over $\ell = 0$, $\ell = 1$ and $\ell = 2$. In the case of the disk and the square, it is proven that the infimum cannot be for $\ell = 0$ and we conjecture that the infimum is for $\ell = 1$ and attained for the punctured domain at the center. For $k = 5$, it seems that the infimum is for $\ell = 4$ in the case of the square and for $\ell = 1$ in the case of the disk.

Let us explain very briefly why this conjecture is natural.
Considering a minimal k-partition $\mathcal{D} = (D_1, \dots, D_k)$, we know that it has a regular representative and we denote by $X^{odd}(\mathcal{D}) := (X_1, \dots, X_\ell)$ the critical points of the partition corresponding to an odd number of meeting half-lines.
Then the guess is that $\mathfrak{L}_k(\Omega) = \lambda_k(\dot{\Omega}_{\mathbf{X}})$ (Courant sharp situation).
One point to observe is that we have proven in [HHOT1] the existence of a family u_i such that u_i is a groundstate of $H(D_i)$ and $u_i - u_j$ is a second eigenfunction of $H(D_{ij})$ when $D_i \sim D_j$.

The hope is to find a sequence $\epsilon_i(x)$ of $\mathbb{S}^1$-valued functions, where ϵ_i is a suitable[1] square root of $e^{i\phi_{\mathbf{X}}}$ in D_i, such that $\sum_i \epsilon_i(x) u_i(x)$ is an eigenfunction of the **ABX**-Hamiltonian associated with the eigenvalue $\mathfrak{L}_k$.

Conversely, any family of nodal domains of an Aharonov-Bohm operator on $\dot{\Omega}_{\mathbf{X}}$ corresponding to L_k gives a k-partition.

[1] Note that by construction the D_i's never contain any pole.

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