bose–einstein condensation on non homogeneous networks

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May 17, 2011

abstract

We investigate the Bose–Einstein Condensation on nonhomogeneous amenable networks for the model describing arrays of Josephson junctions. The resulting topological model, whose Hamiltonian is the pure hopping one given by the opposite of the adjacency operator, has also a mathematical interest in itself. We show that for the nonhomogeneous networks like the comb graphs, particles condensate in momentum and configuration space as well. In this case different properties of the network, of geometric and probabilistic nature, such as the volume growth, the shape of the ground state, and the transience, all play a rôle in the condensation phenomena. The situation is quite different for homogeneous networks where just one of these parameters, e.g. the volume growth, is enough to determine the

appearance of the condensation. The mathematical aspects of the Bose–Einstein Condensation on some nonamenable networks like the Cayley Trees are also briefly discussed.

The present talk is based on the following paper:
–Fidaleo F., Guido D., Isola T. *Bose-Einstein condensation in inhomogeneous amenable graphs*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **14** (2011)
DOI: 10.1142/S0219025711004389

Some results concerning the mathematical aspects of the BEC described in:
–Fidaleo F. *Harmonic analysis on perturbed Cayley trees*, J. Funct. Anal. **261** (2011), 604–634.
are also briefly discussed

the model

The framework is a sea of *Bardeen–Cooper pairs* in arrays of *Josephson junctions* on a network G: particles are located on vertices VG, and edges EG describe the presence of a Josephson junction.

The Hamiltonian of the system is the *Bose Hubbard Hamiltonian*

$$H_{BH} = m \sum_i n_i + \sum_{i,j} A_{i,j} \Big(V n_i n_j - J_0 a_i^\dagger a_j \Big) . \quad (1)$$

Here, $a_i^\dagger$ is the Bosonic creator, and $n_i = a_i^\dagger a_i$ the number operator on the site i. Finally, A is the adjacency operator whose matrix element $A_{i,j}$ in the place ij is the number of the edges connecting the site i with the site j. When m and V are negligible with respect to J_0, it might be expected that the hopping term dominates the physics of the system. Thus, under this approximation, (1) becomes the quadratic *pure hopping* Hamiltonian given by

$$H_{PH} = -J \sum_{i,j} A_{i,j} a_i^{\dagger} a_j \,, \tag{2}$$

where the constant $J > 0$ is a mean field coupling constant which is in general different from the J_0 appearing in the more realistic Hamiltonian (1).

mathematical aspects

Being the previous Hamiltonian a free (quadratic) one, it is enough to study the selfadjoint operator $-A$ on the one–particle space $\ell^2(VG)$. We put $J_0 = 1$ in (2), and normalizing such that the bottom of the spectrum of the energy is zero. The resulting Hamiltonian for the purely topological model under consideration is

$$H = \|A\| \mathbf{1} - A \,, \tag{3}$$

where A is the adjacency of the fixed graph G, acting on the Hilbert space $\ell^2(VG)$.

The appearance of the BEC is connected with the asymptotics near zero, of the spectrum of the Hamiltonian. For vectors in the spectral subspace near zero, the Taylor expansion for the "Bose occupation function" relative to the *chemical potential* $\mu < 0$ leads to

$$\begin{aligned}\frac{1}{e^{H-\mu\mathbf{1}}-1}\approx&H^{-1}=((\|A\|-\mu)\mathbf{1}-A)^{-1}\\ \equiv&R_A(\|A\|-\mu)\,.\end{aligned}$$

Then the mathematics of the BEC is reduced to the investigation of the spectral properties of the (more familiar object for mathematicians which is the) resolvent $R_A(\lambda)$, for $\lambda \approx \|A\|$.

The non homogeneous graphs we deal with are *density zero additive perturbations* of periodic lattices (Fig 1: the *comb graph* $\mathbb{Z} \dashv \mathbb{Z}$, see also Fig 2: the *star graph*). We also briefly discuss the mathematical aspects of the BEC on homogeneous Cayley trees (Fig 3 and Fig 4).

We are able to find out when the perturbation is sufficiently big to modify the norm of the adjacency of the perturbed graph. Then if it happens, being the perturbation of density zero, it does not modify the cumulative function describing the density of the eigenvalues (called in physics *integrated density of the states*) up to the shift due to the change of the bottom of the spectrum (the norm of the adjacency is changing). Put $\delta := \|A^X\| - \|A^Y\| < 0$ (it has the meaning of a chemical potential, see below), we get for the integrated density of the states of $-A^Y$,

$$F_Y(x) = F_X(x + \delta)\,. \tag{4}$$

This always leads to the *hidden spectrum*, that is the part of the spectrum close to the bottom of the Hamiltonian which does not contribute to the density of the states. The immediate consequence is

$$\rho_c^Y(\beta) = \int \frac{\mathrm{d}F(x)}{e^{\beta(x-\delta))} - 1} = \rho^X(\beta, \delta) < +\infty \,. \quad (5)$$

Namely, **in presence of the hidden spectrum the critical density of the model is always finite** independently on the geometrical dimension of the network.

We are able to write down the formula for the resolvent of the adjacency matrix of the perturbed graph then investigate the *transience character* of the adjacency, that is when

$$\lim_{\lambda \downarrow \|A\|} \langle R_A(\lambda)\delta_x, \delta_x \rangle < +\infty \,,$$

which does not depend on the point $x \in VG$.*

*If the generator of the process is the Laplacian, the transience character is connected with probabilistic properties of the *random walk* on the graph under consideration.

We can prove that the finite volume sequence of the *Perron–Frobenius* eigenvectors, normalized to 1 in a "root", converges pointwise to a (generalized) PF eigenvector for the adjacency.† The surprising fact is that it decays exponentially far away from the perturbed zone of the graph.

All the above properties have a precise physical meaning as we are going to explain.

physical applications

We already explained that the appearance of the hidden spectrum always leads to finite critical density. **it is then possible to find nonhomogeneous networks exhibiting the BEC, even if their geometrical dimension is "small"**.

†When the graph is transient the subspace of Perron–Frobenius eigenvectors might be not one dimensional.

The transience character is connected with the possibility to exhibit *locally normal* states enjoying BEC. A locally normal state ω describes a situation for which the local density of the particles

$$\rho_\Lambda(\omega) := \frac{1}{|\Lambda|}\sum_{j\in\Lambda}\omega(a^\dagger(\delta_j)a(\delta_j))$$

is finite. If the adjacency is recurrent we prove that, for each choice of a sequence of chemical potentials $\mu_{\Lambda_n}\uparrow 0$ for the finite volume *Gibbs grand canonical ensamble* state ω_{Λ_n}, $\Lambda_n\uparrow G$, we get that the two–point function diverges:

$$\lim_n \omega_{\Lambda_n}(a^\dagger(\delta_j)a(\delta_j)) = +\infty\,.$$

Namely, **it is impossible to construct any locally normal state exhibiting BEC if the adjacency is recurrent**. Conversely, in the transient case we are able to construct locally normal states describing BEC.

The PF (generalized) eigenvector is nothing but the (generalized) wave function of the physical ground state.[‡]. Then it describes the distribution of the condensate in the configuration space (due to nonhomogeneity, particle condensate on the network as well). As it exponentially decreases far away to the perturbation, the condensate distribution is well described by the Perron–Frobenius dimension d_{PF}. Consider the ball $\Lambda_n \uparrow G$ of radius n centered in any fixed root of the graph. Consider the Perron–Frobenius eigenvector v, previously described. The *geometrical dimension* d_G of G is defined to be a if $|\Lambda_n| \sim n^a$. The *Perron–Frobenius dimension* $d_{PF}(G)$ of G is defined to be b if $\left\| v\lceil_{\ell^2(\Lambda_n)} \right\| \sim n^{b/2}$.

If the critical density is finite and the graph is transient (condition under which it is possible to exhibit locally normal states describing

[‡]Here "generalized" stands for non normalizable.

BEC), we look at d_{PF}. If $d_{PF} < d_G$ it is impossible to exhibit such states whose particle density

$$\rho(\omega) := \lim_{\Lambda \uparrow G} \rho_\Lambda(\omega)$$

is greater than ρ_c. In this situation **we are able to construct only locally normal states** ω **exhibiting BEC for which** $\rho(\omega) = \rho_c$. If $d_{PF} = d_G$ we can exhibit locally normal state ω describing BEC such that $\rho(\omega) > \rho_c$.

In addition, we remark that, if we use the perturbed Laplacian as the Hamiltonian of the system, it is impossible to change the character of the graph. Indeed the hidden spectrum never happen.

The situation described is quite different from the homogeneous case for which the condensation phenomena are uniquely described by the dimension of the graph.

We end with the description of the properties described above for some pivotal example of the graphs under consideration.

finite additive perturbations (see Fig 1): finite critical density (provided the perturbation is sufficiently big to modify the norm of the adjacency), recurrent (as the PF eigenvector is normalizable), $d_{PF} = 0$.

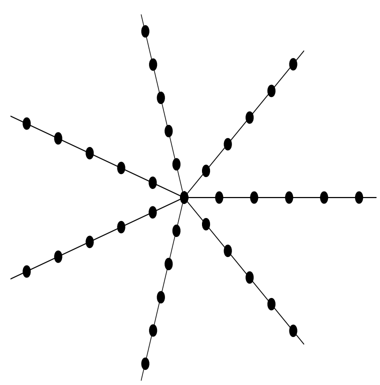

Fig 1: the star graph.

Comb graphs $G^d := \mathbb{Z}^d \dashv \mathbb{Z}$ (see fig): finite critical density, recurrent if and only if $d \leq 2$, $d = d_{PF} < d_G = d + 1$.

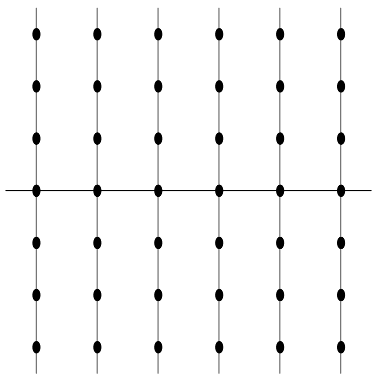

Fig 2: the comb graph $\mathbb{Z} \dashv \mathbb{Z}$.

$\mathbb{N}$:
infinite critical density, transient, $3 = d_{PF} > 1$.

Comb graphs $H^2 := \mathbb{N} \dashv \mathbb{Z}^2$:
finite critical density, transient, $3 = d_{PF} = d_G$.

The mathematical aspects of the BEC are extended to exponentially growing graphs such as the perturbed Cayley tree.

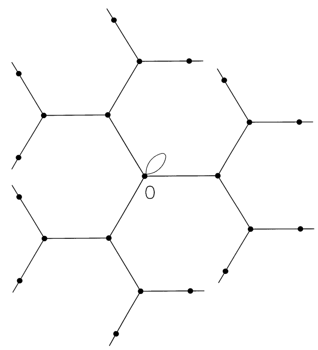

Fig 3: finite perturbation of the Cayley Tree of order 3.

It is recurrent and PF–"0" dimensional.

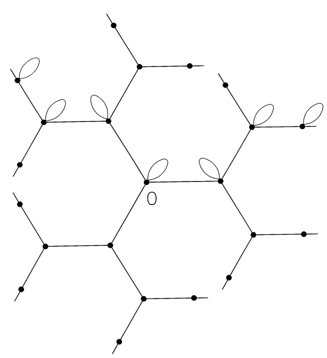

Fig 4: perturbation of the Cayley Tree of order 3 along $\mathbb{Z}$.

It is recurrent and PF–"1" dimensional.

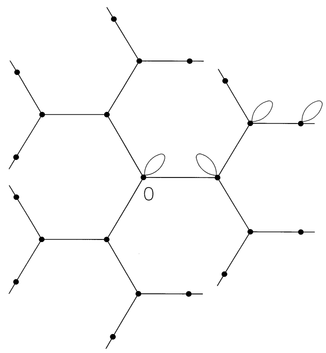

Fig 4: perturbation of the Cayley Tree of order 3 along $\mathbb{N}$.
It is transient and PF–"3" dimensional.

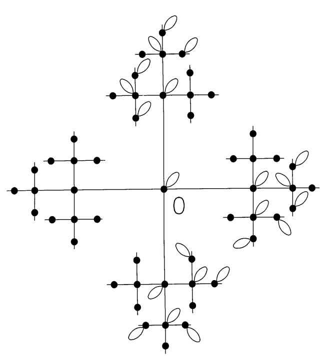

Fig 5: perturbation of the Cayley Tree of order 4 along a Cayley subtree of order 3.
It is transient and has the same PF–behavior as the basepoint.