harmonic analysis on perturbed Cayley Trees and the bose einstein condensation

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May 11, 2011

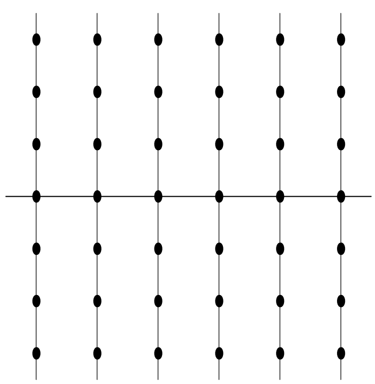
introduction

We study some spectral properties of the adjacency operator of non homogeneous networks. The graphs under investigation are obtained by adding density zero perturbations to the homogeneous Cayley Trees. Apart from the natural mathematical meaning, such spectral properties are relevant for the Bose Einstein Condensation for the pure hopping model describing a sea of Bardeen–Cooper pairs in arrays of Josephson junctions on non homogeneous networks. The resulting topological model is described by a one particle Hamiltonian which is, up to an additive constant, the opposite of the adjacency operator on the graph. It is known that the Bose Einstein condensation already occurs for unperturbed homogeneous Cayley Trees. However, the particles condensate even in the configuration space, because of the nonhomogeneity. Even if the graphs under consideration are exponentially growing, we show that it is enough to perturb in a negligible way the original homogeneous graph, in order to obtain a new network whose mathematical and physical properties dramatically change. Among such results, we mention the following ones. The appearance of the *Hidden Spectrum* near the zero of the Hamiltonian, or equivalently below the norm of the adjacency. The latter is related to the value of the critical density and then with the appearance of the condensation phenomena. The investigation of the *recurrence/transience character* of the adjacency, which is connected to the possibility to construct locally normal states exhibiting the Bose Einstein condensation. Finally, the study of the *volume growth of the wave function of the ground state* of the Hamiltonian, which is nothing but a suitable generalized Perron Frobenius eigenvector of the adjacency. This Perron Frobenius weight describes the spatial distribution of the condensate and its shape is connected with the possibility to construct locally normal states exhibiting the Bose Einstein condensation at a fixed density greater than the critical one.

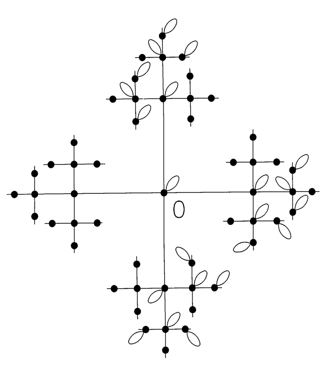
The present talk is based on the following paper:
–Fidaleo F. *Harmonic analysis on perturbed Cayley trees*, J. Funct. Anal. **261** (2011), 604–634.
Some results from the previous paper:
–Fidaleo F., Guido D., Isola T. *Bose-Einstein condensation in inhomogeneous amenable graphs*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **14** (2011)
DOI: 10.1142/S0219025711004389
are also briefly discussed

the model

The framework is a sea of *Bardeen–Cooper pairs* in arrays of *Josephson junctions* on a network G: particles are located on vertices VG, and edges EG describe the presence of a Josephson junction. The networks consist of *density zero* additive perturbations of homogeneous ones



The comb graph $G_1 = \mathbb{Z}^1 \dashv \mathbb{Z}$



$\mathbb{G}^{4,3}$: The Cayley Tree $\mathbb{G}^4$ perturbed along $\mathbb{G}^3$

The Hamiltonian of the system is the *Bose Hubbard Hamiltonian*

$$H_{BH} = m\sum_i n_i + \sum_{i,j} A_{i,j}\Big(Vn_in_j - J_0 a_i^\dagger a_j\Big)\,. \quad (1)$$

Here, $a_i^\dagger$ is the Bosonic creator, and $n_i = a_i^\dagger a_i$ the number operator on the site i. Finally, A is the adjacency operator whose matrix element $A_{i,j}$ in the place ij is the number of the edges connecting the site i with the site j. When m and V are negligible with respect to J_0, the hopping term dominates the physics of the system. Thus, under this approximation, (1) becomes the quadratic *pure hopping* Hamiltonian given by

$$H_{PH} = -J\sum_{i,j} A_{i,j} a_i^\dagger a_j\,, \quad (2)$$

where the constant $J > 0$ is a mean field coupling constant which might be different from the J_0 appearing in the more realistic Hamiltonian (1).

mathematical aspects

The previously described model is a free theory (the Hamiltonian (2) is quadratic). Then it is enough to study the selfadjoint operator $-A$ on the one–particle space $\ell^2(VG)$. We put $J_0 = 1$ in (2), and normalize such that the bottom of the spectrum of the energy is zero. The resulting Hamiltonian for the purely topological model under consideration is

$$H = \|A\|\mathbf{1} - A\,, \tag{3}$$

where A is the adjacency of the fixed graph G, acting on the Hilbert space $\ell^2(VG)$.

The appearance of the BEC is connected with the asymptotics, close to zero, of the spectrum of the Hamiltonian. For free Bosonic models, mathematically described by the Canonical Commutation Relations, most of the physical relevant quantities are computed by using the functional calculus of suitable functions of the Hamiltonian. The critical density (cf. (4)) is one of them. But, the asymptotic behavior of the Hamiltonian (3) near zero corresponds to the asymptotic of the spectrum of A close to $\|A\|$. Indeed, by using the Taylor expansion, we heuristically get for the function appearing in the Bose Gibbs occupation number for the chemical potential $\mu < 0$ at small energies,

$$\frac{1}{e^{H-\mu\mathbf{1}}-1} \approx (H-\mu\mathbf{1})^{-1}$$
$$=((\|A\|-\mu)\mathbf{1}-A)^{-1} \equiv R_A(\|A\|-\mu)\,.$$

By following the previous considerations, it is proved that the mathematics of the BEC is reduced to the investigation of the behavior of the more familiar object for mathematicians, the resolvent $R_A(\lambda)$, for $\lambda \approx \|A\|$.

By following the lines of the previous paper (Fidaleo F., Guido D., Isola T.: Bose Einstein condensation on inhomogeneous amenable graphs, cited above), the non homogeneous graphs we deal with are density zero additive perturbations of homogeneous Cayley trees. The emerging results are quite surprising even if the graphs under consideration are exponentially growing, and even when the additive perturbation is only finite.

hidden spectrum

The appearance of the hidden spectrum is the combination of two opposite phenomena arising from the perturbation. If the perturbation is sufficiently big (even if in many cases it is enough a finite perturbation), the norm $\|A_p\|$ of the adjacency of the perturbed graph becomes bigger than the analogous one $\|A\|$ of the unperturbed adjacency. On the other hand, as the perturbation is sufficiently small (i.e. density zero), the part of the spectrum $\sigma(A_p)$ in the segment $(\|A\|, \|A_p\|]$ does not contribute to the density of the states. This allows us to compute any function of the perturbed adjacency by using the integrated density of the states F of the unperturbed one. For example, we get for the critical density $\rho_c(\beta)$ at the inverse temperature β for the perturbed model,

$$\rho_c(\beta) = \int \frac{\mathrm{d}F_X(x)}{e^{\beta(x+(\|A_p\|-\|A\|))} - 1}\,. \tag{4}$$

The resulting effect on the critical density of the perturbed model exhibiting the hidden spectrum (i.e. when $\|A_p\| - \|A\| > 0$) is that it is always finite. This is because

$$F_Y(x) = F_X(x+\delta)\,, \tag{5}$$

being F_X, F_Y be the integrated density of the states of the adjacency and the perturbed adjacency, respectively, and $\delta := \|A^X\| - \|A^Y\| < 0$.*

*δ has the meaning of a chemical potential, see (4).

Notice that in presence of the hidden spectrum the critical density of the model is always finite independently on the geometrical dimension of the network.†

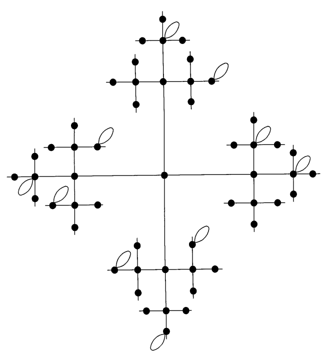
for finite additive perturbation, in order to check if the adjacency exhibits hidden spectrum it is enough to find out whether $\|A_p\| > \|A\|$

This can be done by solving the *secular equation* (see FGI).

To simplify the computations, we deal with perturbations by self loops. On the other hand,

†The linearized part of (1) has the form $-\Delta + V(x)$, where Δ is the discrete Laplacian given by $\Delta = A - D$, with $D_{i,j} := (\deg i)\delta_{i,j}$. Notice that $-\Delta$ is positive but not positive preserving, whereas A is not positive but positive preserving. They differ by a diagonal term which is constant for homogeneous graphs. Thus, when the graph is not homogeneous they are completely different operators. The possibility to have BEC shall depend on V. For example, if $V = 0$ we can show (cf. FGI) that we cannot have hidden spectrum.

it is expected (cf. FGI) that our simplified model captures all the qualitative phenomena appearing in more complicated examples relative to general additive negligible perturbations.



(finite) additive perturbations by self loops

In our situation, secular equation is written as

$$\|P_{\ell^2(S)}R_{A_{\mathbb{G}^Q}}(\lambda)P_{\ell^2(S)}\| = 1\,, \tag{6}$$

where $S \in \mathbb{G}^Q$ is the density zero set of vertices where are localized the self loops. As $\|S(\lambda)\|$ is decreasing in $\lambda \in (\|A\|, +\infty)$, where

$$S(\lambda) := P_{\ell^2(S)}R_{A_{\mathbb{G}^Q}}(\lambda)P_{\ell^2(S)}\,,$$

(6) has at most one solution which is proven to be precisely $\|A_p\|$. In addition, $S(\lambda)$ is analytic for $\lambda > \lambda_*$ where λ_* is such a solution of (6) and allows us to write the explicit formula for $R_{A_p}(\lambda)$.[‡]

transience character

By using the explicit formula for the resolvent of the perturbed adjacency, we are able to investigate the *transience character* of the adjacency, that is when

$$\lim_{\lambda \downarrow \|A\|} \langle R_A(\lambda)\delta_x, \delta_x \rangle < +\infty \,,$$

which does not depend on the point $x \in VG$.[§] In our situation, the matter is reduced to the

[‡]By using the Neumann series, it is shown that the Krein formula for the perturbed resolvent can be then analytically continued on the complex plane for $\|\lambda\| > \lambda_*$.

[§]If the generator of the process is the Laplacian, the transience character is connected with probabilistic properties of the *random walk* on the graph under consideration.

investigation of the limit when $\lambda\downarrow\lambda_*$ of

$$\left\langle R_{A_p}(\lambda)\delta_0,\delta_0\right\rangle=\left\langle S(\lambda)\Big(\mathbf{I}_{\ell^2(S)}-S(\lambda)\Big)^{-1}\delta_0,\delta_0\right\rangle.$$

The above limit can be computed, via the Spectral Mapping, by using Complex Analysis and then the Residue Theorem in all the cases under consideration.¶

The transience character is connected to the possibility to exhibit *locally normal* states enjoying BEC. A locally normal state ω describes a situation for which the local density of the particles

$$\rho_\Lambda(\omega):=\frac{1}{|\Lambda|}\sum_{j\in\Lambda}\omega(a^\dagger(\delta_j)a(\delta_j))$$

is finite. If the adjacency is recurrent it is expected that, for each choice of a sequence of

¶It is straightforward for $\mathbb{G}^{Q,2}$ (Fourier Analysis by using the Poisson Kernel). It is quite complicated for $\mathbb{H}^Q$ (same techniques as before). It is heavy (Harmonic Analysis on Cayley trees: Figà–Talamanca, Picardello book) for the last situation $\mathbb{G}^{Q,q}$ considered here.

chemical potentials $\mu_{\Lambda_n} \uparrow 0$ for the finite volume *Gibbs grand canonical ensemble* state ω_{Λ_n}, $\Lambda_n \uparrow G$, we get that the two–point function diverges:

$$\lim_n \omega_{\Lambda_n}(a^\dagger(\delta_j)a(\delta_j)) = +\infty\,.$$

Namely, **it is impossible to construct any locally normal state exhibiting BEC if the adjacency is recurrent**. Conversely, in the transient case we are able to construct locally normal states describing BEC. The last result holds true in the amenable cases treated in the FGI article cited above.

the Perron Frobenious weight

Let B be a bounded matrix with positive entries acting on $\ell^2(VX)$. Such an operator is called *positive preserving* as it preserves the elements of $\ell^2(VX)$ with positive entries. A sequence $\{v(x)\}_{x\in VX}$ is called a *(generalized) Perron Frobenius eigenvector* (or equivalently Perron Frobenius weight) if it has positive entries and

$$\sum_{y\in VX} B_{xy}v(y) = \|B\|v(x)\,, \quad x \in VX\,.$$

For finite additive perturbations the (ℓ^2) Perron Frobenius (normalized at 1 on a fixed root) can be explicitly written (cf. FGI). If $S \in \mathbb{G}^Q$ is a finite connected set supporting the perturbation by self loops, we get

$$v_S(x) = a(\lambda_S)^{d(x,S)} w_S(y(x))\,.$$

Here,

$$a(\lambda) := \frac{1 - \sqrt{1 - \frac{4(Q-1)}{\lambda^2}}}{\frac{2(Q-1)}{\lambda}}\,, \tag{7}$$

λ_S is the unique solution of the secular equation, w_S is the unique Perron Frobenius eigenvector for the convolution operator T_a^S on S

by the function $f_a := a^{d(\,\cdot\,,0)}$ (0 is a fixed root on S), and finally $y(x)$ is the unique nearest element of S to x.‖

Consider an infinite, connected and density zero set $S \in \mathbb{G}^Q$, together with the elements $S_n := S \bigcap B_n$ of S in the ball of the radius n centered in a fixed root 0. As $a(\lambda_{S_n}) \to a(\lambda_S)$, we can prove that

$$v_{S_n}(x) \to a(\lambda_S)^{d(x,S)} w_S(y(x))\,,$$

provided the sequence $\{w_{S_n}\}$ of the Perron Frobenius eigenvectors of the convolution operator $T_a^{S_n}$ converges to a Perron Frobenius weight of T_a^S.** We show that this is the case for all the situation under consideration.

‖Notice that $S(\lambda)$ appearing in the secular equation is expressed in terms of such a convolution operator.

**Notice that such a Perron Frobenius weight is unique if T_a^S is recurrent.

We can prove that the finite volume sequence of the *Perron–Frobenius* eigenvectors, normalized to 1 in a "root", converges pointwise to a (generalized) PF eigenvector for the adjacency.[††] The surprising fact is that it decays exponentially far away from the perturbed zone of the graph.

The PF (generalized) eigenvector is nothing but the (generalized) wave function of the physical ground state.[‡‡]. Then it describes the distribution of the condensate in the configuration space (due to nonhomogeneity, particle condensate on the network as well). As it exponentially decreases far away to the perturbation (even for the exponentially growing networks under consideration here), the condensate distribution is concentrated nearby the

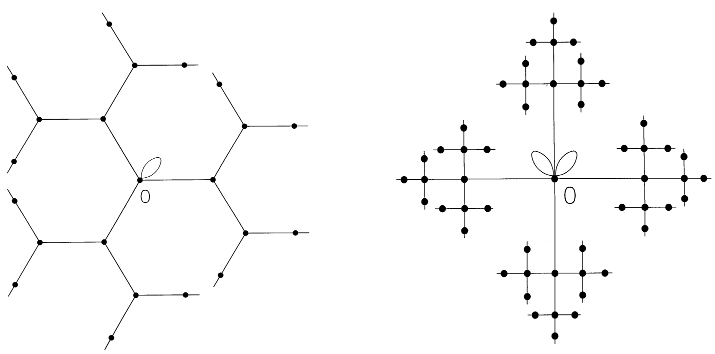
[††] When the graph is transient the set of Perron–Frobenius eigenvectors might be not unique.

[‡‡] Here "generalized" stands for non normalizable.

base space S supporting the perturbation. Such results are in accordance with the other amenable models previously considered in FGI.

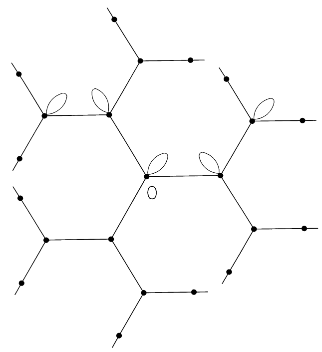
the graphs under consideration

We briefly describe the networks under considerations.



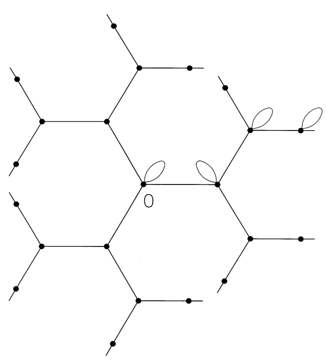
the perturbation of $\mathbb{G}^3$ and $\mathbb{G}^4$ by self loops in a fixed root

When $Q = 3$ it is enough only one self loop, for $Q = 4$ we need at least two. Such graphs are recurrent and the Perron Frobenius weight is normalizable and unique.



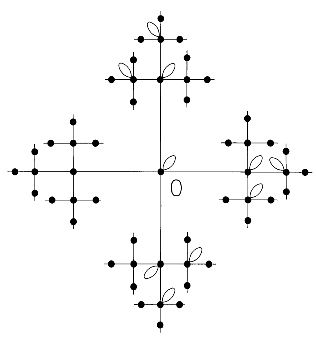
$\mathbb{G}^{3,2}$: the perturbation of $\mathbb{G}^3$ along $S \sim \mathbb{Z}$

Such a graph is recurrent, then the Perron Frobenius weight is unique.



$\mathbb{H}^3$: the perturbation of $\mathbb{G}^3$ along $S \sim \mathbb{N}$

Such a graph is transient. For the case $\mathbb{G}^{Q,2}$ and $\mathbb{H}^Q$, we get hidden spectrum for $2 < Q < 8$.



$\mathbb{G}^{4,3}$: the perturbation of $\mathbb{G}^4$ along $S \sim \mathbb{G}^3$

For such graphs, after fixing $q \geq 2$ we can compute $Q(q) \geq q$ such that $\mathbb{G}^{Q,q}$ exhibits hidden spectrum. Such graphs are transient.