Perturbations of Brownian motion: existence, uniqueness and representations

Mihai N. Pascu

Simion Stoilow Institute of Mathematics of the Romanian Academy

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A strong solution of (1) on a given probability space $(\Omega, \mathcal{F}, P)$ with respect to a fixed Brownian motion B_t and with initial condition ξ is a continuous process $(X_t)_{t\geq 0}$ with the following properties:

i) X_t is adapted to the augmented filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by B and ξ;

ii) $P(X_0 = \xi) = 1$;

iii) $P\left(\int_0^t \left|\sigma^2(X_s)\right| + |b(X_s)|\, ds < \infty\right) = 1$ holds for every $t \geq 0$.

iv) X_t satisfies (1) for every $t \geq 0$.

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- iv) X_t satisfies (1) for every $t \geq 0$.

A weak solution of (1) is a triple (X, W), $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_t)_{t\geq 0}$, where

- i) $(\Omega, \mathcal{F}, P)$ is a probability space and $(\mathcal{F}_t)_{t\geq 0}$ is a filtration on $\mathcal{F}$;
- ii) X_t is a continuous $\mathcal{F}_t$-adapted $\mathbb{R}$-valued process;
- iii) B_t is a $\mathcal{F}_t$-adapted 1-dimensional Brownian motion starting at 0;
- iv) (1) and $P\left(\int_0^t \left|\sigma^2(X_s)\right| + |b(X_s)|\, ds < \infty\right) = 1$ hold for every $t \geq 0$.

Remarks

The main difference between the two notions is that the strong solutions require the measurability of X_t with respect to the filtration $\mathcal{F}^B$ of the driving Brownian motion B_t, whereas the weak solution requires only the measurability of X_t and B_t with respect to some filtration (not necessarily $\mathcal{F}^B$).

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The value $X_t(\omega)$ of a strong solution at time $t \geq 0$ is completely determined (a measurable functional) of the path $\{B_s(\omega), 0 \leq s \leq t\}$ and the value of the initial condition $\xi(\omega)$.

Weak and strong uniqueness

We say that strong uniqueness holds for (1) if whenever B_t is a Brownian motion on some probability space $(\Omega, \mathcal{F}, P)$ and $X_t, \widetilde{X}_t$ are two strong solutions relative to B_t and with the same initial condition ξ, then

$$P\left(X_t = \widetilde{X}_t \quad \text{for} \quad t \geq 0\right) = 1.$$

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We say that weak uniqueness holds for (1) if whenever (X_t, B_t), $(\Omega, \mathcal{F}, P)$, $(\mathcal{F}_t)_{t \geq 0}$ and $\left(\widetilde{X}_t, \widetilde{B}_t\right)$, $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P}\right)$, $\left(\widetilde{\mathcal{F}}_t\right)_{t \geq 0}$ are two weak solutions with the same initial distribution (i.e. $P(X_0 \in \Gamma) = P\left(\widetilde{X}_0 \in \Gamma\right)$ for any $\Gamma \in \mathcal{B}(\mathbb{R})$), the processes X and $\widetilde{X}$ have the same law.

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Note that strong existence implies weak existence, and strong uniqueness implies weak uniqueness (Yamada and Watanabe, 1971). Also, weak existence and strong uniqueness implies strong existence.

Results on weak existence and uniqueness

Consider the stochastic differential equation

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dB_s, \qquad t \geq 0, \tag{2}$$

and define the zero set of σ by

$$Z(\sigma) = \{x \in \mathbb{R} : \sigma(x) = 0\}, \tag{3}$$

and the non-local integrability set of σ^{-2} by

$$I(\sigma) = \left\{ x \in \mathbb{R} : \int_{-\varepsilon}^{\varepsilon} \frac{dy}{\sigma^2(x+y)} = \infty, \ \forall \varepsilon > 0 \right\}. \tag{4}$$

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Theorem (Engelbert and Schmidt, 1984)

The equation (2) has a weak solution if and only if $I(\sigma) \subset Z(\sigma)$.
Moreover, the solution is weakly unique if and only if $I(\sigma) = Z(\sigma)$.

Results on strong existence and uniqueness

Consider the SDE

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dB_s + \int_0^t b(X_s)\, ds, \qquad t \geq 0, \tag{5}$$

and the following hypotheses:

(A) There exists an increasing function $\rho : [0, \infty) \to [0, \infty)$ with $\int_{0+} \frac{du}{\rho(u)} = +\infty$ such that

$$(\sigma(x) - \sigma(y))^2 \leq \rho(|x - y|), \qquad x, y \in \mathbb{R}. \tag{6}$$

(B) There exists an increasing, bounded function $f : \mathbb{R} \to \mathbb{R}$ such that

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Theorem (Le Gall, 1983)

Suppose that σ and b are bounded measurable functions which satisfy one of the following hypotheses:

i) *σ satisfies (A) and b is Lipschitz;*

ii) *σ satisfies (A) and there exists $\varepsilon > 0$ such that $|\sigma| \geq \varepsilon$;*

iii) *σ satisfies (B) and there exists $\varepsilon > 0$ such that $\sigma \geq \varepsilon$.*

Then the solution of (5) is pathwise unique.

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Remark: condition (A) does not allow for jump discontinuities of σ, and (B) can only be used if σ is bounded below away from 0, so there is a GAP between the weak and the strong uniqueness.
What can be said about uniqueness in this case?

φ-strong solution and φ-strong uniqueness of SDE

The *principle of causality* for dynamical systems shows that a SDE can be thought as a machinery which given the “input” B_t and the initial condition ξ produces the “output” X_t. The fact that the SDE does not have a strong solution shows that X_t cannot be determined from the input B_t.

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Definition

Consider $\varphi : \mathbb{R} \to \mathbb{R}$ a measurable function.
A φ-strong solution to (1) on a probability space $(\Omega, \mathcal{F}, P)$ with respect to the Brownian motion B_t is a continuous process X_t for which (1) holds a.s., and such that $\varphi(X_t)$ is adapted to the augmented filtration generated by B_t and $P\left(\int_0^t |b(X_s)| + \sigma^2(X_s)\, ds < \infty\right) = 1$ holds for every $t \geq 0$.

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We say that φ-strong uniqueness holds for (1) if whenever X_t and $\widetilde{X}_t$ are two φ-strong solutions relative to the same driving Brownian motion B_t, then

$$P\left(\varphi\left(X_t\right) = \varphi(\widetilde{X}_t), \quad t \geq 0\right) = 1.$$

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If φ is an injective function, φ-strong solution/φ-strong uniqueness is the same as the classical notion of strong solution/strong uniqueness.

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In the previous definition, the process X_t is not required to be adapted with respect to the filtration of the Brownian motion B_t. If X_t is adapted with respect to the σ-algebra generated by $\sigma\left(B_s : s \leq t\right) \cup \mathcal{G}_t$, then the σ-algebra $\mathcal{G}_t$ can be viewed as the extra source of randomness needed to predict the output of the solution of (1).

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The above notion of solution/uniqueness interpolates between the classical notions of weak and strong solution, and is meant to show the amount of information that can be uniquely determined from the SDE.

A classical example

Consider the (singular) SDE

$$X_t = \int_0^t \operatorname{sgn}(X_s)\, dB_s, \qquad t \geq 0. \tag{7}$$

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Also, since $\operatorname{sgn}^{-2}(x)$ is everywhere locally integrable and $\operatorname{sgn}(x)$ has no zeroes, by the results of Engelbert and Schmidt, (7) has a weak solution and the solution is weakly unique.

Weak, but not strong existence and uniqueness

First note that if $\left(X, B, (\mathcal{F}_t)_{t\geq 0}\right)$ is a weak solution for (7), then X_t is a continuous, square integrable martingale with quadratic variation $\langle X\rangle_t = t$, so by Lévy's theorem X_t is a Brownian motion, and therefore weak uniqueness holds for (7).

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To prove the existence of a weak solution of (7), consider a 1-dimensional Brownian motion X_t, and define $B_t = \int_0^t \operatorname{sgn}(X_s)\, dX_s$, so B_t is a 1−dimensional Brownian motion.
Then $\int_0^t \operatorname{sgn}(X_s)\, dB_s = \int_0^t (\operatorname{sgn}(X_s))^2\, dX_s = X_t$, so $\left(X_t, B_t, (\mathcal{F}_t)_{t\geq 0}\right)$ is a weak solution of (7).

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If (7) had a strong solution X_t (so $\mathcal{F}_t^X \subset \mathcal{F}_t^B$ for all $t \geq 0$), then $B_t = \int_0^t \operatorname{sgn}(X_s)\, dX_s$, so $\mathcal{F}_t^B \subset \mathcal{F}_t^X$, and therefore $\mathcal{F}_t^X = \mathcal{F}_t^B$.
By Tanaka formula we obtain $B_t = \int_0^t \operatorname{sgn}(X_s)\, dX_s = |X_t| - L_t^0$, where
$L_t^0 = \lim_{\varepsilon\searrow 0} \frac{1}{2\varepsilon}\operatorname{meas}\{s\in[0,t] : |X_s| \leq \varepsilon\}$ is the local time of X at the origin, so $\mathcal{F}_t^B \subset \mathcal{F}_t^{|X|}$, which leads to a contradiction: $\mathcal{F}_t^X \subset \mathcal{F}_t^{|X|}$.
The contradiction shows that (7) does not have a strong solution.

$|x|$-strong uniqueness for (7)

If X_t verifies (7), by the Tanaka formula we have

$$|X_t| = \int_0^t \operatorname{sgn}(X_s)\, dX_s + L_t^0 = \int_0^t (\operatorname{sgn}(X_s))^2\, dB_s + L_t^0 = B_t + L_t^0, \qquad t \geq 0.$$

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By classical results on the pathwise uniqueness of reflecting Brownian motion on $[0, \infty)$, $|X_t|$ is pathwise unique (it is the reflecting Brownian motion on $[0, \infty)$ with driving Brownian motion B_t).

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To prove the existence of a $|x|$-strong solution, we introduce the notion of sign choice of a process.

Sign choice for a process

Definition

Given a non-negative continuous process $(Y_t)_{t\geq 0}$, a sign choice for Y_t is a process $(U_t)_{t\geq 0}$ taking the values ± 1, such that $(U_t Y_t)_{t\geq 0}$ is a continuous process.

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Remark (Construction of a sign choice)

Given a non-negative continuous process $(Y_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, P)$, consider a process $(V_t)_{t\geq 0}$ taking the values ± 1 with probability $P(V_t = 1) = 1 - P(V_t) = p \in [0,1]$. The process $(U_t)_{t\geq 0}$ defined by $U_t = V_t$ if $Y_t = 0$ and $U_t = U_s$ if $Y_t \neq 0$, where $s = \sup\{u \leq t : Y_u = 0\}$, is a sign choice for Y_t.

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Remark

If U_t is a sign choice for Y_t, then U_t is constant on each open connected component of $\{t \geq 0 : Y_t \neq 0\}$, and the process $U_t Y_t$ is obtained from Y_t by flipping with probability $1 - p$ the excursions of Y_t away from zero.

$|x|$-strong existence and uniqueness for (7)

Theorem

$|x|$-strong uniqueness holds for (7).
A $|x|$-strong solution (and a weak solution) of (7) exists, and its is explicitly given by $\left(U_t Y_t, B_t, (\mathcal{F}_t)_{t\geq 0}\right)$, where Y_t is the reflecting Brownian motion on $[0,\infty)$ with driving Brownian motion B_t, U_t is a sign choice for Y_t taking the values ± 1 with equal probability, and $(\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by B_t and U_t which satisfies the usual conditions.
Conversely, any weak solution $\left(X_t, B_t, (\mathcal{F}_t)_{t\geq 0}\right)$ of (7) has the representation $X_t = U_t Y_t$, where U and Y are as above.

A singular SDE

Consider the SDE

$$X_t = \int_0^t \sigma_{a,b}(X_s)\, dB_s, \qquad t \geq 0, \tag{8}$$

where $\sigma_{a,b}(x) = \begin{cases} a, & x \geq 0 \\ b, & x < 0 \end{cases}$, and $a, b \in \mathbb{R}^*$ are arbitrary constants.

Theorem

For any $a > 0 > b$, $\varphi_{a,b}$-strong uniqueness holds for (8), where $\varphi_{a,b}(x) = \begin{cases} \frac{1}{a}x, & x \geq 0 \\ \frac{1}{b}x, & x < 0 \end{cases}$.
A $\varphi_{a,b}$-strong solution (and a weak solution) of (8) exists, and it is explicitly given by $\left(\sigma_{ab}(U_t)\, Y_t, B_t, (\mathcal{F}_t)_{t\geq 0}\right)$, where Y_t is the reflecting Brownian motion on $[0, \infty)$ with driving Brownian motion B_t, U_t is a sign choice for Y_t taking the values 1 and -1 with probabilities $\frac{-b}{a-b}$ respectively $\frac{a}{a-b}$, and $(\mathcal{F}_t)_{t\geq 0}$ is the augmentation of the filtration generated by B and U which satisfies the usual conditions.
Conversely, any weak solution $\left(X_t, B_t, (\mathcal{F}_t)_{t\geq 0}\right)$ of (8) has the representation $X_t = \sigma_{ab}(U_t)\, Y_t$, where U and Y are as above.

Proof.

(φ-strong uniqueness) If X_t satisfies (8), applying the Tanaka-Itô formula to the function $\varphi_{a,b}$ and to the process X_t we obtain

$$Y_t := \varphi_{a,b}(X_t) = \int_0^t \varphi'_{a,b}(X_s)\, dX_s + \frac{1}{2}\left(\varphi'_{a,b}(0+) - \varphi'_{a,b}(0-)\right) L_t^0(X) = B_t + \frac{1}{2}\left(\frac{1}{a} - \frac{1}{b}\right) L_t^0(X),$$

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It can be shown that $L_t^0(Y) = \frac{1}{2}\left(\frac{1}{a} - \frac{1}{b}\right) L_t^0(X)$, so the process Y_t verifies the SDE

$$Y_t = B_t + L_t^0(Y), \qquad t \geq 0,$$

that is, Y_t is the reflecting Brownian motion on $[0, \infty)$ with driving Brownian motion B_t.

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(**φ-strong existence**) It can be shown that $X_t = \sigma_{a,b}(U_t)\, Y_t$ is a weak solution of (8), by using the representation

$$\begin{aligned}\int_0^t \sigma_{a,b}(X_s)\, dB_s - X_t \quad &= \quad -\varepsilon \sum_{i=1}^{D_t(\varepsilon)} \sigma_{a,b}\left(U_{\sigma_i}\right) + \sigma_{a,b}(U_t)\, Y_t \sum_{i\geq 1} 1_{[\tau_{i-1},\sigma_i)}(t) \\ &\quad -\varepsilon\sigma_{a,b}(U_t) \sum_{i\geq 1} 1_{[\sigma_i,\tau_i)}(t) + \sum_{i\geq 1} \int_{\tau_{i-1}\wedge t}^{\sigma_i\wedge t} \sigma_{a,b}(U_s)\, 1_{R^*}(X_s)\, dY_s,\end{aligned} \tag{9}$$

and by showing that all the terms on the right converge in L^2 to zero as $\varepsilon \searrow 0$ (Levy's characterization of local time, Wald's second identity, aso). □

Extensions

Theorem

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a measurable ***odd*** *function on $\mathbb{R}^*$ such that $|\sigma|$ is bounded above and below by positive constants and suppose there exists a strictly increasing function f on $\mathbb{R}$ such that*

$$(\sigma(x) - \sigma(y))^2 \leq |f(x) - f(y)|, \qquad x, y \in \mathbb{R}. \tag{10}$$

Given a 1-dimensional Brownian motion B_t, $|x|$-strong uniqueness holds for the SDE

$$X_t = \int_0^t \sigma(X_s)\, dB_s, \qquad t \geq 0. \tag{11}$$

A $|x|$-strong solution (and a weak solution) exists, and is given by $(U_t Y_t, B_t, \mathcal{F}_t)$ where Y_t is the pathwise unique solution to

$$Y_t = \int_0^t |\sigma(Y_s)|\, dB_s + L_t^0(Y), \qquad t \geq 0,$$

and U_t is a sign choice for Y_t taking the values ± 1 with equal probability.
Moreover, any weak solution has the representation $X_t = U_t Y_t$ where U_t and Y_t are as above.

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